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Strong local nondeterminism and exact modulus of continuity for spherical Gaussian fields

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Abstract

In this paper, we are concerned with sample path properties of isotropic spherical Gaussian fields on $\mathbb{S}^2$. In particular, we establish the property of strong local nondeterminism of an isotropic spherical Gaussian field based on the high-frequency behaviour of its angular power spectrum; we then exploit this result to establish an exact uniform modulus of continuity for its sample paths. We also discuss the range of values of the spectral index for which the sample functions exhibit fractal or smooth behaviour.

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1. Introduction and overview

1.1. Motivations

The analysis of sample path properties of random fields has been considered by many authors, see, for instance, [4,7,14,15,21,22,25,26,30,31] and their combined references. These papers have covered a wide variety of circumstances, including scalar and vector valued random fields,

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isotropic and anisotropic behaviour, analytic and geometric properties. The parameter space of the random fields in these references, however, has been typically considered to be Euclidean, i.e., $\mathbb{R}^k$, for $k \geq 1$.

From the point of view of applications, however, there is of course a lot of interest in investigating sample path properties of random fields defined on manifolds. In particular, we shall focus here on isotropic random fields defined on the unit sphere $\mathbb{S}^2$; these fields have considerable mathematical interest by themselves, and arise very naturally in a number of scientific areas, i.e., geophysics, astrophysics and cosmology, atmospheric sciences, image analysis, to name only a few, see [17] for a systematic account. To the best of our knowledge, very little is currently known on the sample path properties of these fields, even under Gaussianity and Isotropy assumptions; the only currently available references seem to be [11,13], which investigate differentiability and Hölder continuity properties of the sample functions in terms of the so-called spectral index, to be defined below.

Our aim in this paper is to pursue this line of investigation further and to provide two main results. The first of these results is to establish a property of strong local nondeterminism for a large class of isotropic spherical Gaussian fields. In the Euclidean setting, the notion of strong local nondeterminism has played a pivotal role to establish a number of characterizations for sample trajectories, see again [22,25,26,30–32] for more discussions and review of recent papers; we thus believe that our result will open a way for similar developments in the area of spherical Gaussian fields. In particular, by exploiting this property, we are able to establish our second main result, i.e. the exact uniform modulus of continuity for isotropic spherical Gaussian fields. The exact form of the scaling depends in a very explicit way on the behaviour of the angular power spectrum (to be recalled below) of the field, and we can hence identify the class of models that lead to fractal properties. In order to state more precisely these results, we need to introduce however some more notation and background material, which we do in the following subsection.

1.2. Background and notation

We start by recalling some background from [17] on second order spherical random fields, by which we mean as usual measurable applications $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$, where $\{\Omega, \Im, \mathbb{P}\}$ is some probability space, such that for all $x \in \mathbb{S}^2$,

$$\mathbb{E}\big(T^2(x,\omega)\big) = \int_\Omega T^2(x,\omega)d\mathbb{P}(\omega) < \infty.$$

Without loss of generality, in the sequel we shall always assume the field to have zero-mean, $\mathbb{E}\big(T(x,\omega)\big) = 0$. Also, as usual, by (strong) isotropy we mean that the random fields $T = \{T(x), x \in \mathbb{S}^2\}$ and $T^g = \{T(gx), x \in \mathbb{S}^2\}$ have the same law, for all rotations $g \in SO(3)$. T is called 2-weakly isotropic if $\mathbb{E}\big(T(x)T(y)\big) = \mathbb{E}\big(T(gx)T(gy)\big)$ for all $g \in SO(3)$.

Given a 2-weakly isotropic random field $T = \{T(x), x \in \mathbb{S}^2\}$, the following spectral representation is well known to hold (cf. [17, Theorem 5.13]):

$$T(x;\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\omega)Y_{\ell m}(x), \tag{1}$$

where $\{Y_{\ell m},\ \ell \geq 0; m = 0, \pm 1, \ldots, \pm\ell\}$ are the spherical harmonic functions on $\mathbb{S}^2$ and $a_{\ell m} = \int_{\mathbb{S}^2} T(x)\overline{Y_{\ell m}(x)}\,dx$; we are adopting here the so-called Condon–Shortley phase convention, entailing that for $m > 0$

$$Y_{\ell m}(x) = (-1)^m\overline{Y_{\ell,-m}}(x) \ \text{ and consequently } \ a_{\ell m}(\omega) = (-1)^m\overline{a_{\ell,-m}}(\omega).$$

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The equality in (1) holds both in $L^2(\Omega)$ at every fixed x, and in $L^2(\Omega \times \mathbb{S}^2)$, i.e.

$$\lim_{L\to\infty} \mathbb{E}\left[T(x) - \sum_{\ell}^{L}\sum_{m} a_{\ell m}(\omega) Y_{\ell m}(x)\right]^2 = 0,$$

and

$$\lim_{L\to\infty} \mathbb{E}\left[\int_{\mathbb{S}^2}\left(T(x;\omega) - \sum_{\ell}^{L}\sum_{m} a_{\ell m}(\omega) Y_{\ell m}(x)\right)^2 dx\right] = 0.$$

We recall that the finite-variance condition $\mathbb{E}\big(T^2(x)\big) < \infty$ under isotropy automatically entails the mean-square continuity; the spectral representation hence follows without further assumptions, see [17,18].

If $T = \{T(x), x \in \mathbb{S}^2\}$ is a Gaussian random field, then its strong isotropy and 2-weak isotropy are equivalent. The distribution of an isotropic zero-mean Gaussian field $T = \{T(x), x \in \mathbb{S}^2\}$ is fully characterized by the covariance function $\mathbb{E}\big(T(x)T(y)\big)$. By a theorem of Schoenberg [24], the latter can be expanded as follows:

$$\mathbb{E}\big(T(x)T(y)\big) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell P_\ell(\langle x, y\rangle); \tag{2}$$

here, $\langle\cdot,\cdot\rangle$ represents the standard inner product in $\mathbb{R}^3$, whereas $P_0 \equiv 1$ and $P_\ell : [-1, 1] \to \mathbb{R}$, for $\ell = 1, 2, \ldots,$ denote the Legendre polynomials, which satisfy the normalization condition $P_\ell(1) = 1$ and can be recovered by Rodrigues' formula as

$$P_\ell(t) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dt^\ell}(t^2-1)^\ell, \quad \ell = 1, 2, \ldots.$$

On the other hand, the sequence $\{C_\ell,\ \ell = 0, 1, \ldots\}$ of nonnegative weights represents the so-called angular power spectrum of the field, and the ℓ's are referred to as frequencies (also labelled multipoles). In terms of the spectral representation, we have the identification

$$\mathbb{E}\big(a_{\ell m}\overline{a}_{\ell' m'}\big) = C_\ell \delta_\ell^{\ell'} \delta_m^{m'}, \tag{3}$$

so that the angular power spectrum provides the variance of the (uncorrelated) Gaussian random coefficients $\{a_{\ell m},\ \ell = 0, 1, 2, \ldots; m = -\ell, \ldots, \ell\}$. By standard Fourier arguments, the small scale behaviour of the covariance is determined by the behaviour of the angular power spectrum at high frequencies; namely, the behaviour of C_ℓ for as $\ell \to \infty$.

It is known that for $\ell = 0$, $Y_{00}(x)$ in (1) is a constant function on $\mathbb{S}^2$, which does not affect the sample path regularity of $T(x)$. Hence, for simplicity of notation, we will remove the term for $\ell = m = 0$ from (1) and (2) (i.e., we consider $T(x) - a_{00}Y_{00}(x)$) throughout the rest of this paper. Furthermore, we shall impose the following condition on the behaviour of the angular power spectrum, which we consider in every respect as minimal.

Condition (A): The random field $T = \{T(x), x \in \mathbb{S}^2\}$ is zero-mean, Gaussian and isotropic, with angular power spectrum such that:

$$C_\ell = G(\ell)\,\ell^{-\alpha} > 0, \quad \forall\, \ell = 1, 2, \ldots, \tag{4}$$

where $\alpha > 2$ is a constant and, moreover, there exists a finite constant $c_0 \geq 1$, such that

$$c_0^{-1} \leq G(\ell) \leq c_0.$$

The assumption $\alpha > 2$ is necessary to ensure that the field has finite variance (recall the identity $\mathbb{E}\big(T^2(x)\big) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell$). On the other hand, we stress that we are imposing no regularity condition on the function $G(\ell)$, on the contrary of much of the literature on spherical random fields, which typically requires $\lim_{\ell\to\infty} G(\ell) = const.$ or other forms of additional regularity conditions (see i.e., [3,12,19,20]). We believe that Condition (A) covers the vast majority of models which seems of interest from a theoretical or applied point of view; for instance, it fits very well with the theoretical and observational evidence on Cosmic Microwave Background radiation data (see [5,6,23]), which has been one of the main motivating areas for the analysis of spherical fields over the last decade. Most of our results to follow will depend in a simple analytic way from the value of the parameter α, which we refer to as the spectral index of T.

