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Front velocity and directed polymers in random medium

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Abstract

We consider a stochastic model of N evolving particles studied by Brunet and Derrida. This model can be seen as a directed polymer in random medium with N sites in the transverse direction. Cook and Derrida use heuristic arguments to obtain a formula for the ground state energy of the polymer. We formalize their argument and show that there is an additional term in the formula in the critical case. We also consider a generalization of the model, and show that in the noncritical case the behavior is basically the same, whereas in the critical case a new correction appears.

Keywords: Interacting particle systems; Front propagation; Traveling wave; Directed polymer in random medium; Ground state

1. Introduction

A relatively simple formulation for the problem of directed polymer in random medium can be found in [7,13]. The lattice consists of L planes in the transversal direction. In every plane there are N points that are connected to all points of the previous plane and the next one. For each edge ij, connecting the tth plane to the $(t+1)$th plane, a random energy $\xi_{ij}(t+1)$ is sampled from a common probability distribution ξ. With a slight abuse of notations we write ξ for both the distribution and a random variable with distribution ξ. For $\omega = [\omega_1, \ldots, \omega_L]$ a standard random walk on $\mathbb{G}_N$ the complete graph on N vertices, we define the energy E_ω of the directed path by summing the energies of the visited bounds

$$E_\omega := \sum_{s=1}^{L} \xi_{\omega_s\,\omega_{s+1}}(s+1).$$

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We define the probability measure μ_L on the space of all directed paths of length L by

$$\mu_L(\omega) := Z_L(T)^{-1} \exp(-E_\omega/T),$$

where T is the temperature and $Z_L(T)$ is the partition function. The directed path $\big(\omega_i, i\big)_{i\geq 0}$ can be interpreted as a polymer chain living on $\mathbb{G}_N \times \mathbb{N}$, constrained to stretch in one direction and governed by the Hamiltonian $\exp(-E_\omega/T)$.

We will focus on the case where the random energies ξ_{ij} depend on N the number of vertices of $\mathbb{G}_N$ and we will work at zero temperature. When $T = 0$, we are faced with an optimization problem: computing the ground state energy of the model i.e. the lowest energy of all possible walks.

In [7] Section 7.3, Cook and Derrida consider the particular case of zero temperature and ξ distributed according to a Bernoulli of parameter $1/N^{1+r}$, with $r \geq 0$, which they call the percolation distribution. In this case, the energy E_ω of a directed path of length L is equal to the number of times $\xi_{\omega_s\,\omega_{s+1}} = 1$ along this path.

For N fixed, the ratio E_L/L converges a.s. as $L \to \infty$ to a constant, which depends on N. In [7], the authors call this limit the ground state energy per unity of length and they derive the following asymptotic for it, as $N \to \infty$

$$E = \Big(1 + \lfloor 1/r \rfloor\Big)^{-1}, \tag{1.1}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Their statement is based on the observation that the typical number of sites on the tth plane connected to the first plane by a path of zero energy is N^{1-tr}. Hence, if N is large enough and $1-tr$ positive there is a path of zero energy (which is necessarily a ground state) from 0 to t, whereas when $1-tr$ is negative there is no such path. Their argument, although informal, is correct, but the case where $1/r$ is an integer (the critical case) requires a more careful analysis. In this paper we formalize their argument and show that there is an additional term in (1.1) when $1/r$ is an integer.

We will approach the polymer problem described above through the point of view of an interacting particles system. It consists in a constant number N of evolving particles on the real line initially at the positions $X_1(0), \ldots, X_N(0)$. Then, given the positions $X_i(t)$ of the N particles at time $t \in \mathbb{N}$, we define the positions at time $t+1$ by:

$$X_i(t+1) := \max_{1\leq j\leq N}\big\{X_j(t) + \xi_{j,i}(t+1)\big\}, \tag{1.2}$$

where $\big\{\xi_{i,j}(s);\, 1 \leq i, j \leq N, s \in \mathbb{N}\big\}$ are i.i.d. real random variables of common law ξ. The N particles can also be seen as the fitness of a population under reproduction, mutation and selection keeping the population size constant. Moving fronts are used to model some problems in biology and physics. It describes, for example, how the fitness of a gene propagates through a population. In physics they appear in non-equilibrium statistical mechanics and in the theory of disordered systems [11].

One can check by induction that

$$X_i(t) = \max\left\{X_{\omega_0}(0) + \sum_{s=1}^{t} \xi_{\omega_{s-1}\omega_s}(s);\, 1 \leq \omega_s \leq N,\ \forall s = 0, \ldots, t-1 \text{ and } \omega_t = i\right\}.$$

We will often assume that all particles start from zero, i.e. $X_i(0) = 0$ for all $1 \leq i \leq N$. Hence, if we sample $-\xi_{ij}(t)$ from a Bernoulli of parameter $1/N^{1+r}$, then $-X_i(t)$ is equal to the lowest energy among all paths ω such that $\omega_t = i$. It corresponds to the ground state energy of the polymer in [7] conditioned to be on i at t. Therefore, the ground state is obtained by taking the maximum over all possible positions.

Definition 1.1 (*Front Speed*)**.** Let $\phi\big(X(t)\big) := \max_{1\le i\le N}\big\{X_i(t)\big\}$. The front speed v_N is defined as

$$v_N := \lim_{t\to\infty} \frac{\phi\big(X(t)\big)}{t}. \tag{1.3}$$

For N fixed, the limit (1.3) exists and is a constant a.s. see [6] for more details and a rigorous proof. It is not difficult to see that the ground state energy per unit of length is equal to $-v_N$ as defined in (1.3).

The model defined in (1.2) was introduced by Brunet and Derrida in [4] to get a better understanding of the behavior of some noisy traveling-wave equations, that arise from microscopic stochastic models. By the selection mechanism, the particles remain grouped, they are essentially pulled by the leading ones, and the global motion is similar to a front propagation in reaction–diffusion equations with traveling waves. In [4], Brunet and Derrida solve the microscopic dynamics for a specific choice of the disorder (ξ_{ij} are sampled from a Gumbel distribution) and calculate exactly the velocity and diffusion constant. Comets, Quastel and Ramirez in [6] prove that if ξ is a small perturbation of the Gumbel distribution the expression in [5] for the velocity of the front remains sharp and that the empirical distribution function of particles converges to the Gumbel distribution as $N \to \infty$. They also study the case of bounded jumps, for which a completely different behavior is found and finite-size corrections to the speed are extremely small.

Traveling fronts pulled by the farmost particles are of physical interest and not so well understood, see [15] for a survey from a physical perspective. It is conjectured that, for a large class of such models where the front is pulled by the farmost particles, the motion and the particle structure have universal features, depending mainly on the tails distribution [4,5]. We mention recent results that confirm some of these conjectures. Bérard and Gouéré [2] study the binary Branching Random Walk (BRW) under the effect of selection (keeping the N right-most particles) and they prove that the asymptotic velocity converges to a limiting value at rate $(\log N)^{-2}$. Couronné and Gerin [8] study a particular case of BRW with selection where the corrections to the speed are extremely small. Maillard in [14] shows that there exists a killing barrier for the branching Brownian motion such that the population size stays almost constant. He also proves that the recentered position of this barrier converges to a Levy process as N diverges. In the case where there are infinitely many competitors evolving on the line, called the Indy-500 model, quasi-stationary probability measures are superposition of Poisson point processes [1].

In the first part of this paper, we study the model presented in [7] and described above. We consider the case where the distribution of the ξ_{ij} depends on N and is given by

$$\begin{aligned} &\mathbb{P}\big(\xi(N)=0\big) = p_0(N) \sim \rho/N^{1+r} \\ &\mathbb{P}\big(\xi(N)=-1\big) = 1-\mathbb{P}\big(\xi(N)=0\big), \end{aligned} \tag{1.4}$$

where $r>0$, $\rho>0$ and for sequences a_N, b_N we write $a_N \sim b_N$ if $a_N/b_N \to 1$. We will often omit N in the notation. Since ξ is non-positive, the front moves backwards. As a consequence of the selection mechanism and the features of ξ, all particles stay at a distance of at most one from the leaders. And when the front moves, i.e. $\phi\big(X(t)\big) = \phi\big(X(t-1)\big)-1$, all particles are at the same position. This particular behavior hides a renewal structure that will be used when computing the front speed.

The case $1/r \in \mathbb{N}$ is critical and the system displays a different behavior. For N large enough, at time $t = 1/r$, we show that there is a Poissonian number of particles X_i that remain in position zero. Then, at the $1/r$th plane there exists a finite number (possibly zero) of sites that can still be connected to the first plane through a path of zero energy. Whereas, when $1/r \notin \mathbb{N}$ the typical number of such sites is of order N^{1-tr}. This difference of behavior leads to an additional term in (1.1) and the following Theorem holds.

Theorem 1.2. *Let ξ be distributed according to (1.4). Then the front speed v_N satisfies*

$$\lim_{N\to\infty} v_N = \begin{cases} -\big(1 + \lfloor 1/r \rfloor\big)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -\big(1 + \lfloor 1/r \rfloor - e^{-\rho^{1/r}}\big)^{-1}, & \text{if } 1/r \in \mathbb{N}, \end{cases} \tag{1.5}$$

In the case where $r = 0$

$$\lim_{N\to\infty} v_N = 0. \tag{1.6}$$

In Section 3 we generalize (1.4) and consider ξ taking values in the lattice $\mathbb{Z}_0 = \{l \in \mathbb{Z}; l \le 0\}$. Then we set for $i \in \mathbb{N}$

$$p_i(N) = \mathbb{P}(\xi(N) = -i), \tag{1.7}$$

and assume that $p_0 \sim \rho/N^{1+r}$ where r and ρ are non-negative. Let

$$q_2(N) := \mathbb{P}\big(\xi(N) \le -2\big) = 1 - p_0 - p_1. \tag{1.8}$$

We also assume that for $i \ge 2$

$$\mathbb{P}\,(\xi = -i \mid \xi \le -2) = \frac{p_i(N)}{q_2(N)} = \mathbb{P}(\vartheta = -i), \tag{1.9}$$

where ϑ is an integrable distribution on the lattice $\mathbb{Z}_{-2}$ that does not depend on N. We then compute the asymptotic of v_N as $N \to \infty$, that resembles the expression in (1.5), but a different correction appears in the critical case, *see Theorem* 1.3.

As we explain in Section 4, we can further generalize the model and consider ξ distributed as

$$\xi = p_0(N)\delta_{\lambda_0} + p_1(N)\delta_{\lambda_1} + q_2(N)\vartheta(dx), \tag{1.10}$$

where $\lambda_1 < \lambda_0$, $\vartheta(dx)$ is an integrable probability distribution over $\big(-\infty, \lambda_1 - (\lambda_0 - \lambda_1)\big]$ and δ_{λ_i} is the mass distribution. Then, if we assume that $p_0(N) \sim \rho/N^{1+r}$, the velocity v_N obeys the following asymptotic.

