



Optimal expulsion and optimal confinement of a Brownian particle with a switching cost

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Abstract

We solve two stochastic control problems in which a player tries to minimize or maximize the exit time from an interval of a Brownian particle, by controlling its drift. The player can change from one drift to another but is subject to a switching cost. In each problem, the value function is written as the solution of a free boundary problem involving second order ordinary differential equations, in which the unknown boundaries are found by applying the principle of smooth fit. For both problems, we compute the value function, we exhibit the optimal strategy and we prove its generic uniqueness.

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1. Description of the problem

Consider a game in which the player's goal is to force a Brownian particle out of an interval (say $[0, 1]$) as quickly as possible. At each instant, the player selects one of two opposite constant forces, either upwards or downwards, which adds or subtracts a constant drift μ to the Brownian motion. The player is allowed to switch between the two forces at any time, but at each switch,

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he incurs a penalty of c units of time ($c > 0$). The goal is to find a strategy that minimizes the expected penalized time, that is, the sum of the time needed for the particle to exit the interval and the switching penalties (“optimal expulsion problem”).

We also solve the “opposite” problem, in which the goal is to keep the particle inside the interval for as long as possible, subject to the same kind of switching penalty, which is now subtracted from the time to exit the interval (“optimal confinement problem”).

These two abstract problems can be viewed in the context of various applications, such as maintaining an inventory between certain bounds by controlling the production rate [9,22], or maintaining an insurance company’s capital reserve between two bounds by controlling the insurance premium [2]. In certain asymptotic limits, these quantities may behave like a Brownian motion, and a change of production rate or of premium may entail a switching cost. The two boundaries may represent certain levels that one may want to reach as soon (or as late) as possible.

The presence of the switching cost is the key issue here: for instance, Prokhorov [21] solves a similar problem but without cost penalty, and Mandl [15] treats a control problem for a Brownian motion under a constraint on the number of switchings. When there is no switching cost, then the solutions of these problems are well-known (see [10, p.167–168]).

There is a well-developed literature for studying this kind of stochastic control problem, including [8,10,12,16,26]. Most frequently, these problems involve terminal costs and running costs. More recently, even more general kinds of performance criteria have been considered, as in [17], where the criterion also involves the running maximum of the observed process. In the presence of a switching cost, the problem falls into the theory of impulse control, as described for instance in [16, Chapter 6].

In order to solve our two control problems, we begin by formulating a free boundary problem for the value function. This involves splitting the state space into two regions, a *continuation* region and a *switching* region. The particular form of the regions is guessed from the description of the problem. In the continuation region, the value function solves certain ordinary differential equations, and in the complement of this region, the value function satisfies a relationship related to the switching cost. There are also boundary conditions at the extremities of the interval. In general, this system of equations is *not* sufficient to characterize the value function, and this is indeed the case here: it is necessary to specify appropriate additional conditions at the free boundaries between the regions, which we do using the so-called *principle of smooth fit* (see for example [19, p.147] and [20, Section 5.3.4]).

This approach has a long history, going back to [4,11,23,24], and, more recently, [19], and has proved to be quite successful in a wide range of problems, including, in addition to those in the references just mentioned, the problem of optimal switching (without cost) between two Brownian motions [14]. Other examples of optimal switching problems related to ours and with explicit solutions can be found in [1,6,13]. The problems considered in these papers differ from ours in particular because the state space is either the real line or the half-line, and there are no boundary conditions.

In our problem, if the switching cost is high enough, then, obviously, one should switch drifts rarely or not at all, and in fact, it turns out that there is a critical value $c^*(\mu)$, which turns out to be the same in both the expulsion and confinement problems, above which it is optimal never to switch drifts. We compute this value explicitly, and then we show that for costs $c < c^*(\mu)$, the optimal strategy is determined by four thresholds that are the endpoints of the switching regions. We determine these thresholds explicitly, up to the resolution of a single transcendental equation (in each problem).

This paper is organized as follows. In Section 2, we give the precise formulation of the optimal expulsion and confinement problems, we guess the form of the continuation and switching regions in order to state the free boundary problems and their solutions, we identify the critical cost $c^*(\mu)$, and we present our main results concerning the value functions and the description of the optimal controls. In Section 3, we solve the free boundary problems, by considering first the case $c = c^*(\mu)$, and then use this for the cases $c < c^*(\mu)$ and $c > c^*(\mu)$. Using the *verification method*, we prove that the solutions are indeed the value functions, by identifying a process that plays the role of Snell’s envelope [7]: it is a sub- (or super-) martingale for all strategies and a martingale for the optimal strategy. This proof makes use of the local time–space formula of [18]. Finally, in Section 4, we identify the consequences of a suboptimal action, which allows us to show that the optimal solutions are generically unique (except in the critical case $c = c^*(\mu)$, where there are two distinct optimal solutions), and we study the limiting case $c \downarrow 0$, from which we recover the zero-cost case of [10].

2. Formulation of the problem and main results

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $B_0 = 0$ a.s., let $(\mathcal{F}_t)_{t \geq 0}$ be its natural filtration and let \mathcal{A} denote the set of all \mathcal{F}_t -adapted processes that are right continuous, piecewise constant and take values in $\{-1, 1\}$. The elements of \mathcal{A} are the *strategies* available to the player. We consider a control model in which the system’s state is given by the stochastic differential equation

$$dX_t^A = A_t \mu dt + dB_t, \tag{2.1}$$

where $A = (A_t)_{t \geq 0} \in \mathcal{A}$ and $\mu > 0$ is a given positive constant. The random variable X_t^A denotes the *position of the particle at time t* if the player is using the strategy A , and A_t gives the direction in which the player is pushing at time t . The initial conditions are given by a family of probability measures $\{\mathbb{P}_{x,a}, x \in [0, 1], a \in \{\pm 1\}\}$ defined by $\mathbb{P}_{x,a}(X_0^A = x, A_{0-} = a) = 1$, with associated expectations $\mathbb{E}_{x,a}$. Here, A_{0-} is the drift that applies just before time 0, and which can change at time 0 precisely, if desirable. Let $c > 0$ be the *switching cost* and let

$$N_t(A) = \#\{s \in [0, t] : A_{s-} \neq A_s\} \tag{2.2}$$

be the number of switches of drift of the process X^A up to time t . Notice that $N_0(A) > 0$ is possible. The *cost function* for the minimization (resp. maximization) problem is then given by

$$J_c(x, a, A) = \mathbb{E}_{x,a}(\tau^A + cN_{\tau^A}(A)),$$

respectively,

$$J_c^{\max}(x, a, A) = \mathbb{E}_{x,a}(\tau^A - cN_{\tau^A}(A)),$$

where

$$\tau^A = \inf\{t \geq 0 : X_t^A \notin]0, 1[\} \tag{2.3}$$

and the *value functions* are respectively given by

$$V_c(x, a) = \inf_{A \in \mathcal{A}} J_c(x, a, A), \tag{2.4}$$

$$V_c^{\max}(x, a) = \sup_{A \in \mathcal{A}} J_c^{\max}(x, a, A). \tag{2.5}$$

The goal is then to compute these value functions and to find *optimal controls* $A^* \in \mathcal{A}$ and $G^* \in \mathcal{A}$ such that $V_c(x, a) = J_c(x, a, A^*)$ and $V_c^{\max}(x, a) = J_c^{\max}(x, a, G^*)$, for all $c > 0$ and for all $\mu > 0$.

2.1. Properties of the solution

In order to formulate a free boundary problem for each value function, we will assume that the solution will satisfy three properties. The validity of these properties will be established in Section 3.

Property 1. *The optimal strategy is symmetric with respect to the initial drift and the value functions satisfy $V_c(x, a) = V_c(1-x, -a)$ and $V_c^{\max}(x, a) = V_c^{\max}(1-x, -a)$, for all $x \in [0, 1]$ and $a \in \{\pm 1\}$.*

Property 2. *There exists a critical value $c^*(\mu) > 0$ for which the optimal strategy is the constant strategy if $c > c^*(\mu)$.*

Indeed, for a given μ , the expected exit time of a Brownian motion with constant drift $\pm\mu$ is a bounded function of the starting point $x \in [0, 1]$. Thus, if the cost exceeds a certain value, then a reasonable player will never pay this cost to change the initial drift. This value is given by the maximal difference between the expected exit time from $[0, 1]$ of a Brownian motion starting at x with a constant drift μ or $-\mu$. Namely, for $v \in \mathbb{R}$, set $\sigma^v = \inf\{t \geq 0 : B_t + vt \notin]0, 1[\}$ and $f^v(x) = \mathbb{E}_x(\sigma^v)$. Then, by taking the derivative at 0 of the moment-generating function of a Brownian motion with drift (see [3, II.2.3]), we find, after tedious calculations, that

$$f^v(x) = -\frac{x}{v} + \frac{1 - e^{-2vx}}{v(1 - e^{-2v})} \tag{2.6}$$

and that

$$\begin{aligned} c^*(\mu) &:= \max_{x \in [0,1]} (f^\mu(x) - f^{-\mu}(x)) = \max_{x \in [0,1]} \left(-\frac{2x}{\mu} + \frac{1}{\mu} + \frac{e^{2\mu x} - e^{2\mu(1-x)}}{\mu(e^{2\mu} - 1)} \right) \\ &= \frac{-1}{\mu^2} \left\{ \log \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right) \right) + \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right\}, \end{aligned} \tag{2.7}$$

for any $\mu > 0$ (note that $\lim_{\mu \downarrow 0} c^*(\mu) = 0$ and $\lim_{\mu \rightarrow +\infty} c^*(\mu) = 0$). This maximum is attained at

$$x^* = \frac{1}{2\mu} \left(\log \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right) \right) + \mu \right). \tag{2.8}$$

The third property concerns the general shape of the optimal strategy. Indeed, consider two scenarios in the minimization problem. Assume first that one starts near 1 with a positive drift. The player will keep this favorable drift for a while. If the particle goes down, then when it reaches $\frac{1}{2}$, both drifts are equivalent because of the symmetry property. Since the player is subject to a switching penalty, he will keep the positive drift. If the particle keeps going down, then it will become more advantageous to change to a negative drift so that the particle will exit more quickly through 0.

Secondly, if one starts close to 0 with a positive drift, then the diffusive behavior of the particle makes it very likely that it will rapidly hit 0 even if the drift is in the unfavorable direction. Thus, it

is probably not worthwhile to pay the penalty to change the drift. These two facts are summarized by the following property:

Property 3. *There exist two barriers a_c and b_c satisfying $0 < a_c \leq b_c < \frac{1}{2}$ and such that it is optimal to keep a positive drift above b_c or below a_c and it is optimal to switch to a negative drift within $[a_c, b_c]$.*

In the case of the maximization problem, this property becomes:

Property 3^{max}. *There exist two barriers a_c^{\max} and b_c^{\max} satisfying $\frac{1}{2} < a_c^{\max} \leq b_c^{\max} < 1$ and such that it is optimal to keep a positive drift above b_c^{\max} or below a_c^{\max} and it is optimal to switch to a negative drift within $[a_c^{\max}, b_c^{\max}]$.*

2.2. Solution of the minimization problem

Via the *dynamic programming principle* and the Hamilton–Jacobi–Bellman equation (see e.g. [16, Section 3.1], [26, Section 3], or [20, Chapter 5]), the value function $V_c(x, \pm 1)$ of the stochastic control problem (2.4) should satisfy the following system of variational inequalities:

$$\min \left\{ 1 \pm \mu \frac{\partial V_c}{\partial x}(x, \pm 1) + \frac{\partial^2 V_c}{\partial x^2}(x, \pm 1), -V_c(x, \pm 1) + V_c(x, \mp 1) + c \right\} = 0, \quad x \in]0, 1[,$$

$$V_c(0, \pm 1) = V_c(1, \pm 1) = 0.$$

We formulate this system as a *free boundary problem* for the value function. In the region where it is optimal to keep the current drift, the value function must satisfy the ordinary differential equation (2.9a). In the region where it is optimal to switch to the other drift, we have Eq. (2.9b). With the three properties of Section 2.1, we expect that the value function should satisfy the following problem, in which $V_c(x+, a)$ (resp. $V_c(x-, a)$) denotes $\lim_{y \downarrow x} V_c(y, a)$ (resp. $\lim_{y \uparrow x} V_c(y, a)$):

$$\mu \frac{\partial V_c}{\partial x}(x, 1) + \frac{1}{2} \frac{\partial^2 V_c}{\partial x^2}(x, 1) = -1, \quad x \in [0, a_c[\cup]b_c, 1] \tag{2.9a}$$

$$V_c(x, 1) = V_c(x, -1) + c, \quad x \in [a_c, b_c] \tag{2.9b}$$

$$V_c(0, 1) = V_c(1, 1) = 0, \tag{boundary conditions} \tag{2.9c}$$

$$V_c(a_c-, 1) = V_c(a_c+, 1), \tag{continuous fit} \tag{2.9d}$$

$$V_c(b_c-, 1) = V_c(b_c+, 1), \tag{continuous fit} \tag{2.9e}$$

$$\frac{\partial V_c}{\partial x}(a_c-, 1) = \frac{\partial V_c}{\partial x}(a_c+, 1), \tag{smooth fit} \tag{2.9f}$$

$$\frac{\partial V_c}{\partial x}(b_c-, 1) = \frac{\partial V_c}{\partial x}(b_c+, 1), \tag{smooth fit} \tag{2.9g}$$

$$V_c(x, -1) = V_c(1 - x, 1), \quad x \in [0, 1] \tag{symmetry} \tag{2.9h}$$

where a_c and b_c are two unknowns satisfying $0 < a_c \leq b_c < \frac{1}{2}$.

