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Occupation times of intervals until first passage times for spectrally negative Lévy processes

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Abstract

In this paper, we identify Laplace transforms of occupation times of intervals until first passage times for spectrally negative Lévy processes. New analytical identities for scale functions are derived and therefore the results are explicitly stated in terms of the scale functions of the process. Applications to option pricing and insurance risk models are also presented.

Keywords: Occupation times; Spectrally negative Lévy processes; Fluctuation theory; Scale functions

1. Introduction and main results

In this paper, we are interested in the joint Laplace transforms of

$$\left(\tau_0^-, \int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s\right) \quad \text{and} \quad \left(\tau_c^+, \int_0^{\tau_c^+} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s\right),$$

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where $X = (X_t)_{t\geq 0}$ is a spectrally negative Lévy process, where

$$\tau_0^- = \inf\{t > 0 \colon X_t < 0\} \quad \text{and} \quad \tau_c^+ = \inf\{t > 0 \colon X_t > c\},$$

and where $0 \leq a \leq b \leq c$. Recently, Landriault et al. [10] and Kyprianou et al. [9] have studied occupation times of half lines for spectrally negative Lévy processes, though the latter article considers a more general process, namely a refracted spectrally negative Lévy process. The main difference between this paper and the papers [10,9] is that by using some of the techniques in [8], we find considerably simpler expressions, which further allow us to establish a more general set of identities involving occupation times of spectrally negative Lévy processes. Note that occupation times appear both in option pricing and in insurance risk models; we will mention two applications later on.

We now briefly introduce spectrally negative Lévy processes and the associated scale functions, before stating our main results. Let $X = (X_t)_{t\geq 0}$ on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a spectrally negative Lévy process, that is a process with stationary and independent increments and no positive jumps. Hereby we exclude the case that X is the negative of a subordinator, i.e. we exclude the case of X having decreasing paths. The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. As the Lévy process X has no positive jumps, its Laplace transform exists and is given by

$$\mathbb{E}\left[e^{\lambda X_t}\right] = e^{t\psi(\lambda)},$$

for $\lambda \geq 0$, where

$$\psi(\lambda) = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty \left(e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{(0,1]}(z)\right) \Pi(dz),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and where Π is a σ-finite measure on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\Pi(dz) < \infty.$$

We call the measure Π the Lévy measure of X, while we refer to (γ, σ, Π) as the Lévy triplet of X. Note that for convenience we define the Lévy measure in such a way that it is a measure on the positive half line instead of the negative half line. Further, note that $\mathbb{E}[X_1] = \psi'(0+)$. The process X has paths of bounded variation if and only if $\sigma = 0$ and $\int_0^1 z\Pi(dz) < \infty$. In that case we denote by $\mathtt{d} := \gamma + \int_0^1 z\Pi(dz)$ the so-called drift of X.

For an arbitrary spectrally negative Lévy process, the Laplace exponent ψ is strictly convex and $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Thus, there exists a function $\Phi \colon [0, \infty) \to [0, \infty)$ defined by $\Phi(q) = \sup\{\lambda \geq 0 \mid \psi(\lambda) = q\}$ (the right inverse of ψ) such that

$$\psi(\Phi(q)) = q, \quad q \geq 0.$$

We have that $\Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) \geq 0$.

We now recall the definition of the q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of the process X is defined on $[0, \infty)$ as the continuous function with Laplace transform given by

$$\int_0^\infty e^{-\lambda y} W^{(q)}(y) dy = \frac{1}{\psi(\lambda) - q}, \quad \text{for } \lambda > \Phi(q), \tag{1}$$

with the following definition for the initial value: $W^{(q)}(0) := \lim_{x\downarrow 0} W^{(q)}(x)$. This function is unique, positive and strictly increasing for $x \geq 0$ and is further continuous for $q \geq 0$. We extend

$W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. We write $W = W^{(0)}$ when $q = 0$. We will also frequently use the following function

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}. \tag{2}$$

We recall some of the properties of the q-scale function $W^{(q)}$ and its use in fluctuation theory. Most results are taken, or can easily be derived, from [7]. The initial value of $W^{(q)}$ is known to be

$$W^{(q)}(0) = \begin{cases} 1/\mathtt{d} & \text{when } \sigma = 0 \quad \text{and} \quad \int_0^1 z\Pi(\mathrm{d}z) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $a \in \mathbb{R}$, define

$$\tau_a^- = \inf\{t > 0 \colon X_t < a\},$$

and

$$\tau_a^+ = \inf\{t > 0 \colon X_t > a\},$$

with the convention $\inf \emptyset = \infty$. It is well known that, if $a > 0$ and $x \le a$, then the solution to the two-sided exit problem is given by

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^+}; \tau_a^+ < \tau_0^-\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \tau_a^+\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{4}$$

where, for a random variable Y and an event A, $\mathbb{E}[Y; A] := \mathbb{E}[Y\mathbf{1}_A]$. Also, it is known that, for $a \le x \le b$ and f a positive, measurable function, we have

$$\begin{aligned} &\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} f(X_{\tau_a^-}); \tau_a^- < \tau_b^+\right] \\ &\quad = f(a)\frac{\sigma^2}{2}\left[W^{(q)\prime}(x-a) - W^{(q)}(x-a)\frac{W^{(q)\prime}(b-a)}{W^{(q)}(b-a)}\right] \\ &\qquad + \int_0^{b-a} \mathrm{d}y \int_y^\infty f(y-\theta+a)\Pi(\mathrm{d}\theta) \\ &\qquad \times \left[\frac{W^{(q)}(b-a-y)}{W^{(q)}(b-a)}W^{(q)}(x-a) - W^{(q)}(x-a-y)\right], \end{aligned} \tag{5}$$

where $W^{(q)\prime}(x)$ is the derivative of $W^{(q)}(x)$, which is well-defined if $\sigma > 0$. The first term of this identity corresponds to the case when $X_{\tau_a^-} = a$, a behaviour called creeping.

For more details on spectrally negative Lévy processes and fluctuation identities, the reader is referred to [7]. Further information, examples and numerical techniques related to the computation of scale functions can be found in [6], see also Remark 1.1 below.

1.1. Main results

For our main results we first need to introduce three auxiliary functions. We note that by taking Laplace transforms on both sides and using (1) we can easily check that the following two equalities hold:

$$\begin{aligned}(q-p)\int_0^a W^{(p)}(a-y)W^{(q)}(y)\mathrm{d}y &= W^{(q)}(a)-W^{(p)}(a),\\ (q-p)\int_0^a W^{(p)}(a-y)Z^{(q)}(y)\mathrm{d}y &= Z^{(q)}(a)-Z^{(p)}(a).\end{aligned} \tag{6}$$

We now introduce the following two functions for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\mathcal{W}_a^{(p,q)}(x) &:= W^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y\\ &= W^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y,\\ \mathcal{Z}_a^{(p,q)}(x) &:= Z^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-y)Z^{(p)}(y)\mathrm{d}y\\ &= Z^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)Z^{(p)}(y)\mathrm{d}y,\end{aligned} \tag{7}$$

where the second representations of $\mathcal{W}_a^{(p,q)}(x)$ and $\mathcal{Z}_a^{(p,q)}(x)$ follow from (6). We will use both representations throughout the text. We further introduce, for $p \geq 0$ and $q \in \mathbb{R}$ such that $p+q \geq 0$, the function

$$\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x}\left(1 + q\int_0^x \mathrm{e}^{-\Phi(p)y}W^{(p+q)}(y)\mathrm{d}y\right), \quad x \in \mathbb{R}.$$

Note that $\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x}$ for $x \leq 0$ and that the Laplace transform of $\mathcal{H}^{(p,q)}$ on $[0,\infty)$ is explicitly given by

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mathcal{H}^{(p,q)}(x)\mathrm{d}x = \frac{1}{\lambda-\Phi(p)}\left(1+\frac{q}{\psi(\lambda)-p-q}\right), \quad \lambda > \Phi(p+q).$$

We now state our two main results.

Theorem 1. *For* $0 \leq a \leq b \leq c$, $p, q \geq 0$ *and* $0 \leq x \leq c$,

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^- - q\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_0^- < \tau_c^+\right] = \mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z\\ &\quad - \frac{\mathcal{Z}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z}{\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}\\ &\quad \times\left(\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\right).\end{aligned} \tag{8}$$

Theorem 2. *For $0 \leq a \leq b \leq c$, $p, q \geq 0$ and $0 \leq x \leq c$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_c^+ < \tau_0^-\right] = \frac{\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}{\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}. \tag{9}$$

Note that the two theorems generalize (3) and (4). From these two theorems we can derive the following corollaries.

Corollary 1. (i) *For $0 \leq a \leq b$ and $p, q, x \geq 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_0^- < \infty\right] = \mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z$$
$$- \frac{\frac{p}{\Phi(p)} + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{Z}_a^{(p,q)}(y)\mathrm{d}y}{1 + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{W}_a^{(p,q)}(y)\mathrm{d}y}$$
$$\times\left(\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\right),$$

where $\lim_{p\to 0} p/\Phi(p) = \psi'(0+) \vee 0$ in the case $p = 0$.

(ii) *For $a, p, q, x \geq 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s}; \tau_0^- < \infty\right] = \mathcal{Z}_a^{(p,q)}(x) - \frac{\frac{p+q}{\Phi(p+q)} - q\int_0^a \mathrm{e}^{-\Phi(p+q)y}Z^{(p)}(y)\mathrm{d}y}{1 - q\int_0^a \mathrm{e}^{-\Phi(p+q)y}W^{(p)}(y)\mathrm{d}y}\mathcal{W}_a^{(p,q)}(x).$$

Corollary 2. (i) *For $-\infty < a \leq b \leq c$, $p, q \geq 0$ and $x \leq c$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_c^+ < \infty\right] = \frac{\mathcal{H}^{(p,q)}(x-a) - q\int_b^x W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y-a)\mathrm{d}y}{\mathcal{H}^{(p,q)}(c-a) - q\int_b^c W^{(p)}(c-y)\mathcal{H}^{(p,q)}(y-a)\mathrm{d}y}.$$

(ii) *For $b \leq c$, $p, q \geq 0$ and $x \leq c$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+} \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s}; \tau_c^+ < \infty\right] = \frac{\mathcal{H}^{(p+q,-q)}(x-b)}{\mathcal{H}^{(p+q,-q)}(c-b)}.$$

Corollary 3. (i) *Assume $\psi'(0+) > 0$. Then for $-\infty < a \leq b$, $q \geq 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}\right] = \frac{Z^{(q)}(x-a) - q\int_b^x W(x-y)Z^{(q)}(y-a)\mathrm{d}y}{1 + \frac{q}{\psi'(0+)}\int_0^{b-a} Z^{(q)}(y)\mathrm{d}y}.$$

(ii) *Assume $\psi'(0+) > 0$. Then for $q \geq 0$ and $b, x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s}\right] = \frac{\psi'(0+)\Phi(q)}{q}\mathcal{H}^{(q,-q)}(x-b).$$

(iii) *Assume* $\psi'(0+) < 0$. *Then for* $-\infty < a \le b,\ q \ge 0$ *and* $x \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}\right] &= Z^{(q)}(x-a) - q\int_b^x W(x-y)Z^{(q)}(y-a)\mathrm{d}y \\ &\quad - \frac{q\int_0^{b-a}\mathrm{e}^{-\Phi(0)y}Z^{(q)}(y)\mathrm{d}y}{\psi'(\Phi(0)) + q\int_0^{b-a}\mathrm{e}^{-\Phi(0)y}\mathcal{H}^{(0,q)}(y)\mathrm{d}y} \\ &\quad \times \left(\mathcal{H}^{(0,q)}(x-a) - q\int_b^x W(x-y)\mathcal{H}^{(0,q)}(y-a)\mathrm{d}y\right).\end{aligned}$$

(iv) *Assume* $\psi'(0+) < 0$. *Then for* $q \ge 0$ *and* $a, x \in \mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s}\right] = Z^{(q)}(x-a) - \frac{\Phi(q)-\Phi(0)}{\Phi(q)}\mathcal{H}^{(0,q)}(x-a).$$

We remark that Corollary 3(ii) was derived earlier in [10, Corollary 1]. Note that regarding Corollary 3, due to the long-term behaviour of X, if $\psi'(0+) \le 0$, then $\int_0^\infty \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s = \infty$ a.s., if $\psi'(0+) \ge 0$, then $\int_0^\infty \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s = \infty$ a.s. and if $\psi'(0+) = 0$, then $\int_0^\infty \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s = \infty$ a.s.

