



Normal approximations for discrete-time occupancy processes

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Abstract

We study normal approximations for a class of discrete-time occupancy processes, namely, Markov chains with transition kernels of product Bernoulli form. This class encompasses numerous models which appear in the complex networks literature, including stochastic patch occupancy models in ecology, network models in epidemiology, and a variety of dynamic random graph models. Bounds on the rate of convergence for a central limit theorem are obtained using Stein's method and moment inequalities on the deviation from an analogous deterministic model. As a consequence, our work also implies a uniform law of large numbers for a subclass of these processes.

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1. Introduction

Treating a complex system as a large collection of interacting entities has become a standard modelling paradigm [8,22,33,34]. Growing interest in binary interacting particle systems and agent-based modelling, where entities are treated as nodes with a binary state, has led to the development of general and highly detailed discrete-time models of use in a

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wide variety of fields. Ecologists have been captivated by the capacity of *stochastic patch occupancy models* (SPOMs) to help explain the influence of spatial heterogeneity on population dynamics [24,25,36]. *Probabilistic cellular automata* (PCA) have enjoyed similar popularity in statistical mechanics [16,48,51], and *network models* and related *random graph models* have seen numerous applications in epidemiology, and social and computer science [6,7,14,17].

Encompassing many of these models is a class of processes, called here *occupancy processes*, whose state records the occupancy at each of n nodes, and whose transitions at the various nodes are independent conditional on the state. More precisely, an occupancy process is a discrete-time Markov chain $X_t = (X_{1,t}, \dots, X_{n,t})$, $t = 0, 1, \dots$, taking values in $\{0, 1\}^n$ (1 denoting occupancy) such that, given $X_t = \mathbf{x}$, $X_{1,t+1}, \dots, X_{n,t+1}$ are independent. Under this assumption, the transition probabilities of X_t are given in terms of functions $P_{i,t} : \{0, 1\}^n \rightarrow [0, 1]$ ($i = 1, \dots, n$; $t = 0, 1, \dots$) given by

$$\mathbb{P}(X_{i,t+1} = 1 \mid X_t = \mathbf{x}) = P_{i,t}(\mathbf{x}).$$

This includes all finite PCA and all SPOMs, as well as any network model where individuals behave independently in the short-term. In the random graph framework, the nodes become the edges of the graph, with occupancies dictating the corresponding adjacency matrix. Here, the occupancy process subset of dynamic random graphs are those which evolve at each time-step according to a set of edge-independent rules, which can depend (in an arbitrary way) on the state of the graph at the previous time-step.

Borrowing from the terminology of cellular automata [51], we refer to the collection of functions $P_t = (P_{i,t})_{i=1}^n$, $t = 0, 1, \dots$, as the *global rule* of X_t , and to each $P_{i,t}$ as the *local rule* of $X_{i,t}$. It is convenient to write each local rule in terms of functions $S_{i,t}, C_{i,t} : \{0, 1\}^n \rightarrow [0, 1]$, called the *survival* and *colonisation functions* respectively, satisfying

$$P_{i,t}(\mathbf{x}) = x_i S_{i,t}(\mathbf{x}) + (1 - x_i) C_{i,t}(\mathbf{x}), \quad \mathbf{x} \in \{0, 1\}^n. \quad (1.1)$$

The increased precision afforded by an occupancy process often comes at the expense of tractability. Models of practical interest are not usually amenable to traditional finite-state Markov chain analysis, for the state space is often prohibitively large. Even the efficient simulation of very large systems of this kind presents an ongoing challenge [10]. Instead, it is common to rely on approximations, the global rule suggesting a natural deterministic model for the evolution of occupancy probabilities (see for example [44]). Assuming the domain of P_t is extended to the interior of the hypercube $[0, 1]^n$, we may define $\mathbf{p}_t = (p_{1,t}, \dots, p_{n,t})$ by

$$p_{i,t+1} = P_{i,t}(\mathbf{p}_t), \quad i = 1, \dots, n. \quad (1.2)$$

One clear advantage of working with (1.2) over its corresponding occupancy process is that the long-term dynamics are easier to elucidate, especially in the time-homogeneous case (this is discussed in greater detail in Section 7). On the other hand, (1.2) captures none of the variability present in the original system, limiting its applicability as a predictive model. To address this, we shall instead consider a distributional approximation, analysing the fluctuations of X_t about its deterministic approximation.

Now, it is unclear *a priori* whether the dynamics of (1.2) and its corresponding occupancy process X_t are indeed similar, or in what sense; the extension of the global rule is not unique, $\mathbb{P}(X_{i,t} = 1)$ rarely equates with $p_{i,t}$, and, in terms of total variation at least, \mathbf{p}_t is never a good approximation of X_t (see §2.3 of [4]). Instead, it is traditional to develop limit theorems of particle systems in the weak topology, and a common approach is to embed the process X_t in a sequence of empirical random measures over some parameter space [2,15,39]. For example, if

each node i is assigned a location $z_i \in \Omega \subset \mathbb{R}^d$, the occupancy of every node may be naturally encoded through a single point process on Ω . Mechanically, under suitable assumptions, it is common to show that for large n and any $t \geq 0$, the integrals of measures μ_t and π_t given by

$$\int h(z)\mu_t(dz) = \frac{1}{n} \sum_{i=1}^n h(z_i)X_{i,t}, \quad \text{and} \quad \int h(z)\pi_t(dz) = \frac{1}{n} \sum_{i=1}^n h(z_i)p_{i,t}, \quad (1.3)$$

are close, for any bounded, continuous function $h : \Omega \rightarrow \mathbb{R}$; see [39], for example. This approach is commonly taken in the study of mean field approximations [18,19,32], and establishes that macroscopic properties of the occupancy process (such as the total proportion of occupied nodes, by taking $h \equiv 1$) are well-represented by those of the deterministic approximation when the number of nodes n is large. Furthermore, in light of the conditional independence feature of the occupancy process, it would seem reasonable to expect that fluctuations are approximately normal, and so the integrals $n^{1/2} \int h(z)(\mu_t - \pi_t)(dz)$ should converge to some normal random variable as the size of the system increases.

There are a few challenges with this approach at full generality. To establish a limit theorem, one must show that π_t converges weakly as $n \rightarrow \infty$, which often requires additional structural assumptions. Furthermore, while convergence of the integrals (1.3) is sufficient to infer weak convergence of measures [29, Theorem 16.16], this is not true for establishing a central limit theorem, which instead requires an embedding of the random measure into the dual of a certain Sobolev space [2]. Consequently, these limit theorems frequently yield convergence to a computationally intractable limiting object, which is itself often approximated by the finite case anyway.

Instead, we choose to take an abstract, non-asymptotic approach, and directly compare the weighted sums

$$\frac{1}{n} \sum_{i=1}^n h_i X_{i,t} \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n h_i p_{i,t}, \quad (1.4)$$

for arbitrary $h \in \mathbb{R}^n$ and $t \geq 0$, with a fixed number of nodes n . These are equivalent to the integrals (1.3), without appealing to specific parameterisation. Using the method of bounded differences and Stein's method, our primary objective is to analyse the discrepancy between fluctuations $n^{-1/2} \sum_{i=1}^n h_i (X_{i,t} - p_{i,t})$, and a normal random variable that is the realisation of an autoregressive Gaussian process at time t . By involving only assumptions on the derivatives of the global rule, this affords a clearer picture of the subclass of occupancy processes exhibiting approximate Gaussian tendencies. Stein's method is known to be remarkably flexible for handling sums of random variables with complex dependencies, and is well-suited for our purposes. See [12] for a comprehensive exposition of normal approximation techniques involving Stein's method. The application of Stein's method to the one-point distributions of a discrete-time process was considered by Goldstein [20] to develop normal approximations for *hierarchical structures*. While the contraction principle used in his analysis does not apply here, the one-step linearisation approach in our analysis is of a similar flavour. Other applications of Stein's method for mean field approximations can be seen in [52,53]. We remark that our setting differs in so far that the global rule need not be time-homogeneous, nor strictly dependent on any particular weighted sum (1.4) of the occupancy states, although we do restrict ourselves to binary state spaces.

Although our approximations focus on short time scales, by imposing additional assumptions on the rate of convergence of the Gaussian process, we are able to develop normal approximations for the long-term behaviour of certain occupancy processes as well. In this regard,

our approach is conceptually simpler, sidestepping the usual Sobolev embedding techniques for establishing closeness of normal approximations. Nevertheless, in [Example 2](#), we shall see how our abstract approach can be reframed in the random measure approach to obtain more familiar limit theorems when sufficient structural assumptions are imposed.

1.1. Paper outline

The remainder of this document is structured as follows: first, we present our main results in [Section 2](#). Then, in [Section 3](#), we discuss implications of our results in the context of spreading processes from epidemiology ([Example 1](#)), Hanski's incidence function model from population ecology ([Example 2](#)), and dynamic random graph models ([Example 3](#)). Furthermore, [Example 2](#) demonstrates the implications of [Corollary 2.5](#) on the convergence of empirical random measures associated with an occupancy process, and [Example 3](#), in [Proposition 3.6](#), shows how our main results, in conjunction with the approximation (2.2) and [Proposition 5.5](#), lead to normal approximations for non-linear functions of an occupancy process. Technical estimates derived from the method of bounded differences are summarised and derived in [Section 4](#). In [Section 5](#), as a critical precursor to [Theorem 2.1](#), the quality of the deterministic approximation (1.2) is considered, culminating in a general uniform law of large numbers ([Corollary 2.5](#)), which we consider interesting in its own right. Next, in [Section 6](#), we outline the proofs of our main results ([Theorem 2.1](#) and [Corollary 2.2](#)) using Stein's method and the estimates from [Section 5](#). Finally, the long-term behaviours of an occupancy process and its approximations are considered in [Section 7](#).