Proposition 1. *Let $c^*(\mu)$ be given by (2.7). There exists a unique solution $\{\bar{V}_c, a_c, b_c\}$ to the free boundary problem (2.9).*

1. *If $c = c^*(\mu)$, then the solution is given by*

$$a_{c^*} = b_{c^*} = \frac{1}{2\mu} \left(\log \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right) \right) + \mu \right) \tag{2.10}$$

and

$$\bar{V}_{c^*}(x, a) = f^{a\mu}(x) = J_{c^*}(x, a, \tilde{A}), \quad x \in [0, 1], a \in \{\pm 1\}, \tag{2.11}$$

where $\tilde{A} = (\tilde{A}_t \equiv a)_{t \geq 0}$ is the constant strategy and $f^{a\mu}$ is defined by (2.6). Moreover,

$$a_{c^*} = \operatorname{argmax}_{x \in [0, 1]} \{f^\mu(x) - f^{-\mu}(x)\} \quad \text{and} \quad f^\mu(a_{c^*}) = f^{-\mu}(a_{c^*}) + c^*(\mu). \tag{2.12}$$

2. If $0 < c < c^*(\mu)$, then the solution is given by

$$\bar{V}_c(x, 1) = \begin{cases} -\frac{x}{\mu} + \beta_c \left(e^{-2\mu x} - 1 \right), & x \in [0, a_c[, \\ \frac{x}{\mu} + \alpha_c \left(e^{2\mu x} - 1 \right) + c, & x \in [a_c, b_c], \\ \frac{1-x}{\mu} + \alpha_c \left(e^{2\mu(1-x)} - 1 \right), & x \in]b_c, 1], \end{cases} \tag{2.13}$$

$$\bar{V}_c(x, -1) = \bar{V}_c(1-x, 1), \quad x \in [0, 1],$$

where $b_c \in [0, \frac{1}{2}[$ is the unique solution x of the transcendental equation

$$e^{4\mu x - 2\mu} (2\mu x - \mu + c\mu^2 - 1) + 2\mu x - \mu + c\mu^2 + 1 = 0, \tag{2.14}$$

$$\alpha_c = \frac{-e^{-\mu}}{2\mu^2 \cosh(2\mu b_c - \mu)}, \tag{2.15}$$

a_c is the unique solution $y \in]0, b_c[$ of the transcendental equation

$$\mu^2 \alpha_c e^{4\mu y} + (1 - 2\mu^2 \alpha_c) e^{2\mu y} - 2\mu y + \mu^2 \alpha_c - c\mu^2 - 1 = 0 \tag{2.16}$$

and

$$\beta_c = -\alpha_c e^{4\mu a_c} - \frac{1}{\mu^2} e^{2\mu a_c}. \tag{2.17}$$

We now use the value of the barriers a_c and b_c to define the following four subsets of $[0, 1]$:

$$\begin{aligned} C_1 &= [0, a_c[\cup]b_c, 1], & C_{-1} &= [0, 1 - b_c[\cup]1 - a_c, 1], \\ D_1 &= [a_c, b_c], & D_{-1} &= [1 - b_c, 1 - a_c]. \end{aligned}$$

The subsets C_a and D_a are called respectively the *continuation* and the *switching* region for the drift a . For $0 < c \leq c^*$, we define the candidate optimal strategy A^c as follows. Let $(x, a) \in [0, 1] \times \{-1, 1\}$ be the initial conditions, define inductively an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times by $\tau_0 = 0$ and for $n \geq 0$,

$$\tau_{n+1} = \begin{cases} \inf \{t \geq \tau_n : X_t^n \in D_{(-1)^n a}\}, & \text{if } \{\dots\} \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $X_0^{-1} = x$, and for $n \geq 0$, X^n is the process defined as the solution of

$$\begin{aligned} dX_t^n &= (-1)^n a \mu dt + dB_t, \quad t \in [\tau_n, +\infty[, \\ X_{\tau_n}^n &= X_{\tau_n}^{n-1}. \end{aligned}$$

Set $\tau = \inf \{t \geq 0 : \exists n \geq 0 \text{ with } X_t^n \notin]0, 1[\text{ and } t \in [\tau_n, \tau_{n+1}[\}$. Then define $A_{0-}^c = a$, $A_t^c = a$ for $t \in [\tau_0, \tau_1[$, and for $n \geq 1$,

$$A_t^c = -A_{\tau_n-}^c, \quad \text{for } t \in [\tau_n, \tau_{n+1}[. \tag{2.18}$$

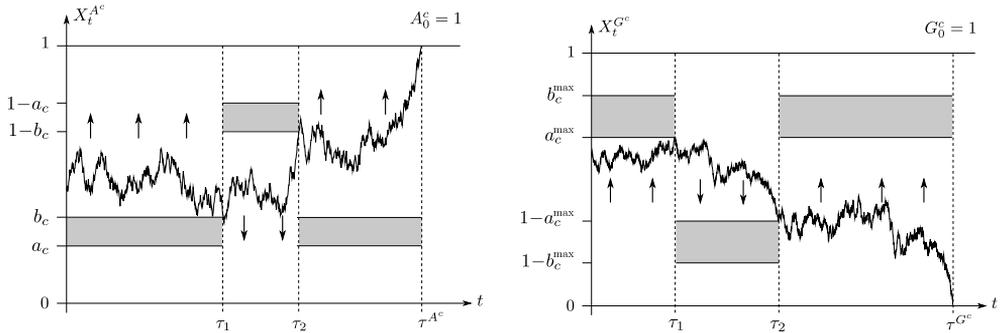


Fig. 1. Illustration of the control A^c on the left-hand side and of G^c on the right-hand side in the case where $0 < c < c^*(\mu)$. Here we assume that the initial drift is positive.

This construction implies that A^c satisfies for all $t \geq 0$

$$A_t^c = \begin{cases} A_{t-}^c, & \text{if } X_t^{A^c} \in C_{A_{t-}^c}, \\ -A_{t-}^c, & \text{if } X_t^{A^c} \in D_{A_{t-}^c}, \end{cases} \tag{2.19}$$

and the controlled process X^{A^c} is the solution of $dX_t^{A^c} = A_t^c \mu dt + dB_t$ and $X_0^{A^c} = x$. Observe that the sequence of the stopping times (τ_n) corresponds to the jump times of the strategy A^c , that $X_t^{A^c} = X_t^n$ on $[\tau_n, \tau_{n+1}[$ and that $\tau = \tau^{A^c}$, the exit time from $]0, 1[$ of X^{A^c} . This candidate strategy is pictured in Fig. 1 and it satisfies the following properties.

Proposition 2. *If $0 < c \leq c^*(\mu)$, then for all $x \in [0, 1]$ and $a \in \{-1, 1\}$, we have*

$$\mathbb{E}_{x,a}(\tau^{A^c}) < +\infty \quad \text{and} \quad \mathbb{E}_{x,a}(N_{\tau^{A^c}}(A^c)) < +\infty.$$

Now that we have exhibited a candidate strategy and a candidate value function as the solution of a free boundary problem, we can state the optimality theorem.

Theorem 3. *Let $c^*(\mu)$ be given by (2.7). If A^c is the control satisfying (2.18) (and (2.19)) and if \bar{V}_c denotes the unique solution of the free boundary problem (2.9) given in Proposition 1, then:*

1. *for $0 < c \leq c^*(\mu)$, the value function is $V_c = \bar{V}_c$ and A^c is an optimal control for the problem (2.4);*
2. *for $c > c^*(\mu)$, the value function is $V_c = \bar{V}_{c^*(\mu)}$ and $\tilde{A} = (\tilde{A}_t \equiv a)_{t \geq 0}$ is $\mathbb{P}_{x,a}$ -a.s. the unique optimal control for the problem (2.4).*

Remark 4. In the case where $c = c^*(\mu)$, we see in Proposition 1 that the switching regions consist of one single point, at which a change of drift can be considered as insignificant. Indeed at this point, the price to pay for a change of drift is equal to the maximum expected profit provided by this change itself. Moreover, the same proposition together with Theorem 3 show that the constant strategy and the candidate A^{c^*} are both optimal in this case. The same remark will apply to the maximization problem. For a uniqueness result when $0 < c < c^*(\mu)$, see Proposition 17.

2.3. Solution of the maximization problem

Similar to the minimization problem, the value function $V_c^{\max}(x, \pm 1)$ should satisfy

$$\max \left\{ 1 \pm \mu \frac{\partial V_c^{\max}}{\partial x}(x, \pm 1) + \frac{\partial^2 V_c^{\max}}{\partial x^2}(x, \pm 1), -V_c^{\max}(x, \pm 1) + V_c^{\max}(x, \mp 1) - c \right\} = 0,$$

$$x \in]0, 1[,$$

$$V_c^{\max}(0, \pm 1) = V_c^{\max}(1, \pm 1) = 0.$$

In view of the three properties of Section 2.1, we formulate this system as the following free boundary problem for the value function:

$$\mu \frac{\partial V_c^{\max}}{\partial x}(x, 1) + \frac{1}{2} \frac{\partial^2 V_c^{\max}}{\partial x^2}(x, 1) = -1, \quad x \in [0, a_c^{\max}[\cup]b_c^{\max}, 1] \tag{2.20a}$$

$$V_c^{\max}(x, 1) = V_c^{\max}(x, -1) - c, \quad x \in [a_c^{\max}, b_c^{\max}] \tag{2.20b}$$

$$V_c^{\max}(0, 1) = V_c^{\max}(1, 1) = 0, \tag{boundary conditions} \tag{2.20c}$$

$$V_c^{\max}(a_c^{\max}-, 1) = V_c^{\max}(a_c^{\max}+, 1), \tag{continuous fit} \tag{2.20d}$$

$$V_c^{\max}(b_c^{\max}-, 1) = V_c^{\max}(b_c^{\max}+, 1), \tag{continuous fit} \tag{2.20e}$$

$$\frac{\partial V_c^{\max}}{\partial x}(a_c^{\max}-, 1) = \frac{\partial V_c^{\max}}{\partial x}(a_c^{\max}+, 1), \tag{smooth fit} \tag{2.20f}$$

$$\frac{\partial V_c^{\max}}{\partial x}(b_c^{\max}-, 1) = \frac{\partial V_c^{\max}}{\partial x}(b_c^{\max}+, 1), \tag{smooth fit} \tag{2.20g}$$

$$V_c^{\max}(x, -1) = V_c^{\max}(1-x, 1), \quad x \in [0, 1] \tag{symmetry} \tag{2.20h}$$

where a_c^{\max} and b_c^{\max} are two unknowns satisfying $\frac{1}{2} < a_c^{\max} \leq b_c^{\max} < 1$. Even though the solution of the free boundary problem (2.20) is similar to the one in the minimization problem, the two problems are not symmetric as we shall see in the next proposition.

Proposition 5. *Let $c^*(\mu)$ be given by (2.7). There exists a unique solution $\{\bar{V}_c^{\max}, a_c^{\max}, b_c^{\max}\}$ to the free boundary problem (2.20).*

1. *If $c = c^*(\mu)$, then the solution is given by $a_c^{\max} = b_c^{\max} = 1 - a_c$ and*

$$\bar{V}_{c^*}^{\max}(x, a) = \bar{V}_{c^*}(x, a) = f^{a\mu}(x), \quad x \in [0, 1], a \in \{\pm 1\}, \tag{2.21}$$

where a_c and \bar{V}_{c^*} are given by Proposition 1 and $f^{a\mu}$ is defined by (2.6).

2. *If $0 < c < c^*(\mu)$, then the solution is given by*

$$\bar{V}_c^{\max}(x, 1) = \begin{cases} -\frac{x}{\mu} + \gamma_c \left(e^{-2\mu x} - 1 \right), & x \in [0, a_c^{\max}[, \\ -\frac{1-x}{\mu} + \gamma_c \left(e^{-2\mu(1-x)} - 1 \right) - c, & x \in [a_c^{\max}, b_c^{\max}], \\ \frac{1-x}{\mu} + \delta_c \left(e^{2\mu(1-x)} - 1 \right), & x \in]b_c^{\max}, 1], \end{cases} \tag{2.22}$$

$$\bar{V}_c^{\max}(x, -1) = \bar{V}_c^{\max}(1-x, 1), \quad x \in [0, 1],$$

where $a_c^{\max} = 1 - b_c$ and $\gamma_c = \alpha_c e^{2\mu}$ with b_c and α_c given by Proposition 1, b_c^{\max} is the unique solution $x \in]a_c^{\max}, 1[$ of the transcendental equation

$$\gamma_c \mu^2 e^{4\mu x - 4\mu} + e^{2\mu x - 2\mu} \left(1 - 2\gamma_c \mu^2 \right) - 2\mu x + 2\mu - 1 + \gamma_c \mu^2 + c\mu^2 = 0 \tag{2.23}$$

and

$$\delta_c = -\frac{1}{\mu^2} e^{-2\mu(1-b_c^{\max})} - \gamma_c e^{-4\mu(1-b_c^{\max})}. \tag{2.24}$$

Moreover, $b_c^{\max} > 1 - a_c$.

We construct the candidate strategy for the maximization problem as we did to get to (2.19).

Let

$$\begin{aligned} C_1^{\max} &= [0, a_c^{\max} \cup]b_c^{\max}, 1], & C_{-1}^{\max} &= [0, 1 - b_c^{\max} \cup]1 - a_c^{\max}, 1], \\ D_1^{\max} &= [a_c^{\max}, b_c^{\max}], & D_{-1}^{\max} &= [1 - b_c^{\max}, 1 - a_c^{\max}], \end{aligned}$$

with a_c^{\max} and b_c^{\max} given in Proposition 5. We denote by G^c the strategy constructed using the ideas that led to (2.18) and that satisfies

$$G_t^c = \begin{cases} G_{t-}^c, & \text{if } X_t^{G^c} \in C_{G_{t-}^c}^{\max}, \\ -G_{t-}^c, & \text{if } X_t^{G^c} \in D_{G_{t-}^c}^{\max}, \end{cases} \tag{2.25}$$

where X^{G^c} is the process controlled by G^c . This strategy is pictured in Fig. 1 and it satisfies the following properties.

Proposition 6. *If $0 < c \leq c^*(\mu)$, then for all $x \in [0, 1]$ and $a \in \{-1, 1\}$, we have*

$$\mathbb{E}_{x,a}(\tau^{G^c}) < +\infty \quad \text{and} \quad \mathbb{E}_{x,a}(N_{\tau^{G^c}}(G^c)) < +\infty.$$

Theorem 7. *Let $c^*(\mu)$ be given by (2.7). If G^c is the control satisfying (2.25) and if \bar{V}_c^{\max} denotes the unique solution of the free boundary problem (2.20) given in Proposition 5, then:*

1. *for $0 < c \leq c^*(\mu)$, the value function is $V_c^{\max} = \bar{V}_c^{\max}$ and G^c is an optimal control for the problem (2.5);*
2. *for $c > c^*(\mu)$, the value function is $V_c^{\max} = \bar{V}_{c^*(\mu)}^{\max}$, and $\tilde{A} = (\tilde{A}_t \equiv a)_{t \geq 0}$ is $\mathbb{P}_{x,a}$ -a.s. the unique optimal control for the problem (2.5).*

3. Proofs

In this section, we solve the boundary problems (2.9), (2.20) and prove that the solution of each is the value function of the associated control problem. We begin with the optimal expulsion problem.

