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# Existence and convergence results for infinite dimensional nonlinear stochastic equations with multiplicative noise

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## Abstract

The solution $X_n$ to a nonlinear stochastic differential equation of the form $dX_n(t) + A_n(t)X_n(t)\,dt - \frac{1}{2}\sum_{j=1}^{N}(B_j^n(t))^2 X_n(t)\,dt = \sum_{j=1}^{N} B_j^n(t)X_n(t)d\beta_j^n(t) + f_n(t)\,dt$, $X_n(0) = x$, where $\beta_j^n$ is a regular approximation of a Brownian motion $\beta_j$, $B_j^n(t)$ is a family of linear continuous operators from $V$ to $H$ strongly convergent to $B_j(t)$, $A_n(t) \to A(t)$, $\{A_n(t)\}$ is a family of maximal monotone nonlinear operators of subgradient type from $V$ to $V'$, is convergent to the solution to the stochastic differential equation $dX(t) + A(t)X(t)\,dt - \frac{1}{2}\sum_{j=1}^{N} B_j^2(t)X(t)\,dt = \sum_{j=1}^{N} B_j(t)X(t)\,d\beta_j(t) + f(t)\,dt$, $X(0) = x$. Here $V \subset H \cong H' \subset V'$ where $V$ is a reflexive Banach space with dual $V'$ and $H$ is a Hilbert space. These results can be reformulated in terms of Stratonovich stochastic equation $dY(t) + A(t)Y(t)\,dt = \sum_{j=1}^{N} B_j(t)Y(t) \circ d\beta_j(t) + f(t)\,dt$.


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## 1. Introduction

Consider the stochastic differential equation

$$\begin{cases} dX(t) + A(t)X(t)\,dt - \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} B_j^2(t)X(t)\,dt = \sum_{j=1}^{N} B_j(t)X(t)\,d\beta_j(t) + f(t)\,dt, \\ \quad t \in [0,T] \\ X(0) = x \end{cases} \tag{1.1}$$

where $A(t): V \to V'$ is a nonlinear monotone operator, $B_j(t) \in L(V,H)$, $\forall t \in [0,T]$ and $\beta_j$ are independent Brownian motions in a probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$.

Eq. (1.1) is of course equivalent with the Stratonovich stochastic differential equation

$$dY(t) + A(t)Y(t)\,dt = \sum_{j=1}^{N} B_j(t)Y(t) \circ d\beta_j(t) + f(t)\,dt, \quad t \in [0,T]. \tag{1.1$'$}$$

Here $V$ is a reflexive Banach space with dual $V'$ such that $V \subset H \subset V'$ algebraically and topologically, where $H$ (the *pivot* space) is a real Hilbert space. (The assumptions on $A(t)$, $B_j(t)$ will made precise later on.)

We associate with (1.1) the random differential equation

$$\begin{cases} \dfrac{dy}{dt} + \Lambda(t)y = g(t), \quad t \in [0,T], \mathbb{P}\text{-a.s.} \\ y(0) = x, \end{cases} \tag{1.2}$$

where $g(t) = e^{\sum_{j=1}^{N} \beta_j(t) B_j^*(t)} f(t)$, $B_j^*(t)$ is the adjoint of $B_j(t)$ and $\Lambda(t)$ is the family of operators

$$\begin{aligned} \Lambda(t)y &= e^{-\sum_{j=1}^{N} \beta_j(t)\, B_j(t)} A(t) e^{\sum_{j=1}^{N} \beta_j(t)\, B_j(t)} y \\ &\quad + \sum_{j=1}^{N} \int_0^{\beta_j(t)} e^{-s\, B_j(t)} \dot{B}_j(t) e^{s\, B_j(t)} y\, ds, \quad \forall y \in V \end{aligned} \tag{1.3}$$

where $\dot{B}_j$ is the derivative of $t \to B_j(t) \in L(V,H)$ and $(e^{sB_j(t)})_{s\in\mathbb{R}}$ is the $C_0$-group generated by $B_j(t)$ on $H$ and $V$.

It is well known (see e.g., [18, p. 202], [14,21]) that assuming that $B_j B_k(t) = B_k(t)B_j(t)$ for all $j,k$, at least formally Eqs. (1.1) and (1.2) are equivalent via the transformation

$$X(t) = e^{\sum_{j=1}^{N} \beta_j(t)\, B_j(t)} y(t), \quad \mathbb{P}\text{-a.s.}, t \in [0,T], \tag{1.4}$$

and this is indeed the case if (1.2) has a strong, progressively measurable solution $y: [0,T] \times \Omega \to H$.

We consider also the family of approximating stochastic equations

$$\begin{cases} \dfrac{d}{dt}X_n + A_n(t)X_n = \displaystyle\sum_{j=1}^{N} B_j^n(t)X_n(t)\dot{\beta}_j^n(t) + f_n(t), \quad \mathbb{P}\text{-a.s.} \\ X_n(0) = x \end{cases} \tag{1.5}$$

where $\{\beta_j^n\}$ is a sequence of smooth stochastic processes convergent to $\beta_j$, that is $\beta_j^n(t) \to \beta_j(t)$ uniformly on $[0, T]$, $\mathbb{P}$-a.s. and $A_n \to A$, $B_j^n \to B_j$, $f_n \to f$ as $n \to \infty$ in a sense to be made precise below.

Eq. (1.5) is just an approximation of Stratonovich equation (1.1′) where $\beta_j^n$ is a regularization of $\beta_j$. One must emphasize that $\{\beta^n\}$ might be adapted to a filtration different from $\{\mathscr{F}_t\}$.

Eq. (1.5) reduces via (1.4), that is

$$X_n(t) = e^{\sum_{j=1}^N \beta_j^n(t)\, B_j^n(t)} y_n(t) \tag{1.4$_n$}$$

to a random differential equation of the form (1.2) that is

$$\begin{cases} \dfrac{dy_n}{dt} + \Lambda_n(t) y_n = g_n(t), \quad t \in [0, T], \mathbb{P}\text{-a.s.} \\ y_n(0) = x \end{cases} \tag{1.2$_n$}$$

where $g_n(t) = e^{\sum_{j=1}^N \beta_j^n(t) B_j^{n*}(t)} f_n(t)$ and $\Lambda_n$ are given by

$$\Lambda_n(t) = e^{-\sum_{j=1}^N \beta_j^n(t)\, B_j^n(t)} A_n(t) e^{\sum_{j=1}^N \beta_j^n(t)\, B_j^n(t)} + \sum_{j=1}^N \int_0^{\beta_j^n(t)} e^{-s\, B_j^n(t)} \dot{B}_j^n(t) e^{s\, B_j^n(t)} y\, ds. \tag{1.3$_n$}$$

The main result (see Theorems 2.1–2.3 below) is that under suitable assumptions Eqs. (1.2), (1.2$_n$) have unique solutions $y$ and $y_n$ which are progressively measurable processes and for $n \to \infty$, we have $y_n \to y$, $X_n \to X$ in a certain precise sense. In the linear case such a result was established by a different method in [14], (we refer also to [17,21] to other results in this direction). The variational method we use here allows to treat a general class of nonlinear equations (1.1) possibly multi-valued (On these lines see also [3,4]).

The applications given in Section 4.1 refer to stochastic porous media equations and nonlinear stochastic diffusion equations of divergence type but of course the potential area of applications is much larger.