2. Main results

For the extension of the global rule $P_t : \{0, 1\}^n \rightarrow [0, 1]^n$ to the domain $[0, 1]^n$, we assume (I) $P_t \in \mathcal{C}^3$, providing some regularity to the approximation, and so that (1.1) holds with $S_{i,t}$, $C_{i,t}$ independent of their i th argument, (II) $\partial_i^2 P_{i,t}(x) = 0$ for each $i = 1, \dots, n$, where ∂_i is the partial derivative with respect to the i th component. In the sequel, we let $Df = (\partial_j f)_{ij}$ be the Jacobian matrix of a vector-valued differentiable function f and let $D^{(2)}f = (\partial_j^2 f)_{ij}$ be the corresponding matrix of second derivatives. We also assume throughout that X_0 is fixed and $\mathbf{p}_0 = X_0$, although the case where the $X_{i,0}$ are independent random variables with $\mathbb{P}(X_{i,0} = 1) = p_{i,0}$ can be treated by taking $P_{i,0}(\mathbf{x}) = p_{i,0}$ and starting the process from time $t = 1$ instead. Additional assumptions would be required to treat dependent X_0 .

Let $v_{i,t}(\mathbf{x}) = C_{i,t}(\mathbf{x})[1 - C_{i,t}(\mathbf{x})](1 - x_i) + S_{i,t}(\mathbf{x})[1 - S_{i,t}(\mathbf{x})]x_i$ for each $i = 1, \dots, n$, and define the autoregressive Gaussian process $\mathbf{Z}_t = (Z_{1,t}, \dots, Z_{n,t})$ by the recursion

$$Z_{i,t} = p_{i,t} + \sum_{j=1}^n \partial_j P_{i,t}(\mathbf{p}_{t-1})(Z_{j,t-1} - p_{j,t-1}) + z_{i,t} \sqrt{v_{i,t}(\mathbf{p}_{t-1})}, \quad (2.1)$$

where each $z_{i,t}$ is an independent standard normal random variable. The process (2.1) can be considered a discrete-time analogue to the linear noise approximation for diffusion processes treated in [\[31\]](#), and shall serve as the Gaussian process approximation to its corresponding occupancy process. Letting $\zeta_t = n^{-1/2}(\mathbf{X}_t - \mathbf{p}_t)$ and $\xi_t = n^{-1/2}(\mathbf{Z}_t - \mathbf{p}_t)$ denote the normalised fluctuations of \mathbf{X}_t and \mathbf{Z}_t about \mathbf{p}_t , we show under a few additional assumptions that, for large n , denoting $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$, the projections $\langle \zeta_t, h \rangle$ and $\langle \xi_t, h \rangle$ over $h \in \mathbb{R}^n$, are close in law. For a distributional approximation of $f(\mathbf{X}_t)$ where $f \in \mathcal{C}^3([0, 1]^n)$ is arbitrary, we appeal to the linear approximation

$$f(\mathbf{X}_t) \approx f(\mathbf{p}_t) + \sqrt{n} \langle \zeta_t, \nabla f(\mathbf{p}_t) \rangle. \quad (2.2)$$

An L^1 estimate for the error in (2.2) in terms of the derivatives of f is provided in Proposition 5.5.

The cross-covariances of $\langle \xi_t, h \rangle$ are established by polarisation: for each $0 \leq s \leq t$, letting $D_{s,t} = D_s \cdots D_{t-1}$ with $D_t = DP_t(\mathbf{p}_t)^\top$ (interpreting the empty product as unity),

$$\text{Cov}[\langle \xi_s, h \rangle, \langle \xi_t, h' \rangle] = \sum_{r=1}^{s \wedge t} \sigma_r[D_{r,s}h, D_{r,t}h'],$$

where σ_t is the symmetric bilinear form defined for $h, h' \in \mathbb{R}^n$ by

$$\sigma_t[h, h'] := \frac{1}{n} \sum_{i=1}^n h_i h'_i v_{i,t}(\mathbf{p}_{t-1}),$$

for $t \geq 1$ and $\sigma_t \equiv 0$ for $t = 0$. The quadratic case $\sigma_t[h, h] =: \sigma_t^2[h]$ provides the approximate variance introduced in the t th step of the occupancy process; indeed $\text{Var}\langle \xi_t, h \rangle = \sigma_t^2[h] + \text{Var}\langle \xi_{t-1}, D_{t-1}h \rangle$ for each $t \geq 1$.

For $1 \leq q \leq \infty$ and a vector $h \in \mathbb{R}^n$, let $\|h\|_q$ denote the traditional ℓ^q norm; for a matrix A , $\|A\|_q$ the induced ℓ^q matrix norm. For a random variable X , $\|X\|_q = (\mathbb{E}|X|^q)^{1/q}$ is the L^q norm, and similarly, for any function f on \mathbb{R}^d , $\|f\|_q$ is the L^q norm under Lebesgue measure. We exploit the celebrated method of Stein [12] to bound the difference in law between $\langle \zeta_t, h \rangle$ and $\langle \xi_t, h \rangle$ under the L^q metric (defined for random variables X and Y by $\|\mathcal{L}(X) - \mathcal{L}(Y)\|_q$, where $\mathcal{L}(X)$ denotes the distribution function (or law) of X). As a consequence of Hölder's inequality and integration by parts, if $q^{-1} + r^{-1} = 1$, then

$$\begin{aligned} \|\mathcal{L}(X) - \mathcal{L}(Y)\|_q &= \left(\int_{\mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|^q dx \right)^{1/q} \\ &= \sup\{|\mathbb{E}g(X) - \mathbb{E}g(Y)| : \|g'\|_r \leq 1\}, \end{aligned} \quad (2.3)$$

where the derivative g' is understood in the absolutely continuous sense. For particular choices of q this reduces to other commonly used metrics; for example, the case $q = 1$ coincides with the Wasserstein metric: by the Kantorovich–Rubinstein formula [30]

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_1 = \inf\{\mathbb{E}|X' - Y'| : (X', Y') \text{ is a coupling of } X, Y\},$$

In contrast, the case $q = \infty$,

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_\infty = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|$$

is the Kolmogorov metric. Denoting by $\|\cdot\|_\infty$ the supremum norm, define, for each $t = 0, 1, \dots$, quantities

$$\begin{aligned} \alpha_t &= \max_{j=1, \dots, n} \sum_{\substack{i=1 \\ i \neq j}}^n \|\partial_j P_{i,t}\|_\infty, & \beta_t^2 &= \frac{1}{n} \sum_{\substack{i,j=1 \\ i \neq j}}^n \|\partial_j P_{i,t}\|_\infty^2, \\ \Gamma_t &= \max_{j,k=1, \dots, n} \sum_{i=1}^n \|\partial_j \partial_k P_{i,t}\|_\infty, & \gamma_t &= \frac{1}{n} \sum_{i,j=1}^n \|\partial_j^2 P_{i,t}\|_\infty, \\ \delta_t &= \max_{j=1, \dots, n} \sum_{i,k=1}^n \|\partial_j \partial_k^2 P_{i,t}\|_\infty, & \alpha_{s,t} &= \sum_{r=s+1}^{t-1} \alpha_r, \end{aligned}$$

and let $\psi_t = \beta_t + \gamma_t$. The quantities α_t and β_t are not altogether unusual, as the speed of the system and the dependence between nodes is well quantified in the derivatives of each local rule. For systems whose general dynamics strongly depend upon the state of a single node, α_t will be large, and so there is little hope in expecting the deterministic process (1.2) to be representative. This is the case for the mainland-island metapopulation model considered in [38], which is instead well-approximated by a *semi-deterministic* system. Moreover, γ_t , Γ_t , and δ_t provide some essential measures of the regularity of the global rule.

Our main approximation result is encapsulated in [Theorem 2.1](#).

Theorem 2.1. *There exists a universal constant $0 < C < 700$ such that, for any $h \in \mathbb{R}^n$, any integer $t \geq 1$ and $1 \leq q \leq \infty$,*

$$\|\mathcal{L}\langle \zeta_t, h \rangle - \mathcal{L}\langle \xi_t, h \rangle\|_q \leq C \|h\|_\infty^{4-1/q} \sqrt{\frac{1 + \log n}{n}} \sum_{s=0}^{t-1} \frac{\kappa_s \cdot e^{(4-1/q)\alpha_{s,t}}}{\sigma_{s+1}^{4-2/q} [D_{s+1,t} h]}, \quad (2.4)$$

where, for every $t = 0, 1, \dots$,

$$\kappa_t = (1 + \alpha_t + n\Gamma_t + n^{1/2}\delta_t) \sum_{s=0}^t (1 + n\psi_s^2) t e^{16\alpha_{s,t}}.$$

The logarithmic factor in (2.4) is an unfortunate artifact of our proof. For many special cases, including certain mean-field models, the order may be improved to $\mathcal{O}(n^{-1/2})$ with minimal effort (see [Remark 6.3](#)). An obvious example is the case where each node transitions without interactions (so that $\psi_t = 0$), as a consequence of the Berry–Esseen bound [12, Theorem 3.6, Corollary 4.2]. We expect this to hold in complete generality. Furthermore, as encapsulated in the results of [23], the $\mathcal{O}(n^{-1/2})$ rate of convergence is expected to be optimal. On the other hand, it is likely that the dependence on time, as well as the upper bound on the constant C in [Theorem 2.1](#) can be improved. This is especially true for occupancy processes in the stationary regime.