We also mention the following useful identities,

$$\begin{aligned}&\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z \\ &\quad = W^{(p)}(x) + q\int_a^b W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z, \\ &\mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z \\ &\quad = Z^{(p)}(x) + q\int_a^b W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z.\end{aligned} \tag{10}$$

These two identities can be proved easily by setting first $x = a = b$ in Theorem 2 and comparing with (3) and then setting $x = a = b$ in Theorem 1 and comparing with (4). Similarly,

$$\begin{aligned}&\mathcal{H}^{(p,q)}(x-a) - q\int_b^x W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y-a)\mathrm{d}y \\ &\quad = \mathrm{e}^{\Phi(p)(x-a)} + q\int_a^b W^{(p)}(x-y)\mathcal{H}_a^{(p,q)}(y-a)\mathrm{d}y,\end{aligned} \tag{11}$$

which can be proved easily by setting $x = a = b$ in Corollary 2(i) and comparing with the identity in Corollary 2(i) for $q = 0$. Note that (10) and (11) lead to alternative identities for the main theorems and corollaries. Further, (10) and (11) will also be used to prove Corollary 1(i) and Corollary 3(i) respectively.

Remark 1.1. The expressions appearing in Theorems 1 and 2 and Corollaries 1–3 are all given in terms of scale functions for which in general only the Laplace transform is known. However, there are examples of spectrally negative Lévy processes for which an explicit formula (though the degree of explicitness can vary case by case) exists for the scale function $W^{(q)}$. For instance, in the case where X is a compound Poisson process plus drift whose jump distribution has a rational Laplace transform (i.e. the Laplace transform is a ratio of two polynomials), then the Laplace transform of the scale function $1/(\psi(\lambda) - q)$ is also a rational function and an explicit expression (in terms of the roots of $\lambda \mapsto \psi(\lambda) - q$) for the scale function $W^{(q)}$, for any $q \geq 0$, can then be found by the method of partial fractions. We treat a specific example of this class in Example 1.1 below. Another example is the case where $\psi(\lambda) = (\lambda + \eta)^\alpha - \eta^\alpha$ for $1 < \alpha < 2$ and $\eta \geq 0$, which corresponds to a so-called spectrally negative tempered stable process, which is a zero-mean Lévy process with no Gaussian component and Lévy measure $\Pi(\mathrm{d}z) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \mathrm{e}^{-\eta z} z^{-\alpha-1} \mathrm{d}z$, where $\Gamma(\cdot)$ denotes the Gamma function. In this case, the scale function is given by

$$W^{(q)}(x) = \mathrm{e}^{-\eta x} x^{\alpha-1} E_{\alpha,\alpha}\left((q + \eta^\alpha) x^\alpha\right), \quad x, q \geq 0,$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag-Leffler function defined by $E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}$, see Chazal et al. [3, Example 3.2]. For the above mentioned examples one is then able to get a more explicit expression for the functions appearing in the aforementioned theorems and corollaries.

On the other hand (when considering other examples for which the scale function is not explicit), there are good numerical methods for dealing with Laplace inversion (cf. [6, Section 5] which deals specifically with Laplace inversion of the scale function and its derivative) and these can be used to numerically evaluate the expressions in Theorems 1 and 2 and Corollaries 1–3. Although the Laplace transforms of $Z^{(q)}$ and $\mathcal{H}^{(p,q)}$ are known and thus these functions can be computed via a single Laplace inversion, this is not true in general for the functions $\mathcal{W}_a^{(p,q)}$ and $\mathcal{Z}_a^{(p,q)}$ due to the appearance of incomplete convolutions. This also means that several of our identities cannot be computed via a single Laplace inversion and more complicated numerical procedures involving Laplace inversion and computation of iterated integrals are needed.

Our results improve the results from [10,9] (in the non refraction case) in several ways. First, we consider occupation times of an arbitrary interval, not just intervals of the form $(-\infty, b)$. Second, we deal with the case $p > 0$. Third, we deal with a general starting point x; note that the expressions simplify when $x \leq b$ or $x \leq a$. Finally, our expressions are considerably simpler than the ones derived in [10,9]. To illustrate this consider Corollary 1(i) with $p = 0, a = 0$ and $x = b$. Then

$$\mathbb{E}_b\left[\mathrm{e}^{-q \int_0^{\tau_0^-} \mathbf{1}_{(0,b)}(X_s)\mathrm{d}s}; \tau_0^- < \infty\right] = Z^{(q)}(b) - \frac{\left(\psi'(0+) \vee 0\right) + q \int_0^b \mathrm{e}^{-\Phi(0)y} Z^{(q)}(y)\mathrm{d}y}{1 + q \int_0^b \mathrm{e}^{-\Phi(0)y} W^{(q)}(y)\mathrm{d}y} W^{(q)}(b),$$

which is a more compact expression and easier to evaluate than the one in Theorem 2 of [10] and Corollary 1(ii) of [9] (in the no refraction case). Below we give an example for which we compute explicitly the functions appearing in Theorems 1 and 2.

Example 1.1. In this example we let X be a compound Poisson process plus drift, and possibly perturbed by Brownian motion, with a hyperexponential jump distribution, i.e.

$$X_t = \mathtt{d}t + \sigma B_t - \sum_{i=1}^{N_t} Y_i,$$

where $\sigma \geq 0, \mathtt{d} > 0$ if $\sigma = 0$ and $\mathtt{d} \in \mathbb{R}$ if $\sigma > 0$, $\{B_t : t \geq 0\}$ is a Brownian motion, $\{N_t : t \geq 0\}$ is a Poisson process with intensity $\eta > 0$ independent of $\{B_t : t \geq 0\}$ and $Y_1, Y_2, \ldots$

are i.i.d. positive random variables, independent of $\{B_t : t \geq 0\}$ and $\{N_t : t \geq 0\}$, with common probability density function given by

$$z \mapsto \sum_{i=1}^{n} a_i \alpha_i \mathrm{e}^{-\alpha_i z}, \quad z > 0,$$

where n is a positive integer, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $\sum_{i=1}^{n} a_i = 1$, where $a_i > 0$ for all $i = 1, \ldots, n$. The Laplace exponent ψ of X is given by

$$\psi(\lambda) = \log \mathbb{E}\left[\mathrm{e}^{\lambda X_1}\right] = \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 - \eta + \eta \sum_{i=1}^{n} \frac{a_i \alpha_i}{\lambda + \alpha_i}, \quad \lambda > -\alpha_1.$$

Denote (with abuse of notation) by $\psi(\lambda)$ the right hand side of the above equation and note that this expression is well defined for all $\lambda \in \mathbb{R} \setminus \{-\alpha_1, \ldots, -\alpha_n\}$. We assume for convenience that $q > 0$ or $\psi'(0) \neq 0$. Then by the method of partial fractions,

$$\begin{aligned}
\frac{1}{\psi(\lambda) - q} &= \frac{1}{\psi(\lambda) - q} \times \frac{\prod_{i=1}^{n} (\lambda + \alpha_i)}{\prod_{i=1}^{n} (\lambda + \alpha_i)} \\
&= \frac{\prod_{i=1}^{n} (\lambda + \alpha_i)}{A \prod_{i=1}^{N} (\lambda - \theta_i^{(q)})} \\
&= \frac{1}{A} \sum_{i=1}^{N} \frac{\left(\prod_{j=1}^{n} (\theta_i^{(q)} + \alpha_j)\right) \Big/ \prod_{j=1, j \neq i}^{N} (\theta_i^{(q)} - \theta_j^{(q)})}{\lambda - \theta_i^{(q)}} \\
&= \sum_{i=1}^{N} \frac{1/\psi'(\theta_i^{(q)})}{\lambda - \theta_i^{(q)}}, \quad \lambda \in \mathbb{R} \setminus \left(\{\theta_1^{(q)}, \ldots, \theta_N^{(q)}\} \cup \{\alpha_1, \ldots, \alpha_n\}\right),
\end{aligned} \tag{12}$$

where $N = n + 1 + \mathbf{1}_{\{\sigma > 0\}}$, $A = \begin{cases} \mathrm{d} & \text{if } \sigma = 0 \\ \frac{1}{2}\sigma^2 & \text{if } \sigma > 0 \end{cases}$ and $\theta_1^{(q)} > \theta_2^{(q)} > \cdots > \theta_N^{(q)}$ are the roots of $\lambda \mapsto \psi(\lambda) - q$, which, one can easily show, satisfy $\theta_1^{(q)} = \Phi(q)$ and

$$\begin{cases} -\alpha_n < \theta_{n+1}^{(q)} < -\alpha_{n-1} < \theta_n^{(q)} \cdots < -\alpha_1 < \theta_2^{(q)} < \theta_1^{(q)} & \text{if } \sigma = 0, \\ \theta_{n+2}^{(q)} < -\alpha_n < \theta_{n+1}^{(q)} < -\alpha_{n-1} < \theta_n^{(q)} \cdots < -\alpha_1 < \theta_2^{(q)} < \theta_1^{(q)} & \text{if } \sigma > 0, \end{cases}$$

provided $q > 0$ or $\psi'(0) \neq 0$, which we will assume throughout this example. Note that the third line in (12) follows because

$$\sum_{i=1}^{N} \left(\frac{\prod_{j=1}^{n} (\theta_i^{(q)} + \alpha_j)}{\prod_{j=1, j \neq i}^{N} (\theta_i^{(q)} - \theta_j^{(q)})} \prod_{j=1, j \neq i}^{N} (\lambda - \theta_j^{(q)}) \right)$$

and $\prod_{i=1}^{n}(\lambda+\alpha_i)$ are both polynomials in λ of maximum degrees $N-1$ and n respectively, which coincide for $\lambda=\theta_1^{(q)},\ldots,\theta_N^{(q)}$ and thus by the unisolvence theorem, they coincide for all $\lambda\in\mathbb{R}$. The fourth line in (12) follows because by the third line

$$\frac{1}{A}\frac{\prod_{j=1}^{n}(\theta_i^{(q)}+\alpha_j)}{\prod_{j=1,j\neq i}^{N}(\theta_i^{(q)}-\theta_j^{(q)})}=\lim_{\lambda\to\theta_i^{(q)}}\frac{\lambda-\theta_i^{(q)}}{\psi(\lambda)-q}=1/\psi'(\theta_i^{(q)}).$$