**Notation.** If $Y$ is a Banach space we denote by $L^p(0, T; Y)$, $1 \le p \le \infty$ the space of all (equivalence classes of) strongly measurable functions $u: (0, T) \to Y$ with $\|u\|_Y \in L^p(0, T)$ (here $\|\cdot\|_Y$ is the norm of $Y$). Denote by $C([0, T]; Y)$ the space of all continuous $Y$-valued functions on $[0, T]$ and by $W^{1,p}([0, T]; Y)$ the infinite dimensional Sobolev space $\{y \in L^p(0, T; Y), \frac{dy}{dt} \in L^p(0, T; Y)\}$ where $\frac{d}{dt}$ is considered in the sense of vectorial distributions. It is well known that $W^{1,p}([0, T]; Y)$ coincides with the space of absolutely continuous functions $y: [0, T] \to Y$, a.e. differentiable and with derivative $y'(t) = \frac{dy}{dt}(t)$ a.e. $t \in (0, T)$ and $\frac{dy}{dt} \in L^p(0, T; Y)$ (see e.g., [5]). If $p \in [1, \infty]$ is given we denote by $p'$ the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

## 2. The main results

We shall study here Eq. (1.2) under the following assumptions.

(i) *$V$ is a separable real reflexive Banach space with the dual $V'$ and $H$ is a separable real Hilbert space such that $V \subset H \subset V'$ algebraically and topologically.*

We denote by $|\cdot|$, $\|\cdot\|_V$ and $\|\cdot\|_{V'}$ the norms in $H$, $V$ and $V'$, respectively and by $\langle\cdot,\cdot\rangle$ the duality pairing on $V \times V'$ which coincides with the scalar product $(\cdot,\cdot)$ of $H$ on $H \times H$.

(ii) *$A(t)y = \partial\psi(t, y)$, a.e. $t \in (0, T)$, $\forall y \in V$, $\mathbb{P}$-a.s., where $\psi: (0, T) \times V \times \Omega \to \mathbb{R}$ is convex and lower-semicontinuous in $y$ on $V$ and measurable in $t$ on $[0, T]$. There are $\alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1, 2$, $1 < p_1 \le p_2 < \infty$*

$$\gamma_1 + \alpha_1 \|y\|_V^{p_1} \le \psi(t, y) \le \gamma_2 + \alpha_2 \|y\|_V^{p_2}, \quad \forall y \in V. \tag{2.1}$$

(iii) *There are $C_1, C_2 \in \mathbb{R}^+$ such that*

$$\psi(t, -y) \le C_1 \psi(t, y) + C_2, \quad \forall y \in V, t \in (0, T). \tag{2.2}$$

(The constants $C_i, \gamma_i, \alpha_i$ are dependent on $\omega$.)

(iv) *For each $y \in V$ the stochastic process $\psi(t, y)$ is progressively measurable with respect to filtration $\{\mathscr{F}_t\}_{t\ge0}$.*

(v) *$B_j(t)$ is a family of linear, closed and densely defined operators in $H$ such that $B_j(t) = -B_j^*(t)$, $\forall t \in [0, T]$, $B_j(t)$ generates a $C_0$-group $(e^{s B_j(t)})_{s\in\mathbb{R}}$ on $H$ and $V$. Moreover, $B_j \in C^1([0, T]; L(V, H))$, $B_j(t)B_k(t) = B_k(t)B_j(t)$ for all $j, k$.*

(vi) *$f: [0, T] \times \Omega \to V'$ is progressively measurable and $f \in L^{p_1'}(0, T; V')$, $\mathbb{P}$-a.s.*

We note that by (ii) $A(t, \omega): V \mapsto V'$ is, for all $t \in [0, T]$ and $\omega \in \Omega$, maximal monotone and surjective (see [5]) but in general multi-valued if $\psi$ is not Gâteaux differentiable in $y$.

**Theorem 2.1.** *Let $y_0 \in H$. Then under assumptions* (i)–(vi) *there is for each $\omega \in \Omega$ a unique function $y = y(t, \omega)$ to Eq.* (1.2) *which satisfies*

$$y \in L^{p_1}(0, T; V) \cap C([0, T]; H) \cap W^{1,p_2'}([0, T]; V'), \tag{2.3}$$

$$\begin{cases} \dfrac{dy}{dt}(t) + \Lambda(t)y(t) \ni g(t), & \text{a.e. } t \in (0, T), \\ y(0) = y_0. \end{cases} \tag{2.4}$$

*Moreover, the process $y: [0, T] \times \Omega \to H$ is progressively measurable with respect to the filtration $\{\mathscr{F}_t\}_{t\ge0}$.*

By $\xrightarrow{G}$ we denote the variational or $\Gamma$-convergence. This means that for each $y \in V$ and $\xi \in A(t)y$ there are $y_n$ and $\xi_n \in A(t)y$ such that $y_n \to y$ strongly in $V$, $\xi_n \to \xi$ strongly in $V'$ and similarly for $A_n^{-1}(t) \to A^{-1}(t)$. Assumption (2.5) implies and is equivalent to: $\psi_n(t, z) \to \psi(t, z)$, $\psi_n^*(t, \tilde{z}) \to \psi^*(t, \tilde{z})$ for all $z \in V$, $\tilde{z} \in V'$ and $t \in [0, T]$ where $\psi^*$ is the conjugate of $\psi$ (see Appendix).

**Theorem 2.2.** *Assume that for each $n$, $\Lambda_n$, $B_j^n$ and $f_n$ satisfy* (i)–(iv). *Then for any $y_0 \in V$ there is a unique function $y_n = y_n(t, \omega)$ which satisfies* (2.3) *and Eq.* (2.4) *with $\Lambda_n$ instead of $\Lambda$. Moreover, assume that for $n \to \infty$*

$$\begin{aligned} &A_n(t) \xrightarrow{G} A(t), \quad t \in [0, T] \\ &A_n^{-1}(t) \xrightarrow{G} A^{-1}(t), \quad t \in [0, T] \end{aligned} \tag{2.5}$$

$$f_n(\cdot, \omega) \to f(\cdot, \omega), \quad \text{strongly in } L^{p_2'}(0, T; V'), \ \mathbb{P}\text{-a.s. in } \Omega. \tag{2.6}$$

$$B_j^n x \to B_j x, \quad \text{in } C^1([0, T]; H), \forall x \in H. \tag{2.7}$$

*Then for* $n \to \infty$

$$y_n(\cdot, \omega) \to y(\cdot, \omega) \tag{2.8}$$

$\mathbb{P}$-*a.s. weakly in* $L^{p_1}(0, T; V)$, *weakly-star in* $L^\infty(0, T; H)$.

Assumption (2.5) implies and is equivalent to: $\psi_n(t, z) \to \psi(t, z)$, $\psi_n^*(t, \tilde{z}) \to \psi^*(t, \tilde{z})$ for all $z \in V$, $\tilde{z} \in V'$ and $t \in [0, T]$ where $\psi^*$ is the conjugate of $\psi$ (see e.g., [1]).

Coming back to Eq. (1.1) we say that the process $X : [0, T] \to H$ is a solution to (1.1), if it is progressively measurable with respect to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ induced by the Brownian motion,

$$X \in C([0, T], H) \cap L^{p_1}(0, T; V), \quad \mathbb{P}\text{-a.s.} \tag{2.9}$$

and

$$\begin{aligned} X(t) = x - \int_0^t A(s)X(s)\,ds + \frac{1}{2}\sum_{j=1}^N \int_0^t B_j^2(s)X(s)\,ds \\ + \sum_{j=1}^N \int_0^t B_j(s)X(s)d\beta_j(s) + \int_0^t f(s)\,ds, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.10}$$

By Theorems 2.1 and 2.2 we find the following.

**Theorem 2.3.** *Under the assumptions of Theorem 2.1 there exist unique solutions $X$ and $X_n$ to (1.1) and (1.5) respectively given by*

$$X(t) = e^{\sum_{j=1}^N \beta_j(t) B_j(t)} y(t), \qquad X_n(t) = e^{\sum_{j=1}^N \beta_j^n(t) B_j^n(t)} y_n(t), \tag{2.11}$$

*where $y$ and $y_n$ are solutions to (1.2) and (1.2$_n$). Moreover, we have*

$$X, X_n \in L^{p_1}(0, T; V), \quad \mathbb{P}\text{-a.s.} \tag{2.12}$$

$X, X_n: [0, T] \to H$ *are* $\mathbb{P}$-*a.s. continuous and*

$$X_n \to X \quad \text{weakly in } L^{p_1}(0, T; V), \text{weakly-star in } L^\infty(0, T; H), \mathbb{P}\text{-a.s.} \tag{2.13}$$

The precise meaning of Theorems 2.2 and 2.3 is the structural stability of the Itô stochastic differential equation (1.1) and of its Stratonovich counterpart (1.1′). As a matter of fact, as mentioned earlier, all these results can be reformulated in terms of Stratonovich equation (1.1′).