3.1. Free boundary problem for the minimization problem

The general solution to the o.d.e. (2.9a) is $v(x) = -\frac{x}{\mu} + c_1 e^{-2\mu x} + c_2$ where c_1 and c_2 are arbitrary constants. Since (2.9a) is satisfied on the two disjoint intervals $[0, a_c[$ and $]b_c, 1]$, this yields four arbitrary constants to determine. The boundary conditions (2.9c) reduce this to two unknown constants (see [25, Ch.2] for details). The value of $\bar{V}_c(x, 1)$ for $x \in [a_c, b_c]$ is obtained by using (2.9b) together with (2.9h). We then find that \bar{V}_c must satisfy

$$\bar{V}_c(x, 1) = \begin{cases} -\frac{x}{\mu} + \beta_c \left(e^{-2\mu x} - 1 \right), & x \in [0, a_c[, \\ \frac{x}{\mu} + \alpha_c \left(e^{2\mu x} - 1 \right) + c, & x \in [a_c, b_c], \\ \frac{1-x}{\mu} + \alpha_c \left(e^{2\mu(1-x)} - 1 \right), & x \in]b_c, 1], \end{cases} \tag{3.1}$$

where α_c, β_c, a_c and b_c are four unknowns that we have to determine using the equations (2.9d)–(2.9g). They give us after some simplifications, in the same order:

$$\frac{-2a_c}{\mu} + \beta_c \left(e^{-2\mu a_c} - 1 \right) = \alpha_c \left(e^{2\mu a_c} - 1 \right) + c, \tag{3.2}$$

$$\alpha_c \left(e^{2\mu b_c} - 1 \right) + c = \frac{1-2b_c}{\mu} + \alpha_c \left(e^{2\mu(1-b_c)} - 1 \right), \tag{3.3}$$

$$2\mu\alpha_c e^{2\mu a_c} + 2\beta_c \mu e^{-2\mu a_c} + \frac{2}{\mu} = 0, \tag{3.4}$$

$$2\mu\alpha_c \left(e^{2\mu b_c} + e^{2\mu(1-b_c)} \right) + \frac{2}{\mu} = 0. \tag{3.5}$$

Multiply (3.2) by $e^{2\mu a_c}$, (3.3) by $e^{2\mu b_c}$, (3.4) by $\frac{1}{2\mu} e^{2\mu a_c}$ and (3.5) by $\frac{1}{2\mu} e^{2\mu b_c}$, to obtain, after simplifications, respectively the four equations

$$\alpha_c e^{4\mu a_c} + \left(\beta_c - \alpha_c + \frac{2a_c}{\mu} + c \right) e^{2\mu a_c} - \beta_c = 0, \tag{3.6}$$

$$\alpha_c e^{4\mu b_c} + \left(\frac{2b_c-1}{\mu} + c \right) e^{2\mu b_c} - \alpha_c e^{2\mu} = 0, \tag{3.7}$$

$$\alpha_c e^{4\mu a_c} + \frac{e^{2\mu a_c}}{\mu^2} + \beta_c = 0, \tag{3.8}$$

$$\alpha_c e^{4\mu b_c} + \frac{e^{2\mu b_c}}{\mu^2} + \alpha_c e^{2\mu} = 0. \tag{3.9}$$

Subtract (3.7) from (3.9), and solve this equation for α_c , then insert this expression into the sum of (3.6) and (3.9), to get a new equation (3.10) for b_c . Solve (3.8) for β_c , and plug this value into (3.6) to get a new Eq. (3.11) for a_c . These equations are

$$e^{2\mu(2b_c-1)}(\mu(2b_c - 1) + c\mu^2 - 1) + \mu(2b_c - 1) + c\mu^2 + 1 = 0, \tag{3.10}$$

$$\mu^2\alpha_c e^{4\mu a_c} + (1 - 2\mu^2\alpha_c)e^{2\mu a_c} - 2\mu a_c + \mu^2\alpha_c - c\mu^2 - 1 = 0, \tag{3.11}$$

and the formulas for α_c and β_c are

$$\alpha_c = e^{2\mu b_c} \left(\frac{2b_c-1}{\mu} + c - \frac{1}{\mu^2} \right) \frac{1}{2} e^{-2\mu}, \quad \beta_c = -\alpha_c e^{4\mu a_c} - \frac{e^{2\mu a_c}}{\mu^2}.$$

Eqs. (3.10) for b_c and (3.11) for a_c are transcendental. We define

$$h_c(t) = e^{2t}(t + c\mu^2 - 1) + t + c\mu^2 + 1, \tag{3.12}$$

$$\tilde{h}_c(s) = \mu^2\alpha_c e^{2s} + (1 - 2\mu^2\alpha_c)e^s - s + \mu^2\alpha_c - c\mu^2 - 1, \tag{3.13}$$

so that b_c and a_c are respectively solutions of $h_c(\mu(2b_c - 1)) = 0$ and $\tilde{h}_c(2\mu a_c) = 0$. Setting $t_c = \mu(2b_c - 1)$ and $s_c = 2\mu a_c$, we find that solving the system (3.2)–(3.5) with $0 < a_c \leq b_c < \frac{1}{2}$ is equivalent to solving

$$h_c(t_c) = 0, \tag{3.14}$$

$$\tilde{h}_c(s_c) = 0, \tag{3.15}$$

$$\alpha_c = (t_c + c\mu^2 - 1) \frac{1}{2\mu^2} e^{t_c - \mu}, \tag{3.16}$$

$$\beta_c = -\alpha_c e^{2s_c} - \frac{1}{\mu^2} e^{s_c}, \tag{3.17}$$

with $-\mu < t_c < 0$ and $0 < s_c \leq t_c + \mu$.

Remark 8. By computing two derivatives, we see that $h_c''(t) > 0$, for all $t \in \mathbb{R}$, $\lim_{t \downarrow -\infty} h_c'(t) = 1$, therefore $h_c'(t) > 0$, for all $t \in \mathbb{R}$, therefore h_c is strictly increasing with $h_c(-c\mu^2 - 1) = -2e^{-2(c\mu^2+1)} < 0$ and $h_c(-c\mu^2) = -e^{-2c\mu^2} + 1 > 0$, so that (3.14) admits a unique solution $t_c \in]-c\mu^2 - 1, -c\mu^2[$. On the other hand, $\tilde{h}_c(0) = -c\mu^2 < 0$ and we see by direct calculation that $\tilde{h}'_c(s) = 0$ if and only if $s \in \left\{0, \log\left(\frac{-1}{2\mu^2\alpha_c}\right)\right\}$. We set

$$\tilde{m}_c = \log\left(\frac{-1}{2\mu^2\alpha_c}\right) \tag{3.18}$$

and depending on the sign of this value and of $\tilde{h}_c(\tilde{m}_c)$, (3.15) has up to three solutions. This will be discussed later on a case by case basis.

Proof of Proposition 1. Let us start with the case where $c = c^* := c^*(\mu)$. In this case, the unique solution of (3.14) is given by

$$t_{c^*} = -c^*\mu^2 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}. \tag{3.19}$$

Indeed, by (2.7), (3.19) and direct computations,

$$\begin{aligned} h_{c^*}(t_{c^*}) &= e^{2t_{c^*}}(t_{c^*} + c^*\mu^2 - 1) + t_{c^*} + c^*\mu^2 + 1 \\ &= \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}\right)\right)^2 \left(-\sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} - 1\right) - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} + 1 \\ &= 0. \end{aligned}$$

Using the formula for c^* in (2.7), we can write

$$t_{c^*} = \log\left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}\right)\right). \tag{3.20}$$

Moreover, $0 > t_{c^*} > -\mu$. Indeed, the first inequality follows from the fact that $\sinh \mu > \mu$ and the second one is equivalent to $\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}\right) > e^{-\mu}$, which, in turn, is equivalent to $e^{2\mu}(\mu - 1) + \mu + 1 > 0$; this last inequality is satisfied for all $\mu > 0$. Plugging into (3.16) the value of t_{c^*} given by (3.19), or by (3.20) when it appears in an exponential, we obtain

$$\alpha_{c^*} = \frac{-e^{-\mu}}{2\mu \sinh(\mu)}. \tag{3.21}$$

We now observe that the unique solution s_{c^*} of (3.15) such that $0 < s_{c^*} \leq t_{c^*} + \mu$ is given by

$$s_{c^*} = t_{c^*} + \mu. \tag{3.22}$$

Indeed, by definition of t_{c^*} (see (3.19) or (3.20)) and of α_{c^*} in (3.21), we have

$$\begin{aligned} \tilde{h}_{c^*}(t_{c^*} + \mu) &= \mu^2\alpha_{c^*}e^{2(t_{c^*}+\mu)} + (1 - 2\mu^2\alpha_{c^*})e^{t_{c^*}+\mu} - (t_{c^*} + \mu) + \mu^2\alpha_{c^*} - c^*\mu^2 - 1 \\ &= \mu^2\alpha_{c^*}(e^{t_{c^*}+\mu} - 1)^2 + e^{t_{c^*}+\mu} - (t_{c^*} + c^*\mu^2) - \mu - 1 \\ &= \frac{-\mu e^{-\mu}}{2\sinh(\mu)} \left[e^\mu \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}\right)\right) - 1 \right]^2 \end{aligned}$$

$$\begin{aligned}
 &+ e^\mu \left[\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right) \right] + \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} - \mu - 1 \\
 &= 0.
 \end{aligned}$$

The uniqueness on the interval $]0, t_{c^*} + \mu]$ is obtained by considering the position of the extrema of \tilde{h}_{c^*} . Indeed, setting

$$\tilde{m}_{c^*} = \log \left(\frac{-1}{2\mu^2\alpha_{c^*}} \right) = \log \left(e^\mu \frac{\sinh(\mu)}{\mu} \right) = \mu + \log \left(\frac{\sinh(\mu)}{\mu} \right)$$

and since $\sinh(\mu)/\mu > 1$ for all $\mu > 0$, we have that $\tilde{m}_{c^*} > \mu > t_{c^*} + \mu > 0$. Thus, the function $\tilde{h}_{c^*}(s)$ vanishes three times: the first time on the interval $]-\infty, 0[$, then it reaches at 0 a local minimum $\tilde{h}_{c^*}(0) = -c^*\mu^2$, then it increases, vanishes at $t_{c^*} + \mu < \tilde{m}_{c^*}$ and keeps increasing until \tilde{m}_{c^*} where it reaches a local maximum. Therefore $\tilde{h}_{c^*}(\tilde{m}_{c^*}) > 0$. Finally, $\tilde{h}_{c^*}(s)$ vanishes a third time on the interval $]\tilde{m}_{c^*}, +\infty[$.

It remains to determine the parameter β_{c^*} which, by (3.17) and (3.20)–(3.22), is given by

$$\beta_{c^*} = \frac{-e^\mu}{2\mu \sinh(\mu)}. \tag{3.23}$$

We have therefore solved the system (3.14)–(3.17) and we have found that

$$b_{c^*} = \frac{1}{2} \left(\frac{t_{c^*}}{\mu} + 1 \right) \in \left] 0, \frac{1}{2} \right[\quad \text{and} \quad a_{c^*} = \frac{s_{c^*}}{2\mu} = \frac{t_{c^*} + \mu}{2\mu} = b_{c^*}, \tag{3.24}$$

with t_{c^*} given by (3.19). This establishes the existence and uniqueness of the solution \bar{V}_{c^*} of the free boundary problem (2.9). This solution is given by

$$\begin{aligned}
 \bar{V}_{c^*}(x, 1) &= \begin{cases} \frac{-x}{\mu} + \beta_{c^*} \left(e^{-2\mu x} - 1 \right), & x \in [0, a_{c^*}], \\ \frac{1-x}{\mu} + \alpha_{c^*} \left(e^{2\mu(1-x)} - 1 \right), & x \in [a_{c^*}, 1], \end{cases} \\
 &= \begin{cases} \frac{-x}{\mu} + \frac{1-e^{-2\mu x}}{\mu(1-e^{-2\mu})}, & x \in [0, a_{c^*}], \\ \frac{-x}{\mu} + \frac{1-e^{-2\mu x}}{\mu(1-e^{-2\mu})}, & x \in [a_{c^*}, 1], \end{cases} \\
 &= f^\mu(x) = \mathbb{E}_x(\sigma^\mu), \quad x \in [0, 1],
 \end{aligned} \tag{3.25}$$

and $\bar{V}_{c^*}(x, -1) = \bar{V}_{c^*}(1 - x, 1) = f^{-\mu}(x)$. By definition of \tilde{A} , we have $f^{a\mu} = J_{c^*}(x, a, \tilde{A})$ where \tilde{A} is the constant strategy ($\tilde{A}_t \equiv a$). Finally, we find using (3.24) and (3.20) that

$$a_{c^*} = b_{c^*} = \frac{1}{2\mu} \left(\log \left(\frac{\sinh(\mu)}{\mu} \left(1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} \right) \right) + \mu \right),$$

which, according to (2.8), is the location of the global maximum of $f^\mu(x) - f^{-\mu}(x)$ on the interval $[0, 1]$. Therefore, by definition of $c^*(\mu)$, we have $f^\mu(a_{c^*}) = f^{-\mu}(a_{c^*}) + c^*(\mu)$, and this completes the proof of the first part of Proposition 1.

Let us now consider $0 < c < c^*(\mu)$. The form of the solution \bar{V}_c of the free boundary problem (2.9) given in (2.13) has already been discussed starting with (3.1) and reduced to the resolution of the equivalent system (3.14)–(3.17) with $-\mu < t_c < 0$ and $0 < s_c < t_c + \mu$.

We start by showing that the unique solution t_c of (3.14) mentioned in Remark 8 is such that $-\mu < t_{c^*} < t_c < -c\mu^2$. The first inequality is mentioned just after (3.20) and the last inequality, as well as the uniqueness of the solution, has been discussed in Remark 8.

Observe that by definition, $t \mapsto h_c(t)$ and $c \mapsto h_c(t)$ are both strictly increasing. Thus, $h_c(t_{c^*}) < h_{c^*}(t_{c^*}) = 0$ and since, by definition, $h_c(t_c) = 0$, we obtain the last inequality $t_{c^*} < t_c$ and this establishes (2.14) of Proposition 1.