1.3. Statement of the main results

To introduce our first main result (on strong local nondeterminism), we need first to introduce some more notation. In particular, for $\alpha > 2$, let $\rho_\alpha : [0,\pi] \to \mathbb{R}^+$ be the continuous function defined by

$$\rho_\alpha(t) := \begin{cases} t^{(\alpha-2)/2}, & \text{if } 2 < \alpha < 4, \\ t\sqrt{|\ln t|}, & \text{if } \alpha = 4, \\ t, & \text{if } \alpha > 4 \end{cases} \tag{5}$$

and $\rho_\alpha(0) = 0$ for all values of α. As we shall show later, up to a constant factor the functions ρ_α can be related to the canonical (Dudley) metric for the Gaussian processes to be investigated; it is important to note the explicit dependence on the spectral index α. As usual, we take

$$d_{\mathbb{S}^2}(x, y) = \arccos(\langle x, y\rangle)$$

as the standard spherical (or geodesic) distance on $\mathbb{S}^2$. The following result establishes the property of strong local nondeterminism for spherical Gaussian fields satisfying Condition (A) with $2 < \alpha < 4$.

Theorem 1. *Let $T = \{T(x), x \in \mathbb{S}^2\}$ be an isotropic Gaussian field that satisfies Condition (A) with $2 < \alpha < 4$. There exist positive and finite constants c_2 and ε_0 such that for all integers $n \geq 1$ and all $x_0, x_1, \ldots, x_n \in \mathbb{S}^2$ with $\min_{1\leq k\leq n} d_{\mathbb{S}^2}(x_0, x_k) \leq \varepsilon_0$ we have*

$$\operatorname{Var}\left(T(x_0) \,|\, T(x_1), \ldots, T(x_n)\right) \geq c_2 \min_{1\leq k\leq n} \rho_\alpha\left(d_{\mathbb{S}^2}(x_0, x_k)\right)^2. \tag{6}$$

The proof of Theorem 1 is presented in Section 3. The argument does not seem to work for the critical case of $\alpha = 4$, we expect that (6) still holds, but a new method may be needed.

In the following we show how the strong local nondeterminism property can be exploited to develop a number of important characterizations for the sample path behaviour of spherical random fields. Among these characterizations, in this paper we shall focus on the asymptotic behaviour of the uniform modulus of continuity. Lang and Schwab [13, Theorem 4.5] have studied this problem for a class of isotropic Gaussian field on $\mathbb{S}^2$ and obtained an upper bound for uniform modulus of continuity. Under Condition (A), Theorem 4.5 of Lang and Schwab [13] implies that for every $\gamma < (\alpha-2)/2$, there exists a finite constant c such that a.s.

$$|T(x) - T(y)| \leq c\, d_{\mathbb{S}^2}(x, y)^\gamma \quad \text{for all } x, y \in \mathbb{S}^2. \tag{7}$$

Our theorem below significantly improves the theorem of Lang and Schwab [13] by providing the exact uniform modulus of continuity for $T = \{T(x), x \in \mathbb{S}^2\}$.

Theorem 2. *Let $T = \{T(x), x \in \mathbb{S}^2\}$ be an isotropic Gaussian field that satisfies Condition (A).*

(i). *If $2 < \alpha < 4$, then there exists a positive and finite constant K_1 such that, with probability one*

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{\rho_\alpha\left(d_{\mathbb{S}^2}(x,y)\right)\sqrt{\left|\ln d_{\mathbb{S}^2}(x,y)\right|}} = K_1. \tag{8}$$

(ii). *If $\alpha = 4$, then there exists a positive and finite constant K_2 such that, with probability one*

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{d_{\mathbb{S}^2}(x,y)\left|\ln d_{\mathbb{S}^2}(x,y)\right|} \le K_2. \tag{9}$$

The proof of Theorem 2 will be given in Section 4. In the following, we provide some remarks.

- It is important to note the fractal behaviour that occurs for $2 < \alpha < 4$, when the modulus of continuity decays slower than linearly with respect to the angular distance (hence the sample function $T(x)$ is nondifferentiable). We note that this range of values of α is typical for many applied fields, for instance for Cosmic Microwave Background data α is known to be very close to 2, from theoretical arguments and from experimental data (see e.g., [23]).
- For the case of $\alpha = 4$, (9) implies that the sample function $T(x)$ is almost Lipschitz. We believe that the equality in (9) actually holds and the sample function presents subtle fractal properties. However, we have not been able to prove these results, due to the unsolved case in Theorem 1.

Next we consider the case of $\alpha > 4$. Let $k \ge 1$ be the unique integer such that $2 + 2k < \alpha < 4 + 2k$. It follows from Lang and Schwab [13, Theorem 4.6] that $T = \{T(x), x \in \mathbb{S}^2\}$ has a modification, still denoted by T, such that its sample function is almost surely k-times continuously differentiable. Moreover, the k th (partial) derivatives of $T(x)$ are Hölder continuous on $\mathbb{S}^2$ with exponent $\gamma < \frac{\alpha-2}{2} - k$.

In the following, we adapt the approach of Lang and Schwab [13] (see also [11]) to study the regularity properties of higher-order derivatives of T based on pseudo-differential operators, as described in the classical monograph [27]. In particular, for a real $k \in \mathbb{R}$, introduce $(1 - \Delta_{\mathbb{S}^2})^{k/2}$ as the pseudo-differential operator whose action on functions $f(\cdot) := \sum f_{\ell m} \in L^2(\mathbb{S}^2)$ is defined by

$$(1 - \Delta_{\mathbb{S}^2})^{k/2} f := \sum_{\ell m} f_{\ell m}(1 + \ell(\ell+1))^{k/2} Y_{\ell m}, \tag{10}$$

provided the right-hand side converges in $L^2(\mathbb{S}^2)$. In the above, $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian, also called Laplace–Beltrami operator which, in spherical coordinates $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$, is defined by

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left\{\sin\vartheta\frac{\partial}{\partial\vartheta}\right\} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}. \tag{11}$$

Recall that, for every $x \in \mathbb{S}^2$, it can be written as $x = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$. In this paper, with slight abuse of notation, we always identify the Cartesian and angular coordinates of the point $x \in \mathbb{S}^2$.

It is shown in [27, Chapter XI] that the Sobolev space $\mathcal{W}^{k,2}(\mathbb{S}^2)$ of functions with square-integrable k th derivatives can be viewed as the image of $L^2(\mathbb{S}^2)$ under the operator $(1-\Delta_{\mathbb{S}^2})^{-k/2}$; this and related property are exploited by Lang and Schwab [13] to prove their Theorem 4.6 on regularity of higher-order derivatives. More precisely, consider the Gaussian random field $T^{(k)} = \{T^{(k)}(x), x \in \mathbb{S}^2\}$ defined by

$$T^{(k)} := (1-\Delta_{\mathbb{S}^2})^{k/2} T = \sum_{\ell m} a_{\ell m} (1+\ell(\ell+1))^{k/2} Y_{\ell m}.$$

Lang and Schwab [13, Theorem 4.6] have proved the almost-sure Hö lder continuity of $T^{(k)}$. Namely and analogue of (7) holds for $T^{(k)}$ and $\gamma < (\alpha-2)/2 - k$. We are able to improve their results by establishing the exact modulus of continuity for $T^{(k)}$, for which we provide the following result.

Theorem 3. *If in Condition (A), $2+2k < \alpha \le 4+2k$ for some integer $k \ge 1$, then $T^{(k)} = \{T^{(k)}(x), x \in \mathbb{S}^2\}$ satisfies the following exact uniform modulus of continuity:*

(i). *If $2+2k < \alpha < 4+2k$, then there exists a positive and finite constant K_3 such that*

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2 \\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T^{(k)}(x)-T^{(k)}(y)|}{\rho_{\alpha-2k}\left(d_{\mathbb{S}^2}(x,y)\right)\sqrt{\left|\ln d_{\mathbb{S}^2}(x,y)\right|}} = K_3, \quad \text{a.s.}$$

(ii). *If $\alpha = 4+2k$, then there exists a positive and finite constant K_4 such that*

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2 \\ d_{\mathbb{S}^2}(x,y)<\varepsilon}} \frac{|T^{(k)}(x)-T^{(k)}(y)|}{d_{\mathbb{S}^2}(x,y)\left|\ln d_{\mathbb{S}^2}(x,y)\right|} \le K_4, \quad \textit{a.s.}$$

Throughout this paper, we shall restrict our setting to the unit sphere in $\mathbb{R}^3$. There are no theoretical reasons why our results should not be generalizable to higher-dimensions under conditions more general than Condition (A) (e.g., C_ℓ does not have to be regularly varying); however, some analytic computations will become significantly more involved, so we leave these extensions for further research.

1.4. Plan of the paper

The plan of the paper is as follows. In Section 2 we introduce some auxiliary tools that will be instrumental for our proofs to follow; in particular, a careful analysis of the variogram/covariance function on very small scales, and the construction of the so-called spherical bump function, i.e. a compactly supported function on the sphere satisfying some required smoothness conditions. The latter construction builds upon ideas discussed by Geller and Mayeli [8,9] in the framework of spherical wavelets. In Section 3, we exploit these results to establish the property of strong local nondeterminism for a large class of isotropic spherical Gaussian fields. In Section 4, by applying Gaussian techniques and strong local nondeterminism we prove Theorem 2 on the exact uniform modulus of continuity; while an extension to higher-order derivatives is discussed in Section 5. Some auxiliary results are collected in the Appendix.