We can easily apply Laplace inversion on the right hand side of (12) in order to get via (1) and (2), for $q>0$ or $q=0$ and $\psi'(0)\neq 0$,

$$\begin{aligned}
W^{(q)}(x)&=\sum_{i=1}^{N}\frac{e^{\theta_i^{(q)}x}}{\psi'(\theta_i^{(q)})},\quad x\geq 0,\\
Z^{(q)}(x)&=1+q\sum_{i=1}^{N}\frac{e^{\theta_i^{(q)}x}-1}{\psi'(\theta_i^{(q)})\theta_i^{(q)}}\\
&=\begin{cases} q\sum_{i=1}^{N}\dfrac{e^{\theta_i^{(q)}x}}{\psi'(\theta_i^{(q)})\theta_i^{(q)}} & \text{if } q>0,\\ 1 & \text{if } q=0,\end{cases}\qquad x\geq 0,
\end{aligned}$$

where the last equality follows by (12) with $\lambda=0$. For convenience, we only consider the case $p>0$. Then using the first identities in (7), we get

$$\begin{aligned}
\mathcal{W}_a^{(p,q)}(x)&=\sum_{i=1}^{N}\frac{e^{\theta_i^{(p+q)}x}}{\psi'(\theta_i^{(p+q)})}\left[1-q\sum_{j=1}^{N}\frac{e^{(\theta_j^{(p)}-\theta_i^{(p+q)})a}-1}{(\theta_j^{(p)}-\theta_i^{(p+q)})\psi'(\theta_j^{(p)})}\right],\quad x\geq a\\
\mathcal{Z}_a^{(p,q)}(x)&=\sum_{i=1}^{N}\frac{e^{\theta_i^{(p+q)}x}}{\psi'(\theta_i^{(p+q)})}\\
&\quad\times\left[\frac{p+q}{\theta_i^{(p+q)}}-qp\sum_{j=1}^{N}\frac{e^{(\theta_j^{(p)}-\theta_i^{(p+q)})a}-1}{(\theta_j^{(p)}-\theta_i^{(p+q)})\psi'(\theta_j^{(p)})\theta_j^{(p)}}\right],\quad x\geq a,
\end{aligned}$$

and then using (10), we get for $x\geq b$,

$$\begin{aligned}
&\mathcal{W}_a^{(p,q)}(x)-q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\\
&=\sum_{k=1}^{N}\frac{e^{\theta_k^{(p)}x}}{\psi'(\theta_k^{(p)})}\left\{1+q\left[1-q\sum_{j=1}^{N}\frac{e^{(\theta_j^{(p)}-\theta_i^{(p+q)})a}-1}{(\theta_j^{(p)}-\theta_i^{(p+q)})\psi'(\theta_j^{(p)})}\right]\right.\\
&\quad\left.\times\sum_{i=1}^{N}\frac{e^{(\theta_i^{(p+q)}-\theta_k^{(p)})b}-e^{(\theta_i^{(p+q)}-\theta_k^{(p)})a}}{(\theta_i^{(p+q)}-\theta_k^{(p)})\psi'(\theta_i^{(p+q)})}\right\}
\end{aligned}$$

and

$$\mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z$$
$$= \sum_{k=1}^N \frac{\mathrm{e}^{\theta_k^{(p)}x}}{\psi'(\theta_k^{(p)})}\left\{\frac{p}{\theta_k^{(p)}} + q\left[\frac{p+q}{\theta_i^{(p+q)}} - qp\sum_{j=1}^N \frac{\mathrm{e}^{(\theta_j^{(p)}-\theta_i^{(p+q)})a}-1}{(\theta_j^{(p)}-\theta_i^{(p+q)})\psi'(\theta_j^{(p)})\theta_j^{(p)}}\right]\right.$$
$$\left.\times\sum_{i=1}^N \frac{\mathrm{e}^{(\theta_i^{(p+q)}-\theta_k^{(p)})b}-\mathrm{e}^{(\theta_i^{(p+q)}-\theta_k^{(p)})a}}{(\theta_i^{(p+q)}-\theta_k^{(p)})\psi'(\theta_i^{(p+q)})}\right\}.$$

Note that for other values of x, we have

$$\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z = \begin{cases}\mathcal{W}_a^{(p,q)}(x) & \text{if } x\le b,\\ W^{(p)}(x) & \text{if } x \le a,\end{cases}$$
$$\mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z = \begin{cases}\mathcal{Z}_a^{(p,q)}(x) & \text{if } x\le b,\\ Z^{(p)}(x) & \text{if } x \le a.\end{cases}$$

When $n=0$ and $\sigma>0$, the Lévy process X is a Brownian motion plus drift and in this case the roots $\theta_i^{(q)}$, $i=1,\dots,N=2$, can be found explicitly,

$$\theta_1^{(q)} = -\mathtt{d}/\sigma^2 + \frac{\sqrt{\mathtt{d}^2+2q\sigma^2}}{\sigma^2}, \qquad \theta_2^{(q)} = -\mathtt{d}/\sigma^2 - \frac{\sqrt{\mathtt{d}^2+2q\sigma^2}}{\sigma^2}.$$

When $n=1$ and $\sigma=0$, the Lévy process X is a compound Poisson process plus drift with exponential jumps with parameter α_1 and in this case the roots $\theta_i^{(q)}$, $i=1,\dots,N=2$, are given by

$$\theta_1^{(q)} = \frac{-(\mathtt{d}\alpha_1-\eta-q)+\sqrt{(\mathtt{d}\alpha_1-\eta-q)^2+4q\alpha_1\mathtt{d}}}{2\mathtt{d}},$$
$$\theta_2^{(q)} = \frac{-(\mathtt{d}\alpha_1-\eta-q)-\sqrt{(\mathtt{d}\alpha_1-\eta-q)^2+4q\alpha_1\mathtt{d}}}{2\mathtt{d}}.$$

The rest of the paper is organized as follows. The main lemma needed for the proofs, which is based on some of the techniques used in [8], is given in the next section. It is this lemma which allows us in the end to simplify the expressions obtained in [10,9]. Then in Sections 3–5 the proofs of the theorems and corollaries are given. The arguments used in Sections 3 and 4 (at least for the case where X has paths of bounded variation) are similar to the ones in [10]. Finally, in Section 6 we give two applications of our results.

2. Main lemma

Recall that X is a spectrally negative Lévy process with Lévy triplet (γ,σ,Π). For some particular functions f associated with X, the right hand side of (5) can be written in a much nicer form (namely, (13) below) and this observation is the starting point of what leads in the end to the simple form, compared to the earlier works [10,9], of the identities in the main theorems.

For a positive, measurable function $v^{(q)}(x)$, $x\in(-\infty,\infty)$, consider the following condition:

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-}v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] = v^{(q)}(x) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}v^{(q)}(b), \quad 0\le a\le x\le b. \tag{13}$$

Remark 2.1. Note that (13) implies via the Markov property, (3) and the lack of upward jumps that the process

$$t \mapsto \mathrm{e}^{-q(t\wedge\tau_a^-\wedge\tau_b^+)} v^{(q)}\left(X_{t\wedge\tau_a^-\wedge\tau_b^+}\right),$$

is a $\mathbb{P}_x$-martingale for all $x \in [a, b]$. Conversely, if the above displayed process is a $\mathbb{P}_x$-martingale for $x \in [a, b]$, then by taking expectations and the limit as $t \to \infty$, one can show that (13) is satisfied provided $v^{(q)}$ is sufficiently regular so that switching of the expectation and the limit is justified.

For $q, a \geq 0$, we define $\mathcal{V}_a^{(q)}$ to be the function space consisting of functions $v^{(q)}(x)$ that satisfy (13) for all x and b such that $a \leq x \leq b$. We will now show that several types of functions lie in $\mathcal{V}_a^{(q)}$. Consider first the scale function $W^{(q)}(x)$. We have for all $0 \leq a \leq x \leq b$ by the strong Markov property and (3),

$$\begin{aligned}
\frac{W^{(q)}(x)}{W^{(q)}(b)} &= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_a^-\}}\right] + \mathbb{E}_x\left[\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_a^-<\tau_b^+<\tau_0^-\}}\middle|\mathcal{F}_{\tau_a^-}\right]\right] \\
&= \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right]\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] \\
&= \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-}\frac{W^{(q)}(X_{\tau_a^-})}{W^{(q)}(b)}\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right],
\end{aligned}$$

from which it follows that $W^{(q)}$ satisfies (13) and thus $W^{(q)} \in \mathcal{V}_a^{(q)}$ for all $q, a \geq 0$. By spatial homogeneity it then follows that $x \mapsto W^{(q)}(x-y)$ lies in $\mathcal{V}_a^{(q)}$ for all $q \geq 0$ and $0 \leq y \leq a$. Let now

$$v^{(q)}(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} f(X_{\tau_0^-})\mathbf{1}_{\{\tau_0^-<\infty\}}\right], \quad x \in \mathbb{R}, \tag{14}$$

for some measurable function f such that $|v^{(q)}(x)| < \infty$. Note that $v^{(q)}(x) = f(x)$ for $x < 0$. Then we have for $0 \leq a \leq x$ by using the strong Markov property,

$$v^{(q)}(x) = \mathbb{E}_x\left[\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} f(X_{\tau_0^-})\mathbf{1}_{\{\tau_0^-<\infty\}}\middle|\mathcal{F}_{\tau_a^-}\right]\right] = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\infty\}}\right] \tag{15}$$

and therefore again by the strong Markov property and (3), we have for all $0 \leq a \leq x \leq b$,

$$\begin{aligned}
v^{(q)}(x) &= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\infty\}}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_b^+<\tau_a^-<\infty\}}\middle|\mathcal{F}_{\tau_b^+}\right]\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] \\
&\quad + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_a^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\infty\}}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] + \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} \\
&\quad \times \mathbb{E}_b\left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\infty\}}\right].
\end{aligned}$$

Now using (15) for $x = b$ for the last term on the right hand side of the previous computation, we see that $v^{(q)}(\cdot)$ of the form (14) satisfies (13). In particular, for $f \equiv 1$, $v^{(q)}(\cdot)$ of the form (14) lies in $\mathcal{V}_a^{(q)}$ for all $q, a \geq 0$. As $\mathcal{V}_a^{(q)}$ is a linear space it follows via (4) that we also have for all $q, a \geq 0$,

$$Z^{(q)}(x) = \frac{q}{\Phi(q)} W^{(q)}(x) + \mathbb{E}_x \left[\mathrm{e}^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \infty\}} \right] \in \mathcal{V}_a^{(q)}.$$

The proofs of the theorems and corollaries in Section 1.1 and the next lemma in the case where the process X has paths of unbounded variation use an approximation argument for which we need to introduce a sequence $(X^n)_{n \geq 1}$ of spectrally negative Lévy processes of bounded variation. To this end, suppose X is a spectrally negative Lévy process having paths of unbounded variation with Lévy triplet (γ, σ, Π) and, on the same probability space, form for $n \geq 1$ the spectrally negative Lévy process $X^n = (X_t^n)_{t \geq 0}$ with Lévy triplet $(\gamma, 0, \Pi_n)$, whereby

$$\Pi_n(\mathrm{d}\theta) := \mathbf{1}_{\{\theta \geq 1/n\}} \Pi(\mathrm{d}\theta) + \sigma^2 n^2 \delta_{1/n}(\mathrm{d}\theta),$$

with $\delta_{1/n}(\mathrm{d}\theta)$ standing for the Dirac point mass at $1/n$. Note that X^n has paths of bounded variation with the so-called drift given by $\mathtt{d}_n := \gamma + \int_{1/n}^1 \theta \Pi(\mathrm{d}\theta) + \sigma^2 n^2$, which means that $\mathtt{d}_n$ may be negative for small n. Though we do have that X^n is a true spectrally negative Lévy process for large enough n which is all that we need. By Bertoin [2, p. 210], we can construct the sequence (X^n) so that X^n converges almost surely to X uniformly on compact time intervals. Denote by $\mathcal{V}_{a,n}^{(q)}$ the function space $\mathcal{V}_a^{(q)}$ corresponding to X^n. The following lemma is the main result of this section.