We now establish (2.15). Notice that by definition of t_c , we have $e^{2t_c}(t_c + c\mu^2 - 1) + t_c + c\mu^2 + 1 = 0$, which is equivalent to

$$t_c + c\mu^2 = \frac{-1 + e^{2t_c}}{1 + e^{2t_c}} = \frac{\sinh(t_c)}{\cosh(t_c)} \tag{3.26}$$

and which yields (2.15). Indeed, by (3.16),

$$\alpha_c = (t_c + c\mu^2 - 1) \frac{1}{2\mu^2} e^{t_c - \mu} = \frac{-e^{-\mu}}{2\mu^2} \frac{1}{\cosh(t_c)} = \frac{-e^{-\mu}}{2\mu^2 \cosh(2\mu b_c - \mu)}. \tag{3.27}$$

In order to prove (2.16) or, equivalently, to show that there exists a unique solution s_c of (3.15) such that $0 < s_c < t_c + \mu$, we need to study the function \tilde{h}_c and more particularly the position of its local maximum $\tilde{m}_c(\tilde{m}_c)$ (see Remark 8). Observe that by (3.18) and (3.27),

$$\tilde{m}_c = \log\left(\frac{-1}{2\mu^2\alpha_c}\right) = \mu + \log(\cosh(t_c)) > \mu.$$

As $t_c < 0$, we have

$$0 < 2\mu b_c = t_c + \mu < \tilde{m}_c. \tag{3.28}$$

Plugging $s = 2\mu b_c = t_c + \mu$ into (3.13), we see that

$$\tilde{h}_c(t_c + \mu) = \mu^2\alpha_c (e^{t_c + \mu} - 1)^2 + e^{t_c + \mu} - (t_c + c\mu^2) - \mu - 1.$$

Since $\mu^2\alpha_c = -\frac{e^{-\mu}}{e^{t_c} + e^{-t_c}}$ by (2.15), and expressing $t_c + c\mu^2$ using (3.26), we obtain

$$\begin{aligned} \tilde{h}_c(t_c + \mu) &= \frac{-e^{-\mu}}{e^{t_c} + e^{-t_c}} \left[(e^{t_c + \mu} - 1)^2 - e^{t_c + 2\mu} (e^{t_c} + e^{-t_c}) + e^{t_c + \mu} - e^{-t_c + \mu} \right. \\ &\quad \left. + (\mu + 1) (e^{t_c + \mu} + e^{-t_c + \mu}) \right], \end{aligned}$$

and this simplifies to

$$\tilde{h}_c(t_c + \mu) = -\mu + \frac{\sinh(\mu)}{\cosh(t_c)}. \tag{3.29}$$

Define

$$k(t) = -\mu + \frac{\sinh(\mu)}{\cosh(t)}.$$

Clearly, this function has two zeros $\tilde{t} < 0$ and $-\tilde{t} > 0$. From (3.29), $k(t_{c^*}) = \tilde{h}_{c^*}(t_{c^*} + \mu) = 0$ by (3.22). Therefore, $\tilde{t} = t_{c^*}$ and, since $t_{c^*} < t_c < 0$, we conclude that $k(t_c) > 0$, which implies that

$$\tilde{h}_c(2\mu b_c) = \tilde{h}_c(t_c + \mu) = k(t_c) > 0. \tag{3.30}$$

Since \tilde{h}_c is monotone on $[0, \tilde{m}_c]$ and $\tilde{h}_c(0) = -c\mu^2 < 0$, we conclude that $0 < s_c < t_c + \mu$, which establishes (2.16). Finally, (2.17) is simply a rewriting of (3.17). \square

3.2. Proof of optimality for the minimization problem

We now aim to prove that the solution \bar{V}_c of (2.9) given in Proposition 1 is indeed the value function. For this, we could apply the Verification Theorem 6.2 of [16]. However, we prefer to give a direct proof, in which the main steps correspond to checking assumptions of the Verification Theorem. Indeed, Lemma 9 corresponds to verifying hypotheses (vi) and (x) of [16, Theorem 6.2], and Corollary 11 corresponds to hypothesis (ii) there. However, by detailing the proof, we can identify where any deviation from the strategy A^c leads to a suboptimal cost function, and this will be useful in our result on uniqueness (Proposition 17 in Section 4).

Before proving Proposition 2, we introduce some notations. Let (τ_n) be the sequence of switching times of A^c and let τ^{A^c} be the exit time of X^{A^c} from $[0, 1]$. For a Brownian motion B starting a.s. at x and for $a < x < b$, let

$$p_x^\pm(a, b) = \mathbb{P}_x \{ B_t \pm \mu t \text{ hits } a \text{ before } b \}. \tag{3.31}$$

These quantities satisfy the translation invariance property $p_x^\pm(a, b) = p_{x+h}^\pm(a + h, b + h)$ for all $h \in \mathbb{R}$ and the symmetry property $p_x^-(a, b) = p_{-x}^+(-a, -b)$. We also define

$$\sigma_{a,b}^\pm = \inf\{t \geq 0 : B_t \pm \mu t \notin]a, b[\},$$

$$E_x(y, a, b) = \mathbb{E} \left(\sigma_{a,b}^+ \mid B_{\sigma_{a,b}^+} + \mu \sigma_{a,b}^+ = y, B_0 = x \right), \quad y \in \{a, b\}. \tag{3.32}$$

These expectations and probabilities can all be explicitly computed (see e.g. [3, II.2.3]). In particular, the expectations are finite and $0 < p_x^\pm(a, b) < 1$.

Proof of Proposition 2. Using the above notations, we find that for all $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}_{x,a} (N_{\tau^{A^c}}(A^c) = k) &= \mathbb{P}_{x,a} \left(\tau_k < \tau^{A^c} < \tau_{k+1} \right) \\ &= \mathbb{P}_{x,a} \left(\tau^{A^c} < \tau_{k+1} \mid \tau_k < \tau^{A^c} \right) \mathbb{P}_{x,a} \left(\tau_k < \tau^{A^c} \mid \tau_{k-1} < \tau^{A^c} \right) \\ &\quad \times \cdots \times \mathbb{P}_{x,a} \left(\tau_2 < \tau^{A^c} \mid \tau_1 < \tau^{A^c} \right) \mathbb{P}_{x,a} \left(\tau_1 < \tau^{A^c} \right) \end{aligned}$$

since $\{\tau_j < \tau^{A^c}\} \supset \{\tau_{j+1} < \tau^{A^c}\}$ for all $j \geq 1$.

If $x \in \{0, 1\}$, then $\mathbb{E}_{x,a}(\tau^{A^c}) = \mathbb{E}_{x,a}(N_{\tau^{A^c}}(A^c)) = 0$. Consider now the case where $x \in]b_c, 1[$ and $A_{0-}^c = a = 1$. Notice that for all $k \geq 1$, on $\{\tau_k < \tau^{A^c}\}$, $X_{\tau_k}^{A^c} = b_c$ if k is odd and $X_{\tau_k}^{A^c} = 1 - b_c$ if k is even. Clearly, $\mathbb{P}_{x,1}(\tau_1 < \tau^{A^c}) = p_x^+(b_c, 1)$ and since the process X^{A^c} is strong Markov by construction, for k even (but also, by symmetry, for k odd),

$$\mathbb{P}_{x,1} \left(\tau_{k+1} < \tau^{A^c} \mid \tau_k < \tau^{A^c} \right) = \mathbb{P}_{1-b_c,1} \left(\tau_1 < \tau^{A^c} \right) = p_{1-b_c}^+(b_c, 1).$$

Therefore,

$$\mathbb{P}_{x,1} (N_{\tau^{A^c}}(A^c) = k) = \begin{cases} 1 - p_x^+(b_c, 1), & \text{if } k = 0, \\ p_x^+(b_c, 1) \left(p_{1-b_c}^+(b_c, 1) \right)^{k-1} (1 - p_{1-b_c}^+(b_c, 1)), & \text{if } k \geq 1, \end{cases} \tag{3.33}$$

that is, given $N_{\tau^{A^c}}(A^c) \geq 1$, $N_{\tau^{A^c}}(A^c)$ is a geometric r.v. with parameter $1 - p_{1-b_c}^+(b_c, 1) \in]0, 1[$. Therefore, $\mathbb{E}_{x,1}(N_{\tau^{A^c}}(A^c)) < +\infty$, and this establishes the second statement in Proposition 2.

Turning to the other statement, by the law of total probability,

$$\mathbb{E}_{x,1}(\tau^{A^c}) = \sum_{k=0}^{+\infty} \mathbb{E}_{x,1}(\tau^{A^c} \mid \tau_k < \tau^{A^c} < \tau_{k+1}) \mathbb{P}_{x,1}(N_{\tau^{A^c}}(A^c) = k). \tag{3.34}$$

For $k = 0$,

$$\mathbb{E}_{x,1}(\tau^{A^c} \mid 0 < \tau^{A^c} < \tau_1) = E_x(1, b_c, 1).$$

For $k \geq 1$, on $\{\tau_k < \tau^{A^c} < \tau_{k+1}\}$,

$$\tau^{A^c} = \tau_1 + (\tau_2 - \tau_1) + \dots + (\tau_k - \tau_{k-1}) + (\tau^{A^c} - \tau_k).$$

By the strong Markov property,

$$\mathbb{E}_{x,1}(\tau_1 \mid \tau_k < \tau^{A^c} < \tau_{k+1}) = \mathbb{E}_{x,1}(\tau_1 \mid \tau_1 = \sigma_{b_c,1}^+) = E_x(b_c, b_c, 1),$$

and for $2 \leq \ell \leq k$,

$$\mathbb{E}_{x,1}(\tau_\ell - \tau_{\ell-1} \mid \tau_k < \tau^{A^c} < \tau_{k+1}) = E_{1-b_c}(b_c, b_c, 1)$$

and

$$\mathbb{E}_{x,1}(\tau^{A^c} - \tau_k \mid \tau_k < \tau^{A^c} < \tau_{k+1}) = E_{1-b_c}(1, b_c, 1).$$

Therefore, for $k \geq 1$,

$$\begin{aligned} \mathbb{E}_{x,1}(\tau^{A^c} \mid \tau_k < \tau^{A^c} < \tau_{k+1}) \\ = E_x(b_c, b_c, 1) + (k - 1)E_{1-b_c}(b_c, b_c, 1) + E_{1-b_c}(1, b_c, 1), \end{aligned}$$

and we conclude from (3.33) and (3.34) that $\mathbb{E}_{x,1}(\tau^{A^c}) < +\infty$.

The cases $x \in [0, a_c[$, $x \in [a_c, b_c]$ and $a = -1$ are treated similarly. This completes the proof of Proposition 2. \square

Lemma 9. Let \mathbb{L}_a denote the infinitesimal generator of a Brownian motion with drift $a\mu$, $a \in \{\pm 1\}$; that is, for $f \in \mathcal{C}^2([0, 1], \mathbb{R})$

$$\mathbb{L}_a f(x) = a\mu \frac{\partial f}{\partial x}(x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x). \tag{3.35}$$

Then, for all $0 < c < c^*(\mu)$,

$$1 + \mathbb{L}_a \bar{V}_c(x, a) = 0, \quad \text{for all } x \in C_a, \tag{3.36}$$

$$1 + \mathbb{L}_a \bar{V}_c(x, a) > 0, \quad \text{for all } x \in D_a \setminus \partial D_a. \tag{3.37}$$

Moreover, if $c = c^*(\mu)$, then

$$1 + \mathbb{L}_a \bar{V}_{c^*}(x, a) = 0, \quad \text{for all } x \in [0, 1]. \tag{3.38}$$

Proof. In the case where $c = c^*(\mu)$, (3.38) is a standard property of $\bar{V}_{c^*}(x, a) = f^{a\mu}(x)$ (see (2.11)), but it is also easily obtained from the explicit expression that we have for $f^{a\mu}$ (see (2.6)).

Assume $0 < c < c^*$ and consider the case where $a = 1$, since the other case is similar. The function $\bar{V}_c(x, 1)$ is C^2 in both C_1 and $D_1 \setminus \partial D_1$. The first equality (3.36) follows from the construction of \bar{V}_c (see (2.9a)). It remains to prove (3.37). By construction (see (2.9b)), for all $x \in D_1 \setminus \partial D_1 =]a_c, b_c[$, we have $\bar{V}_c(x, 1) = c + \bar{V}_c(x, -1)$ and thus,

$$1 + \mathbb{L}_1 \bar{V}_c(x, 1) = 1 + \mathbb{L}_1(c + \bar{V}_c(x, -1)) = 1 + \mu \frac{\partial \bar{V}_c}{\partial x}(x, -1) + \frac{1}{2} \frac{\partial^2 \bar{V}_c}{\partial x^2}(x, -1).$$

Moreover, since $]a_c, b_c[$ belongs to C_{-1} in which (3.36) is satisfied, we have

$$\begin{aligned} 1 + \mathbb{L}_1 \bar{V}_c(x, 1) &= 1 + \mu \frac{\partial \bar{V}_c}{\partial x}(x, -1) + \frac{1}{2} \frac{\partial^2 \bar{V}_c}{\partial x^2}(x, -1) - (1 + \mathbb{L}_{-1} \bar{V}_c(x, -1)) \\ &= 2\mu \frac{\partial \bar{V}_c}{\partial x}(x, -1). \end{aligned} \tag{3.39}$$

Thus, we have to prove that $\mu \frac{\partial \bar{V}_c}{\partial x}(x, -1) > 0$ for all $x \in D_1 \setminus \partial D_1$. If $x \in]a_c, b_c[$, then $1 - x \in]1 - b_c, 1 - a_c[\subset]b_c, 1]$ and according to (2.13),

$$\bar{V}_c(x, -1) = \bar{V}_c(1 - x, 1) = \frac{x}{\mu} + \alpha_c \left(e^{2\mu x} - 1 \right).$$

Therefore, $\mu \frac{\partial \bar{V}_c}{\partial x}(x, -1) = 1 + 2\mu^2 \alpha_c e^{2\mu x}$ for all $x \in]a_c, b_c[$. As $x \in]a_c, b_c[\subset]0, \frac{1}{2}[$ and $\alpha_c < 0$, we get according to (3.27) that

$$1 + 2\mu^2 \alpha_c e^{2\mu x} > 1 + 2\mu^2 \alpha_c e^\mu = 1 - \frac{1}{\cosh(2\mu b_c - \mu)} > 0,$$

which establishes the lemma. \square

Lemma 10. For all $0 < c \leq c^*(\mu)$ and any $a \in \{-1, 1\}$, if $x \in C_a$, then

$$\bar{V}_c(x, a) < c + \bar{V}_c(x, -a).$$

Proof. In the case where $c = c^*(\mu)$, the result follows immediately from (2.12) together with (2.11). Assume $0 < c < c^*(\mu)$ and consider the case where $a = 1$, since the other case is similar. We distinguish four cases.