Theorem 1.3. *Let ξ be distributed according to (1.10). Assume that*

$$p_0(N) \sim \frac{\rho}{N^{1+r}}, \quad \text{and} \quad \lim_{N\to\infty} q_2(N) = \theta,$$

where $r > 0$ and $0 < \theta < 1$. Then the front speed v_N satisfies

$$\lim_{N\to\infty} v_N = \begin{cases} \lambda_0 - (\lambda_0 - \lambda_1)\big(1 + \lfloor 1/r \rfloor\big)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ \lambda_0 - (\lambda_0 - \lambda_1)\big(\lfloor 1/r \rfloor + 1 - 1/g(\theta)\big)^{-1}, & \text{if } 1/r \in \mathbb{N}, \end{cases} \tag{1.11}$$

where $g(\theta) \ge 1$ is a non-increasing function. The conclusion in the case $1/r \notin \mathbb{N}$ still holds if ξ satisfies the weaker assumption $q_2/(1 - p_0) \le \theta'$, for some $0 < \theta' < 1$.

The paper is organized as follows: in Section 2.1 we compute the typical number of leading particles, which corresponds to the number of paths of zero energy and in Section 2.2, we calculate the limit of v_N as $N \to \infty$, exhibiting in particular the additional term appearing in (1.1) in the critical case. In Section 3.1 we compute the typical number of leading particles, when ξ is distributed according to (1.10). Sections 3.2 and 3.3 present some technical results and calculations. In Section 3.4 we compute the front velocity and prove the discrete version of Theorem 1.3. Finally, in Section 4 we sketch the proof of Theorem 1.3.

2. Front speed for the two-state percolation distribution

In this section, we consider the case of ξ_{ij}'s distributed according to (1.4). For this choice of distribution, all N particles meet at a same location at a geometric time T regardless the initial configuration, see [6] Section 5 for a rigorous proof. Due to the choice of ξ_{ij}'s and (1.2), at all later times $t \geq T$ every $X_i(t)$ is either at a leading position or it lies at a unit distance behind the leaders.

Since $\Phi\big(X(T)\big)/t$ converges to 0 a.s. as $t \to \infty$, the front speed v_N is not affected by the particles' positions at time T. So, we may assume that at $t = 0$ the particles' relative positions to the leader satisfy the above property. In this situation, we consider the following process.

Definition 2.1. Let $\phi\big(X(t-1)\big)$ be the front's position as in Definition 1.1. Then, for $t \in \mathbb{N}$ we define the stochastic process $Z(t) := \big(Z_0(t), Z_1(t)\big)$ as follows. For $t = 0$ let $Z_0(0)$ be the number of leading particles and $Z_1(0)$ the number of particles that are at a unit distance behind the leaders. For $t \geq 1$ define

$$\begin{aligned} Z_0(t) &:= \sharp\big\{1 \leq i \leq N;\, X_i(t) = \phi\big(X(t-1)\big)\big\};\\ Z_1(t) &:= \sharp\big\{1 \leq i \leq N;\, X_i(t) = \phi\big(X(t-1)\big) - 1\big\}, \end{aligned} \tag{2.1}$$

where $\sharp$ denotes the number of elements in a set.

Note that for $t \geq 1$ $Z_0(t)$ is equal to the number of leaders if the front has not moved backwards between times $t-1$ and t, and to 0 if the front moved. Z is a homogeneous Markov chain on the set

$$\Omega(N) = \Big\{x \in \{0, 1, \ldots, N\}^2 \,;\, x_0 + x_1 = N\Big\},$$

where x_0 and x_1 are the coordinates of x. The transition rates of the Markov chain $Z_0(t)$ are given by the Binomial distributions

$$\begin{aligned} \mathbb{P}\big(Z_0(t+1) = \cdot \mid Z(t) = x\big) &= \mathbb{P}\big(Z_0(t+1) = \cdot \mid Z_0(t) = x_0\big)\\ &= \begin{cases} \mathcal{B}\left(N, 1-(1-p_0)^{x_0}\right)(\cdot), & x_0 \geq 1\\ \mathcal{B}\left(N, 1-(1-p_0)^{N}\right)(\cdot), & x_0 = 0. \end{cases} \end{aligned} \tag{2.2}$$

We will often consider Markov chains with different starting distributions. For this purpose we introduce the notations $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ for probabilities and expectations given that the Markov chain initial position has distribution given by μ. Often, the initial distribution will be concentrated at a single state x. We will then simply write $\mathbb{P}_x$ and $\mathbb{E}_x$ for $\mathbb{P}_{\delta_x}$ and $\mathbb{E}_{\delta_x}$.

In this section, $\oplus$ denotes the configuration $(N, 0) \in \Omega(N)$. Furthermore, we introduce the notation

$$1/r = m + \eta, \tag{2.3}$$

where m stands for the integer part of $1/r$ and η its fractional part.

2.1. Number of leading particles

In this subsection, we show that under a suitable normalization and initial conditions the process Z_0 converges as N goes to infinity.

We consider the random variable

$$\tau := \inf\big\{t \geq 1; \phi\big(X(t)\big) < \phi\big(X(t-1)\big)\big\}, \tag{2.4}$$

that is a stopping time for the filtration $\mathcal{F}_t = \{\xi_{ij}(s); s \leq t \text{ and } 1 \leq i, j \leq N\}$. It is not difficult to see that τ is also the first time when Z_0 visits zero

$$\tau = \inf\big\{t \geq 1; Z_0(t) = 0\big\},$$

as a consequence $Z(\tau) = \big(0, N\big)$. From (2.2), it is easy to conclude that the distribution of $\big\{Z(\tau + t), t \geq 1\big\}$ is equal to the distribution of $\big\{Z(t), t \geq 1\big\}$ under $\mathbb{P}_\oplus$. It yields the renewal structure that will be used when computing the front speed.

Definition 2.2. Let $Y(t)$ be the number of leading particles at time t if the front has not moved

$$Y(t) := Z_0(t)\mathbf{1}_{\{t \leq \tau\}}. \tag{2.5}$$

Then, Y is a homogeneous Markov chain with absorption state at zero and transition rates given by the Binomial distributions

$$\mathbb{P}\big(Y(t+1) = \cdot \mid Y(t) = k\big) = \mathcal{B}\left(N, 1 - (1 - p_0)^k\right)(\cdot).$$

The advantage of working with Y rather than Z_0 is that the above formula holds even if $Y(t) = 0$.

Proposition 2.3. *Let ξ be distributed according to (1.4). For $k \in \{1, 2, \ldots, N\}$ denote by $G_k(s, t)$ the Laplace transform of $Y(t)$ under $\mathbb{P}_k$ at $s \in \mathbb{R}$. Then,*

$$G_k(s, t) := \mathbb{E}_k\left[e^{s\,Y(t)}\right] = \exp\Big\{(e^s - 1)k\big(Np_0\big)^t\big(1 + o(1)\big)\Big\} \tag{2.6}$$

as $N \to \infty$.

Proof. Conditioning on $\mathcal{F}_{t-1} := \{\xi_{ij}(s); s \leq t-1\}$

$$\begin{aligned}\mathbb{E}_k\left[e^{s\,Y(t)}\right] &= \mathbb{E}_k\Big[\mathbb{E}\left[e^{s\,Y(t)} \mid Y(t-1)\right]\Big] \\ &= \mathbb{E}_k\left[\left(1 + (e^s - 1)\left(1 - (1 - p_0)^{Y(t-1)}\right)\right)^N\right].\end{aligned}$$

Since $p_0 \sim \rho/N^{1+r}$ with $r > 0$ and $Y(t-1) \leq N$, we obtain by first order expansion that

$$\left(1 + (e^s - 1)\left(1 - (1 - p_0)^{Y(t-1)}\right)\right)^N = s_{(1)}(N)^{Y(t-1)},$$

where $s_{(1)}(N) = \exp\big\{(e^s - 1)\big(Np_0 + o(Np_0)\big)\big\}$ and $o(Np_0)/Np_0 \to 0$ as $N \to \infty$ independently from $Y(t-1)$. Repeating the argument

$$\mathbb{E}_k\left[e^{s\,Y(t)}\right] = \mathbb{E}_k\left[s_{(1)}(N)^{Y(t-1)}\right] = \mathbb{E}_k\left[s_{(2)}(N)^{Y(t-2)}\right],$$

with $s_{(2)}(N) = \exp\{(s_{(1)}(N) - 1)(Np_0 + o(Np_0))\}$. Expanding $s_1(N) - 1$

$$\begin{aligned} s_{(1)}(N) - 1 &= \exp\Big\{(e^s - 1)\big(N\,p_0 + o(Np_0)\big)\Big\} - 1 \\ &= (e^s - 1)\big(N\,p_0 + o(Np_0)\big). \end{aligned}$$

Hence, $s_{(2)}(N) = \exp\{(e^s - 1)\big(N\,p_0\big)^2 + o\big((Np_0)^2\big)\}$. We proceed recursively and obtain the expression

$$\mathbb{E}_k\left[e^{s\,Y(t)}\right] = \exp\Big\{k\,(e^s - 1)\big(N\,p_0\big)^t\big(1 + o(1)\big)\Big\}, \qquad \text{as } N \to \infty,$$

which proves the statement. □

We point out that the case $k = N$ corresponds to $Z(0) = \oplus$. We now state two corollaries of Eq. (2.6).

Corollary 2.4. *Let ξ be distributed according to (1.4) and $k \in \{1, \dots, N\}$. Then, for $t \geq m + 1$*

$$\mathbb{P}_k\big(Y(t) = 0\big) \geq 1 - \rho^t\,N^{1-t\,r} + o\big(N^{1-t\,r}\big) \tag{2.7}$$

as $N \to \infty$.

Proof. Since $\mathbb{P}_k\big(Y(t) = 0\big) = \lim_{s\to-\infty} \mathbb{E}_k\left[e^{sY(t)}\right]$, Proposition 2.3 implies that

$$\mathbb{P}_k\big(Y(t) = 0\big) = \exp\Big\{-k\big(Np_0\big)^t\big(1 + o(1)\big)\Big\} \geq \exp\Big\{-N\big(Np_0\big)^t\big(1 + o(1)\big)\Big\}.$$

Then, we obtain (2.7) by first order expansion. □

Corollary 2.5. *Let ξ be distributed according to (1.4) with $\eta = 0$ (i.e. $r = 1/m$) and $\kappa(N)$ be a sequence of random variables in $\{1, \dots, N\}$ with some distribution $\mu(N)$. Suppose that $\kappa(N)/N$ converges in distribution to U a positive random variable.*

Then, under $\mathbb{P}_{\mu(N)}$, $Y(m)$ converges in distribution to Y_∞ a doubly stochastic Poisson random variable characterized by its Laplace transform

$$\mathbb{E}\left[e^{s\,Y_\infty}\right] = \mathbb{E}\big[\exp\big\{U\,(e^s - 1)\rho^m\big\}\big]. \tag{2.8}$$

Proof. We may assume that all $\{\xi_{ij}(t); t, i, j \in \mathbb{N}\}$ and $\{\kappa(N), N \in \mathbb{N}\}$ are constructed on the same probability space in such a way that $\kappa(N)/N$ converges a.s. to U. Then we use (2.6) to get

$$\mathbb{E}_{\kappa(N)}\left[e^{sY(m)}\right] = \exp\big\{(e^s - 1)\,\kappa(N)\rho^m\,N^{-1}\big(1 + o(1)\big)\big\}, \quad \text{as } N \to \infty.$$

The term $o(1)$ converges to zero independently from $\kappa(N)$ the initial position. Then, by dominated convergence, we obtain that

$$\lim_{N\to\infty} \mathbb{E}\left[e^{sY(m)}\right] = \mathbb{E}\left[\exp\big\{U(e^s - 1)\rho^m\big\}\right],$$

which concludes the proof. □

We now prove a large deviation principle for Y. As in [9,10], we denote by

$$\Lambda_{k,\,t}(s) := \lim_{N\to\infty} \frac{1}{kN^{-rt}} \log \mathbb{E}_k\left[e^{s\,Y(t)}\right], \tag{2.9}$$

the cumulant generating function of Y under $\mathbb{P}_k$. From (2.6) we see that $\Lambda_{k,t}(s) = (e^s - 1)\rho^t$. Denoting by

$$\Lambda^*_{k,t}(x) := \sup_{s\in\mathbb{R}}\{xs - \Lambda_{k,t}(s)\}, \tag{2.10}$$

the Legendre transform of $Y(t)$ under $\mathbb{P}_k$, we have that

$$\Lambda^*_{k,\,t}(x) = \begin{cases} x(\log x - \log \rho^t) + \rho^t - x, & \text{if } x \geq 0; \\ \infty, & \text{if } x < 0. \end{cases} \tag{2.11}$$

Proposition 2.6 (*Large Deviation Principle for Y*)**.** *Let ξ be distributed according to (1.4). For $t \leq m$, let $k(N) \leq N$ be a sequence of positive integers such that*

$$\lim_{N\to\infty} k(N)\, N^{-r\,t} = \infty.$$

Then, under $\mathbb{P}_{k(N)}$, $Y(t)/\big(k(N)N^{-r\,t}\big)$ satisfies a large deviation principle with rate function given by $\Lambda^*_{k,t}$ as in (2.11) and speed $k(N)N^{-r\,t}$.