One of the main consequences of Theorem 2.2 is that the Stratonovich stochastic equation is stable with respect to smooth approximations of the process $B(t)Xd\beta(t)$. On the other hand, the general existence theory for infinite dimensional stochastic differential equations with linear multiplicative noise (see e.g., [16,18]) is not applicable in the present situation due to the fact that the noise coefficient $x \to B(t)x$ is not bounded on the basic space $H$. The approach we use here to treat Eq. (2.4) relies on the Brezis–Ekeland variational principle [12,13] which allows to reduce nonlinear evolution equations of potential type to convex optimization problems. (On these lines see also [3,4,10,22,24]).

The more general case of nonlinear monotone and demicontinuous operators $A(t): V \to V'$ is ruled out from present theory and might expect however to extend the theory to this general case by using Fitzpatrick function formalism (see [22,24]).

As in [14], see Corollary on p. 438, we can define a solution to problem (1.1′) for any *deterministic* continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}^N$. The result from [14] was a generalization of a analogous result from Sussmann's well known paper [23]; see also Doss [19]. We will formulate our result in the same fashion as in [14], i.e. the result contains implicitly a definition. Let us observe, that in this case, we prove the existence for any multidimensional continuous signal, thus a signal more general than a rough signal from the theory of rough paths. However, this is due to the assumption of the commutativity of the vector fields $B_j$, $j = 1, \dots, N$. In the result below, we need deterministic versions of assumptions (ii), (vi). Note that the assumption (iv) is now redundant.

(ii') *$A(t)y = \partial\psi(t, y)$, a.e. $t \in (0, T)$, $\forall y \in V$, where $\psi : (0, T) \times V \to \mathbb{R}$ is convex and lower-semicontinuous in $y$ on $V$ and measurable in $t$ on $[0, T]$. There exist $\alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1, 2$, $1 < p_1 \le p_2 < \infty$, such that*

$$\gamma_1 + \alpha_1 \|y\|_V^{p_1} \le \psi(t, y) \le \gamma_2 + \alpha_2 \|y\|_V^{p_2}, \quad \forall y \in V. \tag{2.14}$$

(vi') $f \in L^{p_1'}(0, T; V')$.

**Theorem 2.4.** *Assume that the assumptions* (i), (iii), (v) *as well as* (ii′) *and* (vi′) *are satisfied. Then for every $x \in V$ and every $\beta \in C([0, T]; \mathbb{R}^N)$, the problem*

$$\begin{cases} dX(t) + A(t)X(t)\,dt = \displaystyle\sum_{j=1}^{N} B_j(t)X(t)\,d\beta_j(t) + f(t)\,dt, \quad t \in [0, T] \\ X(0) = x \end{cases} \tag{2.15}$$

*has a unique solution $X \in L^{p_1}(0, T; V) \cap C([0, T]; H)$ in the following sense.*

(i) *For every $\beta \in C^1([0, T]; \mathbb{R}^N)$, the problem (2.15) has a unique solution $X \in L^{p_1}(0, T; V) \cap C([0, T]; H)$.*
(ii) *If $\beta_n \in C^1([0, T]; \mathbb{R}^N)$ and $\beta_n \to \beta$ in $C([0, T]; \mathbb{R}^N)$ and $X_n \in L^{p_1}(0, T; V) \cap C([0, T]; H)$ is the (unique) solution to the problem (2.15) corresponding to $\beta_n$, then $X_n \to X$ weakly in $L^{p_1}(0, T; V)$, weakly-star in $L^\infty(0, T; H)$, $\mathbb{P}$-a.s.*

From Theorem 2.4 we infer that in the framework of Theorem 2.4 but with $\beta$ being a Brownian motion, the problem (1.1) generates a random dynamical system on $H$. In an obvious way we have the following corollary.

**Corollary 1.** *Assume that the assumptions* (i), (iii), (v) *as well as* (ii′) *and* (vi′) *are satisfied. Assume that $\beta$ is a standard canonical two-sided $\mathbb{R}^N$-valued Brownian motion on a filtered probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P}\}$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^N) : \omega(0) = 0\}$. Let us define a map*

$$\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t\omega = \omega(\cdot + t) - \omega(0) \in \Omega.$$

*Then there exists a map*

$$\varphi : \mathbb{R}^+ \times \Omega \times H \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in H$$

*such that a pair $(\phi, \vartheta)$ is a random dynamical system on $H$, see for instance Definition* 2.1 *in* [15]*, and, for each $s \in \mathbb{R}$ and each $x \in H$, the process $X$, defined for $\omega \in \Omega$ and $t \ge s$ as*

$$X(t, s; \omega, x) := \varphi(t - s; \vartheta_s\omega)x, \tag{2.16}$$

*is a solution to problem (1.1) over the time interval $[s, \infty)$ with an initial data given at time $s$.*

**Remark 2.1.** Theorem 2.4 provides a solution to problem (2.15) for every continuous path. Our main result provides a natural interpretation of this solution in the case when $\beta$ is a Brownian motion. One can also provide a similar interpretation when $\beta$ is fractional; see for instance [20].

Corollary allows one to investigate the existence of random attractors; see [15].

These questions will be investigated in the future works.

## 3. Proofs

**Proof of Theorem 2.1.** For simplicity we consider the case $N = 1$, that is $B_j = B$, $\beta_j = \beta$ for all $1 \leq j \leq 1$.

We note first that though the operator $\Lambda(t) = \Lambda(t, \omega)$ is $\mathbb{P}$-a.e. $\omega \in \Omega$, maximal monotone from $V$ to $V'$ the standard existence theory (see e.g., [5, p. 177]) does not apply here. This is, however, due to the general growth condition (2.1) on $\psi(t, \cdot)$ and implicitly on $A(t)$ as well as due to the multivaluedness of $A(t)$.

So we shall use a direct approach which makes use of the variational structure of problem (1.2). (On these lines see also [3], [10, p. 280]). Namely, we can write

$$\Lambda(t) = \partial\varphi(t, \cdot) + \Gamma(t), \quad \forall t \in [0, T]. \tag{3.1}$$

Here $\varphi: [0, T] \times V \to \mathbb{R}$ is given by

$$\varphi(t, y) = \psi(t, e^{\beta(t)B(t)} y) \tag{3.2}$$

and

$$\Gamma(t)y = \int_0^{\beta(t)} e^{-s\,B(t)}\dot{B}(t)e^{s\,B(t)} y\, ds, \quad \forall y \in H, t \in [0, T]$$

where $\dot{B} = \frac{d}{dt}B(t)$. We fix $\omega \in \Omega$.

By the conjugacy formulae (A.3) and (A.4) we now may equivalently write (1.2) (or (2.4)) as

$$\begin{cases} \varphi(t, y(t)) + \varphi^*(t, u(t)) = \langle y(t), u(t)\rangle, & \text{a.e. } t \in [0, T] \\ y'(t) + \Gamma(t)y(t) = -u(t) + g(t), & \text{a.e. } t \in [0, T] \end{cases} \tag{3.3}$$

while

$$\varphi(t, \bar{y}) + \varphi^*(t, \bar{u}) \geq \langle \bar{y}, \bar{u}\rangle$$

for all $(\bar{y}, \bar{u}) \in L^{p_1}(0, T; V) \times L^{p_2'}(0, T; V')$.