As an important corollary, we provide a general central limit result for a subclass of approximable occupancy processes. Consider a sequence of occupancy processes $\{X_t^{(n)}\}_{n=1}^\infty$, $t = 0, 1, \dots$, with corresponding global rules $\{P_t^{(n)}\}_{n=1}^\infty$ with sequences $\{\alpha_t^{(n)}, \psi_t^{(n)}, \Gamma_t^{(n)}, \delta_t^{(n)}\}_{n=1}^\infty$ for each $t \geq 0$, and one-step variances $\{\sigma_{n,t}^2[h]\}_{n=1}^\infty$. Extending $\langle \zeta_t, h \rangle$ to $h \in \ell^\infty$ in the obvious way, we obtain

Corollary 2.2. *Let $t \geq 1$, and suppose that $\sup_n \alpha_s^{(n)} < \infty$, $\sup_n \delta_s^{(n)} < \infty$, $\beta_s^{(n)} = \mathcal{O}(n^{-1/2})$, and $\Gamma_s^{(n)} = \mathcal{O}(n^{-1})$ as $n \rightarrow \infty$, for each $s \leq t$. Suppose also that, for each $s \leq t$, there are continuous functions $\sigma_s^2 : \ell^\infty \rightarrow [0, \infty)$ and $\mathcal{J}_s : \ell^\infty \rightarrow \ell^\infty$ such that $\sigma_{n,s}^2[h] \rightarrow \sigma_s^2[h]$ as $n \rightarrow \infty$ and*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n h_i \partial_j P_{i,s}^{(n)}(\mathbf{p}_s) = (\mathcal{J}_s h)_j, \quad (2.5)$$

for every $j = 1, 2, \dots$. Furthermore, let $\mathcal{V}_t : \ell^\infty \times \ell^\infty \rightarrow [0, \infty)$ be defined by

$$\begin{aligned} \mathcal{V}_0[h_1, h_2] &= 0, \\ \mathcal{V}_{t+1}[h_1, h_2] &= \sigma_{t+1}[h_1, h_2] + \mathcal{V}_t[\mathcal{J}_t h_1, \mathcal{J}_t h_2], \quad t \geq 0. \end{aligned}$$

Then, for any $h_1, \dots, h_m \in \ell^\infty$ and $t_1, \dots, t_m \geq 0$,

$$(\langle \zeta_{t_1}^{(n)}, h_1 \rangle, \dots, \langle \zeta_{t_m}^{(n)}, h_m \rangle) \xrightarrow{\mathcal{D}} \mathcal{N}(0, (\Sigma_{t_i, t_j}[h_i, h_j])_{i,j=1}^m), \quad (2.6)$$

where the elements of the (symmetric) covariance matrix satisfy

$$\Sigma_{s,u}[h_s, h_u] = \mathcal{V}_s[h_s, \mathcal{J}_s \circ \cdots \circ \mathcal{J}_{u-1} h_u], \quad s \geq u.$$

Unfortunately, our results do not imply a bound on the rate of convergence in (2.6). It is possible that this could be achieved by considering Stein's method for multivariate normal approximation [41], obtaining a bound on the distance to a multivariate normal for ζ_t taken at different time points.

Classical examples of models satisfying the assumptions of Corollary 2.2 include those which involve any form of mean-field assumption. In fact, for homogeneous systems, if α_t is bounded away from zero, then to satisfy the assumptions of Corollary 2.2, each local rule must depend on a non-zero proportion of the whole system. Of course, there are many notable types of occupancy processes in the literature that do not satisfy these assumptions. In particular, this effectively rules out the majority of integrable probabilistic cellular automata (which are characterised by strict locality [16]) from our analysis. However, we do not expect even the deterministic process to be representative in these cases. As an example, consider the Domany–Kinzel probabilistic cellular automata (PCA) on the discrete torus of length n :

$$P_i(\mathbf{x}) = (q_2 - q_1)x_i x_{i+1} + q_1(1 - x_i)x_{i+1}, \quad q_1, q_2 \in [0, 1],$$

where $x_{n+1} := x_1$. The application of the deterministic approximation (1.2) (called the *one-site mean-field approximation* in the relevant literature) to this system suggests the existence of a phase transition at $q_1 = \frac{1}{2}$. On the other hand, a two-site approximation (see [48, §15.2]) suggests the location of the phase transition along q_1 varies according to q_2 — at $q_2 = 0$, the transition occurs at $q_1 = \frac{2}{3}$. Indeed, the behaviour of the Domany–Kinzel PCA has been determined quite well via numerics; at $q_2 = 0$, the phase transition has been estimated at $q_1 \approx 0.8$ to at least three decimal precision [54]. The approximation does not fare better in the short term either. If the $X_{i,0}$ are independent and identically distributed with mean p_0 , \mathbf{X}_t is an exchangeable random vector for each $t = 1, 2, \dots$. In this case, one can verify by direct calculation that, for any n ,

$$\mathbb{E}(\zeta_2, h) = n^{1/2} \bar{h} (q_2 - 2q_1)^2 p_0^2 (1 - p_0) (q_1 + q_2 - 2p_0 q_1)$$

(where \bar{h} is the arithmetic mean of $h \in \mathbb{R}^n$), which will often diverge as $n \rightarrow \infty$.

While the quality of the normal approximation in the short-term has been established for a large class of occupancy processes, it is the approximation of the *long-term behaviour* of the process that is often more useful in the population sciences. For this we require time-homogeneity, and, for each n , the deterministic process converges to some fixed point as $t \rightarrow \infty$ (a set of convenient monotonicity conditions which implies this is provided in Theorem 7.1). Under readily verifiable conditions on the global rule, a time-homogeneous occupancy process centred about its deterministic approximation, converges in $\mathcal{O}(\log n)$ time to an approximately normal equilibrium (Corollary 2.3).

Corollary 2.3. Assume the conditions of Corollary 2.2 hold, and that $\limsup_n \|DP^{(n)}(\mathbf{p}_\infty^{(n)})\|_1 < 1$, and $\sup_n \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. Then there exists a constant $c > 0$ independent of n such that for any sequence $\tau_n \leq c \log n$ with $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$,

$$\langle \zeta_{\tau_n}^{(n)}, h \rangle \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathcal{V}_\infty^2[h]),$$

where $\mathcal{V}_\infty^2[h] = \lim_{t \rightarrow \infty} \mathcal{V}_t[h, h]$.

Unfortunately, we have not been able to prove a central limit theorem for the process centred about the deterministic equilibrium for arbitrary initial values of the process under these conditions. Such a result would require much stronger assumptions on the rate of convergence of the deterministic approximation to equilibrium than we impose. In particular, it would suffice to assume that $n^{-1/2} \sum_{i=1}^n h_i(p_{i,t}^{(n)} - p_{i,\infty}^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$ for any $h \in \ell^\infty$.

Our final abstract results concern the quality of the deterministic approximation (1.2) itself. The following is a consequence of Theorem 5.1.

Corollary 2.4. *There exists a universal constant $0 < C \leq 12\sqrt{\pi}$ such that for any $h \in \mathbb{R}^n$, any integer $t \geq 1$ and $1 \leq q \leq \infty$,*

$$\left\| \frac{1}{n} \sum_{i=1}^n h_i(X_{i,t} - p_{i,t}) \right\|_q \leq Cq^{3/2} \|h\|_\infty \sum_{s=0}^{t-1} (n^{-1/2} + \psi_s) e^{4q\alpha_{s,t}}.$$

A further consequence is a uniform law of large numbers for occupancy processes under weaker conditions than Corollary 2.2. Due to its versatility, we choose to express our result in terms of the Rademacher complexity, defined for $\mathcal{H} \subset \mathbb{R}^n$ by

$$\text{Rad}(\mathcal{H}) = \mathbb{E} \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n h_i \sigma_i,$$

where $\sigma_1, \dots, \sigma_n$ are independent Rademacher random variables [$\mathbb{P}(\sigma_i = 1) = \mathbb{P}(\sigma_i = -1) = \frac{1}{2}$]. Rademacher complexity relates to other measures of the size of \mathcal{H} as follows. Let $N_r(\mathcal{H})$ denote the r -covering number of \mathcal{H} , that is, the smallest number of balls of radius r whose union contains \mathcal{H} . Then [46, Lemma 27.4], there exist universal constants $c_1, c_2 > 0$ such that

$$\text{Rad}(\mathcal{H}) \leq \frac{c_1}{\sqrt{n}} \int_0^H \sqrt{\log N_r(\mathcal{H})} dr \leq c_2 H \sqrt{\frac{\log |\mathcal{H}|}{n}}, \quad (2.7)$$

where $H = \sup_{h \in \mathcal{H}} \|h\|_\infty < \infty$. Furthermore, if \mathcal{H} is a set of binary vectors, and writing $\mathcal{V}(\mathcal{H})$ as the Vapnik–Chervonenkis dimension of \mathcal{H} as a set of functions from $\{1, \dots, n\}$ into $\{0, 1\}$ (see [46, Definition 6.5]), then [27, Theorem 1] there exists $c_3 > 0$ such that

$$\text{Rad}(\mathcal{H}) \leq c_3 \sqrt{\frac{\mathcal{V}(\mathcal{H})}{n}}.$$

For more details on the Rademacher complexity, we direct the reader to [46, §26,27]. Now, consider a sequence of occupancy processes $\{X_t^{(n)}\}_{n=1}^\infty$ indexed by number of nodes with corresponding global rules $\{P_t^{(n)}\}_{n=1}^\infty$, and sequences $\{\alpha_t^{(n)}\}_{n=1}^\infty$ and $\{\psi_t^{(n)}\}_{n=1}^\infty$ for each integer $t \geq 0$.