Proof. In fact, it is a direct application of the Gärtner–Ellis Theorem (see e.g. Theorem V.6 in [10]). Since Λ is smooth, it is a lower semi-continuous function, therefore the lower bound in the infimum can be taken over all points. $\square$

The next corollary formalizes the statement of Cook and Derrida in [7].

Corollary 2.7. *Let ξ be distributed according to (1.4) and $\kappa(N)$ be a sequence of random variables in $\{1, 2, \ldots, N\}$. Assume that all $\{\kappa(N); N \in \mathbb{N}\}$ and $\{\xi_{ij}(t); t, i, j \in \mathbb{N}\}$ are constructed in the same probability space in such a way that they are independent and denote by $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ their distributions.*

Suppose also that in this probability space $\kappa(N)/N$ converges a.s. to U a positive random variable. Then, for $t < 1/r$

$$\lim_{N\to\infty} \mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}_{\kappa(N)} \left(\left| \frac{Y(t)}{\rho^t\, U\, N^{1-tr}} - 1 \right| \geq \varepsilon \right) = 0, \tag{2.12}$$

where $\mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}_{\kappa(N)}$ is the distribution of $\{Y(t); t \in \mathbb{N}\}$ started from $\kappa(N)$.

Proof. We first consider the case where $\kappa(N)$ is a deterministic sequence and $\kappa(N)/N \to u$, with $0 < u \leq 1$. Then the conditions of Proposition 2.6 are satisfied and $Y(t)/\big(\kappa(N)\, N^{-t\,r}\big)$ satisfies a large deviation principle with rate function given by (2.11), that has a unique zero at ρ^t. This implies the desired convergence.

The random case is solved by conditioning on $\kappa(N) = Y(0)$.

$$\begin{aligned} &\mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}_{\kappa(N)} \left(\left| \frac{Y(t)}{\rho^t U N^{1-tr}} - 1 \right| \geq \varepsilon \right) \\ &\quad = \int \mathbb{P}^{(2)}_{\kappa(N)(\omega_1)} \left(\left| \frac{Y(t)}{\rho^t U(\omega_1) N^{1-tr}} - 1 \right| \geq \varepsilon \right) \mathbb{P}^{(1)}(\mathrm{d}\omega_1). \end{aligned}$$

Since $\kappa(N)(\omega_1)/N$ converges to $U(\omega_1)$ $\mathbb{P}^{(1)}$-a.s.

$$\lim_{N\to\infty} \mathbb{P}^{(2)}_{\kappa(N)(\omega_1)} \left(\left| \frac{Y(t)}{\rho^t U(\omega_1) N^{1-tr}} - 1 \right| \geq \varepsilon \right) = 0,$$

and we conclude by dominated convergence. $\square$

Cook and Derrida [7] consider the particular case where $\rho = 1$ in (1.4). From Corollary 2.7, we see that $Y(t)/N^{1-rt}$ converges in probability to one. Since under $\mathbb{P}_N$, $Y(t)$ is equal to the number of paths with zero energy at time t, the typical number of such paths is N^{1-rt}.

2.2. Front speed

In this subsection, we give the exact asymptotic for the front speed, proving Theorem 1.2. The front's position can be computed by counting the number of times Z visits $(0, N)$. Indeed, at a given time t either the front moves backwards and $\phi\big(X(t)\big) = \phi\big(X(t-1)\big) - 1$ or it stays still and $\phi\big(X(t)\big) = \phi\big(X(t-1)\big)$. Then,

$$\frac{\phi\big(X(t)\big)}{t} = \frac{-N_t}{t},$$

where N_t is the stochastic process that counts the number of times that Z visited $(0, N)$ until time t. A classic result in renewal theory (see e.g. [12]) states that

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_{\oplus}[\tau]}. \tag{2.13}$$

Hence, to determine the front velocity, it suffices to determine $\mathbb{E}_{\oplus}[\tau]$.

$$\mathbb{E}_{\oplus}[\tau] = \sum_{t=0}^{\infty} \mathbb{P}_{\oplus}(\tau \geq t+1) = \sum_{t=0}^{\infty} \mathbb{P}_{\oplus}(Y(t) \geq 1). \tag{2.14}$$

A consequence of Corollaries 2.4, 2.5 and 2.7 is that if ξ is distributed according to (1.4) with $\eta > 0$, then

$$\lim_{N\to\infty} \mathbb{P}_{\oplus}\big(Y(t) \geq 1\big) = \begin{cases} 1, & \text{if } t \leq m; \\ 0, & \text{if } t \geq m+1. \end{cases}$$

Whereas we have the following limits when $\eta = 0$

$$\lim_{N\to\infty} \mathbb{P}_{\oplus}\big(Y(t) \geq 1\big) = \begin{cases} 1, & \text{if } t \leq m-1; \\ 1 - e^{-\rho^m}, & \text{if } t = m; \\ 0, & \text{if } t \geq m+1. \end{cases}$$

Then, to finish the proof of Theorem 1.2, it suffices to show that

$$\lim_{N\to\infty} \sum_{t\geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) = 0. \tag{2.15}$$

Since Y is a homogeneous Markov chain we use the Markov property at time $m+1$ to obtain

$$\sum_{t\geq m+1} \mathbb{P}_{\oplus}\big(Y(t) \geq 1\big) = \sum_{t=0}^{\infty} \sum_{k=1}^{N} \mathbb{P}_k\big(Y(t) \geq 1\big)\, \mathbb{P}_{\oplus}\big(Y(m+1) = k\big).$$

It is not difficult to see that under $\mathbb{P}_k$, Y is stochastically dominated by Y under $\mathbb{P}_N$, which implies that $\mathbb{P}_k\big(Y(t) \geq 1\big) \leq \mathbb{P}_N\big(Y(t) \geq 1\big)$. Then, applying this inequality in the above expression, we get

$$\sum_{t\geq m+1} \mathbb{P}_{\oplus}\big(Y(t) \geq 1\big) \leq \mathbb{P}_{\oplus}\big(Y(m+1) \geq 1\big)\mathbb{E}_{\oplus}[\tau]. \tag{2.16}$$

Proposition 2.8. *Let ξ be distributed according to (1.4). Then, $\mathbb{E}_x[\tau]$ is bounded in N*

$$\sup_{N\in\mathbb{N}} \sup_{x\in\Omega(N)} \{\mathbb{E}_x[\tau]\} < \infty. \tag{2.17}$$

Proof. By Corollary 2.7, $\lim_{N\to\infty} \mathbb{P}_{\oplus}\big(\tau \geq m+2\big) = 0$. Therefore, there exists a constant $c_1 < 1$ such that for N sufficiently large

$$\mathbb{P}_{\oplus}(\, \tau \geq m + 2\,) \leq c_1.$$

Coupling the chains started from δ_x and $\delta_{\oplus}$ we obtain that $\mathbb{P}_x(\tau \geq m+2) \leq \mathbb{P}_{\oplus}(\tau \geq m+2)$ for every $x \in \Omega(N)$ and therefore

$$\mathbb{P}_x(\tau \geq m + 2) \leq c_1. \tag{2.18}$$

Then, Proposition 2.8 follows as a consequence of the Markov property and (2.18). In Section 3.2 we present an equivalent argument in all details. $\square$

Applying Proposition 2.8 and Corollary 2.4 in (2.16), we conclude that

$$\sum_{t\geq m+1} \mathbb{P}_{\oplus}(Y(t) \geq 1) = \mathcal{O}\left(N^{1-(m+1)r}\right).$$

Hence, from (2.14) we obtain the limits

$$\lim_{N\to\infty} \mathbb{E}_{\oplus}[\tau] = \begin{cases} 1+m, & \text{if } r \neq 1/m; \\ 1+m-e^{-\rho^m}, & \text{if } r = 1/m, \end{cases} \tag{2.19}$$

proving Theorem 1.2 in the case $r > 0$.

To finish the proof of Theorem 1.2 it remains to study the case $r = 0$. For that we use a coupling argument. Up to the end of this subsection we denote by $\xi(r)$

$$\mathbb{P}(\xi(r) = 0) = 1 - \mathbb{P}(\xi(r) = -1) \sim \rho/N^{1+r}.$$

For $r > 0$, the random variables $\xi(0)$ are stochastically larger than $\xi(r)$ for N large enough. Denoting by $X_i^r(t)$ the stochastic process defined by $\xi(r)$ we construct the process in such a way that the following relation holds

$$0 \geq \frac{\phi\big(X^0(t)\big)}{t} \geq \frac{\phi\big(X^r(t)\big)}{t}.$$

From (2.19), if we choose r such that $1/r$ is not an integer, we have the lower bound

$$0 \geq v_N(0) \geq v_N(r) \to -(1 + \lfloor 1/r \rfloor)^{-1},$$

whence taking r to 0, we obtain $\lim_{N\to\infty} v_N(0) = 0$, which concludes the proof of Theorem 1.2.

3. Front speed for the infinitely many states percolation distribution

In this section, we prove a discrete version of Theorem 1.3. We consider the case of ξ_{ij} distributed according to (1.7).

Assumption (A). The random variable ξ distributed according to (1.7) satisfies Assumption (A) if there exists a constant $0 < \theta < 1$ such that

$$\lim_{N\to\infty} q_2 = \theta,$$

and ϑ defined in (1.9) is integrable.

In the non-critical case we do not need to assume the convergence of q_2. We prove Theorem 3.2 under the weaker condition.

Assumption (A′). The random variable ξ distributed according to (1.7) satisfies Assumption (A′) if there exists a constant $0 < \theta' < 1$ such that for N large enough

$$\frac{q_2}{(1-p_0)} \le \theta',$$

and ϑ defined in (1.9) is integrable.