Thus following a well known idea due to Brezis and Ekeland (see e.g., [12,13]) we are lead to the optimization problem

$$\begin{aligned} \text{Min} \Bigg\{ &\int_0^T \big(\varphi(t, y(t)) + \varphi^*(t, u(t)) - \langle u(t), y(t)\rangle\big)\, dt : y' + \Gamma(t)y \\ &= -u + g, \text{a.e. } t \in [0, T];\ y(0) = y_0, y \in L^{p_1}(0, T; V), u \in L^{p_2'}(0, T; V')\Bigg\}. \end{aligned} \tag{3.4}$$

Equivalently

$$\text{Min} \left\{ \int_0^T \big(\varphi(t, y(t)) + \varphi^*(t, u(t)) - \langle g(t), y(t)\rangle\big)\, dt \right.$$

$$+\frac{1}{2}\big(|y(T)|^2-|y_0|^2\big): y'+\Gamma(t)y=-u+g, \text{ a.e. } t\in[0,T];\ y(0)$$

$$\left.=y_0,\ y\in L^{p_1}(0,T;V),\ u\in L^{p_2'}(0,T;V')\right\}. \tag{3.5}$$

Here we have used (for the moment, formally) the integration by parts formula

$$-\int_0^T\langle u(t),y(t)\rangle\,dt=\frac{1}{2}\big(|y(T)|^2-|y_0|^2\big)+\int_0^T\langle\Gamma(t)y(t),y(t)\rangle\,dt$$
$$-\int_0^T\langle g(t),u(t)\rangle\,dt$$

and hypothesis (v) which implies that $\langle\Gamma(t)y,y\rangle=0$. Of course the equivalence of (3.4) and (3.5) is valid only if the above equality is true which is not always the case in the absence of some additional properties of minimizer $y$ to allow integration by parts in $\int_0^T\langle u(t),g(t)\rangle\,dt$. In the following we shall prove however that Eq. (3.5) has at least one solution and show consequently that it is also a solution to Eq. (2.4).

**Lemma 3.1.** *There is a solution $y^*\in L^{p_1}(0,T;V)\cap W^{1,p_2'}([0,T];V')$ to Eq. (3.5).*

**Proof.** We note that by the standard existence theory of linear evolution equations for each $u\in L^{p_2'}(0,T;V')$ there is a unique solution $y\in L^{p_2}(0,T;V)\cap W^{1,p_2'}([0,T];V')\subset C([0,T];H)$ to equation

$$y'+\Gamma(t,y)=-u+g,\quad \text{a.e. } t\in[0,T],\qquad y(0)=y_0.$$

By assumptions (2.1) and (2.2) we have

$$\tilde\gamma_1+\tilde\alpha_1\|y\|_V^{p_1}\le\varphi(t,y)\le\tilde\gamma_2+\tilde\alpha_2\|y\|_V^{p_2},\quad\forall y\in V \tag{3.6}$$

and

$$\bar\gamma_1+\bar\alpha_1\|y\|_{V'}^{p_2'}\le\varphi^*(t,y)\le\bar\gamma_2+\bar\alpha_1\|y\|_{V'}^{p_1'},\quad\forall y\in V' \tag{3.7}$$

where $\tilde\gamma_i,\bar\gamma_i\in\mathbb{R}$, $\frac{1}{p_i}+\frac{1}{p_i'}=1$ and $\tilde\alpha_i,\bar\alpha_i>0$, $i=1,2$. (Recall that $e^{sB(t)}$ is invertible).

Then the infimum $m^*$ in (3.5) is $>-\infty$ and there are the sequences $\{y_j\}\subset L^{p_1}(0,T;V)$, $\{u_j\}\subset L^{p_2'}(0,T;V')$ such that for all $y$

$$m^*\le\int_0^T\big(\langle\varphi(t,y_j)\rangle+\varphi^*(t,u_j)-\langle g,y_j\rangle\big)\,dt+\frac{1}{2}\big(|y_j(T)|^2-|y_0|^2\big)\le m^*+\frac{1}{j} \tag{3.8}$$

$$\begin{cases}y_j'+\Gamma(t)y_j=-u_j+g, & \text{in } [0,T]\\ y_j(0)=y_0.\end{cases} \tag{3.9}$$

Clearly $y_j\in W^{1,p_2'}([0,T];V')$ and by assumption (2.1) and inequality (3.8) it follows that

$$\|y_j\|_{L^{p_1}(0,T;V)}+\|y_j'\|_{L^{p_2'}(0,T;V')}\le C \tag{3.10}$$

because as easily seen by assumption (v), $|\Gamma(t)y|_H\le C\|y\|_V$, $\forall y\in V$.

Hence on a subsequence, again denoted by $y_j$, we have for $j \to \infty$

$$\begin{aligned} & y_j \to y^* \quad \text{weakly in } L^{p_1}(0, T; V) \\ & u_j \to u^* \quad \text{weakly in } L^{p_2'}(0, T; V') \\ & y_j' \to (y^*)' = g - u^* - \Gamma(t) y^* \quad \text{weakly in } L^{p_2'}(0, T; V'). \end{aligned} \tag{3.11}$$

Since the functions $y \mapsto \int_0^T \varphi(t, y)\, dt$ and $u \mapsto \int_0^T \varphi^*(t, u)\, dt$ are weakly lower-semicontinuous on $L^{p_1}(0, T; V)$ and $L^{p_2'}(0, T; V')$ respectively, letting $j$ tend to infinity we obtain that

$$m^* = \int_0^T \left(\varphi(t, y^*) + \varphi^*(t, u^*) - \langle g, y^* \rangle\right) dt + \frac{1}{2}\left(|y^*(T)|^2 - |y_0|^2\right) \tag{3.12}$$

and

$$\begin{cases} (y^*)' + \Gamma(t) y^* = -u^* + g, \quad t \in [0, T] \\ y^*(0) = y_0. \end{cases} \tag{3.13}$$

Therefore $(y^*, u^*)$ is a solution to optimization problem (3.5) as claimed. □

**Proof of Theorem 2.1** (*Continued*)**.** We shall show now that $y^*$ given by Lemma 3.1 is a solution to (2.4). To this end we notice just that without any loss of generality we may assume that $y_0 = 0$. Indeed we can reduce the problem to this case by translating in problem (2.3) $y$ in $y - y_0$.

We prove now that $m^* = 0$. For this purpose we invoke a standard duality result for infinite dimensional convex optimal control problems, essentially due to Rockafeller. Namely, one has (see [10, Theorem 4.6, p. 287])

$$m^* + \min (3.5') = 0 \tag{3.14}$$

where (3.5′) is the dual control problem

$$\begin{aligned} \text{Min} \Bigg\{ \int_0^T & \left(\varphi(t, -p(t)) + \varphi^*(t, v(t)) + \langle g(t), p(t) \rangle\right) dt + \frac{1}{2}|p(T)|^2 : p' + \Gamma(t) p \\ & = v + g, t \in [0, T] \Bigg\}. \end{aligned} \tag{3.5'}$$

If $(p^*, v^*) \in L^{p_1}(0, T; V) \times L^{p_2'}(0, T; V')$ is optimal in (3.5′), we have

$$\langle (p^*)', p^* \rangle \in L^1(0, T) \tag{3.15}$$

$$\int_0^T \langle (p^*)', p^* \rangle\, dt = \frac{1}{2}\left(|p^*(T)|^2 - |p^*(0)|^2\right). \tag{3.16}$$

Here is the argument. First, note that $p'$ solves $p' + \Gamma(t) p = v + g$. We have by the identities (A.3) and (A.4) and the fact that $\langle \Gamma(t) p^*, p^* \rangle = 0$,

$$-\langle (p^*(t))', p^*(t) \rangle \le \varphi^*(t, v^*(t)) + \varphi(t, -p^*(t)) - \langle g(t), p^*(t) \rangle, \quad \text{a.e. } t \in (0, T)$$

and

$$\langle (p^*(t))', p^*(t) \rangle \le \varphi^*(t, v^*(t)) + \varphi(t, p^*(t)) + \langle g(t), p^*(t) \rangle, \quad \text{a.e. } t \in (0, T).$$

Since $\varphi(t, -p^*) \in L^1(0,T)$ and by assumption (2.2), $\varphi(t, p^*) \in L^1(0,T)$ too, we infer that (3.15) holds. Now since $p^* \in W^{1,p_2'}([0,T]; V') \cap L^{p_1}(0,T;V')$ we have

$$\frac{1}{2}\frac{d}{dt}|p^*(t)|^2 = \langle (p^*)'(t), p^*(t)\rangle, \quad \text{a.e. } t \in (0,T)$$

and by (3.15) we get (3.16) as claimed.