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2. Technical tools

2.1. The variogram

It is well-known that, for the investigation of sample properties of a Gaussian field $T = \{T(x),\ x \in \mathbb{S}^2\}$, it is important to introduce the canonical metric

$$d_T(x, y) = \sqrt{\mathbb{E}\big(|T(x) - T(y)|^2\big)},$$

see for instance [1,16] or any other monograph on the modern theory of Gaussian processes. The square of the canonical metric is also known as the variogram of T. Our first technical result is a careful investigation on the behaviour of this metric for pairs of points that are close in the spherical distance $d_{\mathbb{S}^2}(\cdot,\cdot)$; more precisely, we have the following upper and lower bounds, in terms of the function ρ_α which was introduced in (5).

Lemma 4. *Under Condition (A), there exist constants $1 \le c_1 < \infty$ and $0 < \varepsilon < 1$, such that for all $x, y \in \mathbb{S}^2$ with $d_{\mathbb{S}^2}(x, y) \le \varepsilon$, we have*

$$c_1^{-1}\rho_\alpha^2\left(d_{\mathbb{S}^2}(x, y)\right) \le d_T^2(x, y) \le c_1\rho_\alpha^2\left(d_{\mathbb{S}^2}(x, y)\right), \tag{12}$$

where $\rho_\alpha(\cdot) : [0, \pi] \to \mathbb{R}^+$ is defined in (5).

Proof. Recalling Eq. (2), it is readily seen that

$$d_T^2(x, y) = \mathbb{E}\big(|T(x) - T(y)|^2\big) = \sum_{\ell=1}^{\infty} C_\ell \frac{2\ell + 1}{2\pi}\big(1 - P_\ell(\cos\vartheta)\big), \tag{13}$$

where we write for notational convenience $\vartheta = \vartheta_{xy} := d_{\mathbb{S}^2}(x, y)$. Let

$$Q_\alpha(\vartheta) := \sum_{\ell=1}^{\infty} \ell^{-(\alpha-1)}\big(1 - P_\ell(\cos\vartheta)\big).$$

The Cauchy–Schwarz inequality gives $|P_\ell(\cos\vartheta)| \le P_\ell(1) = 1$, hence it follows from (13) and Condition (A) that

$$\frac{c_0^{-1}}{\pi} Q_\alpha(\vartheta) \le d_T^2(x, y) \le \frac{c_0}{\pi} Q_\alpha(\vartheta). \tag{14}$$

By applying Lemma 10 with $s = \alpha - 1$ in the Appendix, we have

$$Q_\alpha(\vartheta) = K\rho_\alpha^2(\vartheta) + o\left(\rho_\alpha^2(\vartheta)\right) \quad \text{as } \vartheta \to 0, \tag{15}$$

where K is a positive constant depending only on α and $o\left(\rho_\alpha^2(\vartheta)\right)$ denotes a higher order infinitesimal than $\rho_\alpha^2(\vartheta)$. Therefore, statement (12) follows from (14) and (15). ■

Remark 5. Anticipating some results to follow, it is important to stress the phase transition that occurs in the behaviour of the canonical metric as a function of α. For $\alpha > 4$, the ratio between the canonical metric and the standard geodesic distance is bounded above and below; for $2 < \alpha < 4$, on the contrary, the ratio between geodesic and canonical distance diverges on small scales and fractal behaviour occurs. The case of $\alpha = 4$ is, in some sense, critical and an extra logarithmic factor appears in the bounds for the variogram in Lemma 4.

2.2. The construction of the spherical bump function

In this section, we work with spherical coordinates (ϑ, φ), $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, and we review the construction of a family of zonal functions $\delta_\varepsilon : \mathbb{S}^2 \to \mathbb{R}$, $\varepsilon > 0$, which shall vanish outside a spherical cap around the North Pole $\vartheta = \varphi = 0$ (we recall that a zonal function satisfies by definition the identity $\delta_\varepsilon(\vartheta, \varphi) = \delta_\varepsilon(\vartheta, \varphi')$ for all $\varphi, \varphi' \in [0, 2\pi)$). The construction follows a proposal by Geller and Mayeli ([8], Lemma 4.1, pages 16–17), see also [9]; we introduce some minimal modifications, to ensure a suitable rate of decay in the spherical harmonic coefficients. More precisely, we shall show that for all $\varepsilon > 0$, there exists a zonal function

$$\delta_\varepsilon(\vartheta, \varphi) := \sum_{\ell=1}^{\infty} b_\ell(\varepsilon) \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \kappa_{\ell m}(\varepsilon) Y_{\ell m}(\vartheta, \varphi) \tag{16}$$

such that for some positive and finite constants c and c', we have

$$\begin{aligned} &\varepsilon^2 \delta_\varepsilon(\vartheta, \varphi) \leq c \quad \text{for all } \ 0 \leq \vartheta \leq \pi, \ 0 \leq \varphi < 2\pi; \\ &\delta_\varepsilon(\vartheta, \varphi) = 0 \quad \text{for all } \vartheta > \varepsilon \end{aligned} \tag{17}$$

and

$$\delta_\varepsilon(0, 0) \sim c' \varepsilon^{-2} \ \text{ as } \varepsilon \to 0. \tag{18}$$

Moreover, the coefficients $\{b_\ell(\varepsilon), \kappa_{\ell m}(\varepsilon)\}$ in (16) can be taken such that they satisfy

$$\begin{aligned} &|b_\ell(\varepsilon)| \leq c'', \qquad \kappa_{\ell m}(\varepsilon) = 0 \text{ for } m \neq 0, \quad \text{and} \\ &|\kappa_{\ell 0}(\varepsilon)| \leq c''' \sqrt{2\ell+1} \end{aligned} \tag{19}$$

for all integers $\ell \geq 1$, where c'' and c''' are positive and finite constants.

It is natural to label $\delta_\varepsilon(\cdot, \cdot)$ a *spherical bump function,* in analogy with the analogous constructions on the Euclidean domains. On the other hand, up to a different normalization factor the function $\delta_\varepsilon(\cdot, \cdot)$ is just a special case of the so-called Mexican needlet frame by [8], in the special case where the latter has bounded support in the real domain. We hence follow as much as possible the notation by these authors.

In particular, we choose a function $\widehat{G}(\cdot) : \mathbb{R} \to \mathbb{R}$ such that it satisfies the following conditions:

(i). *supp* $\widehat{G}(\cdot) \subseteq (-1, 1)$,
(ii). It is piecewise continuously differentiable up to order M, where M is large enough, and
(iii). Its inverse Fourier transform G is non-negative and satisfies $0 < \int_0^\infty G(u) u du < \infty$.

For example, we can take

$$\widehat{G}(\cdot) = p \star p(\cdot) := \int_{-\infty}^{\infty} p(s) p(\cdot - s)\, ds,$$

where $p(s) = \max\{0, 1 - 2|s|\}$. Then $\widehat{G}(\cdot)$ is piecewise smooth and its inverse Fourier transform is $G(u) = (\frac{2}{\pi})^2 (1 - \cos(u/2))^2 u^{-4}$. Functions $G(u)$ with faster decay rate as $u \to \infty$ can be constructed by convoluting more times.

As in Geller and Mayeli [8], we consider the operator $G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}}) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}}) := \int_{-\infty}^{\infty} \widehat{G}(s) \exp(-is\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}})\, ds;$$

recall that $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian in (11), and the operator $G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}})$ converges in weak operator topology. More precisely, the action of this operator can be described as usual by means of the corresponding kernel; i.e., for any $f \in L^2(\mathbb{S}^2)$ we have

$$G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}})f(\cdot) := \int_{\mathbb{S}^2} K_\varepsilon(x,\cdot)f(x)dx,$$

where

$$\begin{aligned} K_\varepsilon(x,y) &:= \sum_{\ell=1}^{\infty} G\big(\varepsilon\sqrt{-\lambda_\ell}\big)\frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle) \\ &= \sum_{\ell=1}^{\infty}\left\{\int_{-\infty}^{\infty}\widehat{G}(s)\exp(-is\varepsilon\sqrt{-\lambda_\ell})ds\right\}\frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle). \end{aligned} \tag{20}$$

In the above, $\{\lambda_\ell,\ \ell=1,2,\ldots\}$ are the eigenvalues of $\Delta_{\mathbb{S}^2}$, i.e., $\lambda_\ell = -\ell(\ell+1)$,

$$\Delta_{\mathbb{S}^2}Y_{\ell m} = \lambda_\ell Y_{\ell m}$$

for $\ell = 1,2,\ldots$ and $m = -\ell,\ldots,\ell$; see i.e, [17] , Chapter 3. More explicitly, note that, for $f = \sum_{\ell m} a_{\ell m}Y_{\ell m}$, the action of the operator can be equivalently defined as

$$G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}})f\,(\cdot) = \sum_{\ell m} G(\varepsilon\sqrt{-\lambda_\ell})a_{\ell m}Y_{\ell m}\,(\cdot)\,,$$

which is clearly in $L^2(\mathbb{S}^2)$, because

$$\left\|G(\varepsilon\sqrt{-\Delta_{\mathbb{S}^2}})f\,(\cdot)\right\|^2_{L^2(S^2)} = \sum_{\ell m}\left|G(\varepsilon\sqrt{-\lambda_\ell})a_{\ell m}\right|^2,$$

and $|G(\varepsilon\sqrt{-\lambda_\ell})|$ is uniformly bounded, while $\{a_{\ell m}\}_{\ell,m}$ is square summable because $f(\cdot) \in L^2(\mathbb{S}^2)$.