Lemma 2.1. *Let $q, a \geq 0$ and $v^{(q)}$ be a positive, measurable function on $\mathbb{R}$. Given a spectrally negative Lévy process X, consider the following assumptions:*

(i) *If X has paths of bounded variation, assume that $v^{(q)} \in \mathcal{V}_a^{(q)}$ and*

$$\int_0^\infty \mathrm{e}^{-\lambda z} v^{(q)}(z) \mathrm{d}z < \infty, \quad \textit{for } \lambda \textit{ large enough}. \tag{16}$$

(ii) *If X has paths of unbounded variation, assume that $v^{(q)}$ is continuous and that there exists a sequence of functions $v_n^{(q)} \in \mathcal{V}_{a,n}^{(q)}$ satisfying (16) such that $v_n^{(q)}$ converges to $v^{(q)}$ uniformly on compact subsets, i.e.,*

$$\lim_{n \to \infty} \sup_{x \in [x_0, x_1]} |v_n^{(q)}(x) - v^{(q)}(x)| = 0, \quad \textit{for all } x_0 < x_1, \tag{17}$$

and such that for all $x_0 \geq 0$ there exists $K_{x_0} > 0$, $n_0 \geq 1$ such that

$$|v_n^{(q)}(x)| \leq K_{x_0} \quad \textit{for all } n \geq n_0,\ x \leq x_0. \tag{18}$$

If (i) *or* (ii) *holds, then we have for all $p \geq 0$ and x, b such that $a \leq x \leq b$,*

$$\begin{aligned} \mathbb{E}_x \left[\mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}} \right] &= v^{(q)}(x) - (q-p) \int_a^x W^{(p)}(x-y) v^{(q)}(y) \mathrm{d}y \\ &\quad - \frac{W^{(p)}(x-a)}{W^{(p)}(b-a)} \left(v^{(q)}(b) - (q-p) \int_a^b W^{(p)}(b-y) v^{(q)}(y) \mathrm{d}y \right). \end{aligned} \tag{19}$$

Proof. We first prove (in three steps) the lemma for the case that X has paths of bounded variation, i.e. $\sigma = 0$ and $\int_0^1 \theta \Pi(\mathrm{d}\theta) < \infty$. Recall that $\mathtt{d} = \gamma + \int_0^1 \theta \Pi(\mathrm{d}\theta) > 0$ is the drift of X.

Step 1. We have by (13) and (5), for $a \le x \le b$,

$$\begin{aligned}
v^{(q)}(x) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} v^{(q)}(b) &= \mathbb{E}_x \left[\mathrm{e}^{-q\tau_a^-} v^{(q)}(X_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}} \right] \\
&= \int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) \\
&\quad \times \left[\frac{W^{(q)}(b-a-y)}{W^{(q)}(b-a)} W^{(q)}(x-a) - W^{(q)}(x-a-y) \right] \mathrm{d}y \\
&= \int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) \frac{W^{(q)}(b-a-y)}{W^{(q)}(b-a)} W^{(q)}(x-a) \mathrm{d}y \\
&\quad - \int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) W^{(q)}(x-a-y) \mathrm{d}y,
\end{aligned} \tag{20}$$

whereby the splitting of the integral in the last line is possible due to $\int_0^1 \theta \Pi(\mathrm{d}\theta) < \infty$. By putting $x = a$ in (20) and recalling $W^{(q)}(0) = 1/\mathtt{d}$, we get for all $b \ge a$,

$$\begin{aligned}
&\int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) W^{(q)}(b-a-y) \mathrm{d}y \\
&\quad = \mathtt{d} W^{(q)}(b-a) v^{(q)}(a) - v^{(q)}(b).
\end{aligned} \tag{21}$$

Step 2. Let $\lambda_0 > 0$ be large enough such that the Laplace transform of $v^{(q)}(x)$ exists for $\lambda > \lambda_0$, cf. condition (16). Taking Laplace transforms in b on both sides of (21) and using (1) leads to, for $\lambda > \Phi(q) \vee \lambda_0$,

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\lambda y} \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) \mathrm{d}y \\
&\quad = \mathtt{d} v^{(q)}(a) - (\psi(\lambda) - q) \mathrm{e}^{\lambda a} \int_a^\infty \mathrm{e}^{-\lambda b} v^{(q)}(b) \mathrm{d}b.
\end{aligned} \tag{22}$$

Let $p \ge 0$. Then using (22), we get for $\lambda > \Phi(q) \vee \Phi(p) \vee \lambda_0$,

$$\begin{aligned}
&\int_a^\infty \mathrm{e}^{-\lambda b} \int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) W^{(p)}(b-a-y) \mathrm{d}y \mathrm{d}b \\
&\quad = \frac{\mathrm{e}^{-\lambda a}}{\psi(\lambda) - p} \left(\mathtt{d} v^{(q)}(a) - (\psi(\lambda) - q) \mathrm{e}^{\lambda a} \int_a^\infty \mathrm{e}^{-\lambda b} v^{(q)}(b) \mathrm{d}b \right) \\
&\quad = \frac{\mathrm{e}^{-\lambda a}}{\psi(\lambda) - p} \mathtt{d} v^{(q)}(a) - \int_a^\infty \mathrm{e}^{-\lambda b} v^{(q)}(b) \mathrm{d}b + \frac{q-p}{\psi(\lambda) - p} \int_a^\infty \mathrm{e}^{-\lambda b} v^{(q)}(b) \mathrm{d}b.
\end{aligned}$$

Now by Laplace inversion, we get for all $b \ge a$,

$$\begin{aligned}
&\int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a) \Pi(\mathrm{d}\theta) W^{(p)}(b-a-y) \mathrm{d}y \\
&\quad = \mathtt{d} v^{(q)}(a) W^{(p)}(b-a) - v^{(q)}(b) + (q-p) \int_a^b W^{(p)}(b-y) v^{(q)}(y) \mathrm{d}y.
\end{aligned} \tag{23}$$

Step 3. We know by (5) that for $a \le x \le b$,

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right] = \int_0^\infty \int_{(y,\infty)} v^{(q)}(y-\theta+a)\Pi(\mathrm{d}\theta) \times \left[\frac{W^{(p)}(b-a-y)}{W^{(p)}(b-a)} W^{(p)}(x-a) - W^{(p)}(x-a-y)\right]\mathrm{d}y.$$

Hence using (23) twice, we get the identity in (19) when X has paths of bounded variation.

We now prove the lemma for the case that X has paths of unbounded variation. Hereby we assume without loss of generality that $p > 0$ as the case $p = 0$ can be dealt with by taking limits as $p \downarrow 0$ using the fact that $W^{(p)}(x)$ is continuous and increasing (cf. (6)) in $p \ge 0$. We denote by $W_n^{(p)}$ the p-scale function corresponding to the spectrally negative Lévy process X^n. Further, let

$$\tau_{a,n}^- = \inf\{t > 0 : X_t^n < a\}, \qquad \tau_{b,n}^+ = \inf\{t > 0 : X_t^n > b\}.$$

Then since we have proved the lemma for the case of bounded variation,

$$\mathbb{E}_x\left[\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}}\right] = v_n^{(q)}(x) - (q-p)\int_a^x W_n^{(p)}(x-y)v_n^{(q)}(y)\mathrm{d}y - \frac{W_n^{(p)}(x-a)}{W_n^{(p)}(b-a)}\left(v_n^{(q)}(b) - (q-p)\int_a^b W_n^{(p)}(b-y)v_n^{(q)}(y)\mathrm{d}y\right). \tag{24}$$

We aim to prove (19) by taking limits as $n \to \infty$ on both sides of (24).

By p. 210 of Bertoin [2], X^n converges almost surely to X uniformly on compact time intervals, i.e. for all $t > 0, \lim_{n\to\infty} \sup_{s\in[0,t]} |X_s^n - X_s| = 0, \mathbb{P}_x$-a.s. Given $\epsilon > 0$, observe that $\mathbb{P}_x$-a.s.,

$$\tau_{a+\epsilon}^- \le \tau_a^- \le \tau_{a-\epsilon}^-$$

and

$$\tau_{a+\epsilon}^- \wedge t \le \tau_{a,n}^- \wedge t \le \tau_{a-\epsilon}^- \wedge t$$

for n large enough. Since X has paths of unbounded variation, we have $\mathbb{P}_x$-a.s.,

$$\tau_{a+\epsilon}^- \wedge t \uparrow \tau_a^- \wedge t \quad \text{and} \quad \tau_{a-\epsilon}^- \wedge t \downarrow \tau_a^- \wedge t$$

as $\epsilon \to 0+$. This implies that for any $t > 0, \mathbb{P}_x$-a.s.,

$$\tau_{a,n}^- \wedge t \to \tau_a^- \wedge t.$$

Similarly,

$$\tau_{b,n}^+ \wedge t \to \tau_b^+ \wedge t \quad \mathbb{P}_x\text{-a.s.}$$

Next, we aim to show that

$$X^n_{\tau_{a,n}^-\wedge t} \to X_{\tau_a^-\wedge t} \quad \mathbb{P}_x\text{-a.s.}$$

Since $|X^n_{\tau_{a,n}^-\wedge t} - X_{\tau_a^-\wedge t}| \le |X^n_{\tau_{a,n}^-\wedge t} - X_{\tau_{a,n}^-\wedge t}| + |X_{\tau_{a,n}^-\wedge t} - X_{\tau_a^-\wedge t}|$, it remains to show, by the uniform convergence of X^n, that

$$X_{\tau_{a,n}^-\wedge t} \to X_{\tau_a^-\wedge t} \quad \mathbb{P}_x\text{-a.s.} \tag{25}$$

Now (25) is obvious when $\tau_a^- > t$ or $\tau_a^- \le t$ and $X_{\tau_a^-} < a$ because then $\tau_{a,n}^- \wedge t = \tau_a^- \wedge t$ for n large enough. In the remaining case, conditionally on the event that $\tau_a^- \le t$ and $X_{\tau_a^-} = a$, it is well known that the Lévy process must cross the level a by creeping over it (almost surely), meaning that then $\lim_{s\to\tau_a^-} X_s = a$ and so (25) must follow because $\tau_{a,n}^- \wedge t \to \tau_a^- \wedge t$.