By (3.5$'$) and (3.16) we see that

$$\min\ (3.5') = \int_0^T \big(\varphi(t,-p^*) + \varphi^*(t,v^*) + \langle v^*, p^*\rangle\big)\,dt + \frac{1}{2}|p^*(0)|^2 \geq 0.$$

Similarly, by (3.12), (3.13) and by

$$\frac{1}{2}\big(|y^*(T)|^2 - |y^*(0)|^2\big) = \int_0^T \langle (y^*)', y^*\rangle\,dt,$$

(the latter follows exactly as (3.16)) we see that

$$m^* = \int_0^T \big(\varphi(t,y^*) + \varphi^*(t,u^*) - \langle u^*, y^*\rangle\big)\,dt \geq 0.$$

Then by (3.14) we have that $m^* = 0$ and therefore again by (3.12) we have that

$$\varphi(t,y^*) + \varphi^*(t,u^*) = \langle u^*, y^*\rangle, \quad \text{a.e. in } [0,T]$$
$$(y^*)' + \Gamma(t)y^* = g - u^* \quad \text{a.e. in } [0,T]$$

and therefore $y^*$ is a solution to (2.4).

On the other hand, as seen earlier, we have

$$\frac{1}{2}\big(|y^*(t)|^2 - |y^*(s)|^2\big) = \int_s^t \langle (y^*)'(\tau), y^*(\tau)\rangle\,d\tau, \quad \forall\, 0 \leq s \leq t \leq T. \tag{3.17}$$

Hence $y^* \in C([0,T]; H)$. The uniqueness of $y^*$ is immediate by (3.17). It remains to be proven that $y^*$ is progressively measurable.

To this end we note as minimum in (3.5) the pair $(y^*, u^*)$ is the solution to the Euler–Lagrange system (see e.g., [10, p. 263])

$$\begin{aligned}
&(y^*)' + \Gamma(t)y^* = -u^* + g, \quad \text{a.e. } t \in (0,T), \omega \in \Omega\\
&q' - \Gamma'(t)q = -g + A(t)y^*, \quad \text{a.e. } t \in (0,T)\\
&u^*(t) = A(t)q(t), \quad \text{a.e. } t \in (0,T), \omega \in \Omega\\
&y^*(0) = y_0, \qquad q(T) = -y^*(T).
\end{aligned}$$

Since the latter two point boundary value problem has a unique solution $(y^*, q)$ and is of dissipative (accretive) type it can be solved by iteration or more precisely by a gradient algorithm (see [10, p. 252]). In particular, we have $y^* = \lim_{k\to\infty} y_k$, $q = \lim_{k\to\infty} q_k$ weakly in $L^{p_1}(0,T;V)$ and $u^* = \lim_{k\to\infty} u_k$ weakly in $L^{p_2'}(0,T;V')$ where

$$\begin{aligned}
&y_k' + \Gamma(t)y_k = -u_k + g \quad t \in [0,T],\\
&q_k' - \Gamma'(t)q_k = -g + A(t)y_k, \quad t \in [0,T],\\
&u_{k+1} = u_k - A^{-1}(t)u_k + q_k, \quad t \in [0,T],\\
&y_k(0) = y_0, \qquad q_k(T) = 0, \quad k = 0,1,2,\ldots.
\end{aligned}$$

Hence, if we start with a progressively measurable $u_0$, we see that all $u_k$ are progressively measurable and so are $u^*$ and $y^*$. $\square$

**Proof of Theorem 2.2.** As in the previous case it follows that (1.2$_n$) has a unique solution $y_n \in W^{1,p_2'}([0,T];V') \cap L^{p_1}(0,T;V)$ given by the minimization problem (3.5) where $g = g_n$ and $\varphi = \varphi_n$, $\varphi^* = \varphi_n^*$. Here $\varphi_n$ is given as in (3.2) where $\psi = \psi_n$ and $\beta$ is replaced by $\beta_n$ while $\varphi_n^*$ is the conjugate of $\varphi_n$. We have, similarly, $\partial\psi_n = A_n$ and $\varphi_n(t,y) = \psi_n(t, e^{\beta_n(t)y})$

$$(y_n, u_n) = \arg\min\left\{\int_0^T \varphi_n(t,y(t)) + \varphi_n^*(t,u(t)) - \langle g_n(t), u(t)\rangle\, dt \right.$$
$$\left. + \frac{1}{2}(|y(T)|^2 - |y_0|^2);\ y' + \Gamma_n y = -u + g_n,\ y(0) = y_0\right\}.$$

Here $\Gamma_n(t)y = \int_0^{\beta_n(t)} e^{-sB_n(t)}\dot{B}_n(t)e^{sB_n(t)}y\,ds$.

We see that

$$\|y_n\|_{L^\infty(0,T;H)} + \|y_n\|_{L^{p_1}(0,T;V)} + \left\|\frac{dy_n}{dt}\right\|_{L^{p_2'}(0,T;V')} + |y_n(T)| \le C,$$

and this implies that on a subsequence, again denoted by $\{n\}$, we have

$$\begin{cases} u_n \to \tilde{u} & \text{weakly in } L^{p_2'}(0,T;V') \\ y_n \longrightarrow \tilde{y} & \text{weakly in } L^{p_1}(0,T;V) \\ y_n \longrightarrow \tilde{y} & \text{weakly-star in } L^\infty(0,T;H) \\ \dfrac{dy_n}{dt} \longrightarrow \dfrac{d\tilde{y}}{dt} & \text{weakly in } L^{p_2'}(0,T;V') \\ y_n(T) \longrightarrow \tilde{y}(T) & \text{weakly in } H. \end{cases} \tag{3.18}$$

By (2.5), (2.6) we see that $\tilde{y}' + \Gamma\tilde{y} = -\tilde{u} + g$.

Moreover, we have

$$\int_0^T \big(\varphi_n(t,y_n(t)) + \varphi_n^*(t,u_n(t)) - \langle g_n(t), y_n(t)\rangle\big)\,dt + \frac{1}{2}|y_n(T)|^2$$
$$\le \int_0^T \big(\varphi_n(t,y^*(t)) + \varphi_n^*(t,u^*(t)) - \langle g_n(t), u^*(t)\rangle\big)\,dt + \frac{1}{2}|y^*(T)|^2$$

where $(y^*, u^*)$ is the solution to (3.5). Now by assumptions (2.5) and (2.6) we have

$$\begin{aligned} &\varphi_n(t,y^*(t)) \to \varphi(t,y^*(t)) \\ &\varphi_n^*(t,u_n^*(t)) \to \varphi^*(t,u^*(t)) \quad \forall t \in [0,T], \\ &g_n(t) \to g(t) \end{aligned}$$

and this yields

$$\limsup_{n\to\infty} \int_0^T \big(\varphi_n(t,y_n) + \varphi_n^*(t,u_n) - \langle g_n, y_n\rangle\big)\,dt + \frac{1}{2}|y_n^*(T)|^2 - \frac{1}{2}|y_0|^2 = 0. \tag{3.19}$$

In order to pass to limit in (3.19) we shall use (3.18) and the convergence of $\{\varphi_n\}$ and $\{\varphi_n^*\}$ mentioned above. We set $\tilde{z}(t) = e^{\beta(t)B(t)}\tilde{y}(t)$, $z_n(t) = e^{\beta_n(t)B(t)}y_n(t)$. We have

$$\psi_n(t,\tilde{z}(t)) \le \psi_n(t,z_n(t)) + \langle A_n(t,\tilde{z}(t)), \tilde{z}(t) - z_n(t)\rangle,$$

and since $\partial\psi_n^* = A_n^{-1}$ we have also that

$$\psi_n^*(t,\theta(t)) \le \psi_n^*(t,\theta_n(t)) + \langle A_n^{-1}(t)(t,\theta(t)), \theta(t)-\theta_n(t)\rangle, \quad \text{a.e. } t\in(0,T)$$

where $\theta(t) = g(t) - \tilde{y}'(t)$, $\theta_n(t) = g_n(t) - y_n'(t)$.