Case 1. If $x \in [0, a_c[$, then $1 - x \in]1 - a_c, 1] \subset]b_c, 1]$ and, according to (2.9h) and (2.13),

$$c + \bar{V}_c(x, -1) - \bar{V}_c(x, 1) = c + \frac{2x}{\mu} + \alpha_c \left(e^{2\mu x} - 1 \right) - \beta_c \left(e^{-2\mu x} - 1 \right) =: d_1(x).$$

We will show that $d_1(x) > 0$ for $x \in [0, a_c[$. Indeed, $d'_1(x)$ vanishes at most twice on \mathbb{R} , because the equation $d'_1(x) = 0$ is a quadratic equation in $e^{2\mu x}$. The roots of $d'_1(x)$ are given by

$$x_1^\pm = \frac{1}{2\mu} \log \left(-\frac{1}{2\mu^2 \alpha_c} \pm \sqrt{\Delta_1} \right), \tag{3.40}$$

with $\Delta_1 = \frac{1}{4\mu^4 \alpha_c^2} - \frac{\beta_c}{\alpha_c}$. By (2.17) and (2.15), we have

$$1 - 4\mu^4 \alpha_c \beta_c = \left(1 + 2\mu^2 \alpha_c e^{2\mu a_c} \right)^2 = \left(1 - \frac{e^{2\mu a_c - \mu}}{\cosh(2\mu b_c - \mu)} \right)^2 > 0$$

and thus $\Delta_1 > 0$. Using (2.17), (2.15) and the fact that $a_c < b_c$, we see that $\beta_c < 0$, and so $(-2\mu\alpha_c)^{-1} - \sqrt{\Delta_1} > 0$. Therefore, $x_1^\pm \in \mathbb{R}$. From the formula for $d'_1(x)$ and (2.17), we see that

$$\begin{aligned} d'_1(a_c) &= \frac{2}{\mu} + 2\mu\alpha_c e^{2\mu a_c} + 2\mu\beta_c e^{-2\mu a_c} \\ &= \frac{2}{\mu} + 2\mu\alpha_c e^{2\mu a_c} - 2\mu \left(\alpha_c e^{4\mu a_c} + \frac{e^{2\mu a_c}}{\mu^2} \right) e^{-2\mu a_c} = 0. \end{aligned}$$

Therefore, $a_c \in \{x_1^-, x_1^+\}$. By (3.28),

$$0 < 2\mu a_c < 2\mu b_c < \tilde{m}_c = \log \left(\frac{-1}{2\mu^2\alpha_c} \right).$$

Since $2\mu x_1^- < \tilde{m}_c < 2\mu x_1^+$, we find that $x_1^- = a_c < x_1^+$. This implies that the function d_1 is monotone decreasing on $] -\infty, a_c[$ with $d_1(0) = c > 0$ and $d_1(a_c) = 0$ by (2.9b). The function d_1 is thus strictly positive on $[0, a_c[$, which proves the desired inequality.

Case 2. If $x \in]b_c, 1 - b_c[$, then $1 - x \in]b_c, 1 - b_c[$ and according to (2.9h) and (2.13), we have

$$c + \bar{V}_c(x, -1) - \bar{V}_c(x, 1) = c + \frac{2x-1}{\mu} + \alpha_c \left(e^{2\mu x} - e^{2\mu} e^{-2\mu x} \right) =: d_2(x). \tag{3.41}$$

We will show that $d_2(x) > 0$ for $x \in]b_c, 1 - b_c[$. Indeed, its derivative d'_2 vanishes at most twice on \mathbb{R} , at

$$x_2^\pm = \frac{1}{2\mu} \log \left(-\frac{1}{2\mu^2\alpha_c} \pm \sqrt{\Delta_2} \right),$$

where $\Delta_2 = \frac{1}{4\mu^4\alpha_c^2} - e^{2\mu}$. By (2.15), we find that $-2\mu^2\alpha_c = \frac{e^{-\mu}}{\cosh(2\mu b_c - \mu)} < e^{-\mu}$ since $b_c < \frac{1}{2}$. Hence, $\Delta_2 > 0$, and according to (2.9b) and condition (2.9g), we have that $d'_2(b_c) = 0$. Moreover, we notice that $d'_2(1 - b_c) = d'_2(b_c) = 0$. Thus, $x_2^- = b_c < x_2^+ = 1 - b_c$ and the function d_2 is monotone increasing on $]b_c, 1 - b_c[$. From (2.9b), we have that $d_2(b_c) = 0$, thus, the function d_2 is strictly positive on $]b_c, 1 - b_c[$, which proves the desired inequality.

Case 3. If $x \in [1 - b_c, 1 - a_c]$, then $1 - x \in [a_c, b_c] = D_1$ and by (2.9h) and (2.9b),

$$c + \bar{V}_c(x, -1) - \bar{V}_c(x, 1) = c + \bar{V}_c(1 - x, -1) + c - \bar{V}_c(x, 1) = 2c > 0, \tag{3.42}$$

which proves the desired inequality.

Case 4. If $x \in]1 - a_c, 1]$, then $1 - x \in [0, a_c[$ and we have, according to (2.9h) and (2.13),

$$\begin{aligned} c + \bar{V}_c(x, -1) - \bar{V}_c(x, 1) &= c - 2\frac{1-x}{\mu} + \beta_c \left(e^{-2\mu(1-x)} - 1 \right) - \alpha_c \left(e^{2\mu(1-x)} - 1 \right) \\ &=: d_3(x). \end{aligned}$$

We see that $d_3(x) = 2c - d_1(1 - x)$. Thus, d_3 is monotone on $]1 - a_c, 1]$ and since $d_3(1 - a_c) = 2c - d_1(a_c) = 2c$ and $d_3(1) = 2c - d_1(0) = c$, the function d_3 is strictly positive on this interval. This proves the desired inequality in this last case. \square

Let us define $H^c : \mathbb{R}_+ \times [0, 1] \times \{-1, 1\} \times \mathbb{N} \longrightarrow \mathbb{R}_+$ by

$$(t, x, a, n) \mapsto H^c(t, x, a, n) := t + cn + \bar{V}_c(x, a). \tag{3.43}$$

For $A \in \mathcal{A}$, we consider the process $H_t^{c,A} = H^c(t, X_t^A, A_t, N_t(A))$, where $N_t(A)$ is given by (2.2), and we write $\Delta H_s^{c,A} = H_s^{c,A} - H_{s-}^{c,A}$. Observe that $H_{0-}^{c,A} = \bar{V}_c(x, a)$, $\mathbb{P}_{x,a}$ -almost surely.

Corollary 11. Let $0 < c \leq c^*(\mu)$, let A^c denote the control satisfying (2.19) and let $A \in \mathcal{A}$ be any admissible control. Then, for all $x \in [0, 1]$ and $a \in \{-1, 1\}$, $\mathbb{P}_{x,a}$ -a.s., for all $s \in \mathbb{R}_+$: (1) $\Delta H_s^{c,A^c} = 0$; (2) $\Delta H_s^{c,A} \geq 0$; and (3) $\Delta H_s^{c,A} > 0$ if and only if $X_s^A \in C_{A_{s-}}$ and $A_s \neq A_{s-}$.

Proof. Let $A \in \mathcal{A}$ and let $s \in \mathbb{R}_+$. If s is a time of continuity of the strategy A , then clearly $\Delta H_s^{c,A} = 0$. Assume s is such that $A_{s-} \neq A_s$. Then $\Delta N_s(A) = 1$ and

$$\Delta H_s^{c,A} = c + \bar{V}_c(X_s^A, -A_{s-}) - \bar{V}_c(X_s^A, A_{s-}).$$

If $X_s^A \in D_{A_{s-}}$, then $\Delta H_s^{c,A} = 0$ by construction of \bar{V}_c (see (2.9b)). If $X_s^A \in C_{A_{s-}}$, then $\Delta H_s^{c,A} > 0$ by Lemma 10. For the control A^c satisfying (2.19), $A_{s-}^c \neq A_s^c$ if and only if $X_s^{A^c} \in D_{A_{s-}^c}$, and this completes the proof. \square

We can now prove the following proposition.

Proposition 12. Let A^c denote the candidate strategy satisfying (2.19), let H^c be defined by (3.43) and let $c^*(\mu)$ be given by (2.7). If $0 < c \leq c^*(\mu)$, then for all $x \in [0, 1]$ and $a \in \{-1, 1\}$:

1. the process $(H_{t \wedge \tau^{A^c}}^{c,A^c})_{t \geq 0}$ is a martingale under $\mathbb{P}_{x,a}$;
2. for any $A \in \mathcal{A}$, the process $(H_{t \wedge \tau^A}^{c,A})_{t \geq 0}$ is a submartingale under $\mathbb{P}_{x,a}$.

In order to prove this, we first define an extension of \bar{V}_c . Since this function is defined from $[0, 1] \times \{-1, 1\}$ into \mathbb{R} , we let

$$\bar{V}_c(x, y) = \frac{\bar{V}_c(x, 1) - \bar{V}_c(x, -1)}{2}y + \frac{\bar{V}_c(x, 1) + \bar{V}_c(x, -1)}{2}, \quad y \in [-1, 1], \tag{3.44}$$

denote its linear interpolation on $[0, 1] \times [-1, 1]$, so that $\bar{V}_c(x, y)$ becomes a C^∞ -function in the variable $y \in [-1, 1]$. Furthermore, we let $\mathcal{D} = \{a_c, b_c, 1 - a_c, 1 - b_c\}$ be the set of discontinuities of $\frac{\partial^2 \bar{V}_c}{\partial x^2}(x, y)$ in the variable x when $0 < c < c^*$.

Proof of Proposition 12. Using the extension of \bar{V}_c , we see that $H^c(t, x, y, n)$ is C^∞ in the variables t, y and n , in the variable x , it is C^1 on $[0, 1]$ and C^∞ on $[0, 1] \setminus \mathcal{D}$. Applying the local time-space formula [18, Theorem 3.2], we get for all $t \geq 0$,

$$\begin{aligned} H_{t \wedge \tau^A}^{c,A} &= H_0^{c,A} + \int_0^{t \wedge \tau^A} 1 \cdot ds + \int_0^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial x}(X_s^A, A_s) dX_s^A \\ &\quad + \int_0^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial y}(X_s^A, A_{s-}) dA_s \\ &\quad + \int_0^{t \wedge \tau^A} c dN_s(A) + \frac{1}{2} \int_0^{t \wedge \tau^A} \frac{\partial^2 \bar{V}_c}{\partial x^2}(X_s^A, A_s) \mathbf{1}_{\{X_s^A \notin \mathcal{D}\}}(s) ds \\ &\quad + \frac{1}{2} \int_0^{t \wedge \tau^A} \left(\frac{\partial \bar{V}_c}{\partial x}(X_s^A+, A_{s-}) - \frac{\partial \bar{V}_c}{\partial x}(X_s^A-, A_{s-}) \right) \mathbf{1}_{\{X_s^A \in \mathcal{D}\}}(s) d\ell_s^{\mathcal{D}}(X^A) \\ &\quad + \sum_{0 < s \leq t \wedge \tau^A} \left(\Delta H_s^{c,A} - c \Delta N_s(A) - \frac{\partial \bar{V}_c}{\partial y}(X_s^A, A_{s-}) \Delta A_s \right), \end{aligned} \tag{3.45}$$

where $\ell_s^{\mathcal{D}}(X^A)$ is the local time in \mathcal{D} of the process X^A . By the smooth fit conditions, we have $\frac{\partial \bar{V}_c}{\partial x}(x+, a) - \frac{\partial \bar{V}_c}{\partial x}(x-, a) = 0$ for all $x \in [0, 1]$ and for all $a \in \{-1, 1\}$. Thus, the integral with respect to the local time vanishes. If λ stands for the Lebesgue measure, then with probability 1,

$$\mathbf{1}_{\{X_s^A \notin \mathcal{D}\}} = 1, \quad \text{for } \lambda - \text{almost all } s \in \mathbb{R}_+,$$

because X^A is a diffusion and \mathcal{D} is a finite set. Moreover, the semimartingales $(N_t(A))$ and (A_t) are piecewise constant, so (3.45) reduces to

$$\begin{aligned} H_{t \wedge \tau^A}^{c,A} &= H_0^{c,A} + \int_0^{t \wedge \tau^A} (1 + \mathbb{L}_{A_s} \bar{V}_c(X_s^A, A_s)) ds \\ &\quad + \int_0^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial x}(X_s^A, A_s) dB_s + \sum_{0 < s \leq t \wedge \tau^A} \Delta H_s^{c,A} \end{aligned}$$

where \mathbb{L}_a is the operator defined in (3.35) (if we need a value for $\frac{\partial^2 \bar{V}_c}{\partial x^2}(x, a)$ for $x \in \mathcal{D}$, we arbitrarily take the second right derivative).

On one hand, if $A = A^c$, then by Corollary 11, $\Delta H_s^{c,A^c} = 0$ for all $s \geq 0$. Moreover, $\{X_s^{A^c} \in C_1, A_s^c = 1\} = \{A_s^c = 1\}$ and $\{X_s^{A^c} \in C_{-1}, A_s^c = -1\} = \{A_s^c = -1\}$. Therefore,

$$\begin{aligned} H_{t \wedge \tau^{A^c}}^{c,A^c} &= H_0^{c,A^c} + \int_0^{t \wedge \tau^{A^c}} (1 + \mathbb{L}_1 \bar{V}_c(X_s^{A^c}, 1)) \mathbb{1}_{\{X_s^{A^c} \in C_1, A_s^c = 1\}} ds \\ &\quad + \int_0^{t \wedge \tau^{A^c}} (1 + \mathbb{L}_{-1} \bar{V}_c(X_s^{A^c}, -1)) \mathbb{1}_{\{X_s^{A^c} \in C_{-1}, A_s^c = -1\}} ds \\ &\quad + \int_0^{t \wedge \tau^{A^c}} \frac{\partial \bar{V}_c}{\partial x}(X_s^{A^c}, A_s^c) dB_s \\ &= H_0^{c,A^c} + \int_0^{t \wedge \tau^{A^c}} \frac{\partial \bar{V}_c}{\partial x}(X_s^{A^c}, A_s^c) dB_s, \end{aligned}$$

by Lemma 9. Finally, $x \mapsto \frac{\partial \bar{V}_c}{\partial x}(x, \pm 1)$ is bounded on $[0, 1]$, so the stochastic integral above is a martingale, which establishes the first statement.

