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Asymptotic results for the two-parameter Poisson–Dirichlet distribution

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Abstract

The two-parameter Poisson–Dirichlet distribution is the law of a sequence of decreasing nonnegative random variables with total sum one. It can be constructed from stable and gamma subordinators with the two parameters, α and θ, corresponding to the stable component and the gamma component respectively. The moderate deviation principle is established for the distribution when θ approaches infinity, and the large deviation principle is established when both α and θ approach zero.

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1. Introduction

For α in $(0, 1)$ and $\theta > -\alpha$, let $U_k, k = 1, 2, \ldots$, be a sequence of independent random variables such that U_k has $Beta(1-\alpha, \theta+k\alpha)$ distribution. Set

$$X_1^{\alpha,\theta} = U_1, \qquad X_n^{\alpha,\theta} = (1-U_1)\cdots(1-U_{n-1})U_n, \quad n \geq 2. \tag{1.1}$$

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Then with probability one

$$\sum_{k=1}^{\infty} X_k^{\alpha,\theta} = 1,$$

and the law of $(X_1^{\alpha,\theta}, X_2^{\alpha,\theta}, \ldots)$ is called the two-parameter GEM distribution. The law of the descending order statistic $\mathbf{P}(\alpha,\theta) = (P_1(\alpha,\theta), P_2(\alpha,\theta), \ldots)$ of $(X_1^{\alpha,\theta}, X_2^{\alpha,\theta}, \ldots)$ is called the two-parameter Poisson–Dirichlet distribution and is denoted by $\Pi_{\alpha,\theta}$.

The two-parameter Poisson–Dirichlet distribution is a natural generalization to Kingman's one-parameter Poisson–Dirichlet distribution which corresponds to $\alpha = 0$. Many properties of the one-parameter Poisson–Dirichlet distribution have generalizations in the two-parameter setting including but not limited to the sampling formula (cf. [8,17]), the Markov–Krein identity (cf. [7,20]), and subordinator representation (cf. [13,18]), and large deviations (cf. [3,9]). A comprehensive study of the two-parameter Poisson–Dirichlet distribution is found in Pitman and Yor [18] including relations to subordinators, Markov chains, Brownian motion and Brownian bridges. The detailed calculations of moments and parameter estimations were carried out in Carlton [2]. In [6] and the references therein one can find connections between two-parameter Poisson–Dirichlet distribution and models in physics including mean-field spin glasses, random map models, fragmentation, and returns of a random walk to origin. The two-parameter Poisson–Dirichlet distribution has also been used in macroeconomics and finance [1].

The objective of this article is to obtain the two-parameter generalizations to results in [11,10] including the moderate deviation principle and large deviation principle (henceforth, MDP and LDP). The methods used here are similar to that used in [11,10]. The main differences and complications are in the structure of the density function and the subordinator representation where the independency is replaced by exchangeability. Thus the exponential moment is obtained through a combination of Campbell's theorem and the de Finetti type representation obtained in [17].

The paper is organized as follows. Section 2 includes several preliminary distributional results. Two MDPs are obtained in Section 3 when θ goes to infinity. The LDP is established in Section 4 when both α and θ go to zero. Concepts such as local LDP and partial LDP are used as defined in Definition 2.1 in [11] and Definition 2.2 in [3]. The reference [5] is our main source for general theory and techniques on large deviations.

2. Preliminaries

This section begins with the definitions of LDP and MDP. Afterwards we collect several existing results, slightly reformulated to our setting, on the relationship between subordinators and the two-parameter Poisson–Dirichlet distribution. These are then used in deriving the marginal distributions of the two-parameter Poisson–Dirichlet distribution.

Definition 2.1. Let E be a Polish space, and $\{X_\theta : \theta > 0\}$ be a family of E-valued random variables. The law of X_θ is denoted by P_θ. The family of probability measures $\{P_\theta : \theta > 0\}$ (or the family $\{X_\theta : \theta > 0\}$) is said to satisfy a LDP with speed $\lambda(\theta)$ and rate function $I(\cdot)$, if for any closed set F and open set G in E

$$\limsup_{\theta\to\infty} \lambda(\theta) \log P_\theta(F) \le -\inf_{x\in F} I(x),$$
$$\liminf_{\theta\to\infty} \lambda(\theta) \log P_\theta(G) \ge -\inf_{x\in G} I(x),$$
$$\text{for any } c > 0,\ \{x : I(x) \le c\} \quad \text{is compact.}$$

Remark. The speed defined here is reciprocal of the one in [5].

Definition 2.2. Let $\{X_\theta : \theta > 0\}$ be a family of random variables satisfying that there are functions $b(\theta) > 0$, $c(\theta)$, and a finite non-deterministic random variable X such that

$$\lim_{\theta\to\infty} b(\theta) = \infty,$$

and $b(\theta)[X_\theta - c(\theta)]$ converges to Z in distribution as θ tends to infinity. Let $a(\theta)$ satisfy

$$\lim_{\theta\to\infty} a(\theta) = \infty, \qquad \lim_{\theta\to\infty} \frac{a(\theta)}{b(\theta)} = 0.$$

The family $\{P_\theta : \theta > 0\}$ or equivalently the family $\{X_\theta : \theta > 0\}$ is said to satisfy a MDP with speed $\lambda(\theta)$ (depending on $a(\theta)$) and rate function $I(\cdot)$ if the family $\{a(\theta)[X_\theta - c(\theta)] : \theta > 0\}$ satisfies a LDP with speed $\lambda(\theta)$ and rate function $I(\cdot)$. Thus the MDP for $\{X_\theta : \theta > 0\}$ is the LDP for $\{a(\theta)[X_\theta - c(\theta)] : \theta > 0\}$.

A subordinator $\{\rho_s, s \geq 0\}$ is an increasing stochastic process with stationary independent increment. If the drift component is zero, then the Laplace transform of ρ_s is given by

$$\mathbb{E}\left(\exp(-\lambda\rho_s)\right) = \exp\left\{s\int_0^\infty (\mathrm{e}^{-\lambda x} - 1)\Lambda(\mathrm{d}x)\right\}, \quad \lambda \geq 0, \tag{2.1}$$

where Λ is the Lévy measure on $(0, +\infty)$ describing the distribution of the jump sizes. All subordinators considered in this paper have zero drift component.