Then by assumption (2.5) and Eq. (3.19) we have that

$$\limsup_{n\to\infty} \int_0^T \big(\varphi_n(t,\tilde{y}(t)) + \varphi_n^*(t, g(t)-\tilde{y}'(t)) - \langle g(t), \tilde{y}'(t)\rangle\big)\,dt + \frac{1}{2}|\tilde{y}^*(T)|^2 - \frac{1}{2}|y_0|^2 = 0,$$

and so, since as seen earlier (2.5) implies that $\varphi_n(t,z)\to\varphi(t,z)$, $\forall z\in V$, $\varphi_n^*(t,z^*)\to\varphi^*(t,z^*)$, $\forall z^*\in V'$, by the Fatou lemma, we have

$$\int_0^T \big(\varphi(t,\tilde{y}) + \varphi^*(t,\tilde{u}) - \langle g,\tilde{y}\rangle\big)\,dt + \frac{1}{2}|\tilde{y}^*(T)|^2 - \frac{1}{2}|y_0|^2 \le 0,$$

which implies as in the previous case that $\tilde{y}$ is a solution to (3.5) and therefore to (2.4) as claimed. □

## 4. Examples

The specific examples to be presented below refer to nonlinear parabolic stochastic equations which can be written in the abstract form (1.1) where $A(t)$ are subpotential monotone and continuous operators from a separable Banach space $V$ to its dual $V'$.

We briefly present below a few stochastic PDEs to which the above theorems apply. We use here the standard notations for spaces of integrable functions and Sobolev spaces $W_0^{1,p}(\mathcal{O})$, $W^{-1,p'}(\mathcal{O}) = \big(W_0^{1,p}(\mathcal{O})\big)'$, $H^k(\mathcal{O})$, $k=1,2$ on open domains $\mathcal{O}\subset\mathbb{R}^d$.

### *4.1. Nonlinear stochastic diffusion equations*

Consider the stochastic equation

$$\begin{cases} dX_t - \operatorname{div}_\xi a(t,\nabla_\xi X_t)\,dt - \dfrac{1}{2}\, b(t,\xi)\cdot\nabla_\xi(b(t,\xi)\cdot\nabla_\xi X_t)\,dt & \\ \quad = b(t,\xi)\cdot\nabla_\xi X_t\,d\beta(t), & \text{in } (0,T)\times\mathcal{O} \\ X_0 = x & \text{in } \mathcal{O} \\ X_t = 0 & \text{on } (0,T)\times\partial\mathcal{O}. \end{cases} \tag{4.1}$$

Here $a\colon (0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ is a map of gradient type, i.e.,

$$a(t,y) = \partial j(t,y), \quad \forall y\in\mathbb{R}^d,\ t\in(0,T)$$

where $j\colon(0,T)\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is convex in $y$, progressively measurable in $(t,\omega)\in[0,T)\times\Omega$ and

$$\gamma_1 + \alpha_1|y|^{p_1} \le j(t,y) \le \gamma_2 + \alpha_2|y|^{p_2}, \quad \forall y\in\mathbb{R}^d,\ \omega\in\Omega,\ t\in(0,T) \tag{4.2}$$

$$j(t,-y) \le c_1\, j(t,y) + c_2, \quad \forall y\in\mathbb{R}^d,\ t\in(0,T). \tag{4.3}$$

It should be emphasized that the mapping $r\to a(t,r)$ might be multivalued and discontinuous. As a matter of fact if $a(t,\cdot)$ is discontinuous at $r=r_j$, but left and right continuous (as happens by monotonicity) it is replaced by a multivalued maximal monotone mapping $\tilde{a}$ obtained by filling the jumps at $r = r_j$.

Eq. (4.1) is of the form (1.1) where $H = L^2(\mathscr{O})$, $V = W_0^{1,p_1}(\mathscr{O})$, $A(t) = \partial\psi(t,\cdot)$, $2 \le p_1 \le p_2 < \infty$,

$$\psi(t,u) = \int_{\mathscr{O}} j(t, \nabla u)\, d\xi, \quad \forall u \in W_0^{1,p_1}(\mathscr{O})$$

and

$$B(t)u = b(t,\xi) \cdot \nabla_\xi u = \mathrm{div}_\xi(b(t,\xi)u), \quad \forall u \in W_0^{1,p_1}(\mathscr{O}). \tag{4.4}$$

As regards the function $b(t,r)\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we assume that

$$b(t,\cdot), \qquad \frac{\partial b}{\partial r}(t,\cdot) \in \left(C([0,T];\bar{\mathscr{O}})\right)^d \tag{4.5}$$

$$r \to b(t,\cdot) + \alpha r \quad \text{is monotone for some } \alpha \ge 0, \tag{4.6}$$

$$\mathrm{div}_\xi b(t,\xi) = 0, \qquad b(t,\xi)\cdot\nu(\xi) = 0 \quad \forall \xi \in \partial\mathscr{O} \tag{4.7}$$

where $\nu$ is the normal to $\partial\mathscr{O}$. (The boundary $\partial\mathscr{O}$ is assumed to be of class $C^1$.) Here $\mathrm{div}_\xi b$ is taken in the sense of distributions on $\mathscr{O}$.

Then (4.4) defines a linear continuous operator $B(t)$ from $V$ to $H = L^2(\mathscr{O})$ which as early seen is densely defined skew-symmetric, that is $-B(t) \subset B^*(t) \forall t \in [0,T]$. Moreover, $B(t)$ is $m$-dissipative in $L^2(\mathscr{O})$, that is the range of $u \to u - B(t)u$ is all of $L^2(\mathscr{O})$. Indeed for each $f \in L^2(\mathscr{O})$ the equation $u - B(t)u = f$ has the solution

$$u(\xi) = \int_0^\infty e^{-s} f(Z(s,\xi))\, ds, \quad \forall \xi \in \mathscr{O},$$

where $s \to Z(s,\xi)$ is the differential flow defined by equation

$$\frac{dZ}{ds} = b(t,Z), \quad s \ge 0,\ Z(0) = \xi. \tag{4.8}$$

(By assumptions (4.6), (4.7), it follows that $t \to Z(t,\xi)$ is well defined on $[0,\infty)$.)

Hence, for each $t \in [0,T]$, $B(t)$ generates a $C_0$-group $(e^{sB(t)})_{s\in\mathbb{R}}$ on $L^2(\mathscr{O})$ which is given by

$$\left(e^{B(t)s} f\right)(\xi) = f(Z(s,\xi)), \quad \forall f \in L^2(\mathscr{O}), s \in \mathbb{R}.$$

It is also clear that $e^{B(t)s} V \subset V$ for all $s \ge 0$.

**Remark 4.1.** Assumptions (4.5)–(4.7) can be weakened to discontinuous multivalued mappings $\xi \to b(t,\xi)$ satisfying (4.6), (4.7) such that the solution $Z = Z(s,\xi;t)$ to the characteristic system (4.8) is differentiable in $t$. The details are omitted.