On the other hand, for an arbitrary control $A \in \mathcal{A}$, we have by the above that for all $u < t$,

$$\begin{aligned} H_{t \wedge \tau^A}^{c,A} - H_{u \wedge \tau^A}^{c,A} &= \int_{u \wedge \tau^A}^{t \wedge \tau^A} \left(1 + \mathbb{L}_{A_s} \bar{V}_c(X_s^A, A_s)\right) \mathbb{1}_{\{X_s^A \in C_{A_s}\} \cup \{X_s^A \in D_{A_s}\}} ds \\ &\quad + \int_{u \wedge \tau^A}^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial x}(X_s^A, A_s) dB_s + \sum_{u \wedge \tau^A < s \leq t \wedge \tau^A} \Delta H_s^{c,A} \end{aligned} \tag{3.46}$$

$$\geq \int_{u \wedge \tau^A}^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial x}(X_s^A, A_s) dB_s + \sum_{u \wedge \tau^A < s \leq t \wedge \tau^A} \Delta H_s^{c,A} \tag{3.47}$$

$$\geq \int_{u \wedge \tau^A}^{t \wedge \tau^A} \frac{\partial \bar{V}_c}{\partial x}(X_s^A, A_s) dB_s,$$

where we used Lemma 9 and Corollary 11. This shows that $H^{c,A}$ is a submartingale. \square

We are now ready to prove the optimality of our candidate strategy.

Proof of Theorem 3. Let $0 < c \leq c^*(\mu)$. On one hand, by Corollary 11 and by the first statement of Proposition 12, we have that under $\mathbb{P}_{x,a}$,

$$\begin{aligned} \bar{V}_c(x, a) &= H_{0-}^{c,A^c} = H_0^{c,A^c} = \mathbb{E}_{x,a} \left(H_{t \wedge \tau^{A^c}}^{c,A^c} \right) \\ &= \mathbb{E}_{x,a} \left(t \wedge \tau^{A^c} + cN_{t \wedge \tau^{A^c}}(A^c) + \bar{V}_c \left(X_{t \wedge \tau^{A^c}}^{A^c}, A_{t \wedge \tau^{A^c}}^c \right) \right), \end{aligned}$$

for all $t \geq 0$. Since \bar{V}_c is a continuous and bounded function, since $N(A^c)$ is an increasing process and since by Proposition 2, $\mathbb{E}_{x,a}(\tau^{A^c}) < +\infty$ and $\mathbb{E}_{x,a}(N_{\tau^{A^c}}(A^c)) < +\infty$, we get by dominated and monotone convergence that

$$\begin{aligned} \bar{V}_c(x, a) &= \lim_{t \rightarrow \infty} \mathbb{E}_{x,a} \left(t \wedge \tau^{A^c} + cN_{t \wedge \tau^{A^c}}(A^c) + \bar{V}_c \left(X_{t \wedge \tau^{A^c}}^{A^c}, A_{t \wedge \tau^{A^c}}^c \right) \right) \\ &= \mathbb{E}_{x,a} \left(\tau^{A^c} + cN_{\tau^{A^c}}(A^c) \right), \end{aligned} \tag{3.48}$$

as $X_{\tau^{A^c}}^{A^c} \in \{0, 1\}$ and $\bar{V}_c(0, \pm 1) = \bar{V}_c(1, \pm 1) = 0$. On the other hand, let $A \in \mathcal{A}$ be such that $J_c(x, a, A) < +\infty$. Then, by Corollary 11 and by the second statement of Proposition 12,

$$\bar{V}_c(x, a) = H_{0-}^{c,A} \leq H_0^{c,A} \leq \mathbb{E}_{x,a} \left(H_{t \wedge \tau^A}^{c,A} \right),$$

for all $t \geq 0$. As just above, we get by dominated and monotone convergence that

$$\bar{V}_c(x, a) \leq \lim_{t \rightarrow \infty} \mathbb{E}_{x,a} \left(H_{t \wedge \tau^A}^{c,A} \right) = \mathbb{E}_{x,a} \left(\tau^A + cN_{\tau^A}(A) \right). \tag{3.49}$$

If $J_c(x, a, A) = +\infty$, it is then clear that $\bar{V}_c(x, a) < J_c(x, a, A)$. Combining (3.48) and (3.49), we obtain

$$\bar{V}_c(x, a) = \inf_{A \in \mathcal{A}} \mathbb{E}_{x,a}(\tau^A + cN_{\tau^A}(A)) = V_c(x, a),$$

where the infimum is attained by the control A^c . This proves the optimality of the strategy A^c in the case where $0 < c \leq c^*(\mu)$.

Let $c^*(\mu) \leq c$. Then

$$V_{c^*}(x, a) = \inf_{A \in \mathcal{A}} \mathbb{E}_{x,a} \left(\tau^A + c^*N_{\tau^A}(A) \right) \leq \inf_{A \in \mathcal{A}} \mathbb{E}_{x,a} \left(\tau^A + cN_{\tau^A}(A) \right) = V_c(x, a),$$

by definition of the value function. Moreover, if \tilde{A} denotes the constant strategy, then

$$J_{c^*}(x, a, \tilde{A}) = J_c(x, a, \tilde{A}) = \mathbb{E}_{x,a}(\tau^{\tilde{A}}),$$

and, again by definition of the value function, $V_c(x, a) \leq J_c(x, a, \tilde{A})$. Finally, by (2.11) and by the first part of the proof, we know that $V_{c^*}(x, a) = \bar{V}_{c^*}(x, a) = J_{c^*}(x, a, \tilde{A})$. Hence, $V_{c^*}(x, a) = V_c(x, a)$ and the strategy \tilde{A} is optimal for all $c \geq c^*(\mu)$.

Suppose now that $c > c^*(\mu)$ and that there exists another optimal strategy $\bar{A} \in \mathcal{A}$ such that there exists $(x, a) \in [0, 1] \times \{-1, 1\}$ with $\mathbb{P}_{x,a}(N_{\tau^{\bar{A}}}(\bar{A}) > 0) > 0$. Then

$$\mathbb{E}_{x,a} \left(\tau^{\bar{A}} + cN_{\tau^{\bar{A}}}(\bar{A}) \right) > \mathbb{E}_{x,a} \left(\tau^{\bar{A}} + c^*N_{\tau^{\bar{A}}}(\bar{A}) \right) \geq V_{c^*}(x, a) = V_c(x, a),$$

which contradicts the optimality hypothesis. This shows that if $c > c^*(\mu)$, then \tilde{A} is the unique optimal strategy. \square

3.3. Free boundary problem for the maximization problem

The resolution of the free boundary problem (2.20), as well as the proof of the optimality of the candidate control, are similar to what we have already done in the minimization problem; we will however highlight the places where the computations differ.

Proof of Proposition 5. Let $0 < c < c^*(\mu)$. The general solution to (2.20a) is the same as for (2.9a), and since (2.20a) is satisfied in the two intervals $[0, a_c^{\max}[$ and $]b_c^{\max}, 1]$, there are four constants to determine, which are reduced to two by the boundary conditions (2.20c). Then, (2.20b) and (2.20h) give the form of $\tilde{V}_c^{\max}(x, 1)$ given in (2.22). The four unknowns $\delta_c, \gamma_c, a_c^{\max}$ and b_c^{\max} have to be determined using (2.20d)–(2.20g). Using (2.22), these equations give us, after some simplifications, in the same the order:

$$\frac{1-2a_c^{\max}}{\mu} + \gamma_c \left(e^{-2\mu a_c^{\max}} - 1 \right) = \gamma_c \left(e^{-2\mu(1-a_c^{\max})} - 1 \right) - c, \tag{3.50}$$

$$\gamma_c \left(e^{-2\mu(1-b_c^{\max})} - 1 \right) - c = \frac{2-2b_c^{\max}}{\mu} + \delta_c \left(e^{2\mu(1-b_c^{\max})} - 1 \right), \tag{3.51}$$

$$2\mu\gamma_c \left(e^{-2\mu(1-a_c^{\max})} + e^{-2\mu a_c^{\max}} \right) + \frac{2}{\mu} = 0, \tag{3.52}$$

$$2\mu\gamma_c e^{-2\mu(1-b_c^{\max})} + 2\mu\delta_c e^{2\mu(1-b_c^{\max})} + \frac{2}{\mu} = 0. \tag{3.53}$$

Observe that setting $\gamma_c e^{-2\mu} = \alpha_c$ and $1 - a_c^{\max} = b_c$, Eqs. (3.50) and (3.52) become identical to (3.3) and (3.5), respectively. In the proof of Proposition 1, we have already established that (3.3) and (3.5) have a unique solution α_c and $0 < b_c < \frac{1}{2}$. Thus, it remains only to prove the existence of δ_c and b_c^{\max} solution of (3.51) and (3.53) such that δ_c satisfies (2.24) and $1 - a_c < b_c^{\max} < 1$.

Eq. (2.24) follows directly from (3.53). Substituting (2.24) into (3.51), multiplying by μ^2 and rearranging terms, we get (2.23). Now set

$$\tilde{h}_c^{\max}(s) = \gamma_c \mu^2 e^{-2s} + e^{-s} \left(1 - 2\gamma_c \mu^2 \right) + s - 1 + \gamma_c \mu^2 + c \mu^2, \tag{3.54}$$

so that (2.23) is equivalent to $\tilde{h}_c^{\max}(2\mu(1 - b_c^{\max})) = 0$. Observe that $\tilde{h}_c^{\max}(0) = c\mu^2$ and, by (2.15), that

$$\begin{aligned} \tilde{h}_c^{\max}(2\mu b_c) &= -\frac{e^\mu}{2 \cosh(2\mu b_c - \mu)} e^{-4\mu b_c} + e^{-2\mu b_c} \left(1 + \frac{e^\mu}{\cosh(2\mu b_c - \mu)} \right) \\ &\quad + 2\mu b_c - 1 - \frac{e^\mu}{2 \cosh(2\mu b_c - \mu)} + c\mu^2 \\ &= -\frac{\sinh(2\mu b_c - \mu)}{\cosh(2\mu b_c - \mu)} - \frac{\sinh(\mu)}{\cosh(2\mu b_c - \mu)} + 2\mu b_c + c\mu^2 \\ &= \mu - \frac{\sinh(\mu)}{\cosh(2\mu b_c - \mu)}, \end{aligned}$$

where we used (3.26) with $t_c = 2\mu b_c - \mu$ to get the last equality. Looking back to (3.29) and (3.30), we see that

$$\tilde{h}_c^{\max}(2\mu b_c) = \mu - \frac{\sinh(\mu)}{\cosh(2\mu b_c - \mu)} = -\tilde{h}_c(2\mu b_c) < 0.$$

Furthermore, from (3.54), we see that the derivative of \tilde{h}_c^{\max} vanishes twice: at 0 and at $\log(-2\gamma_c\mu^2)$ (compare also with (3.18)). By (2.15), we have that

$$\begin{aligned} \log(-2\gamma_c\mu^2) &= \log(-2\alpha_c e^{2\mu}\mu^2) = \log\left(\frac{e^\mu}{\cosh(2\mu b_c - \mu)}\right) \\ &= \mu - \log(\cosh(2\mu b_c - \mu)). \end{aligned}$$

Since $2\mu b_c - \mu < 0$ because $b_c \in [0, \frac{1}{2}[$ by Proposition 1, we have that $-\log(\cosh(2\mu b_c - \mu)) > 2\mu b_c - \mu$ which is equivalent by the preceding to $2\mu b_c < \log(-2\gamma_c\mu^2)$. Therefore, the function \tilde{h}_c^{\max} is monotone decreasing on $]0, 2\mu b_c[$ and vanishes only once on this interval, at a value which we denote $2\mu(1 - b_c^{\max})$. We have thus established the existence of a number $b_c^{\max} \in]1 - b_c, 1[$ which is the unique solution of (2.23).

Finally, we check that $b_c^{\max} > 1 - a_c$. This is clearly equivalent to showing that $2\mu(1 - b_c^{\max}) < 2\mu a_c$, where $2\mu a_c = s_c$ is the unique solution of $\tilde{h}_c(s) = 0$ on $]0, t_c + \mu[$ (see (3.15)). We have just shown that \tilde{h}_c^{\max} is strictly decreasing on $[0, t_c + \mu]$. The function \tilde{h}_c is strictly increasing on $[0, t_c + \mu]$ (see the end of the proof of Proposition 1). Thus, in order to prove that $2\mu(1 - b_c^{\max}) < s_c$, it suffices to show that $\tilde{h}_c(s) < -\tilde{h}_c^{\max}(s)$ for all $s \in]0, t_c + \mu[$. We have

$$\begin{aligned} \tilde{h}_c(s) + \tilde{h}_c^{\max}(s) &= \mu^2\alpha_c \left(e^{2s} - 2e^s + 1 + e^{2\mu} + e^{2\mu}e^{-2s} - 2e^{2\mu}e^{-s} \right) + e^s + e^{-s} - 2 \\ &= \mu^2\alpha_c (e^s + e^{-s} - 2)(e^s + e^{-s+2\mu}) + e^s + e^{-s} - 2 \\ &= \mu^2\alpha_c \left(4 \sinh^2\left(\frac{s}{2}\right) \right) (2e^\mu \cosh(s - \mu)) + 4 \sinh^2\left(\frac{s}{2}\right), \end{aligned}$$

which is strictly negative on $]0, t_c + \mu[$ since by (2.15) and the fact that $-\mu < t_c < 0$,

$$2\mu^2\alpha_c e^\mu \cosh(s - \mu) + 1 = 1 - \frac{\cosh(s - \mu)}{\cosh(t_c)} < 0, \quad \text{for all } s \in]0, t_c + \mu[.$$