For any $t > 0$, let $V_1^\rho(t) \geq V_2^\rho(t) \geq \cdots$ denote the jump sizes of $\{\rho_s, s \geq 0\}$ over the interval $[0, t]$ in decreasing order. Clearly,

$$\rho_t = \sum_{i=1}^\infty V_i^\rho(t). \tag{2.2}$$

If

$$\Lambda(\mathrm{d}x) = c_\alpha x^{-(1+\alpha)}\mathrm{d}x.$$

for some $c_\alpha > 0$, then the subordinator is called a stable subordinator with index α and is denoted by $\{\tau_s, s \geq 0\}$. Without loss of generality, we choose $c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$ in this paper, where $\Gamma(\cdot)$ is the gamma-function. The gamma subordinator, denoted by $\{\gamma_s : s \geq 0\}$, has Lévy measure

$$\Lambda(\mathrm{d}x) = x^{-1}\mathrm{e}^{-x}\mathrm{d}x, \quad x > 0.$$

For $\theta > 0$, set

$$\zeta(\alpha, \theta) = \gamma_{\theta/\alpha}, \tag{2.3}$$

$$T = T(\alpha, \theta) = \tau_{\zeta(\alpha,\theta)}, \tag{2.4}$$

and

$$V_i(T) = V_i^\tau(\zeta(\alpha, \theta)), \quad i \geq 1. \tag{2.5}$$

For $n \geq 1$, set

$$C_{\alpha,\theta} = \frac{\Gamma(\theta + 1)}{\Gamma\left(\frac{\theta}{\alpha} + 1\right)}, \tag{2.6}$$

$$C_{\alpha,\theta,n} = \frac{\Gamma(\theta+1)\Gamma\left(\frac{\theta}{\alpha}+n\right)\alpha^{n-1}}{\Gamma(\theta+n\alpha)\Gamma\left(\frac{\theta}{\alpha}+1\right)\Gamma(1-\alpha)^n}. \tag{2.7}$$

Theorem 2.1. *Assume that* $\{\tau_s, s \geq 0\}$ *and* $\{\gamma_s, s \geq 0\}$ *are independent.*
(1) *The law of the random sequence*

$$\left(\frac{V_1^\tau(1)}{\tau_1}, \frac{V_2^\tau(1)}{\tau_1}, \ldots\right)$$

is $\Pi_{\alpha,0}$*;*
(2) *The random sequence*

$$\left(\frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \ldots\right)$$

has the two-parameter Poisson–Dirichlet distribution $\Pi_{\alpha,\theta}$ *as the law, and is independent of* T*, which has a* $Gamma(\theta, 1)$ *distribution.*
(3) *For any nonnegative measurable function* f *on the product space* $[0, 1]^\infty$*,*

$$\mathbb{E}^{\Pi_{\alpha,\theta}}[f(P_1, P_2, \ldots)] = C_{\alpha,\theta}\mathbb{E}\left[\tau_1^{-\theta} f\left(\frac{V_1^\tau(1)}{\tau_1}, \frac{V_2^\tau(1)}{\tau_1}, \ldots\right)\right]. \tag{2.8}$$

Proof. Part (1) is obtained in [16]. Part (2) is proposition 21 in [18]. The result in part (3) is essentially Corollary 3.15 in [16]. □

The next result appears as Theorem 5.4 in [12] and we give a different proof here.

Theorem 2.2. *For each* $\beta > 0$ *and any* p*, define*

$$G_{\alpha,\beta}(p) = P\left(P_1(\alpha, \beta) \leq p\right). \tag{2.9}$$

Then for any $n \geq 1$*, the joint density function of* $(P_1(\alpha,\theta), \ldots, P_n(\alpha,\theta))$ *is given by*

$$g_{\alpha,\theta,n}(p_1, \ldots, p_n) = C_{\alpha,\theta,n} \frac{\left(1 - \sum_{i=1}^n p_i\right)^{\theta+n\alpha-1}}{\left(\prod_{i=1}^n p_i\right)^{1+\alpha}} G_{\alpha,\theta+n\alpha}\left(\frac{p_n}{1 - \sum_{i=1}^n p_i}\right), \tag{2.10}$$

where $0 < p_n < \cdots < p_1, \sum_{i=1}^n p_i < 1$.

Proof. It follows from Perman's formula (cf. [15]) that for any $n > 1$, the joint density function of $\left(\tau_1, \frac{V_1^\tau(1)}{\tau_1}, \ldots, \frac{V_n^\tau(1)}{\tau_1}\right)$ is given by

$$\phi_n(t, p_1, \ldots, p_n) = (c_\alpha)^{n-1} \hat{p}_n^{-1} (p_1 \cdots p_{n-1})^{-(1+\alpha)} t^{-(n-1)\alpha} \phi_1\left(t\hat{p}_n, p_n/\hat{p}_n\right) \tag{2.11}$$

where

$$\hat{p}_n = 1 - p_1 - \cdots - p_{n-1}, \tag{2.12}$$

and $\phi_1(t, u)$ satisfies

$$\phi_1(t, u) = c_\alpha t^{-\alpha} u^{-(1+\alpha)} \int_0^{\frac{u}{1-u}\wedge 1} \phi_1(t(1-u), v)\mathrm{d}v. \tag{2.13}$$

Putting together (2.8), (2.11) and (2.13), and integrating out the t coordinate, it follows that

$$\begin{aligned}
&g_{\alpha,\theta,n}(p_1,\ldots,p_n)\\
&= C_{\alpha,\theta}(c_\alpha)^{n-1}\hat{p}_n^{-1}(p_1\cdots p_{n-1})^{-(1+\alpha)}\int_0^\infty t^{-(\theta+(n-1)\alpha)}\phi_1\left(t\hat{p}_n, p_n/\hat{p}_n\right)\mathrm{d}t\\
&= C_{\alpha,\theta}(c_\alpha)^{n-1}\hat{p}_n^{\theta+(n-1)\alpha-2}(p_1\cdots p_{n-1})^{-(1+\alpha)}\int_0^\infty s^{-(\theta+(n-1)\alpha)}\phi_1\left(s, p_n/\hat{p}_n\right)\mathrm{d}s\\
&= C_{\alpha,\theta}(c_\alpha)^{n}\frac{\hat{p}_n^{\theta+n\alpha-1}}{(p_1\cdots p_{n-1}p_n)^{(1+\alpha)}}\int_0^{\frac{p_n}{\hat{p}_{n+1}}\wedge 1}\mathrm{d}x\int_0^\infty s^{-(\theta+n\alpha)}\phi_1\left(s\left(1-p_n/\hat{p}_n\right), x\right)\mathrm{d}s\\
&= C_{\alpha,\theta}(c_\alpha)^{n}\frac{(\hat{p}_{n+1})^{\theta+n\alpha-1}}{(p_1\cdots p_{n-1}p_n)^{(1+\alpha)}}\int_0^{\frac{p_n}{\hat{p}_{n+1}}\wedge 1}\mathrm{d}x\int_0^\infty u^{-(\theta+n\alpha)}\phi_1(u,x)\mathrm{d}u\\
&= \frac{C_{\alpha,\theta}(c_\alpha)^{n}}{C_{\alpha,\theta+n\alpha}}\frac{(\hat{p}_{n+1})^{\theta+n\alpha-1}}{(p_1\cdots p_{n-1}p_n)^{(1+\alpha)}}G_{\alpha,\theta+n\alpha}\left(\frac{p_n}{1-\sum_{i=1}^n p_i}\right),
\end{aligned}\tag{2.14}$$

which leads to (2.10). □

Theorem 2.3. *For any $s > 0$,*

$$\begin{aligned}
F_{\alpha,\theta}(s) &= P(V_1(T)\le s)\\
&= \left(1+c_\alpha\int_s^\infty z^{-(1+\alpha)}\mathrm{e}^{-z}\mathrm{d}z\right)^{-\theta/\alpha}\\
&= \left(1+c_\alpha s^{-\alpha}\int_1^\infty z^{-(1+\alpha)}\mathrm{e}^{-sz}\mathrm{d}z\right)^{-\theta/\alpha}.
\end{aligned}\tag{2.15}$$