Combining the above convergence results with (17), we have for any $t > 0$, $\mathbb{P}_x$-a.s.,

$$\mathrm{e}^{-p(\tau_{a,n}^-\wedge t)} v_n^{(q)}(X^n_{\tau_{a,n}^-\wedge t})\mathbf{1}_{\{\tau_{a,n}^-\wedge t<\tau_{b,n}^+\wedge t\}} \to \mathrm{e}^{-p(\tau_a^-\wedge t)} v^{(q)}(X_{\tau_a^-\wedge t})\mathbf{1}_{\{\tau_a^-\wedge t<\tau_b^+\wedge t\}}$$

or equivalently

$$\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\wedge t\}} \to \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\wedge t\}}. \tag{26}$$

Notice that $\mathbb{P}_x$-a.s.,

$$X^n_{\tau_{a,n}^-}\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}} \le a, \qquad X_{\tau_a^-}\mathbf{1}_{\{\tau_a^-<\tau_b^+\}} \le a, \tag{27}$$

which implies further in combination with the triangle inequality,

$$\begin{aligned}
&|\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}} - \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}|\\
&\quad\le |\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}} - \mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\wedge t\}}|\\
&\qquad + |\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\wedge t\}} - \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\wedge t\}}|\\
&\qquad + |\mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\wedge t\}} - \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}|\\
&\quad= |\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}}\mathbf{1}_{\{t\le\tau_{a,n}^-\}}| + |\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\wedge t\}}\\
&\qquad - \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\wedge t\}}| + |\mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\mathbf{1}_{\{t\le\tau_a^-\}}|\\
&\quad\le \mathrm{e}^{-pt}\left(\mathbf{1}_{\{t\le\tau_{a,n}^-\}}\sup_{y\le a}|v_n^{(q)}(y)| + \mathbf{1}_{\{t\le\tau_a^-\}}\sup_{y\le a}|v^{(q)}(y)|\right)\\
&\qquad + |\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\wedge t\}} - \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\wedge t\}}|.
\end{aligned} \tag{28}$$

By (18) and (26) we can (since we assumed $p > 0$) first choose a t large enough and then choose n large to make the right hand side of (28) arbitrarily small, which means that $\mathbb{P}_x$-a.s.,

$$\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}} \to \mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}.$$

By (27) and (18) in combination with the dominated convergence theorem (DCT) we can then conclude that

$$\lim_{n\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_{a,n}^-} v_n^{(q)}(X^n_{\tau_{a,n}^-})\mathbf{1}_{\{\tau_{a,n}^-<\tau_{b,n}^+\}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-p\tau_a^-} v^{(q)}(X_{\tau_a^-})\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right].$$

It remains to show that the right hand side of (24) converges to the right hand side of (19). It is an easy exercise to show that the Laplace exponent of X^n converges to the Laplace exponent of X which means via (1) that the Laplace transform of $W_n^{(p)}$ converges to the Laplace transform of $W^{(p)}$. Hence by the continuity theorem of Laplace transforms (cf. [4, Theorem 2a in Section XIII.1]), $W_n^{(p)}(x) \to W^{(p)}(x)$ for all $x \ge 0$ and $p \ge 0$. Using the DCT in combination with (17), (18) and the fact that scale functions are increasing, we deduce that indeed the right hand side of (24) converges to the right hand side of (19). □

Lemma 2.2. *The conclusion of Lemma 2.1 holds for* (i) $v^{(q)}(x) = W^{(q)}(x)$ *for any* $q, a \geq 0$, (ii) $v^{(q)}(x) = Z^{(q)}(x)$ *for any* $q, a \geq 0$ *and* (iii) $v^{(q)}(x) = W^{(q)}(x-y)$ *for any* $q \geq 0$ *and* $0 \leq y \leq a$.

Proof. Note that case (iii) will follow from case (i) by spatial homogeneity of a Lévy process. For cases (i) and (ii), from the considerations in the beginning of Section 2 and (1), the assumptions in Lemma 2.1 are clearly satisfied when X has paths of bounded variation. When X has paths of unbounded variation, we let the function $v_n^{(q)} \in \mathcal{V}_{a,n}^{(q)}$ in case (i), respectively case (ii), be $W_n^{(q)}$ (the q-scale function corresponding to X^n), respectively $Z_n^{(q)}(x) := 1 + q\int_0^x W_n^{(q)}(y)\mathrm{d}y$. We have seen in the proof of Lemma 2.1 that $W_n^{(q)}(x)$ converges to $W^{(q)}(x)$ and since the q-scale function is increasing and positive, it follows that (18) is satisfied in case (i). This implies further by the DCT, that $Z_n^{(q)}(x)$ converges to $Z^{(q)}(x)$ and as $Z_n^{(q)}$ is also positive and increasing, (18) is also satisfied in case (ii).

What remains to show is that the convergence of $W_n^{(q)}$ to $W^{(q)}$ and $Z_n^{(q)}$ to $Z^{(q)}$ is actually uniform on compact subsets. Since $x \mapsto \log W_n^{(q)}(x)$ is a concave function (cf. [11, p. 89]) and converges pointwise to $\log W^{(q)}(x)$, it follows that $\log W_n^{(q)}$ converges uniformly on compact subsets to $\log W^{(q)}$, cf. [13, p. 17, Theorem E]. As the exponential function is locally Lipschitz, it is then easy to show that also $W_n^{(q)}$ converges to $W^{(q)}$ uniformly on compact subsets. It then easily follows that also $Z_n^{(q)}$ converges to $Z^{(q)}$ uniformly on compact subsets. $\square$

Remark 2.2. The proof of Lemma 2.1 in the bounded variation case uses very similar steps as the proof of Theorem 16 in [8]. In order to make the connection clear between these two results, let us reformulate the left hand side of (19) in a different setting. Let Y be a spectrally negative Lévy process with Lévy triplet (γ, σ, Π) and killing rate $p \geq 0$, which means that Y is a spectrally negative Lévy process killed at an independent exponentially distributed amount of time with parameter p. Further, let Z be another spectrally negative Lévy process with Lévy triplet (γ', σ', Π') and killing rate $q \geq 0$. Define the first passage times,

$$\tau_a^- = \inf\{t > 0\colon Y_t < a\}, \qquad \tau_b^+ = \inf\{t > 0\colon Y_t > b\},$$
$$\kappa_a^- = \inf\{t > 0\colon Z_t < a\}, \qquad \kappa_b^+ = \inf\{t > 0\colon Z_t > b\}$$

and denote by W_Y the scale function associated to Y, which is defined as the p-scale function $W^{(p)}$ corresponding to the unkilled spectrally negative Lévy process with Lévy triplet (γ, σ, Π). Similarly, define W_Z. Also, let v be a positive, measurable function satisfying

$$\mathbb{E}_x\left[v(Z_{\kappa_a^-})\mathbf{1}_{\{\kappa_a^- < \kappa_b^+\}}\right] = v(x) - \frac{W_Z(x-a)}{W_Z(b-a)}v(b).$$

Then Lemma 2.1 provides, under some additional regularity assumptions on v, an expression for the quantity

$$\mathbb{E}_x\left[v(Y_{\tau_a^-})\mathbf{1}_{\{\tau_a^- < \tau_b^+\}}\right] \tag{29}$$

in the case where $\gamma = \gamma'$, $\sigma = \sigma'$ and $\Pi = \Pi'$ (i.e. only the killing rates differ), whereas Kyprianou and Loeffen [8, Theorem 16] provide a similar-looking expression for (29) with $v = W_Z$ in the case where $\sigma = \sigma'$, $\Pi = \Pi'$ and $p = q$ (i.e. only the first parameters of the Lévy triplets differ).

3. Proof of Theorem 1

We first prove the theorem in the case where X has paths of bounded variation. Fix $0 \le a < b$ and $p, q \ge 0$. For $x \le c$, define

$$w(x) = \mathbb{E}_x \left[\mathrm{e}^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_0^- < \tau_c^+ \right].$$

Using the strong Markov property of X, the fact that X is skip-free upward and (3) and (4), we can write, for $x < a$,

$$\begin{aligned} w(x) &= \mathbb{E}_x \left[\mathrm{e}^{-p\tau_0^-}; \tau_0^- < \tau_a^+ \right] + w(a)\mathbb{E}_x \left[\mathrm{e}^{-p\tau_a^+}; \tau_a^+ < \tau_0^- \right] \\ &= Z^{(p)}(x) + \left(\frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)} \right) W^{(p)}(x). \end{aligned} \tag{30}$$

Similarly, for $a \le x < b$, using (30), we have

$$\begin{aligned} w(x) &= w(b)\mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_b^+}; \tau_b^+ < \tau_a^- \right] + \mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_a^-} w\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] \\ &= w(b)\frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)} + \mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_a^-} Z^{(p)}\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] \\ &\quad + \left(\frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)} \right) \mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_a^-} W^{(p)}\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right]. \end{aligned} \tag{31}$$

Since one can show by the lemmas in Section 2 that

$$\mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_a^-} W^{(p)}\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] = \mathcal{W}_a^{(p,q)}(x) - \frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)}\mathcal{W}_a^{(p,q)}(b) \tag{32}$$

and

$$\mathbb{E}_x \left[\mathrm{e}^{-(p+q)\tau_a^-} Z^{(p)}\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] = \mathcal{Z}_a^{(p,q)}(x) - \frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)}\mathcal{Z}_a^{(p,q)}(b),$$

we get, for $a \le x < b$,

$$\begin{aligned} w(x) &= \frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)} \left(w(b) - \mathcal{Z}_a^{(p,q)}(b) - \left(\frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)} \right) \mathcal{W}_a^{(p,q)}(b) \right) \\ &\quad + \mathcal{Z}_a^{(p,q)}(x) + \left(\frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)} \right) \mathcal{W}_a^{(p,q)}(x). \end{aligned} \tag{33}$$

Recalling (7) one easily sees that (33) also holds for $x < a$. Finally, for $b \le x \le c$, we have using (33),

$$\begin{aligned} w(x) &= \mathbb{E}_x \left[\mathrm{e}^{-p\tau_b^-} w\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+ \right] \\ &= \mathbb{E}_x \left[\mathrm{e}^{-p\tau_b^-} W^{(p+q)}\left(X_{\tau_b^-} - a\right); \tau_b^- < \tau_c^+ \right] \end{aligned}$$

$$
\begin{aligned}
&\times \frac{w(b) - \mathcal{Z}_a^{(p,q)}(b) - \left(\frac{w(a)-Z^{(p)}(a)}{W^{(p)}(a)}\right)\mathcal{W}_a^{(p,q)}(b)}{W^{(p+q)}(b-a)} \\
&+ \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{Z}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] \\
&+ \left(\frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)}\right)\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{W}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right]. \qquad (34)
\end{aligned}
$$

By the lemmas in Section 2 and Fubini's theorem, we have,

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{W}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] \\
&\quad = \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}W^{(p+q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] \\
&\qquad - q\int_0^a W^{(p)}(y)\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}W^{(p+q)}\left(X_{\tau_b^-} - y\right); \tau_b^- < \tau_c^+\right]\mathrm{d}y \\
&\quad = \mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z \\
&\qquad - \frac{W^{(p)}(x-b)}{W^{(p)}(c-b)}\left(\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\right) \qquad (35)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{Z}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] = \mathcal{Z}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z \\
&\quad - \frac{W^{(p)}(x-b)}{W^{(p)}(c-b)}\left(\mathcal{Z}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z\right). \qquad (36)
\end{aligned}
$$

All is left to obtain are the expressions for $w(a)$ and $w(b)$. It is here that we need the assumption that X has paths of bounded variation. Setting $x = a$ in (33), using that $W^{(q)}(0) \neq 0$ because X has paths of bounded variation and noticing that (cf. (7))

$$
\mathcal{W}_a^{(p,q)}(a) = W^{(p)}(a) \quad \text{and} \quad \mathcal{Z}_a^{(p,q)}(a) = Z^{(p)}(a),
$$

leads to

$$
\frac{w(b) - \mathcal{Z}_a^{(p,q)}(b)}{\mathcal{W}_a^{(p,q)}(b)} = \frac{w(a) - Z^{(p)}(a)}{W^{(p)}(a)}.
$$