Corollary 2.5. *Consider a sequence of subsets $\mathcal{H}_n \subset \mathbb{R}^n$ such that $\text{Rad}(\mathcal{H}_n) \rightarrow 0$ and $\sup_n \sup_{h \in \mathcal{H}_n} \|h\|_\infty < \infty$. Suppose that $\sup_n \alpha_s^{(n)} < \infty$ for all $s \leq t$. Then, as $n \rightarrow \infty$,*

- (a) $\sup_{h \in \mathcal{H}_n} n^{-1} \sum_{i=1}^n h_i(X_{i,t}^{(n)} - p_{i,t}^{(n)}) \xrightarrow{\mathbb{P}} 0$ if $\psi_s^{(n)} \rightarrow 0$ for all $s \leq t$;
- (b) $\sup_{h \in \mathcal{H}_n} n^{-1} \sum_{i=1}^n h_i(X_{i,t}^{(n)} - p_{i,t}^{(n)}) \xrightarrow{a.s.} 0$ if $\{\psi_s^{(n)}\}_{n=1}^\infty \in \ell^q$ for all $s \leq t$ and some q .

Corollary 2.5 is a consequence of the log-normal concentration inequality in Corollary 5.4, which, in turn, relies on the functional error bound in Theorem 5.1 in Section 5. In particular, the sequence $\{h_n\}_{n=1}^\infty$ of singletons $h_n \in \mathbb{R}^n$ satisfies the conditions of Corollary 2.5, provided that $\sup_n \|h_n\|_\infty < \infty$.

3. Applications

We now apply our results to a variety of existing models.

Example 1 (Spreading Processes). To demonstrate the utility of [Corollary 2.4](#) in identifying the critical phase of occupancy processes, we consider the time-homogeneous contact-based epidemic *spreading processes* introduced by Wang et al. [49], and generalised in [21]. A survey of more recent applications, and extensions to multi-layer networks can be found in [6, Section 5.2]. This class of processes encompasses those amenable to heterogeneous mean field approaches, and allows for both weighted and unweighted networks. Additionally, there are a number of recent social network [50] and computer science [43] models which fall within this framework. They may be summarised as follows. First it is assumed that the probability of node i (of a total of n) being infected by an infected node j in one time step is r_{ij} , with the convention that $r_{ii} = 0$. The collection $R = (r_{ij})$ is called the *reaction matrix*. For the case of a single-layer network with a weighted adjacency matrix $W = (w_{ij})$, by defining λ_i as the number of contacts from node i per unit time, a reaction matrix R may be derived from W and λ with elements

$$r_{ij} \propto 1 - \left(1 - \frac{w_{ij}}{\sum_{j=1}^n w_{ij}}\right)^{\lambda_i},$$

where the constant of proportionality is assumed to be independent of i and j . For interconnected and multiplex networks, reaction matrices become significantly more complex in form [6]. Assuming that contacts are all independent, the colonisation function C_i is given by

$$C_i(\mathbf{x}) = 1 - \prod_{\substack{j=1 \\ j \neq i}}^n (1 - r_{ij}x_j).$$

If it is assumed that a node recovers with probability μ in one time step, then we may take $S_i(\mathbf{x}) = 1 - \mu$. However, as in [21], we may also consider the possibility of reinfection before the next census, giving a survival function of the form

$$S_i(\mathbf{x}) = 1 - \mu[1 - C_i(\mathbf{x})] = 1 - \mu \prod_{\substack{j=1 \\ j \neq i}}^n (1 - r_{ij}x_j).$$

It is a straightforward exercise to show $\alpha = \|R\|_1$ and $\psi = n^{-1/2}\|R\|_F$.

Let $\bar{X}_t = n^{-1} \sum_{i=1}^n X_{i,t}$ denote the total proportion of infectives at time t , with $\bar{p}_t = n^{-1} \sum_{i=1}^n p_{i,t}$ its deterministic approximation constructed according to (1.2). [Corollary 2.4](#) gives,

$$\mathbb{E}|\bar{X}_t - \bar{p}_t| \leq 12\sqrt{\pi}n^{-1/2}t(1 + \|R\|_F)e^{4t\|R\|_1}. \quad (3.1)$$

Now, Γ is the largest element of the matrix $(R + I)^\top(R + I) - I$, while $\delta = 0$. Assuming $r_{ij} \leq \bar{r}n^{-1}$ for each $i, j = 1, \dots, n$ for some $\bar{r} > 0$, we have $n\Gamma \leq (1 + \bar{r})^2$ and $\|R\|_1, \|R\|_F \leq \bar{r}$. Denoting $v_t = \sum_{s=1}^t \sigma_s^2[D_{s,t}\mathbf{1}] \leq \frac{1}{4}te^{t\|R\|_1}$, and Φ the cumulative distribution

function of a standard normal random variable, [Theorem 2.1](#) implies

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{n}{v_t}} (\bar{X}_t - \bar{p}_t) \leq x \right) - \Phi(x) \right| \\ \leq \frac{700(1 + \log n)^{1/2}}{\sqrt{n}} t e^{20\bar{r}t} (1 + \bar{r})^4 \sum_{s=1}^t \frac{e^{-3\bar{r}s}}{\sigma_s^4[D_{s,t}\mathbf{1}]} \end{aligned} \quad (3.2)$$

Alternatively, for the same occupancy process, the global rule may be extended to $[0, 1]^n$ in the form (6.6) with $f_i(x) = 1 - \mu e^{-x}$, $g_i(x) = 1 - e^{-x}$ and $s_{ij} = c_{ij} = n |\log(1 - r_{ij})|$. In this case, (3.2) may be improved to $\mathcal{O}(n^{-1/2})$ by [Remark 6.3](#), although this comes at the cost of larger $\alpha, \psi, \Gamma, \delta$. Neither (3.1) nor (3.2) are especially tight, and can likely be improved by a more careful treatment of this particular model. However, it is comforting that bounds with a likely optimal rate of convergence in the number of nodes can be so readily obtained from our results.

Gómez et al. [21] showed that if the spectral radius $r(R)$ is strictly less than μ , the disease, as represented through the deterministic system (1.2), cannot become endemic. This is seen by interpreting an epidemic as a non-zero fixed point of the deterministic recurrence. We show that extinction also occurs if $r(R) = \mu$. In fact, since $J_0 = (1 - \mu)I + R$, [Theorem 7.1](#) and (3.1) imply that, when n is large and $\|R\|_1, \|R\|_F$ are not, the process X_t quickly reaches the fixed point $X_t = \mathbf{0}$ (corresponding to the infection dying out) if $r(R) \leq \mu$, and may persist otherwise. In the latter case, using Perron–Frobenius theory, it may be shown that $r(J_\infty) < 1$ holds assuming $r_{ij} > 0$ for all $i, j = 1, \dots, n, i \neq j$. If the deterministic equilibrium satisfies

$$[1 - (1 - \mu)p_{j,\infty}] \sum_{\substack{i=1 \\ i \neq j}}^n \frac{r_{ij}(1 - p_{i,\infty})}{1 - r_{ij}p_{j,\infty}} < \mu,$$

then $\|J_\infty\|_1 < 1$, and so convergence of the recentered process to a normal equilibrium follows from [Corollary 2.3](#).

Example 2 (Hanski’s Incidence Function Model). Arguably the first, and perhaps the most widely used and studied stochastic patch occupancy model, is the *Incidence Function Model* introduced by Hanski [24]. Recent work by McVinish & Pollett [40] has considered the model within the occupancy process framework.

We present a time-inhomogeneous extension of the general formulation of Hanski’s model, which may be realised in our framework in the following way. Let $\Omega \subset \mathbb{R}^d$ be a compact set, and associated with each patch i is a location z_i in Ω . The survival function $S_{i,t}$ for each patch i is chosen to be independent of all other patches, so that $S_{i,t}(x) = s_{i,t}$ for each $i = 1, \dots, n$ and $t = 0, 1, \dots$. For the colonisation function $C_{i,t}$, let $c : [0, \infty) \rightarrow [0, 1]$ be a \mathcal{C}^2 function, and, for each $t = 0, 1, \dots$ and $i = 1, \dots, n$, let $A_{i,t}$ denote the colonisation weight of patch i at time t , corresponding either to patch size or approximate population size. Then, for some kernel $k : \Omega^2 \rightarrow \mathbb{R}$, we let

$$C_{i,t}(x) = c \left(\sum_{j \neq i}^n A_{j,t} k(z_i, z_j) x_j \right). \quad (3.3)$$

The inner sum is often referred to as the *connectivity measure* for patch i .

To investigate the asymptotic behaviour of this system as the number of patches grows, we consider a sequence of (deterministic) patch locations $\{z_i\}_{i=1}^\infty$ that is equidistributed in Ω with

respect to a distribution m , so that, for every $h \in \mathcal{C}(\Omega)$, $n^{-1} \sum_{i=1}^n h(z_i) \rightarrow \int h dm$, as $n \rightarrow \infty$. Random patch locations can also be considered using a conditioning argument.

Suppose that, for each $n \in \mathbb{N}$, patches are placed at locations z_1, \dots, z_n with patch weights $A_{i,t} = n^{-1} a_t(z_i)$, where $a_t \in \mathcal{C}(\Omega)$ describes weight density at time t . Similarly, assume that the survival probabilities $s_{i,t} = s_t(z_i)$ where $s_t \in \mathcal{C}(\Omega)$ describes the probability of survival at time t according to locations in Ω .