The proof of part 2 of Proposition 5 is complete. We note for future reference that

$$2\mu(1 - b_c^{\max}) < t_c + \mu. \tag{3.55}$$

Let us now consider the case $c = c^*(\mu)$. By the preceding, we have immediately that $a_{c^*}^{\max} = 1 - b_{c^*}$ and that $\gamma_{c^*} = \alpha_{c^*}e^{2\mu}$. It remains to see that $b_{c^*}^{\max} = a_{c^*}^{\max}$ or, equivalently, that $2\mu(1 - a_{c^*}^{\max})$ is a solution of $\tilde{h}_{c^*}^{\max}(s) = 0$. Using the formula for c^* in (2.7) and for $\tilde{h}_{c^*}^{\max}$ in (3.54), the expression of α_{c^*} in (3.21) and the formula for $a_{c^*}^{\max} = 1 - b_{c^*}$ in (3.24) via either (3.19) or (3.20), we find that

$$\begin{aligned} \tilde{h}_{c^*}^{\max}(2\mu(1 - a_{c^*}^{\max})) &= \tilde{h}_{c^*}^{\max}(t_{c^*} + \mu) \\ &= \gamma_{c^*}\mu^2 e^{-2(t_{c^*} + \mu)} + e^{-(t_{c^*} + \mu)} \left(1 - 2\gamma_{c^*}\mu^2 \right) + t_{c^*} \\ &\quad + \mu - 1 + \gamma_{c^*}\mu^2 + c^*\mu^2 \\ &= \gamma_{c^*}\mu^2 \left(e^{-(t_{c^*} + \mu)} - 1 \right)^2 + e^{-(t_{c^*} + \mu)} + t_{c^*} + \mu - 1 + c^*\mu^2 \\ &= -\frac{\mu e^\mu}{2 \sinh(\mu)} \left(e^{-\mu} \frac{\mu}{\sinh(\mu)} \frac{1}{1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}} - 1 \right)^2 \end{aligned}$$

$$\begin{aligned}
 &+ e^{-\mu} \frac{\mu}{\sinh(\mu)} \frac{1}{1 - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}} - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} + \mu - 1 \\
 &= -\frac{\mu e^{-\mu}}{2 \sinh(\mu)} \frac{\mu^2}{\sinh^2(\mu)} \frac{1 + 2\sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} + 1 - \frac{\mu^2}{\sinh^2(\mu)}}{\left(\frac{\mu^2}{\sinh^2(\mu)}\right)^2} \\
 &+ \frac{\mu^2}{\sinh^2(\mu)} \frac{1 + \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}}{\frac{\mu^2}{\sinh^2(\mu)}} \\
 &- \frac{\mu e^{\mu}}{2 \sinh(\mu)} + \frac{\mu e^{-\mu}}{\sinh(\mu)} \frac{1 + \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}}}{\frac{\mu^2}{\sinh^2(\mu)}} - \sqrt{1 - \frac{\mu^2}{\sinh^2(\mu)}} + \mu - 1 \\
 &= -\frac{\mu e^{\mu}}{2 \sinh(\mu)} + \frac{\mu e^{-\mu}}{2 \sinh(\mu)} + \mu \\
 &= 0.
 \end{aligned}$$

Therefore, $a_{c^*}^{\max} = b_{c^*}^{\max}$ and by (2.24), the last parameter δ_{c^*} is given by

$$\begin{aligned}
 \delta_{c^*} &= -\frac{1}{\mu^2} e^{-2\mu(1-b_{c^*}^{\max})} - \gamma_{c^*} e^{-4\mu(1-b_{c^*}^{\max})} \\
 &= -\frac{1}{\mu^2} e^{-(t_{c^*}^* + \mu)} - \gamma_{c^*} e^{-2(t_{c^*}^* + \mu)} = -\frac{e^{-\mu}}{2\mu \sinh(\mu)} = \alpha_{c^*}
 \end{aligned}$$

(for the third equality, use (3.20)). Replacing the value of γ_{c^*} and α_{c^*} in the general expression for $\bar{V}_{c^*}^{\max}$ given in (2.22), we finally obtain as in (3.25) that, for all $x \in [0, 1]$,

$$\bar{V}_{c^*}^{\max}(x, 1) = \frac{-x}{\mu} + \frac{1 - e^{-2\mu x}}{\mu(1 - e^{-2\mu})} = \bar{V}_{c^*}(x, 1) \tag{3.56}$$

and $\bar{V}_{c^*}^{\max}(x, -1) = \bar{V}_{c^*}^{\max}(1 - x, 1)$ from (2.20h). This completes the proof. \square

3.4. Proof of optimality for the maximization problem

Now that we have the solution of the free boundary problem (2.20), we shall prove that \bar{V}_c^{\max} is equal to the value function V_c^{\max} and that G^c is an optimal control. This will be similar to the proof of optimality in the minimization problem.

Proof of Proposition 6. The proof is the same as for Proposition 2. \square

Lemma 13. Let \mathbb{L}_a denote the operator defined in (3.35). Then, for all $0 < c < c^*(\mu)$,

$$\begin{aligned}
 1 + \mathbb{L}_a \bar{V}_c^{\max}(x, a) &= 0, \quad \text{for all } x \in C_a^{\max}, \\
 1 + \mathbb{L}_a \bar{V}_c^{\max}(x, a) &< 0, \quad \text{for all } x \in D_a^{\max} \setminus \partial D_a^{\max}.
 \end{aligned}$$

Moreover, if $c = c^*(\mu)$, then

$$1 + \mathbb{L}_a \bar{V}_{c^*}^{\max}(x, a) = 0, \quad \text{for all } x \in [0, 1].$$

Proof. The proof follows the same steps as for Lemma 9. In particular, for $x \in]a_c^{\max}, b_c^{\max}[$, as in (3.39), $1 + \mathbb{L}_1 \bar{V}_c^{\max}(x, 1) = 2\mu \frac{\partial \bar{V}_c^{\max}}{\partial x}(x, -1)$ and $1 - x \in]1 - b_c^{\max}, 1 - a_c^{\max}[\subset]0, a_c^{\max}[$, and so

$$\bar{V}_c^{\max}(x, -1) = \bar{V}_c^{\max}(1 - x, 1) = -\frac{1-x}{\mu} + \gamma_c \left(e^{-2\mu(1-x)} - 1 \right).$$

Therefore,

$$\begin{aligned} \mu \frac{\partial \bar{V}_c^{\max}}{\partial x}(x, -1) &= 1 + 2\mu^2 \gamma_c e^{-2\mu(1-x)} < 1 + 2\mu^2 \alpha_c e^{2\mu a_c^{\max}} \\ &= 1 - \frac{e^{-\mu}}{\cosh(t_c)} e^{-t_c + \mu} = \tanh(t_c) < 0, \end{aligned}$$

where we have used the equalities $\gamma_c = \alpha_c e^{2\mu} < 0$, $a_c^{\max} = 1 - b_c$, (3.27) and $2\mu b_c = t_c + \mu$, and the fact that $t_c < 0$. This proves Lemma 13. \square

Lemma 14. For all $0 < c \leq c^*(\mu)$ and any $a \in \{-1, 1\}$, if $x \in C_a^{\max}$, then

$$\bar{V}_c^{\max}(x, a) > \bar{V}_c^{\max}(x, -a) - c.$$

Proof. We compare the left- and right-hand sides of this inequality on a case by case basis as in the proof of Lemma 10. In the case where $c = c^*(\mu)$, the result follows immediately from (2.21) together with (2.12). Assume $0 < c < c^*(\mu)$ and, without loss of generality, that $a = 1$.

Case 1. If $x \in]0, 1 - b_c^{\max}[$, then $1 - x \in]b_c^{\max}, 1]$ and according to (2.20h) and (2.22),

$$\bar{V}_c^{\max}(x, -1) - \bar{V}_c^{\max}(x, 1) - c = e_1(x),$$

where

$$e_1(x) := \frac{2x}{\mu} + \delta_c \left(e^{2\mu x} - 1 \right) - \gamma_c \left(e^{-2\mu x} - 1 \right) - c. \tag{3.57}$$

The sign of the derivative $e'_1(x) = 2\mu e^{-2\mu x} (\delta_c e^{4\mu x} + \mu^{-2} e^{2\mu x} + \gamma_c)$ is determined by a quadratic function of $e^{2\mu x}$, which vanishes at most twice on \mathbb{R} , at y_1^\pm , which are given by the formula (3.40) for x_1^\pm , but with α_c replaced by δ_c and β_c replaced by γ_c . When $\delta_c \neq 0$, the discriminant $\Delta_1 = \frac{1}{4\mu^4 \delta_c^2} - \frac{\gamma_c}{\delta_c}$ is positive since

$$\Delta_1 \geq 0 \Leftrightarrow 1 - 4\mu^4 \delta_c \gamma_c \geq 0 \Leftrightarrow \left(1 + 2\mu^2 \gamma_c e^{-2\mu(1-b_c^{\max})} \right)^2 \geq 0,$$

by (2.24). Using again (2.24), we see that $e'_1(1 - b_c^{\max}) = 0$ and so $1 - b_c^{\max} \in \{y_1^\pm\}$. If $\delta_c > 0$, then $\sqrt{\Delta_1} > \frac{1}{2\mu^2 \delta_c}$ (recall that $\gamma_c < 0$) and y_1^+ is the only root of e'_1 . The point $1 - b_c^{\max} = y_1^+$ is thus the global minimum of the function e_1 and since $e_1(0) = -c < 0$, we have $e_1(x) < 0$ for all $x \in]0, 1 - b_c^{\max}[$. If $\delta_c < 0$, then e'_1 has two distinct real roots $y_1^- < y_1^+$ such that $2\mu y_1^- < \log\left(-\frac{1}{2\mu^2 \delta_c}\right) < 2\mu y_1^+$. Using (2.24) and (3.27), we see that

$$1 - b_c^{\max} = y_1^- \Leftrightarrow e^{-2\mu(1-b_c^{\max})} > -2\mu^2 \delta_c \Leftrightarrow \cosh(t_c) < e^{\mu - 2\mu(1-b_c^{\max})}.$$

The last inequality is satisfied since $\cosh(t_c) < e^{-t_c}$ because $t_c < 0$, and $2\mu(1 - b_c^{\max}) \in]0, \mu + t_c[$ by (3.55). Thus, e_1 is strictly decreasing on $]0, 1 - b_c^{\max}[$ with $e_1(0) = -c < 0$ and so $e_1(x) < 0$ for all $x \in]0, 1 - b_c^{\max}[$. Finally, if $\delta_c = 0$, then $y_1^+ = y_1^- = 1 - b_c^{\max}$ and $e'_1(x) < 0$ if and only if $x \in]0, 1 - b_c^{\max}[$, and so $e_1(x) < 0$ for all $x \in]0, 1 - b_c^{\max}[$. Therefore, for all possible values of δ_c , we have shown that $e_1(x) < 0$ for all $x \in]0, b_c^{\max}[$.

Case 2. If $x \in [1 - b_c^{\max}, 1 - a_c^{\max}]$, then $1 - x \in [a_c^{\max}, b_c^{\max}] = D_1^{\max}$ and by (2.20h) and (2.20b),

$$\bar{V}_c^{\max}(x, -1) - \bar{V}_c^{\max}(x, 1) - c = \bar{V}_c^{\max}(1 - x, 1) - \bar{V}_c^{\max}(x, 1) - c = -2c < 0.$$

Case 3. If $x \in]1 - a_c^{\max}, a_c^{\max}[$, then $1 - x \in]1 - a_c^{\max}, a_c^{\max}[$ and by (2.22),

$$\bar{V}_c^{\max}(x, -1) - \bar{V}_c^{\max}(x, 1) - c = -c + \frac{2x-1}{\mu} + \gamma_c \left(e^{2\mu(x-1)} - e^{-2\mu x} \right) = -2c + d_2(x),$$

where $d_2(x)$ is given by (3.41). Since $a_c^{\max} = 1 - b_c$ and since we have seen that $d_2(\cdot)$ is strictly increasing on $]b_c, 1 - b_c[$ with $d_2(b_c) = 0$, and $d_2(1 - b_c) = 2c$ by (3.42), we have that $-2c + d_2(x) < 0$ for all $x \in]1 - a_c^{\max}, a_c^{\max}[$.

Case 4. If $x \in]b_c^{\max}, 1[$, then $1 - x \in [0, 1 - b_c^{\max}[$ and by (2.22),

$$\bar{V}_c^{\max}(x, -1) - \bar{V}_c^{\max}(x, 1) - c = -e_1(1 - x) - 2c,$$

where e_1 is defined in (3.57). Notice that

$$-e_1(1 - b_c^{\max}) - 2c = -(\bar{V}_c^{\max}(b_c^{\max}, 1) - \bar{V}_c^{\max}(1 - b_c^{\max}, 1) - c) - 2c = 0,$$

by (2.20b). It follows from the properties of e_1 that we already discussed in part 1 that $-e_1(1 - x) - 2c < 0$ for all $x \in]b_c^{\max}, 1[$. This proves the desired inequality in this last case and completes the proof. \square

Let us define $K^c : \mathbb{R}_+ \times [0, 1] \times \{-1, 1\} \times \mathbb{N} \longrightarrow \mathbb{R}_+$ by

$$(t, x, a, n) \mapsto K^c(t, x, a, n) := t - cn + \bar{V}_c^{\max}(x, a). \tag{3.58}$$

For $A \in \mathcal{A}$, we consider the process $K_t^{c,A} = K^c(t, X_t^A, A_t, N_t(A))$. The next corollary follows immediately from Lemma 14.

Corollary 15. *Let $0 < c \leq c^*(\mu)$, let G^c denote the control satisfying (2.25) and let $A \in \mathcal{A}$ be any admissible control. Then, for all $x \in [0, 1]$ and $a \in \{-1, 1\}$, $\mathbb{P}_{x,a}$ -a.s., for all $s \in \mathbb{R}_+$: (1) $\Delta K_s^{c,G^c} = 0$; (2) $\Delta K_s^{c,A} \leq 0$; (3) $\Delta K_s^{c,A} < 0$ if and only if $X_s^A \in C_{A_s-}^{\max}$ and $A_s \neq A_{s-}$.*

This leads to the following property of the process $K^{c,A}$.

Proposition 16. *Let G^c denote the candidate strategy satisfying (2.25), let K^c be defined by (3.58) and let $c^*(\mu)$ be given by (2.7). If $0 < c \leq c^*(\mu)$, then:*

1. *the process $(K_{t \wedge \tau^{G^c}}^{c,G^c})_{t \geq 0}$ is a martingale under $\mathbb{P}_{x,a}$;*
2. *for any $A \in \mathcal{A}$, the process $(K_{t \wedge \tau^A}^{c,A})_{t \geq 0}$ is a supermartingale under $\mathbb{P}_{x,a}$.*

Proof. This is established by using the local time–space formula, as well as Lemma 13 and Corollary 15, in the same way as in the proof of Proposition 12. \square

Proof of Theorem 7. The proof is exactly the same as the one of Theorem 3, using in this case the supermartingale property given in Proposition 16. \square

4. Further results

In this last section, we present a result on generic uniqueness of the optimal strategy, as well as a scaling property. We also consider the limiting case where $c \downarrow 0$ and verify that it is consistent with the classical result of [10]. These statements are given only for the minimization problem but it is not difficult to see that they are also valid for the maximization problem. At the end of this section, we provide illustrations of the value functions of both problems.