Using the above equation once in (33) and twice in (34),

$$
\begin{aligned}
w(x) &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{Z}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] \\
&\quad + \left(\frac{w(b) - \mathcal{Z}_a^{(p,q)}(b)}{\mathcal{W}_a^{(p,q)}(b)}\right)\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{W}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right], \qquad (37)
\end{aligned}
$$

for $x \leq c$. Setting $x = b$ in (37) and using (35) and (36) then yields

$$\begin{aligned} w(b) &= \mathcal{Z}_a^{(p,q)}(b) - \frac{W^{(p)}(0)}{W^{(p)}(c-b)} \left(\mathcal{Z}_a^{(p,q)}(c) - q \int_b^c W^{(p)}(c-z) \mathcal{Z}_a^{(p,q)}(z) \mathrm{d}z \right) \\ &\quad + \left(\frac{w(b) - \mathcal{Z}_a^{(p,q)}(b)}{\mathcal{W}_a^{(p,q)}(b)} \right) \left(\mathcal{W}_a^{(p,q)}(b) - \frac{W^{(p)}(0)}{W^{(p)}(c-b)} \left(\mathcal{W}_a^{(p,q)}(c) \right.\right. \\ &\quad \left.\left. - q \int_b^c W^{(p)}(c-z) \mathcal{W}_a^{(p,q)}(z) \mathrm{d}z \right) \right). \qquad (38) \end{aligned}$$

Solving (37) for $w(b)$ leads to

$$w(b) = \mathcal{Z}_a^{(p,q)}(b) - \frac{\mathcal{Z}_a^{(p,q)}(c) - q \int_b^c W^{(p)}(c-z) \mathcal{Z}_a^{(p,q)}(z) \mathrm{d}z}{\mathcal{W}_a^{(p,q)}(c) - q \int_b^c W^{(p)}(c-z) \mathcal{W}_a^{(p,q)}(z) \mathrm{d}z} \mathcal{W}_a^{(p,q)}(b).$$

Plugging this into (37), using (35)–(36), cancelling out a few terms and rearranging, we get identity (8) when X has paths of bounded variation.

Now we assume that X has paths of unbounded variation. We assume here without loss of generality that $p > 0$ as the boundary case $p = 0$ can be dealt with by taking limits as $p \downarrow 0$. In order to prove this case, we use a similar argument as in the proof of Lemma 2.1. Using the notation in that proof, we have since Theorem 1 has been proved for the case where the spectrally negative Lévy process has paths of bounded variation,

$$\begin{aligned} &\mathbb{E}_x \left[\mathrm{e}^{-p\tau_{0,n}^- - q \int_0^{\tau_{0,n}^-} \mathbf{1}_{(a,b)}(X_s^n) \mathrm{d}s}; \tau_{0,n}^- < \tau_{c,n}^+ \right] = \mathcal{Z}_{a,n}^{(p,q)}(x) \\ &\quad - q \int_b^x W_n^{(p)}(x-z) \mathcal{Z}_{a,n}^{(p,q)}(z) \mathrm{d}z \\ &\quad - \frac{\mathcal{Z}_{a,n}^{(p,q)}(c) - q \int_b^c W_n^{(p)}(c-z) \mathcal{Z}_{a,n}^{(p,q)}(z) \mathrm{d}z}{\mathcal{W}_{a,n}^{(p,q)}(c) - q \int_b^c W_n^{(p)}(c-z) \mathcal{W}_{a,n}^{(p,q)}(z) \mathrm{d}z} \\ &\quad \times \left(\mathcal{W}_{a,n}^{(p,q)}(x) - q \int_b^x W_n^{(p)}(x-z) \mathcal{W}_{a,n}^{(p,q)}(z) \mathrm{d}z \right), \qquad (39) \end{aligned}$$

where

$$\begin{aligned} \mathcal{W}_{a,n}^{(p,q)}(x) &:= W_n^{(p+q)}(x) - q \int_0^a W_n^{(p+q)}(x-y) W_n^{(p)}(y) \mathrm{d}y, \\ \mathcal{Z}_{a,n}^{(p,q)}(x) &:= Z_n^{(p+q)}(x) - q \int_0^a W_n^{(p+q)}(x-y) Z_n^{(p)}(y) \mathrm{d}y. \end{aligned}$$

As X^n converges $\mathbb{P}_x$-almost surely to X uniformly on compact time intervals, we have, similarly to (26), for all $t > 0$, $\mathbb{P}_x$-a.s.,

$$\mathrm{e}^{-p\tau_{0,n}^- - q \int_0^{\tau_{0,n}^-} \mathbf{1}_{(a,b)}(X_s^n) \mathrm{d}s} \mathbf{1}_{\{\tau_{0,n}^- < \tau_{c,n}^+ \wedge t\}} \to \mathrm{e}^{-p\tau_0^- - q \int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s) \mathrm{d}s} \mathbf{1}_{\{\tau_0^- < \tau_c^+ \wedge t\}}.$$

Further, one can show similarly to (28),

$$\left| e^{-p\tau_{0,n}^- - q\int_0^{\tau_{0,n}^-} \mathbf{1}_{(a,b)}(X_s^n)\mathrm{d}s} \mathbf{1}_{\{\tau_{0,n}^- < \tau_{c,n}^+\}} - e^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s} \mathbf{1}_{\{\tau_0^- < \tau_c^+\}} \right|$$

$$\leq 2e^{-pt} + \left| e^{-p\tau_{0,n}^- - q\int_0^{\tau_{0,n}^-} \mathbf{1}_{(a,b)}(X_s^n)\mathrm{d}s} \mathbf{1}_{\{\tau_{0,n}^- < \tau_{c,n}^+ \wedge t\}} \right.$$

$$\left. - e^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s} \mathbf{1}_{\{\tau_0^- < \tau_c^+ \wedge t\}} \right|,$$

which yields (because $p > 0$),

$$e^{-p\tau_{0,n}^- - q\int_0^{\tau_{0,n}^-} \mathbf{1}_{(a,b)}(X_s^n)\mathrm{d}s} \mathbf{1}_{\{\tau_{0,n}^- < \tau_{c,n}^+\}} \to e^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s} \mathbf{1}_{\{\tau_0^- < \tau_c^+\}}.$$

Thus by the DCT it follows that the left hand side of (39) converges, as $n \to \infty$, to

$$\mathbb{E}_x \left[e^{-p\tau_0^- - q\int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}; \tau_0^- < \tau_c^+ \right].$$

On the other hand, we have seen that $W_n^{(q)}(x)$ and $Z_n^{(q)}(x)$ converge to $W^{(q)}(x)$ and $Z^{(q)}(x)$ respectively for all $q \geq 0$ and since $W_n^{(q)}(x)$, $Z_n^{(q)}(x)$ are increasing, positive functions it follows by the DCT that $\mathcal{W}_{a,n}^{(p,q)}(x) \to \mathcal{W}_a^{(p,q)}(x)$ and $\mathcal{Z}_{a,n}^{(p,q)}(x) \to \mathcal{Z}_a^{(p,q)}(x)$ for any x. Since $\mathcal{W}_{a,n}^{(p,q)}(x)$, $\mathcal{Z}_{a,n}^{(p,q)}(x)$ are also increasing, positive functions, it then follows by the DCT that the right hand side of (39) converges to the right hand side of (8), which proves Theorem 1 also for the case that X has paths of unbounded variation.

4. Proof of Theorem 2

The proof of this theorem is very similar to the proof of Theorem 1. We first prove it in the case where X has paths of bounded variation. Fix $0 \leq a < b \leq c$ and $p, q \geq 0$. For $x \leq c$, define

$$w(x) = \mathbb{E}_x \left[e^{-p\tau_c^+ - q\int_0^{\tau_c^+} \mathbb{I}_{(a,b)}(X_s)\mathrm{d}s}; \tau_c^+ < \tau_0^- \right].$$

Using the strong Markov property of X, the fact that X is skip-free upward and (3), we can write, for $0 \leq x < a$,

$$w(x) = w(a)\mathbb{E}_x \left[e^{-p\tau_a^+}; \tau_a^+ < \tau_0^- \right] = w(a)\frac{W^{(p)}(x)}{W^{(p)}(a)}. \tag{40}$$

Similarly, for $a \leq x < b$, using (40) and (32), we have

$$\begin{aligned} w(x) &= w(b)\mathbb{E}_x \left[e^{-(p+q)\tau_b^+}; \tau_b^+ < \tau_a^- \right] + \mathbb{E}_x \left[e^{-(p+q)\tau_a^-} w\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] \\ &= w(b)\frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)} + \frac{w(a)}{W^{(p)}(a)}\mathbb{E}_x \left[e^{-(p+q)\tau_a^-} W^{(p)}\left(X_{\tau_a^-}\right); \tau_a^- < \tau_b^+ \right] \end{aligned}$$

$$= w(b)\frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)} + \frac{w(a)}{W^{(p)}(a)}\left(\mathcal{W}_a^{(p,q)}(x) - \frac{W^{(p+q)}(x-a)}{W^{(p+q)}(b-a)}\mathcal{W}_a^{(p,q)}(b)\right). \tag{41}$$

Note that via (7) one can easily show that (41) holds also for $x < a$. Finally, for $b \le x \le c$, we have

$$\begin{aligned} w(x) &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+}; \tau_c^+ < \tau_b^-\right] + \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-} w\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right] \\ &= \frac{W^{(p)}(x-b)}{W^{(p)}(c-b)} + \frac{w(b) - \frac{w(a)}{W^{(p)}(a)}\mathcal{W}_a^{(p,q)}(b)}{W^{(p+q)}(b-a)} \\ &\quad \times \mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-} W^{(p+q)}\left(X_{\tau_b^-} - a\right); \tau_b^- < \tau_c^+\right] \\ &\quad + \frac{w(a)}{W^{(p)}(a)}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_b^-}\mathcal{W}_a^{(p,q)}\left(X_{\tau_b^-}\right); \tau_b^- < \tau_c^+\right]. \end{aligned} \tag{42}$$

We need to obtain the expressions for $w(a)$ and $w(b)$. As we assumed that X has paths of bounded variation, we have $W^{(q)}(0) \neq 0$ and thus setting $x = a$ in (41) yields

$$\frac{w(b)}{\mathcal{W}_a^{(p,q)}(b)} = \frac{w(a)}{W^{(p)}(a)}.$$

Plugging this into (41) and (42) using (35) yields,

$$\begin{aligned} w(x) &= w(b)\frac{\mathcal{W}_a^{(p,q)}(x)}{\mathcal{W}_a^{(p,q)}(b)}, \quad x < b, \\ w(x) &= \frac{W^{(p)}(x-b)}{W^{(p)}(c-b)} + \frac{w(b)}{\mathcal{W}_a^{(p,q)}(b)}\left\{\mathcal{W}_a^{(p,q)}(x) - q\int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\right. \\ &\quad \left. - \frac{W^{(p)}(x-b)}{W^{(p)}(c-b)}\left(\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z\right)\right\}, \quad b \le x \le c. \end{aligned} \tag{43}$$

Setting $x = b$ in (43) gives us

$$w(b) = \frac{\mathcal{W}_a^{(p,q)}(b)}{\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}$$

and plugging this into (43) leads to (9) for all $x \le c$. This proves Theorem 2 when X has paths of bounded variation. The case where X has paths of unbounded variation follows using the same arguments as in the proof of Theorem 1.