Proof. For each $s > 0$, it follows from Theorem 2.1 and the property of the Poisson random measure that

$$\begin{aligned}
F_{\alpha,\theta}(s) &= \mathbb{E}(P(V_1(T)\le s|\zeta(\alpha,\theta)))\\
&= \mathbb{E}\left(\exp\left\{-c_\alpha\zeta(\alpha,\theta)\int_s^\infty x^{-(\alpha+1)}\mathrm{e}^{-x}\mathrm{d}x\right\}\right)\\
&= \mathbb{E}\left(\exp\left\{-c_\alpha\gamma_{\theta/\alpha}s^{-\alpha}\int_1^\infty z^{-(\alpha+1)}\mathrm{e}^{-sz}\mathrm{d}z\right\}\right)
\end{aligned}$$

from which (2.15) follows. □

3. MDPs for large θ

Assume $\theta > 0$ in this section and let

$$\beta(\alpha,\theta) = \log\theta - (\alpha+1)\log\log\theta - \log\Gamma(1-\alpha). \tag{3.1}$$

Let $\xi_1 \ge \xi_2 \ge \cdots$ denote the points of a Poisson random measure with intensity measure given by

$$\mathrm{e}^{-x}\mathrm{d}x, \quad x\in\mathbb{R}.$$

It is known (cf. [12]) that, as θ tends to infinity, $\theta\mathbf{P}(\alpha,\theta) - \beta(\alpha,\theta)(1,1,\ldots)$ converges in distribution to $(\xi_1, \xi_2, \ldots)$, and for $m \geq 2$

$$\sqrt{\theta}\left[\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(\mathbf{P}(\alpha,\theta)) - 1\right] \Rightarrow Z_{\alpha,m},$$

where

$$H_m(\mathbf{P}(\alpha,\theta)) = \sum_{i=1}^{\infty} P_i(\alpha,\theta)^m$$

is the homozygosity of order m and $Z_{\alpha,m}$ is a normal random variable with mean zero and variance

$$\sigma^2_{\alpha,m} = \frac{\Gamma(2m-\alpha)\Gamma(1-\alpha)}{\Gamma(m-\alpha)^2} + \alpha - m^2.$$

In this section we establish the MDPs associated with these limiting results.

3.1. MDP for the two-parameter Poisson–Dirichlet distribution

Let $a(\theta)$ satisfy

$$\lim_{\theta\to\infty} \frac{a(\theta)}{\theta} = 0, \qquad \lim_{\theta\to\infty} a(\theta) = \infty. \tag{3.2}$$

It is clear that

$$\lim_{\theta\to\infty} a(\theta)\left(\mathbf{P}(\alpha,\theta) - \frac{\beta(\alpha,\theta)}{\theta}(1,1,\ldots)\right) \to (0,0,\ldots). \tag{3.3}$$

The LDP (cf. definition 2.1 in [11]) associated with (3.3) is called the MDP for $\mathbf{P}(\alpha,\theta)$ and will be established in this subsection. We start with the MDP for $P_1(\alpha,\theta)$.

Lemma 3.1. *The family* $\{a(\theta)\left(P_1(\alpha,\theta) - \frac{\beta(\alpha,\theta)}{\theta}\right) : \theta > 0\}$ *satisfies a LDP on* $\mathbb{R}$ *as* θ *converges to infinity with speed* $\frac{a(\theta)}{\theta}$ *and rate function*

$$I_1(x) = \begin{cases} x, & x \geq 0 \\ \infty, & \text{otherwise}. \end{cases}$$

Proof. By Theorem 2.1, $P_1(\alpha,\theta)$ has the same distribution as $V_1(T)/T$. Thus we only need to establish the MDP for $V_1(T)/T$. By direct calculation,

$$\begin{aligned} a(\theta)\left(\frac{V_1(T)}{T} - \frac{\beta(\alpha,\theta)}{\theta}\right) &= a(\theta)\left(\frac{V_1(T) - \beta(\alpha,\theta) + \beta(\alpha,\theta)}{T} - \frac{\beta(\alpha,\theta)}{\theta}\right) \\ &= \frac{\theta}{T}a(\theta)\left(\frac{V_1(T)}{\theta} - \frac{\beta(\alpha,\theta)}{\theta}\right) + \frac{a(\theta)\beta(\alpha,\theta)}{\theta}\left(\frac{\theta}{T} - 1\right). \end{aligned}$$

It follows from Lemma 2.1 and Corollary 3.1 in [11] that for any $\delta > 0$,

$$\limsup_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left|\frac{V_1(T)}{T} - \frac{V_1(T)}{\theta}\right| \geq \delta\right) = -\infty.$$

This shows that $a(\theta)(\frac{V_1(T)}{T} - \frac{\beta(\alpha,\theta)}{\theta})$ and $a(\theta)(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta})$ are exponentially equivalent. Therefore it suffices to establish the MDP for $V_1(T)/\theta$. First consider x satisfying

$$\lim_{\theta\to\infty}\left[\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right]=+\infty. \tag{3.4}$$

Clearly (3.4) holds for all nonnegative x. It may also hold for negative x depending on the growth rates of $\theta/a(\theta)$ and $\beta(\alpha,\theta)$. For any $s>0$, set

$$M(s)=\left(\int_s^\infty z^{-(1+\alpha)}\mathrm{e}^{-z}\mathrm{d}z\right)^{-1},\qquad N(s)=s^{1+\alpha}\mathrm{e}^{s}.$$

Then it is clear that

$$\lim_{s\to\infty}\frac{N(s)}{M(s)}=1. \tag{3.5}$$

Choosing θ large enough so that $\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)>0$, then it follows from (2.15) that

$$\begin{aligned}
P\left\{a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\le x\right\} &= F_{\alpha,\theta}\left(\frac{\theta}{a(\theta)x+\beta(\alpha,\theta)}\right)\\
&=\left[\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\right)^{M(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta))}\right]^{-\theta/\left(\alpha M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)\right)},
\end{aligned}\tag{3.6}$$

which, combined with (3.5), implies that

$$\begin{aligned}
&\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\le x\right)\\
&=\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\log\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\right)^{M(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta))}\\
&\quad-\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\frac{\theta}{\alpha N\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\\
&=-\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\frac{\theta(\log\theta)^{1+\alpha}\Gamma(1-\alpha)}{\alpha\theta\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)^{1+\alpha}\mathrm{e}^{\frac{\theta}{a(\theta)}x}}\\
&=\begin{cases}0, & x\ge 0\\ -\infty, & x<0.\end{cases}
\end{aligned}\tag{3.7}$$