Now take $x = N = (0,0)$ (the "North Pole"), $y = (\vartheta,\varphi)$ an arbitrary point on the sphere, and define

$$\delta_\varepsilon(\vartheta,\varphi) := K_\varepsilon(N,y).$$

Then the first inequality in (17) follows from an application of Lemma 4.1 in [8] to the case of $\mathbf{M} = \mathbb{S}^2$ (hence $n=2$, $d(x,y) = d_{\mathbb{S}^2}(N,y) = \vartheta$), $t = \varepsilon$ and $j,k,N = 0$. The second statement in (17), namely, $supp\,\delta_\varepsilon \subseteq \{(\vartheta,\varphi) : \vartheta \le \varepsilon\}$ follows from Huygens' principle as in the proof of Lemma 4.1 in [8, page 911].

To verify (18), we use the definition of K_ε in (20) to verify that as $\varepsilon \to 0$,

$$\delta_\varepsilon(0,0) = \sum_{\ell=1}^{\infty} G(\varepsilon\sqrt{\ell(\ell+1)})\frac{2\ell+1}{\sqrt{4\pi}} \sim \frac{1}{\varepsilon^2\sqrt{\pi}}\int_0^\infty G(u)udu = c'\varepsilon^{-2}$$

with $c' = \pi^{-1/2}\int_0^\infty G(u)udu$ which is positive and finite, and $\sim$ denotes convergence to unity of the ratio between the left- and right-hand sides.

Now we define

$$b_\ell(\varepsilon) := \int_{-\infty}^{\infty}\widehat{G}(s)\exp(-is\varepsilon\sqrt{-\lambda_\ell})ds,$$

$$\kappa_{\ell m}(\varepsilon) = \begin{cases} \sqrt{\dfrac{2\ell+1}{4\pi}}\,b_\ell(\varepsilon), & \text{if } m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

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Then $|b_\ell(\varepsilon)| \le c$ for some constant c, and $\{\kappa_{\ell m}(\varepsilon)\}$ satisfies the properties in (19). Moreover, by appealing to the standard identities

$$\frac{2\ell+1}{4\pi} P_\ell(\langle x, y\rangle) = \sum_{m=\ell}^{\ell} \overline{Y}_{\ell m}(x) Y_{\ell m}(y),$$

$$Y_{\ell m}(0,0) = \begin{cases} \sqrt{\dfrac{2\ell+1}{4\pi}}, & \text{for } m = 0, \\ 0, & \text{otherwise}, \end{cases}$$

we see that $\delta_\varepsilon(\vartheta, \varphi)$ can be written as

$$\delta_\varepsilon(\vartheta, \varphi) = \sum_{\ell=1}^{\infty} b_\ell(\varepsilon) \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \kappa_{\ell m}(\varepsilon) Y_{\ell m}(\vartheta, \varphi),$$

which gives the desired representation in (16).

We end this section with some further properties of the spherical bump function $\delta_\varepsilon(\vartheta, \varphi)$ and its coefficients which will be used in the proof of Theorem 1 in Section 3.

To get information on the decay rate of $|b_\ell(\varepsilon)|$ as ℓ increases, we use integration by parts r times ($r \le M$) to get

$$b_\ell(\varepsilon) = \int_{-\infty}^{\infty} \widehat{G}(s) \exp(-is\varepsilon\sqrt{-\lambda_\ell}) ds = \int_{-\infty}^{\infty} \widehat{G}^{(r)}(s) \frac{\exp(-is\varepsilon\sqrt{-\lambda_\ell})}{\left\{i\varepsilon\sqrt{-\lambda_\ell}\right\}^r} ds.$$

Hence for any $r \le M$,

$$\left|b_\ell(\varepsilon)\right| \le \frac{K_r}{\varepsilon^r \ell^r}, \tag{21}$$

where

$$K_r := \sup_{-1\le s\le 1} \left|\widehat{G}^{(r)}(s)\right| < \infty.$$

Note that, by (18), there exists a constant $\varepsilon_0 > 0$ such that

$$\sum_{\ell=1}^{\infty} b_\ell(\varepsilon) \frac{2\ell+1}{4\pi} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \kappa_{\ell m}(\varepsilon) \sqrt{\frac{2\ell+1}{4\pi}} = \delta_\varepsilon(0,0) \ge \frac{c'}{2}\varepsilon^{-2} \tag{22}$$

for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, by (17), we see that for all $\vartheta > \varepsilon$,

$$\begin{aligned} \sum_{\ell=1}^{\infty} b_\ell(\varepsilon) \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta) &= \sum_{\ell m} \kappa_{\ell m}(\varepsilon) \sqrt{\frac{2\ell+1}{4\pi}} Y_{\ell m}(\vartheta, \varphi) \\ &= \delta_\varepsilon(\vartheta, \varphi) = 0. \end{aligned} \tag{23}$$

3. Strong local nondeterminism: Proof of Theorem 1

We are now in the position to prove Theorem 1. Recall that $T = \{T(x), x \in \mathbb{S}^2\}$ is an isotropic Gaussian random field with mean zero and angular power spectrum $\{C_\ell\}$. We prove the following more general theorem which implies Theorem 1 when $2 < \alpha < 4$. For $\alpha \ge 4$, the lower bound given by (24) is strictly smaller than $\rho_\alpha^2(\varepsilon)$. Lemma 4 indicates that (24) can be improved if $n = 1$. However, it is not known if one can strengthen (24) for all $n \ge 2$.

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Theorem 6. *Under Condition (A), there exist positive and finite constants ε_0 and c_2 such that for all $\varepsilon \in (0, \varepsilon_0]$, all integers $n \geq 1$ and all $x_0, x_1, \ldots, x_n \in \mathbb{S}^2$, satisfying $d_{\mathbb{S}^2}(x_0, x_k) \geq \varepsilon$, we have*

$$\mathrm{Var}\left(T\left(x_0\right) | T\left(x_1\right), \ldots, T\left(x_n\right)\right) \geq c_2 \varepsilon^{\alpha-2}. \tag{24}$$

Proof. As before, we work in spherical coordinates (ϑ, φ) and we take without loss of generality $x_0 = (0, 0)$ to be the North Pole, and $x_k = (\vartheta_k, \varphi_k)$ so that $d_{\mathbb{S}^2}(x_0, x_k) = \vartheta_k$. Recall that, since T is a Gaussian random field, we have

$$\begin{aligned}&\mathrm{Var}\left(T\left(0\right) | T\left(x_1\right), \ldots, T\left(x_n\right)\right)\\ &= \inf\left\{\mathbb{E}\left[\left(T(0) - \sum_{j=1}^{n} \gamma_j T(x_j)\right)^2\right] : \gamma_1, \ldots, \gamma_n \in \mathbb{R}\right\}.\end{aligned}$$

Hence, in order to establish (24), it is sufficient to prove that there exists a positive constant c_2 such that for all choices of real numbers $\gamma_1, \ldots, \gamma_n$, we have

$$\mathbb{E}\left\{\left(T(0) - \sum_{j=1}^{n} \gamma_j T(x_j)\right)^2\right\} \geq c_2\, \varepsilon^{\alpha-2}. \tag{25}$$

It follows from (1), (2) or (3) that

$$\begin{aligned}\mathbb{E}\left\{\left(T(0) - \sum_{j=1}^{n} \gamma_j T(x_j)\right)^2\right\} &= \mathbb{E}\left\{\left(\sum_{\ell m} a_{\ell m} Y_{\ell m}(0) - \sum_{j=1}^{n} \gamma_j \sum_{\ell m} a_{\ell m} Y_{\ell m}(x_j)\right)^2\right\}\\ &= \sum_{\ell m} \mathbb{E}(|a_{\ell m}|^2)\left|Y_{\ell m}(0) - \sum_{j=1}^{n} \gamma_j Y_{\ell m}(x_j)\right|^2\\ &= \sum_{\ell}\sum_{m} C_\ell \left|Y_{\ell m}(0) - \sum_{j=1}^{n} \gamma_j Y_{\ell m}(x_j)\right|^2.\end{aligned}$$

Hence, (25) is a consequence of Proposition 7. ■

Proposition 7. *Assume Condition (A) holds. For all $\varepsilon \in (0, \varepsilon_0]$, there exists a constant $c_2 > 0$ such that for all choices of $n \in \mathbb{N}$, all $(\vartheta_j, \varphi_j) : \vartheta_j > \varepsilon$, and $\gamma_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, we have*

$$\sum_{\ell}\sum_{m} C_\ell \left[Y_{\ell m}(0,0) - \sum_{j=1}^{n} \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j)\right]^2 \geq c_2 \varepsilon^{\alpha-2}. \tag{26}$$

Proof. For any fixed $\varepsilon > 0$, let $\delta_\varepsilon(\cdot, \cdot)$ be defined as in (16), with the corresponding coefficients $\left\{b_{\ell m}(\varepsilon)\right\}$ and $\left\{\kappa_{\ell m}(\varepsilon)\right\}$ such that conditions (17)–(19), (21)–(23) hold. Now we consider

$$I = \sum_{\ell}\sum_{m}\left(\frac{\kappa_{\ell m}(\varepsilon)}{\sqrt{C_\ell}}\right)\left\{\sqrt{C_\ell}\left[Y_{\ell m}(0,0) - \sum_{j=1}^{n} \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j)\right]\right\}.$$