In the minimization problem, strictly speaking, we do not have uniqueness of the optimal control in general, since for $c = c^*(\mu)$, the strategy A^* is equivalent to the constant strategy \bar{A} and both are optimal. It turns out, however, that this is the only case where there are two distinct optimal strategies. The case where $c > c^*(\mu)$ has already been discussed in Theorem 3, so we now consider the case $c \leq c^*(\mu)$.

Proposition 17 (Generic Uniqueness). *Let $0 < c \leq c^*(\mu)$ and let $A \in \mathcal{A}$ be a strategy such that for some $(x, a) \in [0, 1] \times \{-1, 1\}$, either $p_1 > 0$ or $0 < c < c^*(\mu)$ and $p_2 > 0$, where*

$$p_1 := \mathbb{P}_{x,a} \left(\exists t \in [0, \tau^A] : X_t^A \in C_{A_t-} \text{ and } A_t \neq A_{t-} \right),$$

$$p_2 := \mathbb{P}_{x,a} \left(\exists t \in [0, \tau^A] : X_t^A \in D_{A_t} \right).$$

Then A is $\mathbb{P}_{x,a}$ -sub-optimal, in the sense that $V_c(x, a) < \mathbb{E}_{x,a} (\tau^A + cN_{\tau^A}(A))$.

Remark 18. The condition $p_1 > 0$ means that with positive probability, the strategy A prescribes at least once to switch drifts in the continuation area of the control A^c . The condition $p_2 > 0$ means that with positive probability, the strategy A prescribes at least once to continue without switching in a switching region of A^c .

Proof. If $\mathbb{E}_{x,a}(\tau^A) = +\infty$, then it is clear that $\mathbb{E}_{x,a}(\tau^A + cN_{\tau^A}(A)) > V_{c^*}(x, a)$. Thus, we can assume that $\mathbb{E}_{x,a}(\tau^A) < +\infty$. Suppose first that $p_1 > 0$. Then by Corollary 11,

$$\mathbb{E}_{x,a} \left(\sum_{0 < s \leq \tau^A} \Delta H_s^{c,A} \right) > 0.$$

Setting $u = 0$ in (3.47), we find that

$$H_{t \wedge \tau^A}^{c,A} \geq H_0^{c,A} + \int_0^{t \wedge \tau^A} \frac{\partial V_c}{\partial x}(X_s^A, A_s) dB_s + \sum_{0 < s \leq t \wedge \tau^A} \Delta H_s^{c,A},$$

and taking the expectations, applying the monotone convergence and the dominated convergence theorems (recall that V is bounded), we get

$$\mathbb{E}_{x,a} \left(H_{\tau^A}^{c,A} \right) \geq H_0^{c,A} + \mathbb{E}_{x,a} \left(\sum_{0 < s \leq \tau^A} \Delta H_s^{c,A} \right) > H_0^{c,A} = V_c(x, a).$$

Since the left-hand side is equal to $\mathbb{E}_{x,a}(\tau^A + cN_{\tau^A}(A))$, this proves the statement when $p_1 > 0$.

Suppose now that $p_2 > 0$. Let λ denote Lebesgue measure. Notice that the interior $\text{int}(D_{A_t})$ of D_{A_t} is not empty for $0 < c < c^*(\mu)$. By right-continuity of $s \mapsto A_s$ and because of the irregular behavior of sample paths of diffusion processes, on the event $\{\exists t \in [0, \tau^A] : X_t^A \in D_{A_t}\}$,

$$\lambda\{s \in [0, \tau^A] : X_s^A \in \text{int}(D_{A_s})\} > 0. \tag{4.1}$$

Setting $u = 0$ in (3.46) and then applying successively Lemma 9 and Corollary 11, we get

$$H_{t \wedge \tau^A}^{c,A} \geq H_0^{c,A} + \int_0^{t \wedge \tau^A} \frac{\partial V_c}{\partial x}(X_s^A, A_s) dB_s + \int_0^{t \wedge \tau^A} \left(1 + \mathbb{L}_{A_s} V_c(X_s^A, A_s)\right) \mathbb{1}_{\{X_s^A \in D_{A_s}\}} ds.$$

Again by Lemma 9, the integrand of the last integral is strictly positive if X_s^A belongs to the interior of D_{A_s} . Taking expectations in the previous inequality and applying successively the dominated and monotone convergence theorems and the hypothesis $p_2 > 0$ with (4.1), we find that

$$\mathbb{E}_{x,a} \left(\tau^A + cN_{\tau^A}(A) \right) = \mathbb{E}_{x,a} \left(H_{\tau^A}^{c,A} \right) > H_0^{c,A} = V_c(x, a),$$

which concludes the proof. \square

If we consider a diffusion coefficient $\sigma \neq 1$ for the particle, we can deduce from Theorem 3 the corresponding value function and optimal control. Let $\sigma > 0$ and let (\hat{B}_t) be a standard Brownian motion, $(\hat{\mathcal{F}}_t)$ be its natural filtration and $\hat{\mathcal{A}}$ be the associated set of strategies. For $\hat{A} \in \hat{\mathcal{A}}$, consider the s.d.e.

$$d\hat{X}_t^{\hat{A}} = \hat{A}_t \mu dt + \sigma d\hat{B}_t, \quad \hat{X}_0^{\hat{A}} = x, \tag{4.2}$$

and the corresponding stochastic control problem whose value function is given by

$$\hat{V}(x, a, c, \mu, \sigma) = \inf_{\hat{A} \in \hat{\mathcal{A}}} \mathbb{E}_{x,a}(\hat{\tau}^{\hat{A}} + c\hat{N}_{\tau^{\hat{A}}}(\hat{A})), \tag{4.3}$$

where $\hat{\tau}^{\hat{A}}$ is the exit time of $\hat{X}^{\hat{A}}$ from $[0, 1]$ and $\hat{N}(\hat{A})$ is the process that counts the discontinuities of \hat{A} .

Proposition 19 (Scaling Property). *The optimal control of problem (4.3) is obtained by the construction that leads to (2.19), but replacing c by $\sigma^2 c$ and μ by μ/σ^2 . The value function satisfies*

$$\hat{V}(x, a, c, \mu, \sigma) = \frac{1}{\sigma^2} \hat{V}\left(x, a, c\sigma^2, \frac{\mu}{\sigma^2}, 1\right),$$

where $\hat{V}(x, a, c\sigma^2, \mu/\sigma^2, 1)$ coincides with the value function of problem (2.4) with c replaced by $c\sigma^2$ and μ replaced by μ/σ^2 .

Proof. Define $B_t = \sigma \hat{B}_{t/\sigma^2}$, so that (B_t) is a standard Brownian motion. Setting

$$d\hat{Z}_t = \pm \mu dt + \sigma d\hat{B}_t, \quad dZ_t = \pm \frac{\mu}{\sigma^2} dt + dB_t,$$

and $\hat{Z}_0 = Z_0$, we see that $\hat{Z}_t = Z_{\sigma^2 t}$.

Let (\mathcal{F}_t) be the natural filtration of (B_t) , that is, $\mathcal{F}_t = \hat{\mathcal{F}}_{t/\sigma^2}$, and let \mathcal{A} be the set of strategies associated with (\mathcal{F}_t) . Given $\hat{A} \in \hat{\mathcal{A}}$, define $A = (A_t)$ by $A_t = \hat{A}_{t/\sigma^2}$. This defines a one-to-one correspondence between $\hat{\mathcal{A}}$ and \mathcal{A} .

Let $dX_t^A = A_t \mu \sigma^{-2} dt + dB_t$, with $X_0^A = x$. Then, $X_{\sigma^2 t}^A = \hat{X}_t^{\hat{A}}$ and we see that $\tau^A = \sigma^2 \hat{\tau}^{\hat{A}}$ and $N_{\sigma^2 t}(A) = \hat{N}_t(\hat{A})$. Therefore,

$$\sigma^2 \left(\hat{\tau}^{\hat{A}} + c \hat{N}_{\hat{\tau}^{\hat{A}}}(\hat{A}) \right) = \tau^A + c \sigma^2 N_{\tau^A}(A).$$

Minimizing the right-hand side is precisely the problem (2.4), with c replaced by $c\sigma^2$ and μ replaced by μ/σ^2 . This proves the proposition. \square

As mentioned in the introduction, in the case where there is no switching cost ($c = 0$), the solution of the control problem corresponding to (2.4) is now classical (see [10, IV.5]). The value function does not depend on the initial drift and is given by

$$V(x) = \frac{\frac{1}{2} - \left| \frac{1}{2} - x \right|}{\mu} - \frac{1}{2\mu^2} e^{-\mu} \left(e^{2\mu \left(\frac{1}{2} - \left| \frac{1}{2} - x \right| \right)} - 1 \right), \quad x \in [0, 1]. \tag{4.4}$$

The optimal control, which would not be admissible in our setting, is given by

$$A_t = \text{sgn} \left(\frac{1}{2} - X_t^A \right), \quad \text{with } dX_t^A = A_t \mu dt + dB_t \text{ on } \{t \leq \tau^A\}, \tag{4.5}$$

where $\text{sgn } x = 1$ if $x \geq 0$ and $\text{sgn } x = -1$ if $x < 0$. We observe that this control A is not piecewise constant, since it corresponds to switching regions given by $D_1 = [0, \frac{1}{2}]$ and $D_{-1} = [\frac{1}{2}, 0]$. Thus, there exists only a weak solution of (4.5) which is given by Tanaka’s formula (see e.g. [5, Sections 7.3 and 10.4]). In the next proposition, we show that (4.4) can be obtained as a limit of $V_c(x, a)$ as $c \downarrow 0$.

Proposition 20. *Let $V(x)$ denote the function defined in (4.4). The solution $\{\bar{V}_c, a_c, b_c\}$ of (2.9) satisfies $\lim_{c \downarrow 0} a_c = 0$, $\lim_{c \downarrow 0} b_c = \frac{1}{2}$ and $\lim_{c \downarrow 0} \bar{V}_c(x, a) = V(x)$, for all $x \in [0, 1]$ and $a \in \{\pm 1\}$.*

Proof. Observe first that when $c \downarrow 0$, Eqs. (2.14) and (2.16) become respectively

$$\begin{aligned} e^{4\mu x - 2\mu} (2\mu x - \mu - 1) + 2\mu x - \mu + 1 &= 0, \\ \mu^2 \alpha_0 e^{4\mu y} + (1 - 2\mu^2 \alpha_0) e^{2\mu y} + \mu^2 \alpha_0 - 2\mu y - 1 &= 0, \end{aligned}$$

where $\alpha_0 = -\frac{1}{2\mu^2} e^{-\mu}$. The unique solution of the first equation is $b_0 = \frac{1}{2}$, so $\lim_{c \downarrow 0} b_c = \frac{1}{2}$. The second equation has exactly two solutions, one of which is $> \frac{1}{2}$ and the other is $a_0 = 0$. Since $0 < a_c < \frac{1}{2}$, we deduce that $\lim_{c \downarrow 0} a_c = 0$. Putting these values into (2.15) and (2.17), respectively, we find that when $c \downarrow 0$, $\alpha_c \rightarrow -\frac{1}{2\mu^2} e^{-\mu}$ and $\beta_c \rightarrow \frac{e^{-\mu}}{2\mu^2} - \frac{1}{\mu^2}$. Therefore, according to (2.13), we find that

$$\lim_{c \downarrow 0} V_c(x, a) = \begin{cases} \frac{x}{\mu} - \frac{1}{2\mu^2} e^{-\mu} \left(e^{2\mu x} - 1 \right), & x \in \left[0, \frac{1}{2} \right], \\ \frac{1-x}{\mu} - \frac{1}{2\mu^2} e^{-\mu} \left(e^{2\mu(1-x)} - 1 \right), & x \in \left[\frac{1}{2}, 1 \right]. \end{cases}$$

These formulas coincide with (4.4) and the proof is complete. \square

In Fig. 2, we give the graph of the value function of both problems (2.4) and (2.5) for two possible values of the switching cost ($c = 0.01$ and $c = 0.04$) when the intensity of the drift is

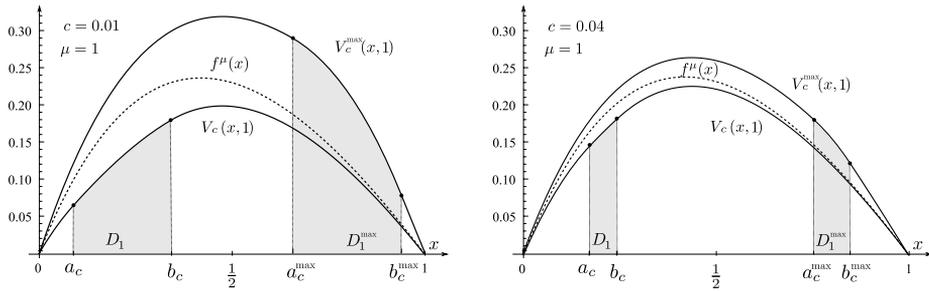


Fig. 2. Graphs of $V_c^{\max}(x, 1)$ and $V_c(x, 1)$ for two different values of the cost. The dotted line is the graph of the function $f^\mu(x)$ defined in (2.6), which coincides with the payoff of the constant strategy.

$\mu = 1$. In this case, the critical value of the cost is $c^*(1) \approx 0.058$. The numerical value of the switching boundaries in the case where $c = 0.01$ are given by:

$$a_c \approx 0.0882, \quad b_c \approx 0.3426, \quad a_c^{\max} = 1 - b_c, \quad b_c^{\max} \approx 0.9387,$$

and in the case where $c = 0.04$, they are given by:

$$a_c \approx 0.1737, \quad b_c \approx 0.2451, \quad a_c^{\max} = 1 - b_c, \quad b_c^{\max} \approx 0.8494.$$

Because of the symmetry, Fig. 2 shows only the value function corresponding to a positive initial drift.

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