If (3.4) fails, then x must be strictly negative. If there exists a subsequence θ' such that the $\lim_{\theta'\to\infty}(\frac{\theta'}{a(\theta')}x+\beta(\alpha,\theta'))=\infty$, then the above argument shows that

$$\lim_{\theta'\to\infty}\frac{a(\theta')}{\theta'}\log P\left(a(\theta')\left(\frac{V_1(T)-\beta(\alpha,\theta')}{\theta'}\right)\le x\right)=-\infty.$$

It remains to consider the case of

$$\limsup_{\theta\to\infty}\left[\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right]<\infty.$$

Since, by Theorem 2.3, $V_1(T)$ converges to infinity as θ converges to infinity, it follows that

$$\lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) \le x\right) = -\infty. \tag{3.8}$$

Putting all these together, we obtain that

$$\lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) \le x\right) = 0, \quad x \ge 0, \tag{3.9}$$

and

$$\limsup_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) \le x\right) = -\infty, \quad x < 0. \tag{3.10}$$

For $x \ge 0$, it follows from (2.15) that

$$\begin{aligned}
&\lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) \ge x\right)\\
&= \lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log\left(1-\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\right)^{-\theta/\alpha}\right)\\
&= \lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log\left(\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\right)^{\theta/\alpha}-1\right)\\
&= \lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log\left(\frac{c_\alpha\theta}{\alpha}N^{-1}\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)\right)\\
&= \lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log\left(\left(\frac{\log\theta}{\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)}\right)^{1+\alpha} \mathrm{e}^{-\frac{\theta}{a(\theta)}x}\right) = -x.
\end{aligned} \tag{3.11}$$

Putting (3.11) and (3.10) together, we obtain that for any $M > 0$ one can find a compact set $K_M = [-M, M]$ such that

$$\limsup_{\theta\to\infty} \frac{a(\theta)}{\theta} \log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) \notin K_M\right) \le -M. \tag{3.12}$$

Hence, the family $\{a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right) : \theta > 0, \alpha \in (0,1)\}$ is exponentially tight.

For any $s > 0$, the density function of $V_1(T)$ is given by

$$F'_{\alpha,\theta}(s) = \frac{\theta c_\alpha}{\alpha N(s)}\left(1+\frac{c_\alpha}{M(s)}\right)^{-(1+\theta/\alpha)}.$$

By arguments similar to those used in (3.7) and (3.11), we obtain that for $x > 0$

$$\lim_{\theta\to\infty} \frac{a(\theta)}{\theta} \log \frac{\theta c_\alpha}{\alpha N\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)} = -x, \tag{3.13}$$

and

$$\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\log\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}x+\beta(\alpha,\theta)\right)}\right)^{-(1+\theta/\alpha)}=0. \tag{3.14}$$

Choosing $\delta>0$ so that $x-\delta>0$. Since for any y in $[x-\delta,x+\delta]$,

$$\begin{aligned}&F'_{\alpha,\theta}\left(\frac{\theta}{a(\theta)}y+\beta(\alpha,\theta)\right)\\&\quad\le\frac{\theta c_\alpha}{\alpha N\left(\frac{\theta}{a(\theta)}(x-\delta)+\beta(\alpha,\theta)\right)}\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}(x+\delta)+\beta(\alpha,\theta)\right)}\right)^{-(1+\theta/\alpha)}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&F'_{\alpha,\theta}\left(\frac{\theta}{a(\theta)}y+\beta(\alpha,\theta)\right)\\&\quad\ge\frac{\theta c_\alpha}{\alpha N\left(\frac{\theta}{a(\theta)}(x+\delta)+\beta(\alpha,\theta)\right)}\left(1+\frac{c_\alpha}{M\left(\frac{\theta}{a(\theta)}(x-\delta)+\beta(\alpha,\theta)\right)}\right)^{-(1+\theta/\alpha)},\end{aligned} \tag{3.16}$$

it follows that

$$\begin{aligned}-x-\delta&\le\liminf_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\in(x-\delta,x+\delta)\right)\\&\le\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\in[x-\delta,x+\delta]\right)\\&\le-x+\delta.\end{aligned} \tag{3.17}$$

The equality (3.9) combined with (3.10) implies that

$$\begin{aligned}&\lim_{\delta\to0}\liminf_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\in(-\delta,\delta)\right)\\&=\lim_{\delta\to0}\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left(a(\theta)\left(\frac{V_1(T)-\beta(\alpha,\theta)}{\theta}\right)\in[-\delta,\delta]\right)\\&=0.\end{aligned} \tag{3.18}$$

This combined with (3.10) and (3.17) implies that the family

$$\left\{a(\theta)\left(P_1(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\right):\theta+\alpha>0,\alpha\in(0,1)\right\}$$

satisfies a local LDP. By Theorem (P) in Pukhalskii [19], the exponential tightness (3.12) leads to a partial LDP for the family. The local LDP combined with the partial LDP implies the result. $\square$

Theorem 3.2. *For each $n\ge1$, the family*

$$\left\{a(\theta)\left(P_1(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta},\ldots,P_n(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta},\ldots\right):\theta>0\right\}$$

satisfies a LDP on $\mathbb{R}^\infty$ with speed $\frac{a(\theta)}{\theta}$ and rate function

$$I(x_1, x_2, \ldots) = \begin{cases} \sum_{i=1}^{\infty} x_i, & x_1 \geq \cdots \geq 0 \\ \infty, & \textit{otherwise}. \end{cases} \tag{3.19}$$

Proof. Since $\mathbb{R}^\infty$ can be identified with the projective limit of $\mathbb{R}^n$, $n = 1, \ldots$, by Theorem 3.3 in [4], it suffices to verify that for each $n \geq 2$, the family

$$\left\{ a(\theta) \left(P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \cdots, P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) : \theta > 0 \right\}$$

satisfies a LDP on $\mathbb{R}^n$ with speed $\frac{a(\theta)}{\theta}$ and rate function

$$I_n(x_1, \ldots, x_n) = \begin{cases} \sum_{i=1}^{n} x_i, & \text{if } 0 \leq x_n \leq \cdots \leq x_1. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.20}$$