As in the "two-state percolation model", the N particles meet at a same location at a geometric time despite the starting configuration. Since $\xi_{ij}(s) \in \mathbb{Z}_0$ (for all $s \in \mathbb{N}$), for all later times t every $X_i(t)$ is at a distance $d_i \in \{0, 1, 2, \dots\}$ behind the leaders. For this reason, we may assume that at $t = 0$ the particles' relative positions to the leader already satisfy this property and we consider the process $Z(t) := \big(Z_l(t); l \in \mathbb{N}\big)$, where

$$Z_l(t) := \sharp\big\{j; 1 \le j \le N, X_j(t) = \phi\big(X(t-1)\big) - l\big\}.$$

Then, Z is a homogeneous Markov chain on the set

$$\Omega(N) := \Big\{x \in \{0, 1, \dots, N\}^{\mathbb{N}}; \sum_{i=0}^{\infty} x_i = N\Big\},$$

where x_i are the coordinates of x. If at time t we have that $Z(t) = x \in \Omega(N)$, for each $k \in \mathbb{N}$ there are x_k particles in position $-k$ with respect to the leader at time $t-1$. In this situation, suppose that $x_0 \ge 1$, then for every $1 \le i \le N$ the probability that $X_i(t+1)$ is in position $-k$ with respect to the leader at time t is equal to

$$s_k(x) := \Big(\sum_{i=1}^{\infty} p_i\Big)^{x_{k-1}} \dots \Big(\sum_{i=k}^{\infty} p_i\Big)^{x_0} - \Big(\sum_{i=1}^{\infty} p_i\Big)^{x_k} \dots \Big(\sum_{i=k+1}^{\infty} p_i\Big)^{x_0}, \tag{3.1}$$

where we define $x_{-1} = 0$. So the probability that $X_i(t+1) = \Phi(X(t))$ is given by

$$s_0(x) := 1 - \big(1-p_0\big)^{x_0}.$$

If $x_0 = 0$, we shift $(x_0, x_1, \dots)$ to get a nonzero first coordinate obtaining a vector $\tilde{x} \in \Omega(N)$ such that $\tilde{x}_0 \ge 1$. Then, one can check that

$$s_k(x) = s_k(\tilde{x}).$$

The transition probability of the Markov chain Z is given by

$$\mathbb{P}\big(Z(t+1) = y \mid Z(t) = x\big) = \mathcal{M}\big(N; s(x)\big)(y), \tag{3.2}$$

where $s(x) = \big(s_0(x), s_1(x) \dots\big)$ and $\mathcal{M}\big(N; s(x)\big)$ denotes a Multinomial distribution with infinitely many classes, we refer to [6] Section 6 for more details on the computations. It is clear that $Z_0(t)$ has the same transition probability as the process studied in the two states model. In particular, the results proved in Section 2.1 hold with the obvious changes.

Definition 3.1. Let $\mathcal{A} \subset \Omega(N)$, we denote by $T_{\mathcal{A}}$ the first time that $Z(t)$ visits $\mathcal{A}$

$$T_{\mathcal{A}} := \inf\{t \ge 1; Z(t) \in \mathcal{A}\}, \tag{3.3}$$

that is a stopping time for the filtration $\mathcal{F}_t = \sigma\{\xi_{ij}(s); s \le t,\ 1 \le i, j \le N\}$. Often $\mathcal{A} = \{x\}$, in this case we will simply write T_x for $T_{\{x\}}$.

For a stopping time T, we define recursively $T^{(0)} = 0$ and for $i \ge 1$

$$T^{(i)}(\omega) := \inf\{t > T^{(i-1)}(\omega); t = T \circ \Theta_{T^{(i-1)}(\omega)}(\omega)\}, \tag{3.4}$$

where Θ_t is the time-shift operator. We adopt the convention that $\inf\{\emptyset\} = \infty$. Once more we denote by τ the stopping time defined as

$$\tau := \inf\big\{t \ge 1;\ \phi\big(X(t)\big) < \phi\big(X(t-1)\big)\big\}. \tag{3.5}$$

In contrast with the previous section, τ is not a renewal time for Z. We adapt the notation of Section 2 and define $\oplus := \big(N, 0, \ldots\big) \in \Omega(N)$ and $\triangle := (0, N, 0, \ldots) \in \Omega(N)$. Finally, we keep notation (2.3) and let m be the integer part of $1/r$ and η its fractional part. We now state the main result of the section.

Theorem 3.2. *Assume that ξ satisfies Assumption (A). Then*

$$\lim_{N\to\infty} v_N = \begin{cases} -\big(1 + \lfloor 1/r \rfloor\big)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -\big(\lfloor 1/r \rfloor + 1 - 1/g(\theta)\big)^{-1}, & \text{if } 1/r = m \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $g(\theta) \ge 1$ is a non-increasing function. The conclusion in the case $r \ne 1/m$ still holds if ξ satisfies the weaker Assumption (A′).

3.1. The distribution of $Z(\tau)$

In this subsection we study the limit distribution of $Z(\tau)$ as $N \to \infty$. When $\eta > 0$ the limit is similar to the one obtained in the previous results.

Proposition 3.3. *Assume that ξ satisfies Assumption (A′) and that $\eta > 0$. Then,*

$$\lim_{N\to\infty} \mathbb{P}_\oplus\big(Z(\tau) = \triangle\big) = 1. \tag{3.7}$$

The case $\eta = 0$ is critical. We show that $Z_1(\tau)/N$ converges in distribution and that the limit distribution is a functional of a Poisson random variable.

Proposition 3.4. *Assume that ξ satisfies Assumption (A) with $\eta = 0$. Then under $\mathbb{P}_\oplus$, $Z_0(m)$ converges in distribution to $\Pi(\rho^m)$ a Poisson random variable with parameter ρ^m.*

Moreover, there exists a function $G : \mathbb{N} \to [0, 1]$ (see Definition (3.16)) such that

$$\left(\frac{Z_1(\tau)}{N}, \sum_{i=2}^{\infty} \frac{Z_i(\tau)}{N}\right) \xrightarrow{d} \Big(G\big(\Pi(\rho^m)\big), 1 - G\big(\Pi(\rho^m)\big)\Big). \tag{3.8}$$

Before analyzing the cases $\eta = 0$ and $\eta > 0$ separately, we prove a technical lemma that holds in both cases. It can be interpreted as follows: if at time t there are sufficiently many leading particles, then at time $t + 1$, with high probability, there is no particle at distance two or more to the leaders at time t.

Lemma 3.5. *Assume that ξ satisfies Assumption (A′). For $x = x(N) \in \Omega(N)$ such that*

$$\log N = o(x_0)$$

as $N \to \infty$, define $s_i(x)$ as in (3.1) and let $\mathcal{M}\big(N; s(x)\big)$ be a Multinomial random variable with infinitely many classes as in (3.2). Then,

$$\lim_{N\to\infty} \mathbb{P}\Big(\mathcal{M}\big(N; s(x)\big) \in \Big\{y \in \Omega(N)\,;\, \sum_{i=2}^{\infty} y_i = 0\Big\}\Big) = 1. \tag{3.9}$$

Proof. We can write

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{M}\big(N; s(x)\big) \in \Big\{y \in \Omega(N); \sum_{i=2}^{\infty} y_i = 0\Big\}\Big) \\
&= \sum_{n=0}^{N} \mathbb{P}\Big(\mathcal{M}\big(N; s(x)\big) \in \Big\{y \in \Omega(N)\,;\, y_0 = n,\, y_1 = N - n\Big\}\Big) \\
&= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} s_0(x)^n\, s_1(x)^{N-n} \\
&\geq \big(1 - \theta'^{\,x_0}\big)^N,
\end{aligned}$$

where the last inequality holds for N large enough as a consequence of Assumption (A′). Since $o(x_0) = \log N$ we obtain that $\big(1 - \theta'^{\,x_0}\big)^N \to 1$, proving the result. □

Case $\eta > 0$

Proof of Proposition 3.3. From Corollaries 2.4 and 2.7 we see that $\mathbb{P}_\oplus\big(\tau \neq m+1\big) \to 0$. Then, it suffices to prove that $\mathbb{P}_\oplus\big(Z(m+1) = \triangle; \tau = m+1\big) \to 1$.

$$\mathbb{P}_\oplus\big(Z(\tau) = \triangle\,;\, \tau = m+1\big) = \sum_{x\in\Omega(N)} \mathbb{P}_\oplus\big(Z(m+1) = \triangle;\, Z(m) = x;\, \tau = m+1\big).$$

Since $\tau = m+1$ it suffices to consider x such that $x_0 \geq 1$. Fix $0 < \varepsilon < \rho^m$ and take $x \in \Omega(N)$ such that $|x_0/N^{r\,\eta} - \rho^m| < \varepsilon$. From (3.2),

$$\begin{aligned}
\mathbb{P}_\oplus\big(Z(m+1) = \triangle | Z(m) = x\big) &= \mathcal{M}\big(N; s(x)\big)(\triangle) = s_1(x)^N \\
&= \Big(\big(1-p_0\big)^{x_0} - \big(1-p_0\big)^{x_1}\big(1-p_0-p_1\big)^{x_0}\Big)^N \\
&\geq \big(1-p_0\big)^{x_0 N}\big(1 - \theta'^{\,x_0}\big)^N,
\end{aligned} \tag{3.10}$$

where the last inequality is a consequence of Assumption (A′). Due to the choice of x_0, (3.10) is bounded from below by

$$\big(1-p_0\big)^{(\rho^m+\varepsilon)N^{1+r\eta}}\big(1 - \theta'^{(\rho^m-\varepsilon)N^{r\eta}}\big)^N,$$

which converges to one as $N \to \infty$. Then, by Proposition 2.6 and Eq. (3.10), we see that

$$\mathbb{P}_\oplus\big(Z(\tau) = \triangle\big) \geq \sum_{|x_0/N^{r\eta} - \rho^m| < \varepsilon} \mathbb{P}_\oplus\big(Z(\tau) = \triangle;\, Z(m) = x;\, \tau = m+1\big)$$

converges to one, proving the result. □

Case $\eta = 0$

In this paragraph, we prove Proposition 3.4 and also a generalization that allows us to compute the distribution of $Z_1(\tau^{(i)})$.