On one hand, by the Cauchy–Schwartz inequality

$$\begin{aligned} I^2 &\le \left\{ \sum_{\ell m} \frac{\kappa_{\ell m}^2(\varepsilon)}{C_\ell} \right\} \left\{ \sum_\ell \sum_m C_\ell \left[Y_{\ell m}(0,0) - \sum_{j=1}^n \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j) \right]^2 \right\} \\ &\le \left\{ \sum_\ell \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} \right\} \left\{ \sum_\ell C_\ell \sum_m \left[Y_{\ell m}(0,0) - \sum_{j=1}^n \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j) \right]^2 \right\}. \end{aligned}$$

This inequality can be rewritten as

$$\sum_\ell C_\ell \sum_m \left[Y_{\ell m}(0,0) - \sum_{j=1}^n \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j) \right]^2 \ge \frac{I^2}{\sum_\ell \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell}}. \tag{27}$$

On the other hand, we can compute I^2 directly. It follows from (22) and (23) that

$$\sum_\ell \sum_m \kappa_{\ell m}(\varepsilon) Y_{\ell m}(0,0) = \sum_\ell \frac{2\ell+1}{4\pi} b_\ell(\varepsilon) = \delta_\varepsilon(0,0) \ge \frac{c'}{2\varepsilon^2},$$

and

$$\begin{aligned} \sum_\ell \sum_m \kappa_{\ell m}(\varepsilon) \left\{ \sum_{j=1}^n \gamma_j Y_{\ell m}(\vartheta_j, \varphi_j) \right\} &= \sum_{j=1}^n \gamma_j \sum_\ell \sum_m \kappa_{\ell m}(\varepsilon) Y_{\ell m}(\vartheta_j, \varphi_j) \\ &= \sum_{j=1}^n \gamma_j \left\{ \sum_\ell \frac{2\ell+1}{4\pi} b_\ell(\varepsilon) P_\ell(\cos(N, x_j)) \right\} \\ &= \sum_{j=1}^n \gamma_j \delta_\varepsilon(\vartheta_j, \varphi_j) = 0, \end{aligned}$$

because $\vartheta_j > \varepsilon$ by assumption. The above two equations imply that $I \ge \frac{c'}{2}\varepsilon^{-2}$ and hence (26) will follow from (27) if we can show that

$$\sum_\ell \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} = O(\varepsilon^{-(\alpha+2)}). \tag{28}$$

Now we verify (28). It follows from (21) that for r large enough there exists a constant $c_r > 0$ such that

$$b_\ell^2(\varepsilon) \le \frac{c_r}{(\ell\varepsilon)^r}.$$

Hence, by choosing an integer $L = L(\varepsilon) = \lfloor \varepsilon \rfloor^{-1}$, we obtain

$$\begin{aligned} \sum_{\ell=1}^\infty \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} &= \sum_{\ell=L}^\infty \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} + \sum_{\ell=1}^L \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} \\ &\le \frac{c_0 c_r}{\varepsilon^{\alpha+2}} \sum_{\ell=L}^\infty (\ell\varepsilon) \frac{1}{(\ell\varepsilon)^r} (\varepsilon\ell)^\alpha \varepsilon + \sum_{\ell=1}^L \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell}. \end{aligned} \tag{29}$$

Now

$$\frac{c_0 c_r}{\varepsilon^{\alpha+2}} \sum_{\ell=L}^\infty (\ell\varepsilon) \frac{1}{(\ell\varepsilon)^r} (\varepsilon\ell)^\alpha \varepsilon \le \frac{c_r'}{\varepsilon^{\alpha+2}} \int_1^\infty x^{\alpha-r+1} dx \le \frac{c_r''}{\varepsilon^{\alpha+2}},$$

for $r > \alpha + 2$, whereas by Condition (A) we can bound the second term in (29) from above by

$$\sum_{\ell=1}^{L} \frac{(2\ell+1)}{4\pi} \frac{b_\ell^2(\varepsilon)}{C_\ell} \le c \sum_{\ell=1}^{L} \frac{(2\ell+1)}{4\pi} \ell^{\alpha} \le c\, L^{\alpha+2} \sim c\varepsilon^{-(\alpha+2)},$$

where c denotes a generic constant which needs not be the same from step to step. Combining (29) with the above verifies (28), which finishes the proof of (26). ■

Remark 8. At this stage we can draw an analogy between the isotropic spherical random fields satisfying Condition (A) with $2 < \alpha < 4$ and a fractional Brownian field with self-similarity parameter H. The analogy can be made clearer by setting the parameter values so that $2H + 2 = \alpha$, and Lemma 4 shows that the variogram of $T = \{T(x), x \in \mathbb{S}^2\}$ is of the order $d_{\mathbb{S}^2}(x, y)^{2H} = d_{\mathbb{S}^2}(x, y)^{\alpha-2}$. This indicates that T shares many analytic and fractal properties with a fractional Brownian field with parameter H. Indeed, by applying Lemma 4 and Theorem 1, we can prove that, for any $u \in \mathbb{R}$, the Hausdorff dimension of the level set $T^{-1}(u)$ is given by

$$\dim_{\mathrm{H}} T^{-1}(u) = 2 - \frac{\alpha - 2}{2}, \qquad \text{a.s.},$$

which shows that, for $2 < \alpha < 4$, $T^{-1}(u)$ is a fractal curve on $\mathbb{S}^2$ of Hausdorff dimension $\in (1, 2)$.

Notice that, $\dim_{\mathrm{H}} T^{-1}(u) = 1$ when $\alpha \ge 4$, but the nature of the level curve is different for $\alpha > 4$ and $\alpha = 4$, respectively. For $\alpha > 4$, the sample function $T(x)$ is differentiable, and thus its level curve $T^{-1}(u)$ is regular. On the other hand, for $\alpha = 4$ we believe that the level curve is not differentiable and possesses subtle fractal properties. Investigation of the topological and geometric properties of $T^{-1}(u)$ and more general excursion sets in more details is left for future research.

4. Modulus of continuity: Proof of Theorem 2

We start by state 0–1 laws regarding the uniform moduli of continuity for an isotropic spherical Gaussian field $T = \{T(x), x \in \mathbb{S}^2\}$. It is a consequence of the representation (1) and Kolmogorov's 0–1 law. We first rewrite Lemma 7.1.1 in Marcus and Rosen [16] as follows.

Lemma 9. *Let $\{T(x), x \in \mathbb{S}^2\}$ be a centred Gaussian random field on $\mathbb{S}^2$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\varphi(0+) = 0$. Then*

$$\lim_{\varepsilon \to 0} \sup_{\substack{x,y \in \mathbb{S}^2 \\ d_{\mathbb{S}^2}(x,y) \le \varepsilon}} \frac{|T(x) - T(y)|}{\varphi(d_{\mathbb{S}^2}(x, y))} \le K, \quad \textit{a.s. for some constant } \ K < \infty$$

implies that

$$\lim_{\varepsilon \to 0} \sup_{\substack{x,y \in \mathbb{S}^2 \\ d_{\mathbb{S}^2}(x,y) \le \varepsilon}} \frac{|T(x) - T(y)|}{\varphi(d_{\mathbb{S}^2}(x, y))} = K', \quad \textit{a.s. for some constant } \ K' < \infty.$$

We remark that Lemma 9 does not exclude the possibility of $K' = 0$. One of the main difficulties in establishing an exact uniform modulus of continuity is to find conditions under which $K' > 0$.

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Proof of Theorem 2. Because of Lemma 9, we see that (8) in Theorem 2 will be proved after we establish upper and lower bounds of the following form: If $2 < \alpha < 4$, then there exist positive and finite constants K_5 and K_6 such that

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{d_{\mathbb{S}^2}(x,y)^{(\alpha-2)/2}\sqrt{\left|\ln d_{\mathbb{S}^2}(x,y)\right|}} \le K_5, \quad \text{a.s.} \tag{30}$$

and

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{d_{\mathbb{S}^2}(x,y)^{(\alpha-2)/2}\sqrt{\left|\ln d_{\mathbb{S}^2}(x,y)\right|}} \ge K_6, \quad \text{a.s.} \tag{31}$$

These and Lemma 9 with $\varphi(r) = r^{(\alpha-2)/2}\sqrt{|\ln r|}$ imply (8) with $K_1 \in [K_6, K_5]$.

We divide the rest of the proof of Theorem 2 into three parts.