For any $x_1 \geq x_2 \cdots \geq x_n$, $\frac{\theta}{a(\theta)} x_n + \beta(\alpha, \theta) > 0$, we can choose θ large enough so that

$$\frac{x_i}{a(\theta)} + \frac{\beta(\alpha, \theta)}{\theta} > 0, \quad i = 1, \ldots, n,$$

and

$$\sum_{i=1}^{n} \left(\frac{x_i}{a(\theta)} + \frac{\beta(\alpha, \theta)}{\theta} \right) < 1.$$

Then it follows from the linear transformation and Theorem 2.2 that the density function

$$h_{\alpha,\theta,n}(x_1, \ldots, x_n)$$

of

$$a(\theta) \left(P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \cdots, P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right)$$

is given by

$$\begin{aligned} & a(\theta)^{-n} g_{\alpha,\theta,n} \left(\frac{x_1}{a(\theta)} + \frac{\beta(\alpha, \theta)}{\theta}, \ldots, \frac{x_n}{a(\theta)} + \frac{\beta(\alpha, \theta)}{\theta} \right) \\ & \quad = \left(\frac{1}{a(\theta)} \right)^n C_{\alpha,\theta,n} \left(\prod_{i=1}^{n} \left(\frac{\theta}{\frac{\theta}{a(\theta)} x_i + \beta(\alpha, \theta)} \right)^{\alpha+1} \right) \\ & \qquad \times \left(1 - \left(\frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n\beta(\alpha, \theta) \right) \Big/ \theta \right)^{\theta+n\alpha-1} \\ & \qquad \times G_{\alpha,\theta+n\alpha} \left(\frac{\frac{\theta}{a(\theta)} x_n + \beta(\alpha, \theta)}{\theta - \left(\frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n\beta(\alpha, \theta) \right)} \right). \end{aligned} \tag{3.21}$$

Since $\beta(\alpha,\theta)/\theta$ is a decreasing function of θ, it follows from direct calculation that for $x_n > 0$

$$
\begin{aligned}
&G_{\alpha,\theta+n\alpha}\left(\frac{\frac{\theta}{a(\theta)}x_n+\beta(\alpha,\theta)}{\theta-\left(\frac{\theta}{a(\theta)}\sum_{i=1}^n x_i+n\beta(\alpha,\theta)\right)}\right) \geq G_{\alpha,\theta+n\alpha}\left(\frac{x_n}{a(\theta)}+\frac{\beta(\alpha,\theta)}{\theta}\right)\\
&\quad\geq G_{\alpha,\theta+n\alpha}\left(\frac{x_n}{a(\theta+n\alpha)}\frac{a(\theta+n\alpha)}{a(\theta)}+\frac{\beta(\alpha,\theta+n\alpha)}{\theta+n\alpha}\right)\\
&\quad= P\left(a(\theta+n\alpha)\left(P_1(\alpha,\theta+n\alpha)-\frac{\beta(\alpha,\theta+n\alpha)}{\theta+n\alpha}\right)\leq\frac{a(\theta+n\alpha)}{a(\theta)}x_n\right).
\end{aligned}
$$

Since

$$\frac{a(\theta+n\alpha)}{a(\theta)}x_n > \frac{x_n}{2}$$

for θ large enough, applying the MDP result in Theorem 2.3 to $P_1(\alpha,\theta+n\alpha)$, we obtain that for $x_n > 0$

$$\frac{a(\theta)}{\theta}\log G_{\alpha,\theta+n\alpha}\left(\frac{\frac{\theta}{a(\theta)}x_n+\beta(\alpha,\theta)}{\theta-\left(\frac{\theta}{a(\theta)}\sum_{i=1}^n x_i+n\beta(\alpha,\theta)\right)}\right)\to 0.$$

For $x_n < 0$, set

$$\psi(x_1,\ldots,x_n;\theta,\alpha)=a(\theta)\left(\frac{\frac{\theta}{a(\theta)}x_n+\beta(\alpha,\theta)}{\theta-\left(\frac{\theta}{a(\theta)}\sum_{i=1}^n x_i+n\beta(\alpha,\theta)\right)}-\frac{\beta(\alpha,\theta+n\alpha)}{\theta+n\alpha}\right).$$

Then

$$
\begin{aligned}
&G_{\alpha,\theta+n\alpha}\left(\frac{\frac{\theta}{a(\theta)}x_n+\beta(\alpha,\theta)}{\theta-\left(\frac{\theta}{a(\theta)}\sum_{i=1}^n x_i+n\beta(\alpha,\theta)\right)}\right)\\
&\quad= P\left(a(\theta)\left(P_1(\alpha,\theta+n\alpha)-\frac{\beta(\alpha,\theta+n\alpha)}{\theta+n\alpha}\right)<\psi(x_1,\ldots,x_n;\theta,\alpha)\right)
\end{aligned}
$$

and

$$\lim_{\theta\to\infty}\psi(x_1,\ldots,x_n;\theta,\alpha)=x_n<0$$

which implies that

$$\lim_{\theta\to\infty}\frac{a(\theta)}{\theta}\log G_{\alpha,\theta+n\alpha}\left(\frac{\frac{\theta}{a(\theta)}x_n+\beta(\alpha,\theta)}{\theta-\left(\frac{\theta}{a(\theta)}\sum_{i=1}^n x_i+n\beta(\alpha,\theta)\right)}\right)=-\infty.$$

Therefore

$$\frac{a(\theta)}{\theta}\log h_{\alpha,\theta,n}(x_1,\ldots,x_n)\to-\sum_{i=1}^{n}x_i,\quad x_n>0, \tag{3.22}$$

$$\frac{a(\theta)}{\theta}\log h_{\alpha,\theta,n}(x_1,\ldots,x_n)\to-\infty,\quad x_n<0. \tag{3.23}$$

For $x_1\geq x_2\cdots\geq x_n$, let $B((x_1,\ldots,x_n),\delta)$ denote the closed ball centered at $(x_1,\ldots,x_n)$ with radius δ, and $B^\circ((x_1,\ldots,x_n),\delta)$ be the corresponding open ball. Let $P_{\alpha,\theta,n}$ denote the law of $a(\theta)\Big(P_1(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\cdots,P_n(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\Big)$. By controlling the density function from below and above using the boundary values of the balls in ways similar to that of (3.15) and (3.16), we obtain that for $x_n>0$,

$$\begin{aligned}
&\lim_{\delta\to0}\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P_{\alpha,\theta,n}\left(B((x_1,\ldots,x_n),\delta)\right)\\
&\quad=\lim_{\delta\to0}\liminf_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P_{\alpha,\theta,n}\left(B^\circ((x_1,\ldots,x_n),\delta)\right)\\
&\quad=-\sum_{i=1}^{n}x_i,
\end{aligned}\tag{3.24}$$

and for any $x_n<0$,

$$\begin{aligned}
&\lim_{\delta\to0}\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P_{\alpha,\theta,n}\left(B((x_1,\ldots,x_n),\delta)\right)\\
&\quad=\lim_{\delta\to0}\liminf_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P_{\alpha,\theta,n}\left(B^\circ((x_1,\ldots,x_n),\delta)\right)\\
&\quad=-\infty.
\end{aligned}\tag{3.25}$$

If $x_{r-1}>0, x_r=0$ for some $1\leq r\leq n$, then the upper estimate is obtained from that of $a(\theta)\Big(P_1(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\cdots,P_{r-1}(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\Big)$. The lower estimates when $x_r=0$ for some $1\leq r\leq n$ are obtained by approximating the boundary with open subsets away from the boundary. These combined with (3.24) and (3.25) imply the local LDP.