Lemma 3.6. *Assume that ξ satisfies Assumption (A′) with $\eta = 0$. Fix $0 < a < b$ and denote by $\Omega_a^b(N)$ the subset of $\Omega(N)$ defined as*

$$\Omega_a^b(N) := \left\{ x \in \Omega(N);\, aN^{1/m} \le x_0 \le bN^{1/m} \right\}.$$

Then the following limit holds

$$\lim_{N\to\infty} \sup_{x\in\Omega_a^b(N)} \mathbb{P}_x\big(Z(1) \ne \triangle \mid Z_0(1) = 0\big) = 0. \tag{3.11}$$

Proof. It is not difficult to obtain the following inequality

$$\mathbb{P}_x\big(Z(1) \ne \triangle \mid Z_0(1) = 0\big) \le \frac{\mathbb{P}_x\Big(Z(1) \in \big\{y \in \Omega(N)\,;\, \sum_{i=2}^{\infty} y_i \ne 0\big\}\Big)}{\mathbb{P}_x\big(Z_0(1) = 0\big)}.$$

Under $\mathbb{P}_x$, $Z(1)$ is distributed according to $\mathcal{M}\big(N, s(x)\big)$, then from the proof of Lemma 3.5

$$\lim_{N\to\infty} \sup_{x\in\Omega_a^b(N)} \mathbb{P}_x\left(Z(1) \in \left\{ y \in \Omega(N)\,;\, \sum_{i=2}^{\infty} y_i \ne 0 \right\} \right) = 0.$$

To finish the proof it suffices to show that $\mathbb{P}_x\big(Z_0(1) = 0\big)$ is bounded away from zero. Indeed, $Z_0(1)$ is distributed according to a Binomial random variable of parameter N and $s_0(x)$. Using the hypotheses of the lemma we obtain the lower bound

$$s_0(x) \ge 1 - (1 - p_0)^{b\,N^{1/m}}.$$

Coupling $Z(1)$ with $\mathcal{B}$ a Binomial of parameter N and $1 - (1 - p_0)^{b\,N^{1/m}}$

$$\mathbb{P}_x\big(Z_0(1) = 0\big) \ge \mathcal{B}\left(N, 1 - (1 - p_0)^{b\,N^{1/m}}\right)(0) \to e^{-\rho b},$$

for every $x \in \Omega_a^b(N)$, which finishes the proof. $\square$

From Corollary 2.7, we see that under $\mathbb{P}_\oplus$, $Z_0(m-1)/N^{1/m}$ converges in probability to ρ^{m-1}, as $N \to \infty$. Hence, from Lemma 3.6, we conclude that

$$\lim_{N\to\infty} \mathbb{P}_\oplus\big(Z(\tau) = \triangle \mid Z_0(m) = 0\big) = 1. \tag{3.12}$$

This is the first step to prove Proposition 3.4. The second step is to study the conditional distribution of $Z(\tau)$ under $Z_0(m) = x_0$ for a positive integer x_0.

Proposition 3.7. *Assume that ξ satisfies Assumption (A) with $\eta = 0$. Let k be a nonzero integer and denote by $\Omega_k(N)$ the subset of $\Omega(N)$ defined as*

$$\Omega_k(N) := \big\{x \in \Omega(N);\, x_0 = k\big\}.$$

Then, for $\varepsilon > 0$ the following limit holds

$$\lim_{N\to\infty} \sup_{x\in\Omega_k(N)} \mathbb{P}_x\left(\left|\left(\frac{Z_1(1)}{N}, \frac{\sum_{i\geq 2} Z_i(1)}{N}\right) - \left(1-\theta^k, \theta^k\right)\right| > \varepsilon\right) = 0. \tag{3.13}$$

Proof. From (3.2), we see that under $\mathbb{P}_x$, $Z(1)$ is distributed according to an infinite class Multinomial of parameters N and $s(x)$. In particular, the triplet

$$\left(Z_0(1), Z_1(1), \sum_{i\geq 2} Z_i(1)\right)$$

is distributed according to a three classes Multinomial of parameters N and $\left(s_0(x), s_1(x), \sum s_i(x)\right)$. If ξ satisfies Assumption (A) and $x \in \Omega_k(N)$, we have that

$$\lim_{N\to\infty} s_0(x) = 0; \quad \lim_{N\to\infty} s_1(x) = 1-\theta^k; \quad \lim_{N\to\infty} \sum s_i(x) = \theta^k, \tag{3.14}$$

and the rate of convergence is uniform on $x \in \Omega_k(N)$. A three classes Multinomial random variable as above satisfies a large deviation principle (see e.g. [9,10]) and the rate function is given by

$$\Lambda^*(y) = \begin{cases} y_1 \log\left(\dfrac{(\theta^k)y_1}{(1-y_1)(1-\theta^k)}\right) - \log\left(\dfrac{\theta^k}{1-y_1}\right), & \text{if } y_1 + y_2 = 1; \\ \infty, & \text{otherwise.} \end{cases} \tag{3.15}$$

The unique zero of Λ^* is at $y = (0, 1-\theta^k, \theta^k)$. Implying the convergence in probability

$$\frac{1}{N}\left(Z_0(1), Z_1(1), \sum_{i\geq 2} Z_i(1)\right) \to \left(0, 1-\theta^k, \theta^k\right),$$

as $N \to \infty$, which proves the statement. $\square$

We now give the definition of the function $G(\cdot)$ appearing in Proposition 3.4.

Definition 3.8. Let $G : \mathbb{N} \longrightarrow [0, 1]$ be defined as

$$G(k) = \begin{cases} 1-\theta^k, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0, \end{cases} \tag{3.16}$$

where θ is given by Assumption (A).

Proof of Proposition 3.4. From Corollary 2.5, we have that under $\mathbb{P}_\oplus$, $Z_0(m)$ converges in distribution to a Poisson random variable of parameter ρ^m. Hence, to prove Proposition 3.4 it suffices to show that

$$\begin{aligned} &\mathbb{P}_\oplus\left(\left|\left(\frac{Z_1(\tau)}{N}, \frac{\sum_{i\geq 2} Z_i(\tau)}{N}\right) - \left(G\big(Z_0(m)\big), 1 - G\big(Z_0(m)\big)\right)\right| > \varepsilon\right) \\ &= \sum_{k=0}^{N} \mathbb{P}_\oplus\left(\left|\left(\frac{Z_1(\tau)}{N}, \frac{\sum_{i\geq 2} Z_i(\tau)}{N}\right) - \left(G(k), 1-G(k)\right)\right| > \varepsilon; Z_0(m) = k\right) \end{aligned}$$

converges to zero. From (3.12) and Proposition 3.7, we know that for each $k \in \mathbb{N}$ the terms in the above sum converge to zero. Then, from the tightness of $Z_0(m)$ we obtain that the sum itself converges to zero, proving the result. □

We finish the present subsection by computing the limit distribution of $Z\left(\tau^{(i)}\right)$ for $i \in \mathbb{N}$. We also prove the convergence of some related processes that will appear when calculating the front velocity in Section 3.4.

Proposition 3.9. *Assume that ξ satisfies Assumption (A) and that $\eta = 0$. Let $\kappa(N)$ be a sequence of random variables in $\Omega(N)$ with some distribution $\mu(N)$ and denote by $\kappa_0(N)$ the first coordinate of $\kappa(N)$. Assume also that $\kappa_0(N)/N$ converges in distribution to U a positive random variable. Then, under $\mathbb{P}_{\mu(N)}$, we have that*

1. *$Z_0(m)$ converges weakly to V a doubly stochastic Poisson random variable, whose distribution is determined by the Laplace transform*
$$\mathbb{E}\left[e^{sV}\right] = \mathbb{E}\left[\exp(e^s - 1)\,\rho^m\, U\right]. \tag{3.17}$$
2. *Furthermore, the joint convergence also holds*
$$\left(Z_0(m), \frac{Z_1(\tau)}{N}, \tau\right) \xrightarrow{d} \left(V, G(V), m + \mathbf{1}_{\{V \neq 0\}}\right). \tag{3.18}$$

Proof. We may assume that all $\xi_{ij}(t)$'s and $\kappa(N)$'s are constructed on the same probability space in such a way that $\kappa_0(N)/N \to U$ a.s. The hypotheses of Corollary 2.5 are satisfied, implying the first statement of the proposition. From Corollaries 2.4 and 2.7, we see that $\mathbb{P}(m \le \tau \le m+1)$ converges to one. Since $\tau = m$ if and only if $Z_0(m) = 0$ and $\tau \notin \{1, \ldots, m-1\}$ we obtain that

$$\lim_{N\to\infty} \mathbb{P}\left(|\tau - m - \mathbf{1}_{\{Z_0(m)\neq 0\}}| > \varepsilon\right) = 0,$$

which implies the convergence in distribution $\tau \xrightarrow{d} m + \mathbf{1}_{\{V\neq 0\}}$. Finally, to prove that $Z_1(\tau)/N$ converges to $G(V)$, we proceed as in Proposition 3.4 and show by dominated convergence that

$$\lim_{N\to\infty} \mathbb{E}\left[\mathbb{P}_{\kappa(N)}\left(\left|\left(\frac{Z_1(\tau)}{N}, \frac{\sum_{i\geq 2} Z_i(\tau)}{N}\right) - \left(G\left(Z_0(m)\right), 1 - G\left(Z_0(m)\right)\right)\right| > \varepsilon\right)\right] = 0,$$

we leave the details to the reader. □

As an application of Proposition 3.9 we can calculate the distribution of $Z\left(\tau^{(2)}\right)$. Indeed we can consider the convergence in Proposition 3.4 as the stronger a.s. convergence. We do not lose any generality since we can construct a sequence of random variables (possibly in an enlarged probability space) $\kappa(N) \stackrel{d}{=} Z(\tau)$ that converges a.s. Details about this construction can be found in [3]. Passing to the appropriate product space we also consider that the κ's and ξ_{ij}'s are independent, which implies that

$$\mathbb{P}_{\kappa(N)}\left(Z \circ \Theta_\tau(t) \in \cdot\right) \stackrel{d}{=} \mathbb{P}_{Z(\tau)}\left(Z \circ \Theta_\tau(t) \in \cdot\right), \tag{3.19}$$

for $t \geq 0$. We apply the strong Markov property and Proposition 3.9 to obtain the weak convergence

$$\left(Z_0(\tau+m), \frac{Z_1\left(\tau^{(2)}\right)}{N}, \tau^{(2)}-\tau^{(1)}\right) \xrightarrow{d} \left(V^{(2)}, G\left(V^{(2)}\right), m+\mathbf{1}_{\{V^{(2)}\neq 0\}}\right),$$

where $V^{(2)}$ is a doubly stochastic Poisson variable governed by $V^{(1)}$ the limit distribution of $Z_0(m)$. This method can be iterated to obtain the following result.

Lemma 3.10. *Assume that ξ satisfies Assumption (A) with $\eta = 0$. Denote $\Delta\tau_N^{(i)} := \tau^{(i)} - \tau^{(i-1)}$ and let $l \in \mathbb{N}$. Then, under $\mathbb{P}_\oplus$*

$$\left\{\left(Z_0\big(\tau^{(i-1)}+m\big), Z_1\big(\tau^{(i)}\big)/N, \Delta\tau_N^{(i)}\right); 1 \leq i \leq l\right\} \tag{3.20}$$

converges weakly to

$$\left\{\left(V^{(i)}, G\left(V^{(i)}\right), m+\mathbf{1}_{\{V^{(i)}\neq 0\}}\right); 1 \leq i \leq l\right\}. \tag{3.21}$$

The distribution of $V^{(i)}$ is determined by

$$\begin{aligned}&\mathbb{P}\left(V^{(i+1)} = t_{i+1} \mid V^{(j)} = t_j,\ j \leq i\right)\\ &\quad = \mathbb{P}\left(V^{(i+1)} = t_{i+1} \mid V^{(i)} = t_i\right) = e^{-G(t_i)\rho^m}\frac{(G(t_i)\rho^m)^{t_{i+1}}}{t_{i+1}!},\end{aligned} \tag{3.22}$$

where $t_1, \ldots, t_{i+1} \in \mathbb{N}$ and $V^{(1)}$ is distributed according to a Poisson variable with parameter ρ^m.