Step 1: *Proof of* (30). We introduce an auxiliary Gaussian field:

$$Y = \{Y(x, y),\ x, y \in \mathbb{S}^2, d_{S^2}(x, y) \le \varepsilon\}$$

defined by $Y(x, y) = T(x) - T(y)$, where $\varepsilon > 0$ is small so that (12) in Lemma 4 holds. By the triangle inequality, we see that the canonical metric d_Y on $\Gamma := \{(x, y) \in \mathbb{S}^2\times\mathbb{S}^2 : d_{\mathbb{S}^2}(x, y) \le \varepsilon\}$ associated with Y satisfies the following inequality:

$$d_Y((x, y), (x', y')) \le \min\{d_T(x, x') + d_T(y, y'),\ d_T(x, y) + d_T(x', y')\}. \tag{32}$$

Denote the diameter of Γ in the metric d_Y by D. Then, by (32) and Lemma 4, we have

$$D \le \sup_{(x,y)\in\Gamma} (d_T(x, y) + d_T(x', y')) \le 2\sqrt{c_1}\,\varepsilon^{(\alpha-2)/2}.$$

For any $\eta > 0$, let $N_Y(\Gamma, \eta)$ be the smallest number of open d_Y-balls of radius η needed to cover Γ. It follows from (32) and Lemma 4 that for $2 < \alpha < 4$,

$$N_Y(\Gamma, \eta) \le K_7\eta^{-\frac{4}{\alpha-2}},$$

for some positive and finite constant K_7, and one can verify that

$$\int_0^D \sqrt{\ln N_Y(T, \eta)}\, d\eta \le K\,\varepsilon^{(\alpha-2)/2}\sqrt{\ln(1+\varepsilon^{-1})}.$$

Hence, by Theorem 1.3.5 in [1], we have

$$\limsup_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{\varepsilon^{(\alpha-2)/2}\sqrt{|\ln\varepsilon|}} \le K, \quad \text{a.s.}$$

for some finite constant K. From here, it is standard to verify (cf. Lemma 7.1.6 in [16]) that this implies (30).

Step 2: *Proof of* (31). For any $n \ge \lfloor|\log_2\varepsilon_0|\rfloor + 1$, where ε_0 is as in Theorem 6, we chose a sequence of 2^n points $\{x_{n,i}, 1 \le i \le 2^n\}$ on $\mathbb{S}^2$ that are equally separated in the following sense: For every $2 \le k \le 2^n$, we have

$$\min_{1\le i\le k-1} d_{S^2}(x_{n,k}, x_{n,i}) = d_{S^2}(x_{n,k}, x_{n,k-1}) = 2^{-n}. \tag{33}$$

There are many ways to choose such a sequence on $\mathbb{S}^2$. Notice that

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in S^2,\\ d_{S^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{d_{S^2}(x,y)^{(\alpha-2)/2}\sqrt{|\ln d_{S^2}(x,y)|}}$$

$$\geq \liminf_{n\to\infty} \max_{2\leq k\leq 2^n} \frac{|T(x_{n,k}) - T(x_{n,k-1})|}{2^{-n(\alpha-2)/2}\sqrt{n}}. \tag{34}$$

It is sufficient to prove that, almost surely, the last limit in (34) is bounded below by a positive constant. This is done by applying the property of strong local nondeterminism in Theorem 6 and a standard Borel–Cantelli argument.

Let $\eta > 0$ be a constant whose value will be chosen later. We consider the events

$$A_m = \left\{ \max_{2\leq k\leq m} \left|T(x_{n,k}) - T(x_{n,k-1})\right| \leq \eta 2^{-n(\alpha-2)/2}\sqrt{n} \right\}$$

for $m = 2, 3, \ldots, 2^n$. By conditioning on A_{2^n-1} first, we can write

$$\begin{aligned} \mathbb{P}\big(A_{2^n}\big) &= \mathbb{P}\big(A_{2^n-1}\big) \\ &\quad \times \mathbb{P}\Big\{ |T(x_{n,2^n}) - T(x_{n,2^n-1})| \leq \eta 2^{-n(\alpha-2)/2}\sqrt{n} | A_{2^n-1} \Big\}. \end{aligned} \tag{35}$$

Recall that, given the random variables in A_{2^n-1}, the conditional distribution of the Gaussian random variable $T(x_{n,2^n}) - T(x_{n,2^n-1})$ is still Gaussian, with the corresponding conditional mean and variance as its mean and variance. By Theorem 6, the aforementioned conditional variance satisfies

$$\mathrm{Var}\big(T(x_{n,2^n}) - T(x_{n,2^n-1})\big|A_{2^n-1}\big) \geq c_2\, 2^{-(\alpha-2)n}.$$

This and Anderson's inequality (see [2]) imply

$$\begin{aligned} &\mathbb{P}\Big\{ |T(x_{n,2^n}) - T(x_{n,2^n-1})| \leq \eta 2^{-n(\alpha-2)/2}\sqrt{n} |\, A_{2^n-1} \Big\} \\ &\leq \mathbb{P}\Big\{ N(0,1) \leq c\,\eta\sqrt{n} \Big\} \\ &\leq 1 - \frac{1}{c\eta\sqrt{n}} \exp\Big(-\frac{c^2\eta^2 n}{2}\Big) \\ &\leq \exp\Big(-\frac{1}{c\eta\sqrt{n}} \exp\Big(-\frac{c^2\eta^2 n}{2}\Big)\Big). \end{aligned} \tag{36}$$

In deriving the last two inequalities, we have applied Mill's ratio and the elementary inequality $1 - x \leq e^{-x}$ for $x > 0$. Iterating this procedure in (35) and (36) for $2^n - 1$ more times, we obtain

$$\mathbb{P}\big(A_{2^n}\big) \leq \exp\Big(-\frac{1}{c\eta\sqrt{n}} 2^n \exp\Big(-\frac{c^2\eta^2 n}{2}\Big)\Big). \tag{37}$$

By taking $\eta > 0$ small enough such that $c^2\eta^2 \log_2 e < 2$, we have $\sum_{n=1}^{\infty} \mathbb{P}\big(A_{2^n}\big) < \infty$. Hence the Borel–Cantelli lemma implies that almost surely,

$$\max_{2\leq k\leq 2^n} \left|T(x_{n,k}) - T(x_{n,k-1})\right| \geq \eta 2^{-n(\alpha-2)/2}\sqrt{n}$$

for all n large enough. This implies that the right-hand side of (34) is bounded from below almost surely by $\eta > 0$.

Step 3: *Proof of* (9) for $\alpha = 4$. This is similar to the proof in Step 1, except that the diameter D of Γ in the metric d_Y is now comparable to $K\varepsilon\sqrt{|\ln\varepsilon|}$ and the covering number $N_Y(\Gamma, \eta) \leq K\eta^{-2}|\ln\eta|$. Hence, in this case,

$$\int_0^D \sqrt{\ln N_Y(T,\eta)}\, d\eta \leq K\,\varepsilon|\ln\varepsilon|.$$

Applying again Theorem 1.3.5 in [1] yields that for $\alpha = 4$,

$$\limsup_{\varepsilon\to 0} \sup_{\substack{x,y\in\mathbb{S}^2\\ d_{\mathbb{S}^2}(x,y)\le\varepsilon}} \frac{|T(x)-T(y)|}{\varepsilon|\ln\varepsilon|} \le K, \quad \text{a.s.}$$

Hence (9) follows from this and Lemma 7.1.6 in [16]. This finishes the proof of Theorem 2.

5. Higher-order derivatives: Proof of Theorem 3

Now we consider the case of $\alpha > 4$. Let $k \ge 1$ be the integer such that $2 + 2k < \alpha \le 4 + 2k$, and let $T^{(k)} = \{T^{(k)}(x), x \in \mathbb{S}^2\}$ be the Gaussian random field defined by $T^{(k)} = (1 - \Delta_{\mathbb{S}^2})^{k/2} T$. It follows from (10) that $T^{(k)}$ is again isotropic and its angular power spectrum is given by

$$\widetilde{C}_\ell = \mathbb{E}\big(|a_{\ell m}|^2\big)(1+\ell(\ell+1))^k = C_\ell(1+\ell(\ell+1))^k, \quad \ell = 1, 2, \ldots.$$

Under Condition (A), we have $\widetilde{C}_\ell = \widetilde{G}(\ell)\,\ell^{2k-\alpha}$ for all $\ell = 1, 2, \ldots$, where

$$c_6^{-1} \le \widetilde{G}(\ell) \le c_6$$

for some finite constant $c_6 \ge 1$. It follows from Theorem 1 that, for all $n \ge 1$ and all $x_0, x_1, \ldots, x_n \in \mathbb{S}^2$ such that $\min_{1\le i\le n} d_{\mathbb{S}^2}(x_0, x_i) \le \varepsilon_0$, we have

$$\operatorname{Var}\left(T^{(k)}(x_0)\,|T^{(k)}(x_1), \ldots, T^{(k)}(x_n)\right) \ge c_2 \min_{1\le i\le n} d_{\mathbb{S}^2}(x_0, x_i)^{(\alpha-2-2k)}.$$

Hence the conclusions of Theorem 3 follow from Theorem 2.

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Appendix

In this Appendix we collect a number of technical results which are mainly instrumental to investigate the behaviour of the canonical Gaussian metric at small angular distances, in terms of the spectral index α.

Let us first recall the Mehler-Dirichlet representation for the Legendre polynomials (see [17, eq. (13.9)] or [28, Section 5.3, eq. (2)]),

$$P_\ell(\cos\vartheta) = \frac{\sqrt{2}}{\pi}\int_0^\vartheta \frac{\cos\left(\left(\ell+\frac{1}{2}\right)\psi\right)}{(\cos\psi - \cos\vartheta)^{1/2}} d\psi, \tag{38}$$

where the integral on the right hand side for $\vartheta = 0$ is understood as the limit as $\vartheta \downarrow 0$.