Noting that $\bigcup_{i=1}^{n}\{a(\theta)\Big(P_i(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\Big)>L\}=\{a(\theta)\Big(P_1(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\Big)>L\}$, it follows that

$$\lim_{L\to\infty}\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left\{\bigcup_{i=1}^{n}\left\{a(\theta)\left(P_i(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\right)>L\right\}\right\}=-\infty. \tag{3.26}$$

On the other hand,

$$\begin{aligned}
&\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left\{\bigcup_{i=1}^{n}\left\{a(\theta)\left(P_i(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\right)<-L\right\}\right\}\\
&\quad\leq\limsup_{\theta\to\infty}\frac{a(\theta)}{\theta}\log P\left\{a(\theta)\left(P_n(\alpha,\theta)-\frac{\beta(\alpha,\theta)}{\theta}\right)\leq-L\right\}=-\infty,
\end{aligned}\tag{3.27}$$

which combined with (3.26) implies the exponential tightness. The theorem then follows from the local LDP and the exponential tightness. □

3.2. MDP for the homozygosity

Let $m \geq 2$ be a fixed integer in this subsection. The scale factor

$$a(\theta) = \theta^r$$

for some constant r in $((m-1)/(2m-1), 1/2)$. One can choose positive constant h and integer $l \geq 3$ so that

$$l \geq \frac{2}{(2m-1)r-(m-1)},$$
$$1-2r < h < \frac{r}{m-1}\frac{l-2}{l}.$$

For any set A, χ_A denotes the indicator function of A. For any $n \geq 1$, set

$$G^{(n)}_{\alpha,\theta,h} = \sum_{i=1}^{\infty} V_i^n(T)\chi_{\{V_i(T)\leq\theta^h\}}, \qquad G^{(n)}_{\alpha,\theta} = \sum_{i=1}^{\infty} V_i^n(T),$$

and define

$$G_{\alpha,\theta,h} = \left(G^{(1)}_{\alpha,\theta,h} - E(G^{(1)}_{\alpha,\theta,h}), G^{(m)}_{\alpha,\theta,h} - E(G^{(m)}_{\alpha,\theta,h})\right),$$
$$G_{\alpha,\theta} = \left(T-\theta, G^{(m)}_{\alpha,\theta} - E(G^{(m)}_{\alpha,\theta})\right).$$

For any s, t in $\mathbb{R}$, define

$$\Lambda(s,t) = \frac{1}{2}\left(s^2 + \frac{2\Gamma(m-\alpha)\Gamma(m+1)}{\Gamma(m)\Gamma(1-\alpha)}st + \left(\frac{\Gamma(2m-\alpha)}{\Gamma(1-\alpha)} + \alpha\left(\frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)}\right)^2\right)t^2\right).$$

It follows by direct calculation that the Fenchel–Legendre transform of $\Lambda(s,t)$ is given by

$$\begin{aligned}\Lambda^*(x,y) &= \sup_{s,t}\{sx+ty-\Lambda(s,t)\}\\ &= \frac{\Gamma(1-\alpha)}{2(\Gamma(1-\alpha)\Gamma(2m-\alpha)+(\alpha-m^2)\Gamma^2(m-\alpha))}\\ &\quad\times\left(\left(\Gamma(2m-\alpha)+\alpha\frac{\Gamma^2(m-\alpha)}{\Gamma(1-\alpha)}\right)x^2 - 2m\Gamma(m-\alpha)xy + \Gamma(1-\alpha)y^2\right), \end{aligned} \tag{3.28}$$

for x, y in $\mathbb{R}$.

Lemma 3.3. *The family $\{\frac{a(\theta)}{\theta}G_{\alpha,\theta,h} : \theta > 0\}$ satisfies a LDP on space $\mathbb{R}^2$ with speed $\frac{a^2(\theta)}{\theta}$ and rate function $\Lambda^*(\cdot,\cdot)$.*

Proof. For any $s, t \in \mathbb{R}$, let

$$g(x) = sx + tx^m, \qquad \varphi_h(x) = \frac{g(x)\chi_{\{x\leq\theta^h\}}}{a(\theta)}.$$

It follows by direct calculation that

$$\int_0^{\theta^h}(\mathrm{e}^{\varphi_h(x)}-1)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x = \int_0^{\theta^h}\frac{g(x)}{a(\theta)}x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^{\theta^h}\frac{g^2(x)}{a^2(\theta)}x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x+\sum_{k=3}^{l}\frac{1}{k!}\frac{1}{a^k(\theta)}\int_0^{\theta^h}(sx+tx^m)^k x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x\\
&+O\left(\sum_{k=l+1}^{\infty}\frac{1}{k!}\frac{1}{a^k(\theta)}(|s|+|t|\theta^{h(m-1)})^k\Gamma(k-\alpha)\right)\\
&=\int_0^{\theta^h}\frac{g(x)}{a(\theta)}x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x+\frac{1}{2}\int_0^{\theta^h}\frac{g^2(x)}{a^2(\theta)}x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x+o\left(\frac{1}{a^2(\theta)}\right),
\end{aligned}
\tag{3.29}
$$

which implies that for θ large enough,

$$
\left|\int_0^{\theta^h}(\mathrm{e}^{\varphi_h(x)}-1)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x\right|<c_\alpha^{-1}.
$$

It follows from Proposition 21 in [18] and Campbell's theorem (cf. page 28 in [14]) that

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left\{\frac{1}{a(\theta)}\left(sG^{(1)}_{\alpha,\theta,h}+tG^{(m)}_{\alpha,\theta,h}\right)\right\}\right)=\mathbb{E}\left(\exp\left\{\sum_{i=1}^{\infty}\varphi_h(V_i(T))\right\}\right)\\
&=\mathbb{E}\left(\mathbb{E}\left(\exp\left\{\sum_{i=1}^{\infty}\varphi_h(V_i(T))\right\}\middle|\,\zeta(\alpha,\theta)\right)\right)\\
&=\mathbb{E}\left(\exp\left\{c_\alpha\gamma\left(\frac{\theta}{\alpha}\right)\int_0^{\theta^h}(\mathrm{e}^{\varphi_h(x)}-1)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x\right\}\right)\\
&=\exp\left\{-\frac{\theta}{\alpha}\log\left(1-c_\alpha\int_0^{\theta^h}(\mathrm{e}^{\varphi_h(x)}-1)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x\right)\right\}.
\end{aligned}
\tag{3.30}
$$