We now consider the measures $\mu_t^{(n)}$ and $\pi_t^{(n)}$ defined for $h \in \mathcal{C}(\Omega)$ by $\int h d\mu_t^{(n)} = n^{-1} \sum_{i=1}^n h(z_i) X_{i,t}^{(n)}$, and similarly for $\pi_t^{(n)}$. Equip the space $\mathcal{M}(\Omega)$ of finite measures acting on the Borel σ -algebra of Ω with the vague topology, so that for $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, and $\mu_n \xrightarrow{v} \mu$ if and only if $\int h d\mu_n \rightarrow \int h d\mu$ for all $h \in \mathcal{C}(\Omega)$ [29, Theorem 16.16]. If k is an equicontinuous function (that is, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $z, z_1, z_2 \in \Omega$ with $\|z_1 - z_2\| < \delta$, then $|k(z_1, z) - k(z_2, z)| < \epsilon$), the proof of [40, Theorem 1] implies the following lemma.

Lemma 3.1. *If $\pi_0^{(n)} \xrightarrow{v} \pi_0$ as $n \rightarrow \infty$ for some $\pi_0 \in \mathcal{M}(\Omega)$ that is absolutely continuous with respect to m , then $\pi_t^{(n)} \xrightarrow{v} \pi_t$ for each $t \in \mathbb{N}$, where $\pi_t \in \mathcal{M}(\Omega)$ is defined recursively in terms of its Radon–Nikodym derivative by*

$$\frac{\partial \pi_{t+1}}{\partial m}(z) = s_t(z) \cdot \frac{\partial \pi_t}{\partial m}(z) + c[C_t(z)] \cdot \left[1 - \frac{\partial \pi_t}{\partial m}(z) \right], \quad (3.4)$$

where $C_t(z) = \int a_t(\tilde{z}) D(z, \tilde{z}) \pi_t(d\tilde{z})$ is the limiting connectivity measure at time t .

Since $\{h(z_i)\}_{i=1}^\infty \in \ell^\infty$ for every $h \in \mathcal{C}(\Omega)$ and $\|k\|_\infty < \infty$, a quick application of Corollary 2.5 with [5, Theorem 2.2] under the assumptions of Lemma 3.1 yields the following result, which improves [40, Theorem 1].

Proposition 3.2. *For every $t = 1, 2, \dots$, $\mu_t^{(n)} \xrightarrow{v} \pi_t$ almost surely.*

Proceeding further, it may be shown that the normalised fluctuations in $\mu_t^{(n)}$ converge to a Gaussian random field, which, to our knowledge, is an entirely new result. For any $h \in \mathcal{C}(\Omega)$, define $\sigma_{n,t}^2 : \mathcal{C}(\Omega) \rightarrow [0, \infty)$ by $\sigma_{n,t}^2[h] := n^{-1} \sum_{i=1}^n h(z_i)^2 p_{i,t}^{(n)} (1 - p_{i,t}^{(n)})$. It is straightforward to check that, for every $t = 0, 1, \dots$ and $h \in \mathcal{C}(\Omega)$, $\sigma_{n,t}^2[h]^2 \rightarrow \int h^2 d\sigma_t$ as $n \rightarrow \infty$, where the measures σ_t on Ω are absolutely continuous with respect to m and are defined recursively according to $\sigma_0 \equiv 0$ and

$$\begin{aligned} \frac{\partial \sigma_{t+1}}{\partial m}(z) &= s_t(z) [1 - s_t(z)] \frac{\partial \pi_t}{\partial m}(z) + c[C_t(z)] (1 - c[C_t(z)]) \left(1 - \frac{\partial \pi_t}{\partial m}(z) \right) \\ &\quad + (s_t(z) - c[C_t(z)])^2 \cdot \frac{\partial \sigma_t}{\partial m}(z). \end{aligned}$$

Additionally, for any $h \in \mathcal{C}(\Omega)$ and $j = 1, 2, \dots$, we have

$$\sum_{i=1}^n h(z_i) \partial_j P_{i,t}^{(n)}(\mathbf{p}_t^{(n)}) \rightarrow (\mathcal{J}_t h)(z_j)$$

as $n \rightarrow \infty$, where $\mathcal{J}_t : \mathcal{C}(\Omega) \rightarrow \mathcal{C}(\Omega)$ is defined by

$$\begin{aligned} \mathcal{J}_t h(z) &= (s_t(z) - c[C_t(z)]) h(z) \\ &\quad + a_t(z) \int k(z, \tilde{z}) h(\tilde{z}) c'[C_t(\tilde{z})] (m - \pi_t)(d\tilde{z}). \end{aligned}$$

Thus, Corollary 2.2 implies the following central limit result.

Proposition 3.3. For each $t = 1, 2, \dots$, and $h \in \mathcal{C}(\Omega)$,

$$\sqrt{n} \left(\int h d\mu_t^{(n)} - \int h d\pi_t^{(n)} \right) \xrightarrow{\mathcal{D}} \sum_{s=0}^t \int \mathcal{Q}_{t-s} h d\tilde{\xi}_s, \quad (3.5)$$

where $\mathcal{Q}_t : \mathcal{C}(\Omega) \rightarrow \mathcal{C}(\Omega)$ is defined by $\mathcal{Q}_t := \mathcal{J}_t \circ \mathcal{J}_{t-1} \circ \dots \circ \mathcal{J}_1$, and each $\tilde{\xi}_s$ is an independent Gaussian white noise on Ω with intensity measure σ_s .

For each $t = 1, 2, \dots$ and $n = 1, 2, \dots$, let $\zeta_t^{(n)}$ and ξ_t denote the random signed measures defined through their integrals with respect to functions $h \in \mathcal{C}(\Omega)$ by the left and right-hand sides of (3.5) respectively. Ideally, the convergence (3.5) could be represented in a concise fashion as in Proposition 3.2. Unfortunately, the space of signed measures endowed with the weak topology is not metrisable [2], prohibiting conventional convergence theorems in this setting. Instead, it is common to embed a signed random measure into the dual of a Sobolev space. For $1 \leq r < \infty$, let $W^r(\Omega)$ denote the Sobolev space of order r on Ω , defined as the closure of $L^2(\Omega) \cap \mathcal{C}^\infty(\Omega)$ under the norm

$$\|u\|_{W^r(\Omega)}^2 := \sum_{|\alpha| \leq r} \int_{\Omega} |\partial^\alpha u(x)|^2 dx,$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| = \sum_i \alpha_i \leq r$, and the derivatives are understood in the weak sense. Assuming $\partial\Omega$ is locally Lipschitz, by the Sobolev embedding theorem [1, Theorem 4.12], $W^r(\Omega) \subset \mathcal{C}(\Omega)$ for all $r > d/2$, and there is a constant $C > 0$ depending only on Ω and r such that

$$\|u\|_\infty \leq C \|u\|_{W^r(\Omega)}, \quad u \in W^r(\Omega), \quad (3.6)$$

with some mild abuse of notation. Let $W^{-r}(\Omega)$ denote the dual space of $W^r(\Omega)$, observing that any signed random measure ζ may be identified as an element in $W^{-r}(\Omega)$ by $(h, \zeta) := \int_{\Omega} h d\zeta$ for $h \in W^r(\Omega)$. Of course, if $\zeta_n \xrightarrow{\mathcal{D}} \zeta$ where $\zeta, \zeta_1, \zeta_2, \dots \in W^{-r}(\Omega)$, then $(h, \zeta_n) \xrightarrow{\mathcal{D}} (h, \zeta)$ for any $h \in W^r(\Omega)$. By virtue of Corollary 2.4 and (3.6), for any $h \in W^r(\Omega)$ and each $t = 1, 2, \dots$,

$$\sup_n \mathbb{E}|(h, \zeta_t^{(n)})| \leq C_t \|h\|_{W^r(\Omega)} \quad \text{and} \quad \sup_n \mathbb{E}\|\zeta_t^{(n)}\|_{W^{-r}(\Omega)} < \infty. \quad (3.7)$$

Since the embedding $W^{r+1}(\Omega) \hookrightarrow W^r(\Omega)$ is Hilbert–Schmidt, any closed ball in $W^{-r}(\Omega)$ is compact in $W^{-r-1}(\Omega)$. Therefore, by Markov’s inequality, (3.7) implies that $\{\zeta_t^{(n)}\}_{n=1}^\infty$ is tight in $W^{-r-1}(\Omega)$, whence Proposition 3.3 implies a concise Corollary 3.4.

Corollary 3.4. For each $t = 1, 2, \dots$, the signed random measures $\zeta_t^{(n)} \xrightarrow{\mathcal{D}} \xi_t$ in $W^{-r}(\Omega)$ for any $r > d/2 + 1$.