In order to study the asymptotic behaviour of $\sum_{\ell=1}^\infty \ell^{-s} P_\ell(\cos\vartheta)$ as $\vartheta \to 0$, we will make use of the following identity: For any $s > 1$,

$$\sum_{\ell=1}^\infty \ell^{-s}\cos\left(\left(\ell+\frac{1}{2}\right)\psi\right) = \operatorname{Re}\left[\sum_{\ell=1}^\infty \ell^{-s} e^{i\left(\ell+\frac{1}{2}\right)\psi}\right] = \operatorname{Re}\left[e^{\frac{i}{2}\psi} Li_s\left(e^{i\psi}\right)\right], \tag{39}$$

where $Li_s(z)$, $z \in \mathbb{C}$ denotes the polylogarithm function defined as

$$Li_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

for $|z| < 1$, and then extended holomorphically to $|z| \geq 1$.

For $|z| \leq 1$, the polylogarithm function can also be viewed as a special example of the *Lerch function* $\Phi(z, s, v)$ at $v = 1$ and $\mathrm{Re}s > 1$, which is defined as

$$\Phi(z, s, v) = \sum_{k=0}^{\infty} \frac{z^k}{(k+v)^s}, \quad |z| < 1, v \neq 0, -1, \ldots$$

and can be extended to $|z| \leq 1$, $z \neq 1$, if $\mathrm{Re}s > 0$ or $z = 1$, if $\mathrm{Re}s > 1$ (see [10, eq. (9.550) and (9.556)]). By exploiting the properties of the Lerch function $\Phi(z, s, v)$, we have that for $s > 1$ and $s = n \in \mathbb{N}$, the following series expansion of $Li_n(z)$ holds (see [29, eq. (9.5)] or [10, eq. (9.554)])

$$Li_n(z) = \frac{(\ln z)^{n-1}}{(n-1)!}\left[H_{n-1} - \ln\left(\ln z^{-1}\right)\right] + \sum_{k=0, k\neq n-1}^{\infty} \frac{\zeta(n-k)}{k!}(\ln z)^k, \tag{40}$$

where $|\ln z| < 2\pi$, H_n denotes the n th harmonic number:

$$H_n = \sum_{j=1}^{n} \frac{1}{j}, \quad H_0 = 0,$$

and $\zeta(m)$ is the so called Riemann zeta function, which is well-defined and holomorphic on the whole complex plane everywhere except for $m = 1$.

For $s \notin \mathbb{N}$, by exploiting Eq. (9.556) in [10], Wood ([29, eq. (9.4)]) proved that for $z \in \mathbb{C}$ satisfying $|\ln z| < 2\pi$,

$$Li_s(z) = \Gamma(1-s)\left(\ln z^{-1}\right)^{s-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!}(\ln z)^k. \tag{41}$$

The following lemma has been used in Section 2 to characterize the small scale behaviours of the canonical metric of T and the function $Q_\alpha(\vartheta)$. The notation $g(\vartheta) = O(f(\vartheta))$ means $|g(\vartheta)/f(\vartheta)| \leq c$ for all $\vartheta \in [0, \pi]$ and $g(\vartheta) = o(f(\vartheta))$ means $g(\vartheta)/f(\vartheta) \to 0$ as $\vartheta \to 0$.

Lemma 10. *For any constant $s > 1$, as $\vartheta \to 0+$, we have*

$$\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) = \begin{cases} \zeta(s) - K_7 \sin^{s-1}\vartheta + o\left(\sin^{s-1}\vartheta\right), & \text{if } 1 < s < 3, \\ \zeta(s) - K_8 \sin^2\vartheta\, |\ln(\sin\vartheta)| + O\left(\sin^2\vartheta\right), & \text{if } s = 3, \\ \zeta(s) - K_9 \sin^2\vartheta + O\left(\sin^3\vartheta\right), & \text{if } s > 3, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta function, K_7, K_8, K_9 are positive constants depending only on s.

Proof. We consider the two cases $s \in \mathbb{N}$ and $s \notin \mathbb{N}$, respectively.

Case 1. For $s \notin \mathbb{N}$, it follows from the representation (41) that for $\vartheta > 0$ small enough, and all $\psi \in (0, \vartheta)$,

$$\begin{aligned} \mathrm{Re}\left[e^{\frac{i}{2}\psi} Li_s\left(e^{i\psi}\right)\right] &= \cos\left(\frac{\psi}{2}\right)\left[A_1 \psi^{s-1} + \zeta(s) - \frac{1}{2}\zeta(s-2)\psi^2\right] \\ &\quad + \sin\left(\frac{\psi}{2}\right)\left[B_1 \psi^{s-1} - \zeta(s-1)\psi + O\left(\psi^3\right)\right], \end{aligned}$$

where

$$A_1 = \Gamma(1-s)\cos\left(\frac{\pi}{2}(s-1)\right) \quad \text{and} \quad B_1 = \Gamma(1-s)\sin\left(\frac{\pi}{2}(s-1)\right)$$

and we have incorporated $O(\psi^4)$ into $O\left(\sin(\frac{\psi}{2})\psi^3\right)$. Then, by (38), (39) and (41) above, we have

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) \\
&= \frac{\sqrt{2}}{\pi}\int_0^{\vartheta} \frac{\cos\frac{\psi}{2}}{(\cos\psi-\cos\vartheta)^{1/2}}\left[A_1\psi^{s-1}+\zeta(s)-\frac{1}{2}\zeta(s-2)\psi^2\right]d\psi \\
&\quad + \frac{\sqrt{2}}{\pi}\int_0^{\vartheta} \frac{\sin\frac{\psi}{2}}{(\cos\psi-\cos\vartheta)^{1/2}}\left[B_1\psi^{s-1}-\zeta(s-1)\psi+O(\psi^3)\right]d\psi \\
&=: J_1+J_2.
\end{aligned} \tag{42}$$

Recall that

$$\cos\psi-\cos\vartheta = 2\sin^2\frac{\vartheta}{2}-2\sin^2\frac{\psi}{2}.$$

A change of variable $x=\sin(\frac{\psi}{2})/\sin(\frac{\vartheta}{2})$ shows that for $\gamma>0$,

$$\begin{aligned}
\int_0^{\vartheta}\frac{\sin^{\gamma-1}\frac{\psi}{2}\,\cos\frac{\psi}{2}}{(\cos\psi-\cos\vartheta)^{1/2}}d\psi &= \sqrt{2}\sin^{\gamma-1}\frac{\vartheta}{2}\int_0^1\frac{x^{\gamma-1}}{\sqrt{1-x^2}}dx \\
&= \frac{\sqrt{2}}{2}B\left(\frac{\gamma}{2},\frac{1}{2}\right)\sin^{\gamma-1}\frac{\vartheta}{2}
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\int_0^{\vartheta}\frac{\sin^{\gamma-1}\frac{\psi}{2}}{(\cos\psi-\cos\vartheta)^{1/2}}d\psi &= \frac{\sqrt{2}}{2}\sin^{\gamma-1}\frac{\vartheta}{2} \\
&\quad\times\left[B\left(\frac{\gamma}{2},\frac{1}{2}\right)+\frac{1}{6}B\left(\frac{\gamma}{2}+1,\frac{1}{2}\right)\sin^2\frac{\vartheta}{2}+O\left(\sin^4\frac{\vartheta}{2}\right)\right],
\end{aligned} \tag{44}$$

where $B(\cdot,\cdot)$ is the standard Beta function. By applying the following asymptotic expansion

$$\frac{\psi^\beta}{\sin^\beta\psi} = 1+\beta\frac{\sin^2\psi}{6}+O\left(\sin^4\psi\right), \qquad \text{if } \beta>0,$$

we can use (43) and (44) to derive

$$\begin{aligned}
J_1 &= A_2\sin^{s-1}\frac{\vartheta}{2}+\frac{1}{\pi}\zeta(s)\,B\left(\frac{1}{2},\frac{1}{2}\right) \\
&\quad -\frac{2}{\pi}\zeta(s-2)\,B\left(\frac{3}{2},\frac{1}{2}\right)\sin^2\frac{\vartheta}{2}+O\left(\sin^{s+1}\frac{\vartheta}{2}\right),
\end{aligned} \tag{45}$$

where A_2 is an explicit positive constant depending on s only. Likewise, we have

$$J_2 = B_2\sin^s\frac{\vartheta}{2}-\frac{2}{\pi}\zeta(s-1)\,B\left(\frac{3}{2},\frac{1}{2}\right)\sin^2\frac{\vartheta}{2}+O\left(\sin^{s+2}\frac{\vartheta}{2}\right), \tag{46}$$

where B_2 is an explicit positive constant depending on s only. By combining (42), (45) and (46), we derive that for $s > 1$ and $s \notin \mathbb{N}$,

$$\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) = \zeta(s) - C_1 \sin^{s-1}\frac{\vartheta}{2} - C_2 \sin^2\frac{\vartheta}{2} + C_3 \sin^s\frac{\vartheta}{2} + O\left(\sin^{(s+1)\wedge 4}\frac{\vartheta}{2}\right),$$

where C_1, C_2 and C_3 are positive constants depending only on s, and $a \wedge b = \min\{a, b\}$. Consequently,

$$\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) = \zeta(s) - C_1 \sin^{s-1}\frac{\vartheta}{2} + O\left(\sin^{s\wedge 2}\frac{\vartheta}{2}\right) \tag{47}$$

for $1 < s < 3,\ s \neq 2$, and

$$\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) = \zeta(s) - C_2 \sin^2\frac{\vartheta}{2} + O\left(\sin^{s\wedge 4}\frac{\vartheta}{2}\right), \tag{48}$$

for $s > 3,\ s \notin \mathbb{N}$.