5. Proof of corollaries

We will prove the corollaries only for $p > 0$ and $q > 0$. The cases where $p = 0$ or $q = 0$, then follow by taking limits as $p \downarrow 0$ or $q \downarrow 0$. For the proofs we will make heavy use of the fact that (cf. [7, Lemma 8.4]) the scale function can be written for $q, x \ge 0$ as

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x), \tag{44}$$

where $W_{\Phi(q)}(x)$ is the 0-scale function of the spectrally negative Lévy process with Laplace exponent $\psi_{\Phi(q)}(\theta) := \psi(\Phi(q)+\theta) - q$. Further (cf. [7, 8.7]),

$$W_{\Phi(q)}(\infty) := \lim_{x\to\infty} W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)} = \frac{1}{\psi'(\Phi(q))},$$

which implies that $W_{\Phi(q)}(\infty) < \infty$ except if simultaneously $q = 0$ and $\psi'(0+) = 0$.

5.1. Proof of Corollary 1

(i) Taking limits as $c \to \infty$ in Theorem 1, we see that we need to show

$$\begin{aligned}
&\lim_{c\to\infty} \frac{\mathcal{Z}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z}{\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z} \\
&= \frac{\frac{p}{\Phi(p)} + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{Z}_a^{(p,q)}(y)\mathrm{d}y}{1 + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{W}_a^{(p,q)}(y)\mathrm{d}y}.
\end{aligned} \tag{45}$$

Using (10) and (44), it follows by the DCT (recalling that we assumed without loss of generality $p > 0$),

$$\lim_{c\to\infty} \frac{\mathcal{W}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}{W^{(p)}(c)} = 1 + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{W}_a^{(p,q)}(y)\mathrm{d}y$$

and similarly, using also [7, Exercise 8.5(i)],

$$\lim_{c\to\infty} \frac{\mathcal{Z}_a^{(p,q)}(c) - q\int_b^c W^{(p)}(c-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z}{W^{(p)}(c)} = \frac{p}{\Phi(p)} + q\int_a^b \mathrm{e}^{-\Phi(p)y}\mathcal{Z}_a^{(p,q)}(y)\mathrm{d}y.$$

Now (45) follows.

(ii) The proof is similar to part (i) and left to the reader.

5.2. Proof of Corollary 2

(i) Using spatial homogeneity and Theorem 2 for sufficiently large m,

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\ \tau_c^+ < \infty\right] = \lim_{m\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\ \tau_c^+ < \tau_{-m}^-\right] \\
&= \lim_{m\to\infty}\mathbb{E}_{x+m}\left[\mathrm{e}^{-p\tau_{c+m}^+ - q\int_0^{\tau_{c+m}^+}\mathbf{1}_{(a+m,b+m)}(X_s)\mathrm{d}s};\ \tau_{c+m}^+ < \tau_0^-\right] \\
&= \lim_{m\to\infty}\frac{\mathcal{W}_{a+m}^{(p,q)}(x+m) - q\int_{b+m}^{x+m} W^{(p)}(x+m-z)\mathcal{W}_{a+m}^{(p,q)}(z)\mathrm{d}z}{\mathcal{W}_{a+m}^{(p,q)}(c+m) - q\int_{b+m}^{c+m} W^{(p)}(c+m-z)\mathcal{W}_{a+m}^{(p,q)}(z)\mathrm{d}z} \\
&= \lim_{m\to\infty}\frac{\mathcal{W}_{a+m}^{(p,q)}(x+m) - q\int_b^x W^{(p)}(x-y)\mathcal{W}_{a+m}^{(p,q)}(y+m)\mathrm{d}y}{\mathcal{W}_{a+m}^{(p,q)}(c+m) - q\int_b^c W^{(p)}(c-y)\mathcal{W}_{a+m}^{(p,q)}(y+m)\mathrm{d}y} \\
&= \frac{\mathcal{H}^{(p,q)}(x-a) - q\int_b^x W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y-a)\mathrm{d}y}{\mathcal{H}^{(p,q)}(c-a) - q\int_b^c W^{(p)}(c-y)\mathcal{H}^{(p,q)}(y-a)\mathrm{d}y},
\end{aligned}$$

where the last line follows by the DCT (noting that $\mathcal{H}^{(p,q)}$ is an increasing function) and

$$\lim_{m\to\infty}\frac{\mathcal{W}^{(p,q)}_{a+m}(x+m)}{W^{(p)}(m)} = \lim_{m\to\infty}\frac{W^{(p)}(x+m)+q\int_a^x W^{(p+q)}(x-y)W^{(p)}(y+m)\mathrm{d}y}{W^{(p)}(m)}$$
$$= \mathrm{e}^{\Phi(p)x}+q\int_a^x W^{(p+q)}(x-y)\mathrm{e}^{\Phi(p)y}\mathrm{d}y$$
$$= \mathrm{e}^{\Phi(p)a}\mathcal{H}^{(p,q)}(x-a),$$

which follows again by the DCT and (44).

(ii) The proof is similar to part (i) and left to the reader.

5.3. *Proof of Corollary 3*

(i) Assume $\psi'(0+)>0$, which implies $\Phi(0)=0$. Then since $\tau_c^+<\infty$ almost surely, we get using Corollary 2(i) with $p=0$, noting that $\mathcal{H}^{(0,q)}(x)=Z^{(q)}(x)$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}\right] = \lim_{c\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^{\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\tau_c^+<\infty\right]$$
$$= \lim_{c\to\infty}\frac{Z^{(q)}(x-a)-q\int_b^x W(x-y)Z^{(q)}(y-a)\mathrm{d}y}{Z^{(q)}(c-a)-q\int_b^c W(c-y)Z^{(q)}(y-a)\mathrm{d}y}.$$

Using (11), we deduce using the DCT,

$$\lim_{c\to\infty}\left(Z^{(q)}(c-a)-q\int_b^c W(c-y)Z^{(q)}(y-a)\mathrm{d}y\right)$$
$$= \lim_{c\to\infty}\left(1+q\int_a^b W(c-y)Z^{(q)}(y-a)\mathrm{d}y\right)$$
$$= 1+qW(\infty)\int_a^b Z^{(q)}(y-a)\mathrm{d}y$$
$$= 1+\frac{q}{\psi'(0+)}\int_0^{b-a} Z^{(q)}(y)\mathrm{d}y,$$

which proves Corollary 3(i).

(ii) Similarly as for part (i), we have now using Corollary 2(ii) with $p=0$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s}\right] = \lim_{c\to\infty}\frac{\mathcal{H}^{(q,-q)}(x-b)}{\mathcal{H}^{(q,-q)}(c-b)}.$$

Further, using (1) and l'Hôpital's rule,

$$\lim_{c\to\infty}\mathcal{H}^{(q,-q)}(c-b) = \lim_{c\to\infty}\mathrm{e}^{\Phi(q)(c-b)}\left(1-q\int_0^{c-b}\mathrm{e}^{-\Phi(q)y}W(y)\mathrm{d}y\right)$$
$$= \lim_{c\to\infty}\frac{q\int_{c-b}^\infty \mathrm{e}^{-\Phi(q)y}W(y)\mathrm{d}y}{\mathrm{e}^{-\Phi(q)(c-b)}}$$
$$= \frac{q}{\psi'(0+)\Phi(q)},$$

which proves Corollary 3(ii).

(iii) Assume $\psi'(0+) < 0$, which implies $\Phi(0) > 0$. Then since $\tau_{-m}^- < \infty$ almost surely for any $m > 0$, we get using spatial homogeneity and Corollary 1(i) for $p = 0$ and sufficiently large m,

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,b)}(X_s)\mathrm{d}s}\right] &= \lim_{m\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^{\tau_{-m}^-}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\tau_{-m}^-<\infty\right]\\
&= \lim_{m\to\infty}\mathbb{E}_{x+m}\left[\mathrm{e}^{-q\int_0^{\tau_0^-}\mathbf{1}_{(a+m,b+m)}(X_s)\mathrm{d}s};\tau_0^-<\infty\right]\\
&= Z^{(q)}(x-a) - q\int_b^x W(x-y)Z^{(q)}(y-a)\mathrm{d}y\\
&\quad - \lim_{m\to\infty}\frac{\mathrm{e}^{-\Phi(0)m}q\mathrm{e}^{-\Phi(0)a}\int_0^{b-a}\mathrm{e}^{-\Phi(0)y}Z^{(q)}(y)\mathrm{d}y}{1+\mathrm{e}^{-\Phi(0)m}q\int_a^b \mathrm{e}^{-\Phi(0)z}\mathcal{W}_{a+m}^{(0,q)}(z+m)\mathrm{d}z}\\
&\quad \times\left(\mathcal{W}_{a+m}^{(0,q)}(x+m) - q\int_b^x W(x-z)\mathcal{W}_{a+m}^{(0,q)}(z+m)\mathrm{d}z\right).
\end{aligned}$$

Note that in the above we used that $\mathcal{Z}_a^{(0,q)}(x) = Z^{(q)}(x-a)$. Now we have by the DCT and (44),

$$\begin{aligned}
\lim_{m\to\infty}\frac{\mathcal{W}_{a+m}^{(0,q)}(x+m)}{W(m)} &= \lim_{m\to\infty}\frac{W(x+m)+q\int_a^x W^{(q)}(x-y)W(y+m)\mathrm{d}y}{W(m)}\\
&= \mathrm{e}^{\Phi(0)x} + q\int_a^x W^{(q)}(x-y)\mathrm{e}^{\Phi(0)y}\mathrm{d}y\\
&= \mathrm{e}^{\Phi(0)a}\mathcal{H}^{(0,q)}(x-a)
\end{aligned}\tag{46}$$

and thus also,

$$\begin{aligned}
&\lim_{m\to\infty}\frac{\mathrm{e}^{\Phi(0)m}+q\int_a^b \mathrm{e}^{-\Phi(0)z}\mathcal{W}_{a+m}^{(0,q)}(z+m)\mathrm{d}z}{W(m)}\\
&= \frac{1}{W_{\Phi(0)}(\infty)} + q\int_a^b \mathrm{e}^{-\Phi(0)z}\mathrm{e}^{\Phi(0)a}\mathcal{H}^{(0,q)}(z-a)\mathrm{d}z\\
&= \psi'(\Phi(0)) + q\int_0^{b-a}\mathrm{e}^{-\Phi(0)y}\mathcal{H}^{(0,q)}(y)\mathrm{d}y.
\end{aligned}$$

Combining all three computations gives us Corollary 3(iii).