Proof. It is a direct consequence of Proposition 3.9 and an induction argument. □

With a very small effort we can state Lemma 3.10 in a more general framework. We consider the space of real valued sequences $\mathbb{R}^{\mathbb{N}}$ where we define the metric

$$d(a,b) = \sum_{n=0}^{\infty}\frac{|a_n - b_n|}{2^n}.$$

A complete description of this topological space can be found in [3]. Since time is discrete, the following proposition holds as a corollary of Lemma 3.10.

Proposition 3.11. *Assume that ξ satisfies Assumption (A) with $\eta = 0$. Then, under $\mathbb{P}_\oplus$ the process*

$$\left\{\left(Z_0\big(\tau^{(i-1)}+m\big), Z_1\big(\tau^{(i)}\big)/N, \Delta\tau^{(i)}\right); i \in \mathbb{N}\right\} \tag{3.23}$$

converges weakly in $\big((\mathbb{R}^{\mathbb{N}})^3, d\big)$. The limit distribution $\mathbb{W}_\theta$ is given by

$$\left\{\left(V^{(i)}, G\left(V^{(i)}\right), m+\mathbf{1}_{\{V^{(i)}\neq 0\}}\right); i \in \mathbb{N}\right\}, \tag{3.24}$$

where $V^{(i)}$ is a Markov chain with initial position at 0 and transition matrix given by

$$\mathbb{P}\big(V^{(i+1)} = l \mid V^{(i)} = k\big) = e^{-G(k)\rho^m}\frac{(G(k)\rho^m)^l}{l!}, \tag{3.25}$$

that is a Poisson distribution with parameter $\rho^m G(k)$.

Process convergence in the case $\eta > 0$

For the sake of completeness, we state the result in the case $\eta > 0$. We omit the proof of the proposition and leave the details to the reader.

Proposition 3.12. *Assume that ξ satisfies Assumption (A′) and that $\eta > 0$. Then under $\mathbb{P}_{\oplus}$ the process*

$$\left\{\left(Z_1(\tau^{(i)})/N, \Delta\tau^{(i)}\right); i \in \mathbb{N}\right\} \tag{3.26}$$

converges weakly in $\left((\mathbb{R}^{\mathbb{N}})^2, d\right)$. The limit distribution is non-random, and concentrated on the sequence

$$\left\{\left(a_i, b_i\right); \quad a_i = 1 \text{ and } b_i = m + 1\ \forall i \in \mathbb{N}\right\}. \tag{3.27}$$

3.2. Uniform integrability and bounds for $T_{\triangle}$

In this subsection, we show that if ξ satisfies Assumption (A′), then $\mathbb{E}_x[T_{\triangle}]$ is bounded independently from the initial configuration x

$$\sup_{N\in\mathbb{N}} \sup_{x\in\Omega(N)} \mathbb{E}_x\left[T_{\triangle}\right] < \infty. \tag{3.28}$$

We prove (3.28) through the following steps.

1. There exists a set $\Xi \subset \Omega(N)$ such that for N large enough and every starting point $x \in \Xi$ there is a positive probability to visit $\triangle$ before $m + 1$
$$\mathbb{P}_x\left(T_{\triangle} \le m + 1\right) > c_2, \tag{3.29}$$
where $c_2 > 0$ does not depend on $x \in \Xi$.
2. For N sufficiently large and every starting point $x \in \Omega(N)$ there is a positive probability to visit Ξ before $m + 1$
$$\mathbb{P}_x\left(T_{\Xi} \le m + 1\right) > c_3, \tag{3.30}$$
where c_3 does not depend on $x \in \Omega(N)$.

Before proving these two statements, we show that they indeed imply (3.28).

Proposition 3.13. *Assume that ξ satisfies Assumption (A′). Then, there exists $K \in \mathbb{R}$ such that*

$$\sup_{N\in\mathbb{N}} \sup_{x\in\Omega(N)} \mathbb{E}_x\left[T_{\triangle}\right] < K. \tag{3.31}$$

Proof. If (3.29) and (3.30) hold, then for N large enough and any starting point $x \in \Omega(N)$

$$\begin{aligned}
&\mathbb{P}_x\left(T_{\triangle} \le 2m + 2\right) \\
&\quad \ge \mathbb{P}_x\left(T_{\triangle} \le 2m + 2; T_{\Xi} \le m + 1\right) \\
&\quad \ge \mathbb{P}_x\left(T_{\triangle} - T_{\Xi} \le m + 1; T_{\Xi} \le m + 1\right) \\
&\quad = \mathbb{E}_x\left[\mathbb{P}_{Z(T_{\Xi})}[T_{\triangle} \le m + 1]\mathbf{1}_{T_{\Xi} \le m+1}\right] \qquad \text{(strong Markov property)} \\
&\quad \ge c_2 c_3 > 0.
\end{aligned}$$

Hence, $\sup_{y\in\Omega(N)} \mathbb{P}_y(T_\triangle \geq 2m+3) \leq 1 - c_2\, c_3$. For $i \in \mathbb{N}$, let j be such that $(2m+3)j \leq i < (2m+3)(j+1)$. Using the Markov property j times we obtain the geometric upper bound for the tail probability

$$\mathbb{P}_x(T_\triangle \geq i) \leq \left(\sup_{y\in\Omega(N)} \left\{ \mathbb{P}_y\big(T_\triangle \geq (2m+3)\big) \right\} \right)^j,$$

that is uniform in $x \in \Omega(N)$. As a consequence, (3.31) holds with $K = (2m+3)/(c_2\, c_3)$. $\square$

We now present the formal definition of Ξ.

Definition 3.14. For $x \in \Omega(N)$ define $I(x) = \inf\{i \in \mathbb{N}; x_i \geq 1\}$. Then, Ξ is the subset of $\Omega(N)$ defined as follows

$$\Xi := \left\{ x \in \Omega(N)\,;\, x_{I(x)} \geq \alpha N \right\},$$

where $0 < \alpha < 1 - \theta'$ and θ' is given by Assumption (A′). Hence, if $Z(t) \in \Xi$ there are at least αN leaders at time t.

We prove (3.29) and (3.30) in the next two lemmas.

Lemma 3.15. *Assume that ξ satisfies Assumption (A′). Then, for Ξ given by Definition 3.14 there exists a positive constant c_2 such that for N sufficiently large*

$$\inf_{x\in\Xi} \mathbb{P}_x\big(T_\triangle \leq m+1\big) > c_2.$$

Proof. Note that

$$\begin{aligned}\mathbb{P}_x\big(T_\triangle \leq m+1\big) &\geq \mathbb{P}_x\big(Z(\tau) = \triangle; \tau \leq m+1\big)\\ &= \mathbb{P}_x(Z(\tau) = \triangle) - \mathbb{P}_x\big(Z(\tau) = \triangle; \tau \geq m+2\big).\end{aligned}$$

From Corollary 2.4, the second term in the lower bound converges to zero as $N \to \infty$ and the rate of decay is uniform on $x \in \Omega(N)$. Hence it suffices to show that there exists a positive constant c_4 such that uniformly on $x \in \Xi$

$$\liminf_{N\to\infty} \mathbb{P}_x(Z(\tau) = \triangle) \geq c_4. \tag{3.32}$$

To prove (3.32) we distinguish between the cases $\eta = 0$ and $\eta > 0$. We start with the latter case $\eta > 0$. Let $Y(t) = Z_0(t)\mathbf{1}_{\{t\leq\tau\}}$ and denote by Y_k the process started from δ_k. Then, for $x \in \Xi$ we can couple the processes in such a way that

$$Y_{\lfloor \alpha N \rfloor}(t) \leq Y_{x_{I(x)}}(t) \leq Y_N(t), \tag{3.33}$$

where $x_{I(x)}$ is the number of leaders when $Z(0) = x$. From the proof of Corollary 2.7 and (3.33) we obtain

$$\lim_{N\to\infty} \mathbb{P}_x\left((\rho^m - \varepsilon)\alpha N^{\eta r} \leq Z_0(m) \leq (\rho^m + \varepsilon) N^{\eta r} \right) = 1. \tag{3.34}$$

Finally, applying the arguments of Lemma 3.5,

$$\lim_{N\to\infty} \mathbb{P}_x(Z(\tau) = \triangle) = 1.$$

In particular, any $0 < c_4 < 1$ satisfies (3.32). The case where $\eta = 0$ is similar but it requires an additional step (3.33) still holds, hence by the same arguments we obtain

$$\lim_{N\to\infty} \mathbb{P}_x\Big((\rho^{m-1} - \varepsilon)\alpha N^{1/m} \le Z_0(m-1) \le (\rho^{m-1} + \varepsilon)N^{1/m}\Big) = 1.$$

From Lemma 3.6, we see that $\mathbb{P}_x\big(Z(\tau) = \triangle \mid Z_0(m) = 0\big) \to 1$ and from the coupling argument (3.33) and Corollary 2.5 we obtain the following limit

$$\mathbb{P}_x\big(Y(m) = 0\big) \ge P_\oplus\big(Y(m) = 0\big) \to 1 - e^{-\rho^m}.$$

It implies that $\liminf_{N\to\infty} \mathbb{P}_x\big(Z_0(m) = 0\big) \ge 1 - e^{-\rho^m}$, hence every $c_4 < 1 - e^{-\rho^m}$ satisfies (3.32), proving the statement. □

Lemma 3.16. *Assume that ξ satisfies Assumption (A′). Then, for Ξ given by Definition 3.14 there exists a positive constant c_3 such that for N large enough*

$$\inf_{x\in\Omega(N)} \mathbb{P}_x\big(T_\Xi \le m + 1\big) > c_3.$$

Proof. Since $\mathbb{P}_x(\tau \ge m + 2)$ converges to zero uniformly on $x \in \Omega(N)$, it sufficient to show that for N sufficiently large

$$\mathbb{P}_x\big(Z(\tau) \in \Xi\big) \ge c_5,$$

and $c_5 > 0$ does not depend on $x \in \Omega(N)$.

$$\begin{aligned}
\mathbb{P}_x(Z(\tau) \in \Xi) &= \sum_{k=1}^{\infty} \mathbb{P}_x\big(Z(k) \in \Xi;\, \tau = k\big) \\
&= \sum_{k=1}^{\infty} \sum_{y\in\Omega(N)} \mathbb{E}_x\left[\mathbb{P}_y(Z(1) \in \Xi;\, \tau = 1)\mathbf{1}_{\{Z(k-1)=y;\ \tau\ge k\}}\right] \quad \text{(Markov property)} \\
&\ge \inf_{y\in\Omega(N)} \big\{\mathbb{P}_y(Z(1) \in \Xi \mid \tau = 1)\big\} \sum_{k=1}^{\infty} \sum_{y\in\Omega(N)} \mathbb{E}_x\Big[\mathbb{P}_y(\tau = 1)\mathbf{1}_{\{Z(k-1)=y;\ \tau\ge k\}}\Big] \\
&= \inf_{y\in\Omega(N)} \big\{\mathbb{P}_y(Z(1) \in \Xi \mid \tau = 1)\big\}.
\end{aligned} \tag{3.35}$$

Then, it suffices to show that the infimum in (3.35) is larger than a $c_5 > 0$. We have that

$$\mathbb{P}_y\big(X_i(1) = \Phi\big(X(0)\big) - 1 \mid \tau = 1\big) = s_1(y)/\big(1 - s_0(y)\big),$$

hence $\mathbb{P}_y\big(Z_1(1) = \cdot \mid \tau = 1\big)$ is binomially distributed with parameters N and $s_1(y)/\,(1 - s_0(y))$. Assuming that $y_0 \ge 1$ (otherwise we must consider $\tilde{y}$ the shifted vector)

$$\begin{aligned}
\frac{s_1(y)}{1 - s_0(y)} &\ge \frac{(1 - p_0)^{y_0} - q_2^{y_0}}{(1 - p_0)^{y_0}} \\
&\ge 1 - (\theta')^{y_0} > \alpha,
\end{aligned}$$

where the lower bound holds for N large enough as a consequence of Assumption (A′) and the definition of α. A large deviation argument allows us to conclude that

$$\mathbb{P}_y\big(Z(1) \in \Xi \mid Z_0(1) = 0\big) \ge \mathbb{P}_y\big(Z_1(1) \ge \alpha N \mid Z_0(1) = 0\big) \to 1.$$

Then, the infimum in (3.35) is larger than any $c_5 < 1$ for N sufficiently large, finishing the proof. □

The next corollary generalizes (3.28) to the later visiting times of $\triangle$.