Case 2. For $s > 1$ and $s = n \in \mathbb{N}$, by making use of the series expansion (40), it follows that

$$\mathrm{Re}\left[e^{\frac{i}{2}\psi} Li_n\left(e^{i\psi}\right)\right] = \mathrm{Re}\left[e^{\frac{i}{2}\psi} \frac{i^{n-1}\psi^{n-1}}{(n-1)!}\left(H_{n-1} - \ln\psi + \frac{\pi}{2}i\right)\right] + \mathrm{Re}\left[\sum_{k=0,k\neq n-1}^{n+1} \frac{\zeta(n-k)}{k!} i^k \psi^k\right] + O\left(\psi^{n+2}\right).$$

If n is an odd integer, then

$$\mathrm{Re}\left[e^{\frac{i}{2}\psi} Li_n\left(e^{i\psi}\right)\right] = (-1)^{(n-1)/2} \frac{\psi^{n-1}}{(n-1)!}\left[\left(H_{n-1} - \ln\psi\right)\cos\frac{\psi}{2} - \frac{\pi}{2}\sin\frac{\psi}{2}\right] + \sum_{k=0,k\neq(n-1)/2}^{(n+1)/2} \frac{\zeta(n-2k)}{k!}(-1)^k \psi^{2k} + O\left(\psi^{n+3}\right).$$

Thus, one can see that

$$\begin{aligned}\sum_{\ell=1}^{\infty} \ell^{-n} P_\ell(\cos\vartheta) &= \frac{\sqrt{2}}{\pi}\frac{(-1)^{(n-1)/2}}{(n-1)!} \\ &\times \int_0^{\vartheta} \frac{\psi^{n-1}}{(\cos\psi - \cos\vartheta)^{1/2}}\left[\left(H_{n-1} - \ln\psi\right)\cos\frac{\psi}{2} - \frac{\pi}{2}\sin\frac{\psi}{2}\right] d\psi \\ &+ \frac{\sqrt{2}}{\pi}\sum_{k=0,k\neq(n-1)/2}^{(n+1)/2} \frac{\zeta(n-2k)}{k!}(-1)^k \int_0^{\vartheta} \frac{\psi^{2k}}{(\cos\psi - \cos\vartheta)^{1/2}} d\psi \\ &+ O\left(\int_0^{\vartheta} \frac{\psi^{n+3}}{(\cos\psi - \cos\vartheta)^{1/2}} d\psi\right).\end{aligned} \tag{49}$$

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Observe that, in (49), the term corresponding to $k = 0$ goes to $\zeta(n)$ as $\vartheta \to 0+$, and the leading integral is

$$J_3 = \int_0^{\vartheta} \frac{\psi^{n-1}\ln\psi}{(\cos\psi - \cos\vartheta)^{1/2}} \cos\left(\frac{\psi}{2}\right) d\psi.$$

By a change of variable $y = \sin^2\frac{\psi}{2}/\sin^2\frac{\vartheta}{2}$, we can write J_3 as

$$\begin{aligned} J_3 =& \frac{2^{n-1}}{\sqrt{2}} \sin^{n-1}\frac{\vartheta}{2} \int_0^1 \frac{y^{\frac{n}{2}-1}\left(1+\sin^2\vartheta\,\frac{n-1}{6}y + O\left(\sin^4\vartheta\, y^2\right)\right)}{(1-y)^{1/2}} \\ &\times \left(\ln y + 2\left(\ln\sin\vartheta + \ln 2\right) + \frac{\sin^2\vartheta}{6} y + O\left(\sin^4\vartheta\, y^2\right)\right) dy. \end{aligned}$$

For $n \geq 3$, we derive

$$\begin{aligned} J_3 =& \frac{2^{n-1}}{\sqrt{2}}(1+2\ln 2) B_{\ln}\left(\frac{n}{2}, \frac{1}{2}\right) \sin^{n-1}\vartheta \\ &+ \frac{2^n}{\sqrt{2}} B\left(\frac{n}{2}, \frac{1}{2}\right) \sin^{n-1}\vartheta \cdot \left(\ln\sin\frac{\vartheta}{2}\right) + O\left(\sin^4\vartheta\right), \end{aligned} \tag{50}$$

where

$$B_{\ln}(a,b) = \int_0^1 \frac{x^{a-1}\ln x}{(1-x)^{1-b}} dx = -\int_0^1 B(y;a,b)\frac{1}{y} dy,$$

and $B(y;a,b)$ is the incomplete Beta function, defined as

$$B(y;a,b) = \int_0^y \frac{x^{a-1}}{(1-x)^{1-b}} dx.$$

By combining (49) and (50) we see that, if $s = n > 1$ is an odd integer, then

$$\begin{aligned} \sum_{\ell=1}^{\infty} \ell^{-n} P_\ell(\cos\vartheta) =& \zeta(n) - D_1 \sin^2\frac{\vartheta}{2} + \delta_n^3 D_2 \sin^2\frac{\vartheta}{2} \cdot \left(\ln\sin\frac{\vartheta}{2}\right) \\ &+ O\left(\sin^3\frac{\vartheta}{2}\right), \end{aligned} \tag{51}$$

where $\delta_i^j = 1$ if $i = j$ and 0 otherwise, D_1 and D_2 are positive constants depending on n only. Consequently, if $n > 1$ is an odd integer, then

$$\sum_{\ell=1}^{\infty} \ell^{-n} P_\ell(\cos\vartheta) = \begin{cases} \zeta(n) + D_2 \sin^2\frac{\vartheta}{2}\left(\ln\sin\frac{\vartheta}{2}\right) + O\left(\sin^2\frac{\vartheta}{2}\right), & \text{if } n = 3, \\ \zeta(n) - D_1\sin^2\frac{\vartheta}{2} + O\left(\sin^3\frac{\vartheta}{2}\right), & \text{if } n \geq 5. \end{cases} \tag{52}$$

Finally, we consider the case when $s = n > 1$ is an even integer. It follows from (40) that

$$\begin{aligned} \mathrm{Re}\left[e^{\frac{i}{2}\psi} Li_n\left(e^{i\psi}\right)\right] =& (-1)^{n/2}\frac{\psi^{n-1}}{(n-1)!}\left[\left(H_{n-1} - \ln\psi\right)\sin\frac{\psi}{2} + \frac{\pi}{2}\cos\frac{\psi}{2}\right] \\ &+ \sum_{k=0}^{n/2+1} \frac{\zeta(n-2k)}{k!}(-1)^k \psi^{2k} + O\left(\psi^{n+4}\right), \end{aligned}$$

which leads to

$$\sum_{\ell=1}^{\infty} \ell^{-n} P_\ell(\cos\vartheta) = \frac{\sqrt{2}}{\pi}\frac{(-1)^{n/2}}{(n-1)!} \times \int_0^\vartheta \frac{\psi^{n-1}}{(\cos\psi-\cos\vartheta)^{1/2}}\left[\left(H_{n-1}-\ln\psi\right)\sin\frac{\psi}{2}-\frac{\pi}{2}\cos\psi\right]d\psi + \frac{\sqrt{2}}{\pi}\sum_{k=0}^{n/2+1}\frac{\zeta(n-2k)}{k!}(-1)^k\int_0^\vartheta \frac{\psi^{2k}}{(\cos\psi-\cos\vartheta)^{1/2}}d\psi + O\left(\int_0^\vartheta \frac{\psi^{n+4}}{(\cos\psi-\cos\vartheta)^{1/2}}d\psi\right).$$

Similarly to the case when s is odd, we can derive that for $s = n$ even,

$$\sum_{\ell=1}^{\infty} \ell^{-n} P_\ell(\cos\vartheta) = \zeta(n) - \delta_n^2\left\{B\left(1,\frac{1}{2}\right)\sin\frac{\vartheta}{2} - D_3\sin^2\frac{\vartheta}{2}\cdot\left(\ln\sin\frac{\vartheta}{2}\right)\right\} - D_4\sin^2\frac{\vartheta}{2} + O\left(\sin^3\frac{\vartheta}{2}\right), \tag{53}$$

where D_3 and D_4 are positive constants depending on s only. That is, for even integer $s > 1$, we have

$$\sum_{\ell=1}^{\infty} \ell^{-s} P_\ell(\cos\vartheta) = \begin{cases} \zeta(s) - 2\sin\frac{\vartheta}{2} + o\left(\sin\frac{\vartheta}{2}\right), & \text{if } s=2, \\ \zeta(s) + D_2\sin^2\frac{\vartheta}{2} + O\left(\sin^3\frac{\vartheta}{2}\right), & \text{if } s \geq 4. \end{cases} \tag{54}$$

This completes the proof of Lemma 10 in view of (47), (48), (52) and (54). ■

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