Putting (3.29) and (3.30) together, we get that

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left\{\frac{1}{a(\theta)}\left(s\left(G^{(1)}_{\alpha,\theta,h}-E(G^{(1)}_{\alpha,\theta,h})\right)+t\left(G^{(m)}_{\alpha,\theta,h}-E(G^{(m)}_{\alpha,\theta,h})\right)\right)\right\}\right)\\
&=\exp\left\{\frac{\theta c_\alpha}{2\alpha a^2(\theta)}\left(c_\alpha\left(\int_0^{\infty}g(x)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x\right)^2\right.\right.\\
&\quad\left.\left.+\int_0^{\infty}g^2(x)x^{-(1+\alpha)}\mathrm{e}^{-x}\mathrm{d}x+o\left(\frac{1}{a^2(\theta)}\right)\right)\right\}\\
&=\exp\left(\frac{\theta}{a^2(\theta)}\left(\Lambda(s,t)+o\left(\frac{1}{a^2(\theta)}\right)\right)\right),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\lim_{\theta\to\infty}\frac{a^2(\theta)}{\theta}\log E\left(\exp\left\{\frac{1}{a(\theta)}\left[s(G^{(1)}_{\alpha,\theta,h}-E(G^{(1)}_{\alpha,\theta,h}))+t(G^{(m)}_{\alpha,\theta,h}-E(G^{(m)}_{\alpha,\theta,h}))\right]\right\}\right)\\
&=\Lambda(s,t).
\end{aligned}
\tag{3.31}
$$

The lemma now follows from (3.28) and the Gärtner–Ellis theorem (cf. page 44 in [5]). □

Next we prove the main result of this subsection.

Theorem 3.4. *The family* $\{a(\theta)\left(\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(\mathbf{P}(\alpha,\theta))-1\right) : \theta > 0\}$ *satisfies a LDP with speed* $\frac{a^2(\theta)}{\theta}$ *and rate function* $\frac{z^2}{2\sigma^2_{\alpha,m}}$.

Proof. By an argument similar to that used in [11], one can show that $G_{\alpha,\theta,r}$ and $G_{\alpha,\theta}$ have the same LDP. On the other hand, by direct calculation,

$$\begin{aligned} &a(\theta)\left(\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(\mathbf{P}(\alpha,\theta))-1\right)\\ &\quad= \frac{a(\theta)}{\theta}(\theta-T)\sum_{k=1}^m\left(\frac{\theta}{T}\right)^k + \left(\frac{\theta}{T}\right)^m \frac{a(\theta)(G^{(m)}_{\alpha,\theta}-E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}. \end{aligned}$$

Since for any fixed integer $i \geq 1$ and any $\delta > 0$

$$\lim_{\theta\to\infty}\frac{a^2(\theta)}{\theta}\log P\left(\left|\left(\frac{\theta}{T}\right)^i - 1\right| \geq \delta\right) = -\infty,$$

it follows from an argument similar to that used in the proof of Lemma 2.1 in [11] that

$$\frac{a(\theta)}{\theta}(\theta-T)\sum_{k=1}^m\left(\frac{\theta}{T}\right)^k \quad \text{and} \quad \left(\frac{\theta}{T}\right)^m \frac{a(\theta)(G^{(m)}_{\alpha,\theta}-E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}$$

are exponentially equivalent to

$$m\frac{a(\theta)}{\theta}(\theta-T) \quad \text{and} \quad \frac{a(\theta)(G^{(m)}_{\alpha,\theta}-E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)},$$

respectively. Hence,

$$a(\theta)\left(\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(\mathbf{P}(\alpha,\theta))-1\right)$$

and

$$\frac{ma(\theta)(\theta-T)}{\theta} + \frac{a(\theta)(G^{(m)}_{\alpha,\theta}-E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}$$

are exponentially equivalent, and thus have the same LDP.

The fact that

$$\inf_{\frac{y\Gamma(1-\alpha)}{\Gamma(m-\alpha)}-mx=z}\Lambda^*(x,y) = \frac{z^2}{2\sigma^2_{\alpha,m}},$$

combined with Lemma 3.3 and the contraction principle, implies the result. □

4. LDP for small parameters

Set

$$a(\alpha,\theta) = \alpha \vee |\theta|, \qquad b(\alpha,\theta) = (-\log(a(\alpha,\theta)))^{-1},$$

and let

$$\nabla = \left\{ \mathbf{p} = (p_1, p_2, \ldots) : p_1 \geq p_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} p_i \leq 1 \right\}$$

be equipped with the subspace topology of $[0, 1]^{\infty}$, and $M_1(\nabla)$ be the space of all probability measures on ∇ equipped with the weak topology. Then $\Pi_{\alpha,\theta}$ belongs to $M_1(\nabla)$.

For any $\delta > 0$, it follows from the GEM representation (1.1) that

$$P\left(X_1^{\alpha,\theta} > 1 - \delta\right) \leq P\left(P_1(\alpha, \theta) > 1 - \delta\right).$$

By direct calculation, we have

$$\lim_{a(\alpha,\theta)\to 0} P\left(X_1^{\alpha,\theta} > 1 - \delta\right) = 1.$$

Therefore, $\Pi_{\alpha,\theta}$ converges in $M_1(\nabla)$ to $\delta_{(1,0,\ldots)}$ as $a(\alpha, \theta)$ converges to zero. In this section, we establish the LDP associated with this limit. This is a two-parameter generalization to the result in [10].

For any $n \geq 1$, set

$$\nabla_n = \left\{ (p_1, \ldots, p_n, 0, 0, \ldots) \in \nabla : \sum_{i=1}^{n} p_i = 1 \right\},$$

$$\nabla_\infty = \bigcup_{i=1}^{\infty} \nabla_i.$$

Lemma 4.1. *The family* $\{P_1(\alpha, \theta) : \alpha + \theta > 0, 0 < \alpha < 1\}$ *satisfies a LDP on* $[0, 1]$ *as* $a(\alpha, \theta)$ *goes to zero with speed* $b(\alpha, \theta)$ *and rate function*

$$S_1(p) = \begin{cases} 0, & p = 1 \\ k, & p \in \left[\dfrac{1}{k+1}, \dfrac{1}{k}\right), \quad k = 1, 2, \ldots \\ \infty, & p = 0. \end{cases} \tag{4.1}$$

Proof. Let $\{X_i^{\alpha,\theta} : i = 1, 2, \ldots\}$ be defined in (1.1). For any $n \geq 1$, set

$$\tilde{P}_1^n(\alpha, \theta) = \max\{X_i^{\alpha,\theta} : 1 \leq i \leq n\}.$$