(iv) Similarly, as for part (iii), we have now using Corollary 1(ii) with $p = 0$ and noting that $\mathcal{Z}_a^{(0,q)}(x) = Z^{(q)}(x-a)$,

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(a,\infty)}(X_s)\mathrm{d}s}\right] = Z^{(q)}(x-a)\\
&\quad - \lim_{m\to\infty}\frac{\frac{q}{\Phi(q)} - q\int_0^{a+m}\mathrm{e}^{-\Phi(q)y}\mathrm{d}y}{1-q\int_0^{a+m}\mathrm{e}^{-\Phi(q)y}W(y)\mathrm{d}y}\mathcal{W}_{a+m}^{(0,q)}(x+m)\\
&= Z^{(q)}(x-a) - \mathrm{e}^{\Phi(0)a}\mathcal{H}^{(0,q)}(x-a)\lim_{m\to\infty}\frac{W(m)\frac{1}{\Phi(q)}\mathrm{e}^{-\Phi(q)(a+m)}}{\int_{a+m}^\infty \mathrm{e}^{-\Phi(q)y}W(y)\mathrm{d}y},
\end{aligned}$$

where in the last line we used (1) and (46). By (44) and l'Hôpital's rule,

$$\lim_{m\to\infty}\frac{W(m)\frac{1}{\Phi(q)}\mathrm{e}^{-\Phi(q)(a+m)}}{\int_{a+m}^{\infty}\mathrm{e}^{-\Phi(q)y}W(y)\mathrm{d}y}=\frac{W_{\Phi(0)}(\infty)\mathrm{e}^{-\Phi(q)a}}{\Phi(q)}\times\lim_{m\to\infty}\frac{\mathrm{e}^{-(\Phi(q)-\Phi(0))m}}{\int_{a+m}^{\infty}\mathrm{e}^{-(\Phi(q)-\Phi(0))y}W_{\Phi(0)}(y)\mathrm{d}y}=\frac{\Phi(q)-\Phi(0)}{\Phi(q)}\mathrm{e}^{-\Phi(0)a}$$

and in combination with the previous computation, this proves Corollary 3(iv).

6. Applications

6.1. Perpetual double knock-out corridor options in an exponential spectrally negative Lévy model

We assume that the price process of an underlying security is given by $(\mathrm{e}^{X_t})_{t\geq 0}$ under the risk-neutral measure $\mathbb{P}$. For this model (which includes the Black–Scholes model) we would like to price a so-called (European) perpetual double knock-out corridor option. In a corridor option (see e.g. Pechtl [12]), the payoff function is the amount of time the underlying spends in a given interval, the so-called corridor. For our option we include the feature, similar to barrier options, that the option expires when the price process leaves a predetermined interval. In particular, if we assume that the corridor is given by $(\mathrm{e}^a, \mathrm{e}^b)$ and the option gets knocked out when the price process leaves the interval $[\mathrm{e}^0, \mathrm{e}^c]$ with $0 \leq a < b \leq c < \infty$, then the price of the option equals

$$V(x) := \mathbb{E}_x\left[\mathrm{e}^{-p(\tau_0^-\wedge\tau_c^+)}\int_0^{\tau_0^-\wedge\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s\right],$$

where $p \geq 0$ is the risk-free interest rate and e^x is the initial price of the security.

From Theorems 1 and 2 in combination with the DCT (which justifies switching derivative and expectation), we have for $x \in [0, c]$,

$$\begin{aligned}V(x) &= \mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^-}\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s;\ \tau_0^-<\tau_c^+\right] \\ &\quad+\mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+}\int_0^{\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s;\ \tau_c^+<\tau_0^-\right] \\ &= \frac{-\mathrm{d}}{\mathrm{d}q}\bigg|_{q=0}\Bigg(\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^- - q\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\ \tau_0^-<\tau_c^+\right] \\ &\quad+\ \mathbb{E}_x\left[\mathrm{e}^{-p\tau_c^+ - q\int_0^{\tau_c^+}\mathbf{1}_{(a,b)}(X_s)\mathrm{d}s};\ \tau_c^+<\tau_0^-\right]\Bigg) \\ &= \frac{-\mathrm{d}}{\mathrm{d}q}\bigg|_{q=0}\Bigg(\mathcal{Z}_a^{(p,q)}(x)-q\int_b^x W^{(p)}(x-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z\end{aligned}$$

$$
- \frac{\mathcal{W}_a^{(p,q)}(x) - q \int_b^x W^{(p)}(x-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}{\mathcal{W}_a^{(p,q)}(c) - q \int_b^c W^{(p)}(c-z)\mathcal{W}_a^{(p,q)}(z)\mathrm{d}z}
\times \left(\mathcal{Z}_a^{(p,q)}(c) - q \int_b^c W^{(p)}(c-z)\mathcal{Z}_a^{(p,q)}(z)\mathrm{d}z - 1 \right) \Bigg).
$$

Using (6) and (7) to compute the derivatives, one can get

$$
V(x) = \int_a^b \left(Z^{(p)}(y) - \frac{Z^{(p)}(c) - 1}{W^{(p)}(c)} W^{(p)}(y) \right) \times \left(\frac{W^{(p)}(x)}{W^{(p)}(c)} W^{(p)}(c-y) - W^{(p)}(x-y) \right) \mathrm{d}y. \tag{47}
$$

The identity (47) can also be derived using the following known formula for the potential measure of X killed on exiting $[0, c]$:

$$
\int_0^\infty \mathrm{e}^{-ps} \mathbb{P}_x(s < \tau_0^- \wedge \tau_c^+, X_s \in \mathrm{d}y)\mathrm{d}s = \left(\frac{W^{(p)}(x)}{W^{(p)}(c)} W^{(p)}(c-y) - W^{(p)}(x-y) \right) \mathrm{d}y,
$$

cf. [7, Theorem 8.7].

Using the above methods, we can of course also price corridor options with a single knock-out feature or with the corridor being an interval of infinite length.

6.2. *Probability of bankruptcy for an Omega Lévy risk process*

Our results can also be applied to the so-called Omega model (for some specific rate functions) introduced in [1] and further investigated in [5]. Intuitively in such a model bankruptcy (instead of ruin) occurs at rate $\omega(x)$ when the surplus process $X = (X_s)_{s \geq 0}$ is at level x. To be more precise, given the rate function $\omega : \mathbb{R} \to [0, \infty)$ the bankruptcy time T_ω can be defined as

$$
T_\omega = \inf \left\{ t > 0 : \int_0^t \omega(X_s)\mathrm{d}s > \mathbf{e}_1 \right\},
$$

where $\mathbf{e}_1$ is an independent exponentially distributed random variable with parameter 1. Typically, the rate function ω is chosen to be a decreasing function equalling zero on the positive half line so that bankruptcy does not occur when the surplus is positive.

In order to connect with the results in Section 1.1, we choose for some $b, q > 0$ the bankruptcy rate as

$$
\omega(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ q & \text{if } -b \leq x < 0, \\ \infty & \text{if } x < -b. \end{cases}
$$

Then bankruptcy occurs at rate q when X is between $-b$ and 0 and bankruptcy occurs immediately when X is below level $-b$. Suppose that the positive loading condition holds, i.e. $\mathbb{E}[X_1] = \psi'(0+) > 0$; this implies that bankruptcy does not happen almost surely. Then for any $x \in \mathbb{R}$, the probability that bankruptcy never occurs is

$$\begin{aligned}
\mathbb{P}_x(T_\omega = \infty) &= \mathbb{P}_x\left(\int_0^\infty \omega(X_s)\mathrm{d}s \le \mathbf{e}_1\right) \\
&= \mathbb{E}_x\left[\mathrm{e}^{-\int_0^\infty \omega(X_s)\mathrm{d}s}\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\int_0^\infty \mathbf{1}_{(-b,0)}(X_s)\mathrm{d}s};\ \tau_{-b}^- = \infty\right].
\end{aligned}$$

Hence by spatial homogeneity, Theorem 2, (10) and (44) in combination with the DCT,

$$\begin{aligned}
\mathbb{P}_x(T_\omega = \infty) &= \lim_{c\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-q\int_0^{\tau_c^+} \mathbf{1}_{(-b,0)}(X_s)\mathrm{d}s};\ \tau_c^+ < \tau_{-b}^-\right] \\
&= \lim_{c\to\infty} \mathbb{E}_{x+b}\left[\mathrm{e}^{-q\int_0^{\tau_{c+b}^+} \mathbf{1}_{(0,b)}(X_s)\mathrm{d}s};\ \tau_{c+b}^+ < \tau_0^-\right] \\
&= \lim_{c\to\infty} \frac{W^{(q)}(x+b) - q\int_b^{b+x} W(x+b-z)W^{(q)}(z)\mathrm{d}z}{W^{(q)}(c+b) - q\int_b^{c+b} W(c+b-z)W^{(q)}(z)\mathrm{d}z} \\
&= \lim_{c\to\infty} \frac{W(x+b) + q\int_0^{b} W(x+b-z)W^{(q)}(z)\mathrm{d}z}{W(c+b) + q\int_0^{b} W(c+b-z)W^{(q)}(z)\mathrm{d}z} \\
&= \frac{W(x+b) + q\int_0^{b} W(x+b-z)W^{(q)}(z)\mathrm{d}z}{\frac{1}{\psi'(0+)} + \frac{q}{\psi'(0+)}\int_0^b W^{(q)}(z)\mathrm{d}z} \\
&= \psi'(0+)\frac{W(x+b) + q\int_0^{b} W(x+b-z)W^{(q)}(z)\mathrm{d}z}{Z^{(q)}(b)}.
\end{aligned}$$

Similarly, the probability that bankruptcy occurs due to the surplus process dropping below the level $-b$ is given by

$$\begin{aligned}
\mathbb{P}_x(X_{T_\omega} < -b, T_\omega < \infty) &= \mathbb{P}_x\left(\int_0^{\tau_{-b}^-} \omega(X_s)\mathrm{d}s \le \mathbf{e}_1,\ \tau_{-b}^- < \infty\right) \\
&= \mathbb{E}_x\left[\mathrm{e}^{-q\int_0^{\tau_{-b}^-} \mathbf{1}_{(-b,0)}(X_s)\mathrm{d}s};\ \tau_{-b}^- < \infty\right] \\
&= \mathbb{E}_{x+b}\left[\mathrm{e}^{-q\int_0^{\tau_0^-} \mathbf{1}_{(0,b)}(X_s)\mathrm{d}s};\ \tau_0^- < \infty\right],
\end{aligned}$$

which, by Corollary 1 and (10), equals

$$\begin{aligned}
\mathbb{P}_x(X_{T_\omega} < -b, T_\omega < \infty) &= Z^{(q)}(x+b) - q\int_b^{x+b} W(x+b-z)Z^{(q)}(z)\mathrm{d}z \\
&\quad - \frac{\psi'(0+) + q\int_0^b Z^{(q)}(y)\mathrm{d}y}{1 + q\int_0^b W^{(q)}(y)\mathrm{d}y} \\
&\quad \times \left(W^{(q)}(x+b) - q\int_b^{b+x} W(x+b-z)W^{(q)}(z)\mathrm{d}z\right)
\end{aligned}$$

$$
\begin{aligned}
&= 1 + q \int_0^b W(x+b-z) Z^{(q)}(z) \mathrm{d}z \\
&\quad - \frac{\psi'(0+) + q \int_0^b Z^{(q)}(y) \mathrm{d}y}{Z^{(q)}(b)} \\
&\quad \times \left(W(x+b) + q \int_0^b W(x+b-z) W^{(q)}(z) \mathrm{d}z \right).
\end{aligned}
$$

In addition, the probability that bankruptcy occurs while the surplus is between $-b$ and 0 is

$$
\begin{aligned}
\mathbb{P}_x(-b \le X_{T_\omega} < 0, T_\omega < \infty) &= 1 - \mathbb{P}_x(T_\omega = \infty) - \mathbb{P}_x(X_{T_\omega} < -b, T_\omega < \infty) \\
&= -q \int_0^b W(x+b-z) Z^{(q)}(z) \mathrm{d}z + \frac{q \int_0^b Z^{(q)}(y) \mathrm{d}y}{Z^{(q)}(b)} \\
&\quad \times \left(W(x+b) + q \int_0^b W(x+b-z) W^{(q)}(z) \mathrm{d}z \right).
\end{aligned}
$$

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