Example 3 (Dynamic Random Graphs). The prototypical representation of a complex network is that of a *random graph* of large size. In a stochastic setting, one can construct very general processes on a space of graphs to model the evolution of large networks [17], but such processes are often difficult to study. It is convenient then that many dynamic random graphs can be formulated as occupancy processes, where now the nodes become the vertices of a line graph, describing the presence of an edge. The natural setting for analysing large dense random graphs is by way of *graphons*, defined as symmetric Borel measurable functions $W : [0, 1]^2 \rightarrow [0, 1]$. Any graph $G = (V(G), E(G))$ with $V(G) = \{1, 2, \dots, m\}$ may be embedded in a graphon W_G by subdividing $[0, 1]$ into intervals I_1, \dots, I_m and taking $W_G(x, y) = 1_{ij \in E}$ for all

$(x, y) \in I_i \times I_j$, $i, j = 1, \dots, m$. In the sequel, the standard graphon of G is to be regarded as the case where each subinterval has equal length. Many properties of a graph may be formulated in terms of its graphon, including the degree function, defined, for vertex i with $x_i \in I_i$, by

$$d_{W_G}(x_i) = \int_0^1 W_G(x_i, y) dy = \frac{\deg i}{n}.$$

But perhaps the most important object in the study of large dense random graphs is the *homomorphism density*: if F is a simple graph and W a graphon, we define the homomorphism density $t(F, W)$ of F into W by

$$t(F, W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{i \in V(F)} dx_i.$$

Naturally, graphons are strict generalisations of graphs, which becomes important for developing limit theorems. In this connection, the homomorphism density provides a good starting point; if a sequence of graphons $W^{(n)}$ converges to W in a reasonable sense, we might expect that $\lim_{n \rightarrow \infty} t(F, W^{(n)}) = t(F, W)$ for any simple graph F . It turns out that to show convergence of the underlying graphs in homomorphism density, it is sufficient to show convergence under the cut metric [35, Lemma 10.23], induced by the norm $\|\cdot\|_\square$, defined for kernels $W : [0, 1]^2 \rightarrow \mathbb{R}$ by

$$\|W\|_\square = \sup_{U, V \subseteq [0,1]} \left| \int_{U \times V} W(x, y) dx dy \right|.$$

For more details on graphons, refer to the comprehensive monograph of Lovász [35].

To illustrate, consider the following sequence of Markov preferential attachment models. For each $n = 1, 2, \dots$, let $G^{(n)}$ be a simple graph with n edges and v_n vertices, and suppose that $G^{(n)} \subset G^{(n+1)}$. Now, let $G_t^{(n)}$, $t = 0, 1, \dots$, be a Markov chain on the space of subgraphs of $G^{(n)}$ whose edges evolve independently in such a way that each edge $ij \in E(G^{(n)})$ is deleted with probability q_t and is added with probability

$$f\left(\frac{1}{2v_n} \deg i + \frac{1}{2v_n} \deg j\right),$$

for some function $f : [0, 1] \rightarrow [0, 1] \in \mathcal{C}^2$. Since $\alpha^{(n)} \leq \|f'\|_\infty$ and $\psi^{(n)} \leq v_n^{-1} \|f'\|_\infty + v_n^{-2} \|f'\|_\infty$, it can be verified that this sequence of occupancy processes satisfies the conditions of Corollaries 2.5 and 2.2. We begin by showing almost sure convergence of the underlying graphs under the cut metric, which is easily achieved with the help of Corollary 2.5.

Proposition 3.5. *Let $W^{(n)}$ and $W_t^{(n)}$ denote the graphons of $G^{(n)}$ and $G_t^{(n)}$ (respectively) and suppose that $\|W^{(n)} - W\|_\square \rightarrow 0$ and $\|W_0^{(n)} - W_0\|_\square \rightarrow 0$ as $n \rightarrow \infty$ for some graphons W and W_0 . Define the sequence of graphons W_t by the recursion*

$$W_{t+1}(x, y) = q_t W_t(x, y) + W(x, y)[1 - W_t(x, y)]f\left(\frac{d_{W_t}(x) + d_{W_t}(y)}{2}\right). \quad (3.8)$$

Then, for each $t = 1, 2, \dots$, $\|W_t^{(n)} - W_t\|_\square \xrightarrow{a.s.} 0$.

Proof. For each $t = 0, 1, \dots$ define the graphon $\tilde{W}_t^{(n)}$ by $\tilde{W}_0^{(n)} = W_0^{(n)}$ and satisfying the same recursion relation (3.8). It is a straightforward exercise to show that $\|\tilde{W}_t^{(n)} - W_t\|_\square \rightarrow 0$ as

$n \rightarrow \infty$. Observe that for $\mathcal{A}_n = \{uv^\top : u, v \in \{0, 1\}^{v_n}\}$ the set of $v_n \times v_n$ binary rank-one matrices,

$$\|W_t^{(n)} - \tilde{W}_t^{(n)}\|_\square = \max_{A \in \mathcal{A}_n} \left| \frac{1}{v_n^2} \sum_{i,j=1}^{v_n} A_{ij} [W_t^{(n)}(x_i, x_j) - \tilde{W}_t^{(n)}(x_i, x_j)] \right|.$$

Since $\log_2 |\mathcal{A}_n| \leq 2v_n$, from (2.7), $\text{Rad}(\mathcal{A}_n) = \mathcal{O}(v_n^{-1/2})$ as $n \rightarrow \infty$, and so the result follows from Corollary 2.5. \square

Proceeding further, a combination of Proposition 5.5 and Corollary 2.2 shows that arbitrary homomorphism densities $t(F, W_t^{(n)})$ of $G_t^{(n)}$ are approximately normally distributed about $t(F, W_t)$. To illustrate, consider the density of triangles in $G_t^{(n)}$ given by $t(\Delta, W_t^{(n)})$ where Δ is the complete graph on three vertices.

Proposition 3.6. Assume that $|t(\Delta, W_0^{(n)}) - t(\Delta, W_0)| = o(v_n^{-1}n^{1/2})$ as $n \rightarrow \infty$. Then, for each $t \geq 1$, as $n \rightarrow \infty$,

$$v_n n^{-1/2} [t(\Delta, W_t^{(n)}) - t(\Delta, W_t)] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathcal{V}_t[A_t]),$$

where $A_t : [0, 1]^2 \rightarrow [0, 3]$ is a kernel defined by

$$A_t(x, y) = 3 \int_0^1 W_t(x, z) W_t(z, y) dz.$$

Here \mathcal{V}_t is a functional acting on kernels $U : [0, 1]^2 \rightarrow \mathbb{R}$ satisfying $\mathcal{V}_0 \equiv 0$ and, for $t = 0, 1, \dots$, the recursion relation $\mathcal{V}_{t+1}U = \sigma_{t+1}^2 U + \mathcal{V}_t \circ \mathcal{J}_t U$, with

$$\begin{aligned} \mathcal{J}_t U(x, y) := & W(x, y) \left\{ U(x, y)(q_t - f[\tfrac{1}{2}(d_U(x) + d_U(y))]) \right. \\ & + \int_0^1 W(x, z)[1 - U(x, z)]f'[\tfrac{1}{2}(d_U(x) + d_U(z))] \\ & \left. + W(y, z)[1 - U(y, z)]f'[\tfrac{1}{2}(d_U(y) + d_U(z))] dz \right\}, \end{aligned}$$

while σ_t^2 is a functional acting on kernels satisfying $\sigma_0^2 \equiv 0$ and for each $t = 0, 1, \dots$,

$$\begin{aligned} \sigma_{t+1}^2[U(x, y)] = & q_t(1 - q_t) \int_{[0,1]^2} U(x, y) W_t(x, y) dx dy \\ & + \int_{[0,1]^2} U(x, y) w_t(x, y)[1 - w_t(x, y)][1 - W_t(x, y)] dx dy \\ & + \sigma_t^2[(q_t - w_t(x, y))U(x, y)], \end{aligned}$$

with $w_t(x, y) = f[\tfrac{1}{2}(d_{W_t}(x) + d_{W_t}(y))]$.

Proof. Defining the triangle density acting on adjacency matrices $A \in \{0, 1\}^{v_n \times v_n}$, and computing derivatives, we find that

$$t(\Delta, A) = \frac{1}{v_n^3} \sum_{i,j,k=1}^{v_n} A_{ij} A_{jk} A_{ik}, \quad \partial_{ij} t(\Delta, A) = \frac{3}{v_n^3} \sum_{k=1}^{v_n} A_{ik} A_{jk}.$$

Constructing $\tilde{W}_t^{(n)}$ once again as in the proof of Proposition 3.5, Proposition 5.5 implies that

$$\mathbb{E}|t(\Delta, W_t^{(n)}) - t(\Delta, \tilde{W}_t^{(n)}) - (\Lambda_t^{(n)}, \zeta_t^{(n)})| = \mathcal{O}(nv_n^{-3}) \quad \text{as } n \rightarrow \infty,$$

where $\Lambda_{ij}^{(n)} = 3n^{1/2}v_n^{-3} \sum_{k=1}^{v_n} W_t^{(n)}(x_i, x_k)W_t^{(n)}(x_j, x_k)$. By induction on the assumed case $t = 0$, it may be shown that $|t(\Delta, \tilde{W}_t^{(n)}) - t(\Delta, W_t)| = o(v_n^{-1}n^{1/2})$ for each $t = 1, 2, \dots$. Therefore, it remains only to show convergence in law of $v_n^2n^{-1/2}(\Lambda_t^{(n)}, \zeta_t^{(n)})$. Taking $n \rightarrow \infty$ and relabelling as necessary, $v_n^2n^{-1/2}\Lambda_{ij,t}^{(n)} \rightarrow \Lambda_t(x_i, x_j)$. The rest is implied by (6.11), following similar computations to those in Example 2. \square

To adapt Proposition 3.6 to homomorphism densities from a different simple graph F , one need only modify the kernel Λ_t —all other objects remain intact. Allowing one final remark, it is also quite possible to consider another occupancy process running on the nodes of a dynamic random graph model as one conglomerate occupancy process. While notation becomes rather unwieldy at this level of complexity, provided that presence/absence of edges and the states of the vertices are not too intimately connected as to violate the assumptions of Corollary 2.2, many of the ideas contained in Examples 1 and 3 should extend to the more general setting.