Corollary 3.17. *Assume that ξ satisfies Assumption (A′). Then, for every $i \in \mathbb{N}$, $\sup_{x\in\Omega} \mathbb{E}_x\left[T_\triangle^{(i)}\right]$ and $\sup_{x\in\Omega} \mathbb{E}_x\left[\tau^{(i)}\right]$ are bounded uniformly on N. In particular, under $\mathbb{P}_x$ the families of random variables $T_\triangle^{(i)}$ and $\tau^{(i)}$ are uniformly integrable.*

Proof. Since $\tau^{(i)} \le T_\triangle^{(i)}$, it suffices to prove the statements for $T_\triangle^{(i)}$. To prove that the expectation is bounded we proceed inductively and apply the strong Markov property at time $T_\triangle^{(i-1)}$. It is clear that

$$\sup_{x\in\Omega(N)} \mathbb{E}_x\left[T_\triangle^{(i)}\right] \le K^i,$$

where K is given by (3.31). We now prove the uniform integrability. Applying the Markov property we obtain the upper bound

$$\mathbb{E}_x\left[T_\triangle^{(i)}; T_\triangle^{(i)} \ge l\right] \le \left(\sup_{y\in\Omega(N)} \mathbb{E}_y\left[T_\triangle^{(i)}\right] + l\right) \mathbb{P}_x\left(T_\triangle^{(i)} \ge l\right). \tag{3.36}$$

It is not difficult to see that the right-hand side of (3.36) converges to zero, finishing the proof. □

3.3. Convergence of some related integrals

To compute the front velocity in Section 3.4, we have to calculate two integrals $\mathbb{E}_\oplus\left[T_\triangle\right]$ and $\mathbb{E}\left[\phi\left(X(T_\triangle)\right)\right]$. As usual, we assume that all particles start from zero, then

$$\phi\left(X(T_\triangle)\right) = -\sum_{i=1}^{\infty} \min\{l \in \mathbb{N};\, Z_l(\tau^{(i)}) \ne 0\} \mathbf{1}_{\{\tau^{(i)} \le T_\triangle\}}.$$

In the next lemma, we use for the first time the condition $\mathbb{E}\left[|\vartheta|\right] < \infty$, which appears in Assumptions (A) and (A′).

Lemma 3.18. *Assume that ξ satisfies Assumption (A′). Then*

$$\sup_{x\in\Omega(N)} \mathbb{E}_x\left[\min\{l \in \mathbb{N};\, Z_l(\tau) \ne 0\}\right] = 1 + o(1) \tag{3.37}$$

as $N \to \infty$.

Proof. By an argument similar to the one used in Lemma 3.16 we obtain that

$$\begin{aligned}
&\mathbb{E}_x\Big[\min\{l \in \mathbb{N};\, Z_l(\tau) \ne 0\}\Big] \\
&\quad\le \sup_{y\in\Omega(N)} \mathbb{E}_y\Big[\min\{l \in \mathbb{N};\, Z_l(1) \ne 0\} \mid \tau = 1\Big] \\
&\quad\le 1 + \sup_{y\in\Omega(N)} \mathbb{E}_y\Big[\min\{l \in \mathbb{N};\, Z_l(1) \ne 0\}\mathbf{1}_{\{\min\{l\in\mathbb{N};\, Z_l(1)\ne 0\}\ge 2\}} \mid \tau = 1\Big].
\end{aligned} \tag{3.38}$$

Under the conditional probability Z is distributed according to a Multinomial with infinitely many classes and parameters $s_i(y)/\big(1 - s_0(y)\big)$, $i \geq 1$. Applying Assumption (A′) we obtain that for N sufficiently large

$$\inf_{y\in\Omega(N)} \Big\{ s_1(y)/\big(1 - s_0(y)\big) \Big\} \geq 1 - \theta'.$$

Moreover, the minimum in (3.38) is bounded from above by some $|\xi_{ij}|$. Indeed, it suffices to choose i such that $X_i(0)$ is a leader, then

$$-\min\{l \in \mathbb{N};\, Z_l(1) \neq 0\} = \phi\big(X(1)\big) - X_i(0) \geq \xi_{ij}.$$

Hence, we can give an upper bound for the right-hand side in (3.38)

$$\begin{aligned}
&\mathbb{E}_y\Big[\min\{l \in \mathbb{N};\, Z_l(1) \neq 0\}\mathbf{1}_{\{\min\{l\in\mathbb{N};\, Z_l(1)\neq 0\}\geq 2\}} \mid \tau = 1\Big] \\
&\quad\leq \mathbb{E}_y\Big[|\xi_{ij}|\mathbf{1}_{\{\min\{l\in\mathbb{N};\, Z_l(1)\neq 0\}\geq 2\}} \mid \tau = 1\Big] \\
&\quad\leq \mathbb{E}\big[|\vartheta_{ij}|\big]\mathbb{P}_y\Big(\min\{l \in \mathbb{N};\, Z_l(1) \neq 0\} \geq 2 \mid \tau = 1\Big) \\
&\quad\leq \mathbb{E}\big[|\vartheta_{ij}|\big](\theta')^N,
\end{aligned}$$

which converges to zero independently from the initial position $y \in \Omega(N)$. □

With Lemma 3.18 at hand we prove the following result in the noncritical case.

Proposition 3.19. *Assume that ξ satisfies Assumption (A′) with $\eta > 0$. Then*

$$\lim_{N\to\infty} \mathbb{E}_{\oplus}[T_\triangle] = (m+1) \quad \textit{and} \quad \lim_{N\to\infty} \mathbb{E}\big[\phi\big(X(T_\triangle)\big)\big] = -1. \tag{3.39}$$

Proof. The first limit is a direct consequence of the uniform integrability of $T_\triangle$ and $\mathbb{P}_{\oplus}(T_\triangle = m+1) \to 1$, as $N \to \infty$. We now prove the second statement.

$$\begin{aligned}
&\mathbb{E}\big[\phi\big(X(T_\triangle)\big)\big] \\
&\quad= -\sum_{i=1}^{\infty} \mathbb{E}_{\oplus}\Big[\min\{l \in \mathbb{N};\, Z_l(\tau^{(i)}) \neq 0\}\mathbf{1}_{\{T_\triangle \geq \tau^{(i)}\}}\Big] \\
&\quad= -\sum_{i=1}^{\infty}\sum_{y\in\Omega(N)} \mathbb{E}_{\oplus}\Big[\mathbb{E}_y\Big[\min\{l \in \mathbb{N};\, Z_l(\tau) \neq 0\}\Big]\mathbf{1}_{\{Z(\tau^{(i-1)})=y;\, T_\triangle\geq\tau^{(i)}\}}\Big] \\
&\quad= \big(-1 + o(1)\big)\sum_{i=1}^{\infty} \mathbb{P}_{\oplus}\Big(T_\triangle \geq \tau^{(i)}\Big), \qquad \text{as } N \to \infty,
\end{aligned} \tag{3.40}$$

the last equality in (3.40) is a consequence of Lemma 3.18. The sum on the right-hand side of (3.40) converges to one as $N \to \infty$. Indeed, $\mathbb{P}_{\oplus}(T_\triangle \geq \tau) = 1$ and

$$\begin{aligned}
\sum_{i\geq 2} \mathbb{P}_{\oplus}\Big(T_\triangle \geq \tau^{(i)}\Big) &= \sum_{i\geq 2}(i-1)\mathbb{P}_{\oplus}\big(T_\triangle = \tau^{(i)}\big) \\
&\leq \mathbb{E}_{\oplus}\big[T_\triangle\mathbf{1}_{\{T_\triangle > \tau\}}\big].
\end{aligned} \tag{3.41}$$

Since $T_\triangle$ is uniform integrable, it follows from Proposition 3.3 that the upper bound in (3.41) converges to zero, which proves the result. □

The critical case is more delicate and we prove the following result.

Proposition 3.20. *Assume that ξ satisfies Assumption (A) and that $\eta = 0$. Then*

$$\lim_{N\to\infty} \mathbb{E}_{\oplus}\left[T_{\triangle}\right] = (m+1)\mathbb{E}_0\left[\mathcal{T}_0\right] - 1 \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}\left[\phi\big(X(T_{\triangle})\big)\right] = -\mathbb{E}_0\left[\mathcal{T}_0\right], \tag{3.42}$$

where $\mathcal{T}_0$ is the stopping time given by $\mathcal{T}_0 := \min\left\{i \geq 1;\, V^{(i)} = 0\right\}$, for $V^{(i)}$ a Markov chain defined as in Proposition 3.11.

We split the proof of Proposition 3.20 into two parts. We first show that $\min\{i \geq 1; Z_1(\tau^{(i)})/N = 1\}$ converges weakly and we characterize the limit distribution. Let $i^* = \min\big\{i \geq 1;\, Z_1\left(\tau^{(i)}\right)/N = 1\big\}$, then $T_{\triangle} = \tau^{(i^*)}$. So in the second part, we show that $\mathbb{E}\big[\tau^{(i^*)}\big]$ and $\mathbb{E}\big[\phi\big(X\big(\tau^{(i^*)}\big)\big)\big]$ converge to the desired limits.