The corresponding random differential equation (1.2) has the form

$$\begin{cases} \dfrac{\partial y}{\partial t} - e^{\beta(t)B(t)} \mathrm{div}_\xi(a(t, \nabla_\xi e^{-\beta(t)B(t)} y)) & \\ \quad + \displaystyle\int_0^{\beta(t)} e^{s\,B(t)} \dot{B}(t) e^{-s\,B(t)} y\, ds = 0, & \text{in } (0,T)\times\mathscr{O}, \\ y(0,\xi) = x(\xi) & \text{in } \mathscr{O}, \\ y(t,\xi) = 0 & \text{on } (0,T)\times\partial\mathscr{O}. \end{cases} \tag{4.9}$$

Then by Theorem 2.1 we have the following.

**Theorem 4.1.** *There exists a solution $X$ to (4.1) such that $\mathbb{P}$-a.s. $X \in L^{p_1}(0,T;W_0^{1,p_1}(\mathcal{O})) \cap L^{p_2'}(0,T;W^{-1,p_2'}(\mathcal{O})) \cap L^\infty(0,T;L^2(\mathcal{O}))$.*

We also note that in line with Theorem 2.2 if $X_n$, $n \in \mathbb{N}$, are solutions to equations

$$\begin{cases} dX_t^n - \mathrm{div}_\xi a^n(t,\nabla_\xi X_t^n)\,dt - \dfrac{1}{2} b_n(t,\xi)\cdot\nabla\left(b_n(t,\xi)X_t^n\right)\,dt \\ \quad = b_n(t,\xi)\cdot\nabla_\xi X_t^n\,d\beta^n(t), \quad (t,\xi) \text{ in } (0,T)\times\mathcal{O}, \\ X_0^n = x \quad \text{in } \mathcal{O}, \\ X_t^n = \text{on } (0,T)\times\partial\mathcal{O}, \end{cases} \tag{4.10}$$

where $b_n \to b$ uniformly on $[0,T]\times\mathcal{O}$ and $a_n(t,y) \to a(t,y)$, $a_n^{-1}(t,y) \to a^{-1}(t,y)$, $\beta_n(t) \to \beta(t)$ for all $y \in \mathbb{R}^d$, $t \in [0,T)$, then $X^n \to X$ weakly in $L^{p_1}(0,T;W_0^{1,p_1}(\mathcal{O}))$. Standard examples refer to structural stability, PDEs as well as to homogenization type results for Eq. (4.1). In the latter case $a_n(t,z) = a(t,nz)$ where $a(t,\cdot)$ is periodic (see e.g., [2]).

Eq. (4.1) is relevant in the mathematical description of nonlinear diffusion processes perturbed by a Brownian distribution with coefficient transport term $b(t,\xi)\cdot\nabla_\xi X$.

The assumption $p_1 \geq 2$ was taken here for technical reason required by the functional framework we work in and this excludes several relevant examples. For instance, the limit case $p_1 = 1$ which corresponds to the nonlinear diffusion function $a(t,y) = \rho\frac{y}{|y|_d}$, $\rho > 0$, which is relevant in material science and image restoring techniques (see e.g. [6,5]) is beyond our approach and requires a specific treatment (see also [7] for the treatment of a similar problem with additive and continuous noise).

In 2-D the appropriate functional setting to treat such a problem is $V = BV(\mathcal{O})$ the space of functions with bounded variation on $\mathcal{O}$ with the norm $\varphi(y)$ and $H = L^2(\mathcal{O})$. Here $\varphi(y) = \|Dy\|(\mathcal{O}) + \int_{\partial\mathcal{O}}|\gamma_0(y)|d\mathcal{H}$, $y \in V$, $\|Dy\|(\mathcal{O})$ is the variation of $y \in V$, $\gamma_0(y)$ is the trace on $\partial\mathcal{O}$ and $\mathcal{H}$ is the Hausdorff measure on $\partial\mathcal{O}$. We recall that the norm $\varphi$ is just the lower semicontinuous closure of the norm of Sobolev space $W_0^{1,1}(\mathcal{O})$ (see e.g., [2, p. 438]). Then the approach developed in Section 3 can be adapted to the present situation though $V$ is not reflexive. We expect to treat this limit case in a forthcoming work. (On these lines see also [4].)

### *4.2. Linear diffusion equations with nonlinear Neumann boundary conditions*

Consider the equation

$$\begin{cases} dX_t - \Delta X_t\,dt - \dfrac{1}{2} b(t,\xi)\cdot\nabla_\xi\left(b(t,\xi)\cdot\nabla_\xi X_t\right)dt \\ \quad = b(t,\xi)\cdot\nabla_\xi X_t\,d\beta(t) & \text{in } [0,T]\times\mathcal{O} \\ \dfrac{\partial}{\partial\nu}X_t + \zeta(t,X_t) \ni 0 & \text{on } [0,T]\times\partial\mathcal{O} \\ X_0 = x & \text{in } \mathcal{O} \end{cases} \tag{4.11}$$

where $\zeta(t,r) = \partial j_0(t,r)$, $\forall t \in (0,T)$, $r \in \mathbb{R}$ and $j_0(t,\cdot)$ is a lower semicontinuous convex function on $\mathbb{R}$ such that

$$\gamma_1 + \alpha_1|y|^2 \leq j_0(t,y) \leq \gamma_2 + \alpha_2|y|^2, \quad \forall y \in \mathbb{R}, t \in (0,T)$$

and $\alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1,2$.

Assume also that (4.3) holds and that $b = b(t, \cdot)$ satisfies conditions (4.5)–(4.7). Then we may apply Theorems 2.1–2.3, where $V = H^1(\mathscr{O})$, $H = L^2(\mathscr{O})$ and

$$\psi(t, y) = \frac{1}{2}\int_{\mathscr{O}} |\nabla y|^2 \, d\xi + \int_{\partial\mathscr{O}} j_0(t, y)\, d\xi, \quad \forall y \in V.$$

It follows so the existence of a solution $X \in L^2(0, T; V) \cap W^{1,2}([0, T]; V')$ to (4.11) and also the structural stability of (4.11) with respect to $b$. Problems of this type arise in thermostat control. In this case

$$\zeta(t, y) = \begin{cases} \big(\alpha_1(t)H(y) + \alpha_2(t)H(-y)\big)y & \text{if } y \neq 0 \\ [-\alpha_2(t), \alpha_1(t)] & \text{if } y = 0 \end{cases}$$

where $\alpha_i > 0$, $\forall t \in [0, T[$ and $H$ is the Heaviside function.

*4.3. Nonlinear stochastic porous media equations*

Consider the equation

$$\begin{cases} dX_t - \Delta_\xi \phi(t, X_t)\, dt - \dfrac{1}{2}\, b(t, \xi) \cdot \nabla_\xi (-\Delta)^{-1}(b(t, \xi) \cdot \nabla_\xi((-\Delta)^{-1}X_t))\, dt \\ \quad = b(t, \xi) \cdot \nabla_\xi (-\Delta)^{-1} X_t \, d\beta(t), \quad \text{in } (0, T) \times \mathscr{O} \\ X_0 = x \quad \text{in } \mathscr{O} \\ X_t = 0 \quad \text{on } (0, \infty) \times \partial\mathscr{O}. \end{cases} \tag{4.12}$$

Here $\mathscr{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$ is a bounded open domain and $(-\Delta)^{-1}$ is the inverse of the operator $A_0 = -\Delta$, $D(A_0) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$. The function $\phi$: $(0, T) \times \mathbb{R}^d \to \mathbb{R}$ is assumed to satisfy the following conditions.