4. The method of bounded differences

The classical *method of bounded differences* provides the simplest and most versatile approach for developing moment estimates and concentration inequalities involving bounded random variables. We recall a few results from the theory which will be greatly useful to us. For any function $f : \{0, 1\}^n \rightarrow \mathbb{R}$, let $\Delta_i f(\mathbf{x}) = |f(\mathbf{x}) - f(\mathbf{x}^i)|$, where \mathbf{x}^i denotes \mathbf{x} with the i th component replaced by its inverse $x_i^i = 1 - x_i$. Let $\Psi(x) = e^{x^2} - 1$ and $\|\cdot\|_\Psi$ be the corresponding Orlicz norm, defined for a random variable X by

$$\|X\|_\Psi = \inf\{t > 0 : \mathbb{E}\Psi(X/t) \leq 1\}.$$

The Orlicz norm is particularly useful for controlling maxima: if X_1, \dots, X_n are random variables, not necessarily independent, then [45]

$$\mathbb{E} \max_i X_i^2 \leq \log 2n \cdot \max_i \|X_i\|_\Psi^2. \quad (4.1)$$

Theorem 4.1 (Method of Bounded Differences). *Let W_1, \dots, W_n be independent $\{0, 1\}$ -valued random variables and $\mathbf{W} = (W_1, \dots, W_n)$. For any $f : \{0, 1\}^n \rightarrow \mathbb{R}$, let $\|\Delta f\|_2^2 = \sum_{i=1}^n \|\Delta_i f\|_\infty^2$. Then, for any $\epsilon > 0$, and $q \geq 1$,*

$$\mathbb{P}(f(\mathbf{W}) > \epsilon + \mathbb{E}f(\mathbf{W})) \leq \exp\left(-\frac{2\epsilon^2}{\|\Delta f\|_2^2}\right) \quad (4.2)$$

$$\|f(\mathbf{W}) - \mathbb{E}f(\mathbf{W})\|_q \leq \sqrt{\frac{\pi q}{2}} \|\Delta f\|_2 \quad (4.3)$$

$$\|f(\mathbf{W}) - \mathbb{E}f(\mathbf{W})\|_\Psi \leq \sqrt{\frac{2}{\pi}} \|\Delta f\|_2. \quad (4.4)$$

The ‘one-sided’ inequality (4.2) is famously due to McDiarmid [37, Theorem 3.1]. Adding (4.2) to itself applied to $-f$ yields the ‘two-sided’ inequality for tail estimates of $|f(\mathbf{W}) - \mathbb{E}f(\mathbf{W})|$, from which (4.3) and (4.4) follow by calculation.

To compare $\mathbb{E}f(\mathbf{W})$ with $f(\mathbb{E}\mathbf{W})$, we can make use of a simple telescoping trick dating back to Lindeberg’s original analytic proof of the central limit theorem. For more recent applications of this trick, we refer the reader to the paper of Chatterjee [11]. Let $f \in \mathcal{C}^2([0, 1]^n)$ and

W_1, \dots, W_n be independent random variables with $p_i = \mathbb{E}W_i$ for each $i = 1, \dots, n$. Now, for each $i = 0, \dots, n$, let $\tilde{\mathbf{W}}_i = (W_1, \dots, W_i, p_{i+1}, \dots, p_n)$, so that, from Taylor's Theorem,

$$\begin{aligned} |\mathbb{E}f(\mathbf{W}) - f(\mathbf{p})| &\leq \sum_{i=1}^n |\mathbb{E}f(\tilde{\mathbf{W}}_i) - \mathbb{E}f(\tilde{\mathbf{W}}_{i-1})| \\ &\leq \sum_{i=1}^n |\mathbb{E}[\partial_i f(\tilde{\mathbf{W}}_{i-1}) \cdot (W_{i,t} - p_{i,t})]| + \frac{1}{2} \sum_{i=1}^n \|\partial_i^2 f\|_\infty. \end{aligned}$$

But, since $\tilde{\mathbf{W}}_{i-1}$ is independent of W_i , the first term is identically 0, and

$$|\mathbb{E}f(\mathbf{W}) - f(\mathbf{p})| \leq \frac{1}{2} \sum_{i=1}^n \|\partial_i^2 f\|_\infty. \quad (4.5)$$

Theorem 4.1 together with (4.5) provides an effective measure on the deviation of $f(\mathbf{W})$ from $f(\mathbf{p})$ for any arbitrary $f \in \mathcal{C}^2([0, 1]^n)$. We shall also find it useful to perform a linear approximation to f as an intermediary to computing $f(\mathbf{W})$, and bound the error incurred in doing so. This requires a tighter estimate than is offered in the moment inequalities of **Theorem 4.1**, for which the Efron–Stein inequality [9, Theorem 3.1] will suffice.

Lemma 4.2. *There exists a universal constant $0 < C \leq \pi + \frac{1}{4}$ such that, for any $f \in \mathcal{C}^3([0, 1]^n)$,*

$$\begin{aligned} \mathbb{E} \left| f(\mathbf{W}) - f(\mathbf{p}) - \sum_{j=1}^n \partial_j f(\mathbf{p})(W_j - p_j) \right|^2 \\ \leq C \sum_{j,k=1}^n \|\partial_j \partial_k f\|_\infty^2 + C \sum_{j=1}^n \left(\sum_{k=1}^n \|\partial_j \partial_k^2 f\|_\infty \right)^2. \end{aligned}$$

Proof. Denoting $F(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^n \partial_j f(\mathbf{p})x_j$, from (4.5), it suffices to consider $\text{Var}F(\mathbf{W})$. Analogous to the Lindeberg argument, let W'_j be an independent copy of W_j and $\mathbf{W}^j = (W_1, \dots, W_{j-1}, W'_j, W_{j+1}, \dots, W_n)$ for each $j = 1, \dots, n$. Then there is a random vector $\tilde{\mathbf{W}}^j$ such that

$$F(\mathbf{W}) - F(\mathbf{W}^j) = [\partial_j f(\mathbf{W}) - \partial_j f(\mathbf{p})](W_j - W'_j) - \frac{1}{2} \partial_j^2 f(\tilde{\mathbf{W}}^j)(W_j - W'_j)^2.$$

But, from [9, Theorem 3.1], $\text{Var}F(\mathbf{W}) \leq \frac{1}{2} \sum_{j=1}^n \mathbb{E}V_j$ where

$$V_j := \mathbb{E}[\{F(\mathbf{W}) - F(\mathbf{W}^j)\}^2 | \mathbf{W}] \leq 2[\partial_j f(\mathbf{W}) - \partial_j f(\mathbf{p})]^2 + \frac{1}{2} \|\partial_j^2 f\|_\infty^2.$$

The lemma now follows from **Theorem 4.1** and (4.5). \square

The final ingredient in the proof of **Lemma 5.3** is a moment inequality for a sum of conditionally independent $\{0, 1\}$ -valued random variables, in which the constant does not depend on the number of variables. The conditional Rosenthal-type inequality in **Lemma 4.3** proves effective, found by modifying the arguments of [28, Theorem 2.5].

Lemma 4.3. *Let X_1, \dots, X_n be $\{0, 1\}$ -valued random variables on a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ which are conditionally independent according to a sub- σ -algebra \mathcal{F} of \mathcal{E} . Then,*

for any $q \geq 1$,

$$\left\| \sum_{i=1}^n X_i \right\|_q \leq 2q \left(1 + \left\| \sum_{i=1}^n \mathbb{E}[X_i | \mathcal{F}] \right\|_q \right). \quad (4.6)$$

In fact, (4.6) can be improved to order $q/\log q$, but this provides no significant improvement to our results.

5. Quality of the deterministic approximation

Our first step is to discern when the macroscopic dynamics of the deterministic system reflect those of the stochastic model in some reasonable sense. Limit theorems under a variety of mean field assumptions have been known for many years; see for example [32]. More recently, [4] considered the problem in a more general framework, where concentration inequalities were obtained for the approximation error between empirical measures of X_t and p_t using the method of bounded differences. Their strategy follows that of [3] by coupling the occupancy process together with another occupancy process, whose nodes transition *independently*. This independence enables the application of techniques discussed in Section 4. By controlling higher-order moments, we extend their approach to cover the general case and improve on their findings, forming the foundation for the rest of this work.

Let $1 \leq q, r \leq \infty$. For a random vector X , we denote the $L^{q,r}$ norm by $\|X\|_{q,r} = [\mathbb{E}(\sum_i |X_i|^{q,r/q})^{1/r}]^{1/r}$, and define the maximal $L^{q,r}$ norm acting on matrices $A = (a_{ij})$ by

$$\|A\|_{q,r} = \begin{cases} \left[\sum_{i=1}^m (\sum_{j=1}^n |a_{ij}|^{q,r/q})^{1/r} \right]^{1/r} & \text{if } q \geq r \\ \left[\sum_{j=1}^n (\sum_{i=1}^m |a_{ij}|^{q,r/r})^{1/r} \right]^{1/r} & \text{if } r > q, \end{cases} \quad (5.1)$$

with the obvious modifications for $q = \infty$ and $r = \infty$. By a Minkowski type inequality [26, Theorem 202], the norm with the cases in (5.1) reversed does not exceed $\|A\|_{q,r}$. If A is an $n \times n$ matrix, the special cases $\|A\|_{2,1}$ and $\|A\|_{1,2}$ are bounded above by $\sqrt{n}\|A\|_F$, where $\|\cdot\|_F$ is the Frobenius (or Hilbert–Schmidt) norm. Finally, for notational convenience, for any matrix-valued function $F = (f_{ij})$ and matrix norm $\|\cdot\|_M$, $\|F\|_M$ shall denote $\|(\|f_{ij}\|_\infty)\|_M$. Our main result for this section bounds the functional error between X_t and p_t under the $L^{q,r}$ norm.