Lemma 3.21. *Assume that ξ satisfies Assumption (A) with $\eta = 0$. Then, under $\mathbb{P}_{\oplus}$, the following convergence in distribution*

$$\min\left\{i \geq 1;\, Z_1(\tau^{(i)})/N = 1\right\} \xrightarrow{d} \min\left\{i \geq 1;\, G(V^{(i)}) = 1\right\}$$

takes place. The process $V^{(i)}$ is the Markov chain defined in Proposition 3.11 and

$$\mathcal{T}_0 := \min\left\{i \geq 1;\, V^{(i)} = 0\right\} = \min\left\{i \geq 1;\, G(V^{(i)}) = 1\right\}.$$

Proof. Since $x \to \min\{i \geq 1; x_i = 1\}$ is not continuous in $\mathbb{R}^{\mathbb{N}}$ Lemma 3.21 is not a direct corollary of Proposition 3.11. On the other hand, $\left\{Z_0\left(\tau^{(i-1)} + m\right); i \in \mathbb{N}\right\}$ converges in distribution to $\left\{V^{(i)}; i \in \mathbb{N}\right\} \in \mathbb{N}^{\mathbb{N}}$ and the minimum becomes continuous when restricted to $\mathbb{N}^{\mathbb{N}}$, so

$$\min\left\{i \geq 1;\, Z_0\big(\tau^{(i-1)} + m\big) = 0\right\} \xrightarrow{d} \mathcal{T}_0 = \min\left\{i \geq 1;\, V^{(i)} = 0\right\},$$

we refer to [3] for more details on convergence in distribution. We use Lemmas 3.6 and 3.10 to obtain that

$$\lim_{N\to\infty} \mathbb{P}_{\oplus}\Big(Z_0\big(\tau^{(j-1)} + m\big) = 0;\, Z\big(\tau^{(j)}\big) \neq \triangle\Big) = 0,$$

for every $j \geq 1$. Then, we deduce from Lemma 3.10 that

$$\min\left\{i \geq 1;\, Z_0\big(\tau^{(i-1)} + m\big) = 0\right\} - \min\left\{i \geq 1;\, Z_1\big(\tau^{(i)}\big)/N = 1\right\}$$

converges in probability to zero. It implies the convergence in distribution of $\min\{i \geq 1; Z_1(\tau^{(i)})/N = 1\}$ to $\mathcal{T}_0$, finishing the proof of the lemma. $\square$

Proof of Proposition 3.20. We may write

$$\begin{aligned}
\mathbb{E}_{\oplus}[T_{\triangle}] &= \sum_{k=1}^{\infty} \mathbb{E}_{\oplus}\left[\tau^{(k)}\, \mathbf{1}_{\{T_{\triangle} = \tau^{(k)}\}}\right] \\
&= \sum_{k=1}^{\infty} \mathbb{E}_{\oplus}\left[\sum_{j=1}^{k}\big(\tau^{(j)} - \tau^{(j-1)}\big)\, \mathbf{1}_{\{\min_{i\geq 1}\{Z_1(\tau^{(i)})/N = 1\} = k\}}\right].
\end{aligned}$$

For a fixed $k \in \mathbb{N}$ the random variable $\sum_{j=1}^{k} \left(\tau^{(j)} - \tau^{(j-1)}\right) \mathbf{1}_{\{\min_{i \geq 1}\{Z_1(\tau^{(i)})/N=1\}=k\}}$ converges in law to

$$\sum_{j=1}^{k} (m + \mathbf{1}_{\{V^{(j)} \neq 0\}}) \mathbf{1}_{\{\min\{i \in \mathbb{N}; G(V^{(i)})=1\}=k\}} = \big((m+1)\, k - 1\big) \mathbf{1}_{\{\mathcal{T}_0 = k\}}.$$

Since $\tau^{(k)}$ is uniformly integrable the convergence also holds in L^1. From the uniform integrability of $T_\triangle$ we obtain the convergence in L^1 of the sum and the following limit holds.

$$\begin{aligned}\lim_{N \to \infty} \mathbb{E}_\oplus[T_\triangle] &= \sum_{k=1}^{\infty} \big((m+1)k - 1\big) \mathbb{P}_0\big(\mathcal{T}_0 = k\big) \\ &= (m+1)\mathbb{E}_0\big[\mathcal{T}_0\big] - 1.\end{aligned}$$

This proves the first statement of Proposition 3.20. We now prove the second limit in (3.42). From the proof of Proposition 3.19 we obtain that

$$\mathbb{E}_\oplus\big[\phi\big(X(T_\triangle)\big)\big] = -\big(1 + o(1)\big) \sum_{i=1}^{\infty} \mathbb{P}_\oplus\big(\tau^{(i)} \leq T_\triangle\big).$$

From the uniform integrability of $T_\triangle$ we obtain that $\sum_{i=1}^{\infty} \mathbb{P}_\oplus\big(\tau^{(i)} \leq T_\triangle\big) \to \mathbb{E}_0\big[\mathcal{T}_0\big]$, which finishes the proof. □

The transition matrix of $V^{(i)}$ depends on G and a fortiori on θ. A coupling argument shows that $\mathbb{E}_0[\mathcal{T}_0]$ is non-increasing in θ. Nevertheless, we do not know how to calculate explicitly the integral. However the asymptotic behaviors as $\theta \to 0$ and 1 are easy to compute.

Proposition 3.22. *Let $V^{(i)}$ be the Markov chain whose transition matrix is given in Proposition 3.11, then*

$$\lim_{\theta \to 0} \mathbb{E}_0\left[\mathcal{T}_0\right] = \exp\left(\rho^m\right).$$

Proof. We write $\mathbb{E}_0[\mathcal{T}_0] = \sum_{k=0}^{\infty} \mathbb{P}_0(\mathcal{T}_0 \geq k+1)$. For $l \geq 1$, then $1 \geq G(l) \geq G(1) = 1 - \theta$, and

$$\begin{aligned}&\mathbb{P}_0(\mathcal{T}_0 \geq k+1) \\ &\quad = \sum_{l_1=1}^{\infty} e^{-\rho^m} \frac{(\rho^m)^{l_1}}{l_1!} \cdots \sum_{l_{k-1}=1}^{\infty} e^{-\rho^m G(l_{k-2})} \frac{(\rho^m G(l_{k-2}))^{l_{k-1}}}{l_{k-1}!} \big(1 - e^{-\rho^m G(l_{k-1})}\big).\end{aligned}$$

The last expression is bounded from above by $(1 - e^{-\rho^m})^k$. Since $G(l) \to 1$ as $\theta \to 0$ we can conclude by dominated convergence. □

We point out here that the case $\theta \to 0$ corresponds to the "two-state percolation distribution" model studied in Section 2. Informally, when θ is very small there is a high probability that $Z(\tau)$ starts afresh from $\triangle$. A similar computation can be done in the case where θ converges to one.

Proposition 3.23. *Let $V^{(i)}$ be the Markov chain whose transition matrix is given in Proposition 3.11. Then*

$$\lim_{\theta \to 1} \mathbb{E}_0[\mathcal{T}_0] = 2 - \exp\left(-\rho^m\right).$$

Proof. The proof follows the same lines as that of Proposition 3.22 and we leave the details to the reader. □

3.4. *Front speed*

As in Section 2.2, we explore the renewal structure of Z that starts afresh from $\triangle$. Let $N(t) = \max\{i \ ; \ T_\triangle^{(i)} \le t\}$. Then

$$\phi\big(X(t)\big) = -\sum_{i=1}^{N(t)} \left[\phi\left(X\left(T_\triangle^{(i+1)}\right)\right) - \phi\left(X\left(T_\triangle^{(i)}\right)\right)\right] + o(t)$$

as $t \to \infty$ almost surely. Taking the limit, as $t \to \infty$, we have that

$$\begin{aligned}\lim_{t\to\infty} \frac{\phi\big(X(t)\big)}{t} &= \lim_{t\to\infty} -\frac{1}{t}\sum_{i=1}^{N(t)} \phi\left(X\left(T_\triangle^{(i+1)}\right)\right) - \phi\left(X\left(T_\triangle^{(i)}\right)\right) \\ &= \frac{\mathbb{E}\big[\phi\big(X(T_\triangle)\big)\big]}{\mathbb{E}_\oplus[T_\triangle]} \quad \text{a.s.}\end{aligned} \tag{3.43}$$

The limit is a consequence of the ergodic Theorem and the renewal structure. In Section 3.3, we calculated the limits of the above expected values. We obtain that

$$\lim_{N\to\infty} v_N = \begin{cases} -\big(1 + \lfloor 1/r \rfloor\big)^{-1}, & \text{if } 1/r \notin \mathbb{N} \\ -\big(\lfloor 1/r \rfloor + 1 - 1/\mathbb{E}_0[\mathcal{T}_0]\big)^{-1}, & \text{if } 1/r = m \in \mathbb{N}, \end{cases}$$

which proves Theorem 3.2 with $g(\theta) = \mathbb{E}_0[\mathcal{T}_0]$.

4. Conclusion and sketch of the proof of Theorem 1.3

Theorem 1.3 follows as a corollary of Theorem 3.2 proved in Section 3. We will not prove it in detail but we give a sketch of the proof. The constants λ_0 and $\lambda_1 - \lambda_0$ appearing in Theorem 1.3 are justified by an affine transformation. Then, it remains to explain how we pass from the distribution over the lattice to the more general one. In the proof of Theorem 3.2 we see that in the discrete case ϑ contributes to the position of the leaders only in rare events. Indeed, if there are k leaders at time t the position of the front is determined by ϑ at $t + 1$ only in the case where $\xi_{ij}(t+1) \le -2$ for at least N^k random variables. The probability of this event is of order θ^{N^k}, as a consequence of Assumption (A). This behavior still holds in the general case. For a complete proof we refer to [6] Theorem 1.3, which applies also to our case with the obvious changes.

The position of the front depends basically on the tail distribution of ξ, that is determined by the point masses λ_0 and λ_1. The only case where ϑ could contribute to the position of the front in long time scales is in the non-integrable case. Then the mechanism responsible for propagation is of a very different nature and the front is no longer pulled by the leading edge. In the rare events, when the front moves backwards more than $\lambda_0 - \lambda_1$ the contribution of ϑ would be non-negligible depending on its tail and the global front profile. This problem is still open and much harder to solve.

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References

[1] M. Aizenman, A. Ruzmaikina, Characterization of invariant measures at the leading edge for competing particle systems, Ann. Probab. 33 (1) (2005) 82–113.
[2] J. Bérard, J.-B. Gouéré, Brunet–Derrida behavior of branching-selection particle systems on the line, Commun. Math. Phys. 298 (2) (2010) 323–342.
[3] P. Billingsley, Convergence of Probability Measures, Wiley-Interscience, 1999.
[4] E. Brunet, B. Derrida, Exactly soluble noisy traveling-wave equation appearing in the problem of directed polymers in a random medium, Phys. Rev. E 70 (1) (2004) 016106.
[5] E. Brunet, B. Derrida, A.H. Mueller, S. Munier, Noisy traveling waves: effect of selection on genealogies, Europhys. Lett. 76 (1) (2006) 1–7.
[6] F. Comets, J. Quastel, A.F. Ramírez, Last passage percolation and traveling fronts, J. Stat. Phys. 152 (2013) 419–451.
[7] J. Cook, B. Derrida, Directed polymers in a random medium: 1/d expansion and the n-tree approximation, J. Phys. 23 (9) (1990) 1523–1554.
[8] O. Couronné, L. Gerin, A branching-selection process related to censored Galton–Watson process, Ann. l'Institut Henri Poincaré 50 (1) (2014) 84–94.
[9] A. Dembo, O. Zeitouni, Large Deviations Techniques and Applications, Springer-Verlag, 2010.
[10] F. Den Hollander, Large Deviations, American Mathematical Society, 2000.
[11] B. Derrida, H. Spohn, Polymers on disordered trees, spin glasses, and traveling waves, Journal of Statistical Physics 51 (5-6) (1988) 817–840.
[12] R. Durrett, Probability: Theory and Examples, Cambridge University Press, 2010.
[13] G. Giacomin, Random Polymer Models, Imperial College Press, 2007.
[14] P. Maillard, Branching Brownian motion with selection of the N right-most particles: An approximate model. arXiv preprint arXiv:1112.0266, 2011.
[15] D. Panja, Effects of fluctuations on propagating fronts, Phys. Rep. 393 (2004) 87–174.