(k) *$\phi = \phi(t, r)$ is monotonically decreasing in $r$, measurable in $t$ and its potential*

$$j(t, r) = \int_0^r \phi(t, \tau)\, d\tau, \quad t \in (0, T)$$

*satisfies the growth conditions*

$$\gamma_1 + \alpha_1 |r|^{p_1} \le j(t, r) \le \gamma_2 + \alpha_2 |r|^{p_2}, \quad \forall r \in \mathbb{R}, \omega \in \Omega, t \in [0, T] \tag{4.13}$$

$$j(t, -r) \le c_1\, j(t, r) + c_2, \quad \forall r \in \mathbb{R}, t \in (0, T) \tag{4.14}$$

*where $\frac{6}{5} \le p_1 \le p_2 < \infty$ if $d = 3$, $1 < p_1 \le p_2 < \infty$ if $d = 1, 2$.*

Then Eq. (4.12) can be written as (1.1), where $H = H^{-1}(\mathscr{O})$, $V = L^{p_1}(\mathscr{O})$ and $A(t) = \partial\psi(t, \cdot)$ where $\psi(t, \cdot)$: $H \to \bar{\mathbb{R}}$ is defined by

$$\psi(t, y) = \begin{cases} \displaystyle\int_{\mathscr{O}} j(t, y)\, d\xi & \text{if } y \in H^{-1}(\mathscr{O}),\ j(t, y) \in L^1(\mathscr{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

and $B(t)$, $t \in \mathbb{R}^+$ is defined by

$$B(t)u = b(t, \xi) \cdot \nabla((-\Delta)^{-1}u), \quad u \in V. \tag{4.15}$$

The space $V'$ is in this case the dual of $V = L^{p_1}(\mathscr{O})$ with $H^{-1}(\mathscr{O})$ as the *pivot* space. By the Sobolev embedding theorem it is easily seen that since $p_1 \ge \frac{6}{5}$ we have $V \subset H^{-1}(\mathscr{O})$. The scalar product on $H$ is defined by

$$\langle u, v\rangle_{H^{-1}(\mathscr{O})} = u(z), \quad z = (-\Delta)^{-1} v.$$

It is well known that $A(t)X = -\Delta_\xi \phi(t, X)$ is indeed the subdifferential of $\psi(t, \cdot)$ in $H^{-1}(\mathscr{O})$ (see e.g., [5, p. 68]).

As regards $b$: $[0, T] \times \bar{\mathscr{O}} \to \mathbb{R}^d$ we assume that conditions (4.5)–(4.7) hold.

We note that for each $t \in [0, T]$, $B(t) \in L(V, H^{-1}(\mathscr{O}))$ is densely defined and skew-symmetric on $H^{-1}(\mathscr{O}) = H$. Indeed we have

$$\begin{aligned}\langle B(t)u, u\rangle &= \int_{\mathscr{O}} \operatorname{div}(b(t, \xi)(-\Delta)^{-1}u) \cdot (-\Delta)^{-1}u \, d\xi \\ &= \frac{1}{2}\int_{\mathscr{O}} b(t, \xi) \cdot \nabla|(-\Delta)^{-1}u(\xi)|^2 \, d\xi = 0,\end{aligned}$$

because $\operatorname{div}_\xi b = 0$ and $b(t, \xi) \cdot \nu(\xi) = 0$ on $\partial\mathscr{O}$.

Moreover, for each $t \in [0, T]$, $B(t)$ is $m$-dissipative on $H^{-1}(\mathscr{O})$. Indeed for each $f \in H^{-1}(\mathscr{O})$, the equation $u - B(t)u = f$ can be equivalently written as $v = (-\Delta)^{-1}u$, where

$$\begin{aligned}&-\Delta v - b(t, \cdot) \cdot \nabla v = f \quad \text{in } \mathscr{O}, \\ &v = 0 \quad \text{on } \partial\mathscr{O}.\end{aligned}$$

By the Lax–Milgram lemma the latter has a unique solution $v \in H_0^1(\mathscr{O})$ and therefore $u \in H^{-1}(\mathscr{O})$ as claimed. Moreover, if $f \in L^{p_1}(\mathscr{O})$ and $\partial\mathscr{O}$ is of class $C^2$ then by the Agmon–Douglis–Nirenberg theorem $v \in W^{2,p_1}(\mathscr{O}) \cap W_0^{1,p_1}(\mathscr{O})$ and so $u \in V$.

Hence, $B(t)$ generates a $C_0$-group $(e^{sB(t)})_{s\in\mathbb{R}}$ on $H = H^{-1}(\mathscr{O})$ which leaves $V = L^{p_1}(\mathscr{O})$ invariant.

Then we may apply Theorem 2.1 as well as the approximation Theorem 2.2 to the present situation. We obtain the following.

**Theorem 4.2.** *There is a unique solution $X$ to (4.12) such that $\mathbb{P}$-a.s. $X \in L^{p_1}(0, T; L^{p_1}(\mathscr{O})) \cap L^\infty(0, T; H^{-1}(\mathscr{O}))$. Moreover, the solution $X$ is a limit of approximating solutions when the Brownian motion $\beta$ is approximated by a sequence of smooth processes.*

Moreover, if $\phi_n \to \phi$ and $\phi_n^* \to \phi^*$, $b_n \to b$ we find by Theorem 2.2 that the corresponding solutions $X_n$ to (4.12) are convergent to solution $X$ to (4.11). The details are omitted.

The existence for the stochastic porous media equation of the form

$$\begin{cases} dX_t - \Delta_\xi \phi(X_t)\, dt = \sigma(X_t)\, dW(t) & \text{in } (0, T) \times \mathscr{O} \\ X_0 = x & \text{in } \mathscr{O} \\ X_t = 0 & \text{on } (0, T) \times \partial\mathscr{O}, \end{cases}$$

when $W_t$ is a Wiener process of the form

$$W(t, \xi) = \sum_{k=1}^{\infty} \mu_k \, e_k(\xi)\, \beta_k(t)$$

with $\sum_{k=1}^\infty \mu_k^2 \lambda_k^2 < \infty$, $\Delta e_k = -\lambda_k e_k$ in $\mathscr{O}$, $e_k \in H_0^1(\mathscr{O})$, and $\sigma = \sigma(x)$ is a linear continuous operator, was studied in [7–9,11]. Note that in this case the noise term can also be written in our form with commuting and, contrary to our paper, bounded operators $B_j$. Here the multiplicative term $\sigma(X_t) = b \cdot \nabla(-\Delta)^{-1} X_t$ is however discontinuous on the space $H^{-1}(\mathscr{O})$ and so Theorem 4.2 is from this point of view different and in this sense more general.

Eq. (4.12) models diffusion processes and the motion of fluid flows in porous media.

**Remark 4.2.** Theorem 2.4 and Remark 2.1 are also valid in the current setup.

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## Appendix. Convex functions

We summarize in this paragraph some facts about convex functions, which we have used in our paper.

Given a convex and lower-semicontinuous function $\phi : Y \to \bar{\mathbb{R}} = (-\infty, \infty]$ we denote by $\partial\phi : Y \to Y'$ (the dual space) the *subdifferential* of $\phi$, i.e.

$$\partial\phi(y) := \left\{ z \in Y' : \phi(y) - \phi(u) \le \langle y - u, z\rangle, \forall u \in Y \right\}. \tag{A.1}$$

(Here $\langle\cdot,\cdot\rangle$ is the duality pairing between $Y$ and $Y'$). The function $\phi^* : Y' \to Y$ defined by

$$\phi^*(z) = \sup\left\{\langle y, z\rangle - \phi(y) : y \in Y\right\}, \tag{A.2}$$

is called the conjugate of $\phi$. Similarly to $\phi$ it is convex lower semicontinuous function on $Y'$. Also we notice the following key conjugacy formulae (see e.g. [10, p. 89]). If $y \in Y$, and $z \in Y'$

$$\phi(y) + \phi^*(z) \ge \langle y, z\rangle \tag{A.3}$$

$$\phi(y) + \phi^*(z) = \langle y, z\rangle \quad \text{iff} \quad z \in \partial\phi(y). \tag{A.4}$$

A vector $x^*$ is said to be a subgradient of a convex function $\phi$ at a point $x$ if

$$\phi(z) \ge \phi(x) + \langle x^*, z - x\rangle. \tag{A.5}$$

Moreover, straightforward calculations give

$$\phi^*(x^*) = \phi_y^*(x^*) - \langle y, x^*\rangle, \tag{A.6}$$

whenever $\phi(x) = \phi_y(x + y)$.

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