Then it follows from direct calculation that for any $\delta > 0$

$$\begin{aligned} P\{P_1(\alpha, \theta) - \tilde{P}_1^n(\alpha, \theta) > \delta\} &\leq P\{(1 - U_1)\cdots(1 - U_n) \geq \delta\} \\ &\leq \delta^{-1} \prod_{i=1}^{n} \frac{\theta + \mathrm{i}\alpha}{\theta + \mathrm{i}\alpha + 1 - \alpha}, \end{aligned}$$

which leads to

$$\limsup_{a(\alpha,\theta)\to 0} b(\alpha, \theta) \log P\{P_1(\alpha, \theta) - \tilde{P}_1^n(\alpha, \theta) > \delta\} \leq -n.$$

Thus the families $\{\tilde{P}_1^n(\alpha,\theta) : 0 < \alpha < 1, \theta+\alpha > 0\}_{n=1,2,\ldots}$ are exponential good approximations to the family $\{P_1(\alpha,\theta) : 0 < \alpha < 1, \theta+\alpha > 0\}$. By the contraction principle, the family $\{\tilde{P}_1^n(\alpha,\theta) : 0 < \alpha < 1, \theta+\alpha > 0\}$ satisfies a LDP on [0, 1] as $a(\alpha,\theta)$ goes to zero with speed $b(\alpha,\theta)$ and rate function

$$I_n(p) = \begin{cases} 0, & p = 1 \\ k, & p \in \left[\dfrac{1}{k+1}, \dfrac{1}{k}\right), \quad k = 1, 2, \ldots, n-1 \\ n, & \text{else.} \end{cases}$$

The lemma now follows from the fact that

$$S_1(p) = \sup_{\delta>0} \liminf_{n\to\infty} \inf_{|q-p|<\delta} I_n(q). \quad \square$$

Theorem 4.1. *The family $\{\Pi_{\alpha,\theta} : \alpha+\theta > 0, 0 < \alpha < 1\}$ satisfies a LDP on ∇ as $a(\alpha,\theta)$ goes to zero with speed $b(\alpha,\theta)$ and rate function*

$$S(\mathbf{p}) = \begin{cases} n-1, & \mathbf{p} \in \nabla_n, p_n > 0, n \geq 1 \\ \infty, & \mathbf{p} \notin \nabla_\infty. \end{cases} \tag{4.2}$$

Proof. It suffices to establish the LDP for finite dimensional marginal distributions since the infinite dimensional LDP can be derived from the finite dimensional LDP through approximation.

It follows from the theorem in section 2 of [21] and Theorem 2.2 that for any $n \geq 2$, $(P_1(0, \alpha+\theta), P_2(0, \alpha+\theta), \ldots, P_n(0, \alpha+\theta))$ and $(P_1(\alpha,\theta), P_2(\alpha,\theta), \ldots, P_n(\alpha,\theta))$ have respective joint density functions

$$g_{0,\alpha+\theta,n}(p_1,\ldots,p_n) = (\alpha+\theta)^n \frac{\left(1-\sum_{i=1}^n p_i\right)^{\theta+\alpha-1}}{\prod_{i=1}^n p_i} P\left(P_1(0,\alpha+\theta) \leq \frac{p_n}{1-\sum_{i=1}^n p_i}\right),$$

and

$$g_{\alpha,\theta,n}(p_1,\ldots,p_n) = C_{\alpha,\theta,n} \frac{\left(1-\sum_{i=1}^n p_i\right)^{\theta+n\alpha-1}}{\left(\prod_{i=1}^n p_i\right)^{1+\alpha}} P\left(P_1\left(\alpha, n\alpha+\theta\right) \leq \frac{p_n}{1-\sum_{i=1}^n p_i}\right),$$

where $p_1 > p_2 > \cdots > p_n > 0$, $\sum_{i=1}^n p_i < 1$.

For any $n \geq 2$, set

$$\triangle_n = \left\{(p_1,\ldots,p_n) : p_1 \geq p_2 \geq \cdots \geq p_n \geq 0, \sum_{i=1}^n p_i \leq 1\right\}, \tag{4.3}$$

and

$$S_n(p_1,\ldots,p_n) = \begin{cases} 0, & (p_1, p_2, \ldots, p_n) = (1, 0, \ldots, 0) \\ l-1, & 2 \le l \le n, \sum_{k=1}^{l} p_k = 1, p_l > 0 \\ n + S_1\left(\dfrac{p_n}{1-\sum_{i=1}^{n} p_i} \wedge 1\right), & \sum_{k=1}^{n} p_k < 1, p_n > 0 \\ \infty, & \text{else.} \end{cases} \tag{4.4}$$

It follows from Lemma 2.4 in [10] that for any $n \ge 2$, the family

$$\{(P_1(0, \alpha+\theta), P_2(0, \alpha+\theta), \ldots, P_n(0, \alpha+\theta)) : \theta+\alpha > 0, 0 < \alpha < 1\}$$

satisfies a LDP on space $\triangle_n$ with speed $b(\alpha,\theta)$ and rate function S_n as $a(\alpha,\theta)$ tends to zero. The main idea in the proof is to establish the local LDP since $\triangle_n$ is compact. This follows from the establishment of the equality

$$\lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log g_{0,\alpha+\theta,n}(p_1,\ldots,p_n) = -S_n(p_1,\ldots,p_n) \tag{4.5}$$

in five different cases.

In our current setting, everything is the same except the density function. By Lemma 4.1, the LDP estimations for $P\left(P_1(\alpha, n\alpha+\theta) \le \frac{p_n}{1-\sum_{i=1}^n p_i}\right)$ is the same as the LDP estimations for $P\left(P_1(0, \alpha+\theta) \le \frac{p_n}{1-\sum_{i=1}^n p_i}\right)$. By direct calculation, we have

$$\lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log(\alpha+\theta)^n = \lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log C_{\alpha,\theta,n} = -n,$$

$$\lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log\left(\frac{\left(1-\sum_{i=1}^n p_i\right)^{\theta+\alpha-1}}{\prod_{i=1}^n p_i}\right)$$

$$= \lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log\left(\frac{\left(1-\sum_{i=1}^n p_i\right)^{\theta+n\alpha-1}}{\left(\prod_{i=1}^n p_i\right)^{1+\alpha}}\right) = 0.$$

Therefore,

$$\begin{aligned} \lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log g_{\alpha,\theta,n}(p_1,\ldots,p_n) &= \lim_{a(\alpha,\theta)\to 0} b(\alpha,\theta) \log g_{0,\alpha+\theta,n}(p_1,\ldots,p_n) \\ &= -S_n(p_1,\ldots,p_n) \end{aligned}$$

and the family $\{(P_1(\alpha,\theta), P_2(\alpha,\theta), \ldots, P_n(\alpha,\theta)) : \theta+\alpha > 0, 0 < \alpha < 1\}$ satisfies a LDP as $a(\alpha,\theta)$ goes to zero with speed $b(\alpha,\theta)$ and rate function S_n. $\square$

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