Theorem 5.1. For any $f = (f_1, \dots, f_m) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$, $q, r \geq 1$, and integers $t \geq 1$ and $s < t$,

$$\begin{aligned} \|f(X_t) - f(p_t)\|_{q,r} &\leq 6\sqrt{\pi} nr^{3/2} \|Df\|_1 \sum_{s=0}^{t-1} \left(\frac{1}{n} + \psi_s \right) e^{4r\alpha_{s,t}} \\ &\quad + \sqrt{\pi(q+r)} \|Df\|_{2,q} + \frac{1}{2} \|D^{(2)}f\|_{1,q}. \end{aligned} \quad (5.2)$$

To show Theorem 5.1, as in [3,4], our approach for comparing X_t and p_t is through a coupling with an intermediate occupancy process approximation W_t , whose nodes evolve independently and satisfy $\mathbb{E}W_{i,t} = p_{i,t}$ for every $i = 1, \dots, n$ and $t \geq 0$. By measuring the total variation between X_t and W_t , the method of bounded differences (see Section 4) allows for the approximation of functionals of X_t by the deterministic process p_t . Observe that the

decomposition (1.1) implies that, for each $t \geq 0$,

$$X_{i,t+1} = X_{i,t} \mathbb{1}\{U_{i,t} \leq S_{i,t}(X_t)\} + (1 - X_{i,t}) \mathbb{1}\{U_{i,t} \leq C_{i,t}(X_t)\},$$

where $(U_{i,t})_{i=1}^n$ is a collection of independent uniformly distributed random variables on $[0, 1]$. We construct \mathbf{W}_t , on the same probability space, by setting $\mathbf{W}_0 = \mathbf{X}_0$ and

$$W_{i,t+1} = W_{i,t} \mathbb{1}\{U_{i,t} \leq S_{i,t}(\mathbf{p}_t)\} + (1 - W_{i,t}) \mathbb{1}\{U_{i,t} \leq C_{i,t}(\mathbf{p}_t)\}.$$

For each $i = 1, \dots, n$ and $t \geq 1$, let $J_{i,t} := \max_{1 \leq s \leq t} \mathbb{1}\{X_{i,s} \neq W_{i,s}\}$ and $\bar{J}_t := n^{-1} \sum_{i=1}^n J_{i,t}$. Using the independence of each $W_{i,t}$ in conjunction with the method of bounded differences outlined in Section 4, we obtain our first approximation result.

Lemma 5.2. For any $f = (f_1, \dots, f_m) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$, $q, r \geq 1$, and integer $t \geq 1$,

$$\|f(X_t) - f(\mathbf{p}_t)\|_{q,r} \leq n \|Df\|_1 \|\bar{J}_t\|_r + \sqrt{\pi(q+r)} \|Df\|_{2,q} + \frac{1}{2} \|D^{(2)}f\|_{1,q}.$$

Proof. Define the function w_q on $[0, 1]^n$ by $w_q(\mathbf{x})^q = \sum_{i=1}^m |f_i(\mathbf{x}) - \mathbb{E}f_i(\mathbf{W}_t)|^q$. From the triangle inequality for the $L^{q,r}$ norm, $\|f(X_t) - f(\mathbf{p}_t)\|_{q,r} \leq T_1 + T_2 + \|w_q(\mathbf{W}_t)\|_r$, where

$$T_1 = \|f(X_t) - f(\mathbf{W}_t)\|_{q,r} \quad \text{and} \quad T_2 = \|\mathbb{E}f(\mathbf{W}_t) - f(\mathbf{p}_t)\|_q.$$

T_1 is straightforward to bound using $J_{i,t}$, because

$$\left(\sum_{i=1}^m |f_i(X_t) - f_i(\mathbf{W}_t)|^q \right)^{1/q} \leq \sum_{i=1}^m \sum_{j=1}^n \|\partial_j f_i\|_\infty J_{j,t}$$

implies that $T_1 \leq n \|Df\|_1 \|\bar{J}_t\|_r$. By the Lindeberg argument (4.5), $T_2 \leq \frac{1}{2} \|D^{(2)}f\|_{1,q}$. Using Jensen's inequality,

$$[\mathbb{E}w_q(\mathbf{W}_t)]^q \leq \sum_{i=1}^m \mathbb{E}|f_i(\mathbf{W}_t) - \mathbb{E}f_i(\mathbf{W}_t)|^q,$$

so Theorem 4.1 implies $\mathbb{E}w_q(\mathbf{W}_t) \leq \sqrt{\frac{\pi q}{2}} \|Df\|_{2,q}$. By the reverse triangle inequality, for each $j = 1, \dots, n$,

$$|\Delta_j w_q(\mathbf{x})| \leq \left(\sum_{i=1}^m |\Delta_j f_i(\mathbf{x})|^q \right)^{1/q} \leq \left(\sum_{i=1}^m \|\partial_j f_i\|_\infty^q \right)^{1/q},$$

and so another application of Theorem 4.1 gives $\|w_q(\mathbf{W}_t)\|_r \leq \sqrt{\frac{\pi}{2}}(q^{1/2} + r^{1/2}) \|Df\|_{2,q}$. We conclude the proof with Cauchy's inequality: $q^{1/2} + r^{1/2} \leq \sqrt{2(q+r)}$. \square

The proof of Theorem 5.1 has now been reduced to obtaining appropriate bounds on the moments of \bar{J}_t . By construction, for each $t = 0, 1, \dots$,

$$\begin{aligned} J_{i,t+1} &\leq J_{i,t} + |\mathbb{1}\{U_{i,t} \leq C_{i,t}(X_t)\} - \mathbb{1}\{U_{i,t} \leq C_{i,t}(\mathbf{p}_t)\}| X_{i,t} \\ &\quad + |\mathbb{1}\{U_{i,t} \leq S_{i,t}(X_t)\} - \mathbb{1}\{U_{i,t} \leq S_{i,t}(\mathbf{p}_t)\}| (1 - X_{i,t}), \end{aligned} \tag{5.3}$$

and so $J_{i,t+1} - J_{i,t}$ may be bounded above by the sum of two conditionally independent $\{0, 1\}$ -valued random variables. Fortunately, each of these terms can be controlled using Lemma 5.2. Thus, we have the following lemma which, together with Lemma 5.2, implies Theorem 5.1.

Lemma 5.3. For any $q \geq 1$ and $t \geq 1$,

$$\|\bar{J}_t\|_q \leq 2q \sum_{s=0}^{t-1} (2n^{-1} + 3\beta_s \sqrt{\pi q} + \gamma_s) e^{4q\alpha_{s,t}}. \quad (5.4)$$

Proof. Letting $S_t = (S_{i,t})_{i=1}^n$ and $C_t = (C_{i,t})_{i=1}^n$, (5.3) together with Lemma 4.3 implies that, for any integer $t \geq 0$,

$$\|\bar{J}_{t+1}\|_q \leq \|\bar{J}_t\|_q + \frac{2q}{n} \{2 + \|S_t(\mathbf{X}_t) - S_t(\mathbf{p}_t)\|_{1,q} + \|C_t(\mathbf{X}_t) - C_t(\mathbf{p}_t)\|_{1,q}\}.$$

Applying Lemma 5.2,

$$\|\bar{J}_{t+1}\|_q \leq (1 + 4q\alpha_t) \|\bar{J}_t\|_q + 2q(2n^{-1} + 2\beta_t \sqrt{2\pi q} + \gamma_t),$$

and, since $1 + 4q\alpha_t \leq e^{4q\alpha_t}$, the lemma follows. \square

Theorem 5.1 is sufficient for proving our main results. However, it seems prudent to examine its asymptotics in the number of nodes and, in particular, demonstrate the law of large numbers result seen in Corollary 2.5. For any vector $\mathbf{x} \in \mathbb{R}^n$, we let $\bar{\mathbf{x}} = n^{-1}\mathbf{x}$, so that $\langle \bar{\mathbf{x}}, h \rangle = n^{-1} \sum_{i=1}^n h_i x_i$ becomes a weighted average of the components of \mathbf{x} , appropriately normalised to remain bounded as $n \rightarrow \infty$ for bounded h . For fixed $h \in \mathbb{R}^n$, by directly applying Theorem 5.1 to $\langle \bar{\mathbf{X}}_t, h \rangle$ and $\langle \bar{\mathbf{p}}_t, h \rangle$, it is found that the variation (or, indeed, any one of the higher-order moments) of the empirical measure of the occupancy process decays with order $\mathcal{O}(\sum_{s \leq t} \psi_s \vee n^{-1/2})$. Proceeding further in this direction, the proof of Theorem 5.1 implies a general log-normal concentration inequality (Corollary 5.4) on the maximal deviation between $\langle \bar{\mathbf{X}}_t, h \rangle$ and $\langle \bar{\mathbf{p}}_t, h \rangle$ over $h \in \mathcal{H} \subset \mathbb{R}^n$, extending the result of [4]. Furthermore, Corollary 2.5 is an immediate consequence of Corollary 5.4.

Corollary 5.4. Let $\mathcal{H} \subset \mathbb{R}^n$ with $H = \sup_{h \in \mathcal{H}} \|h\|_\infty$. Then, for each $t \geq 0$, denoting $\Psi_t = 12\sqrt{\pi}(n^{-1} + \max_{s \leq t} \psi_s)$, for any $x > 1$,

$$\begin{aligned} \mathbb{P} \left(\sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{X}}_t - \bar{\mathbf{p}}_t, h \rangle| > Ht \Psi_t x + \text{Rad}(\mathcal{H}) \right) \\ \leq e^{-\frac{1}{2} n t^2 x^2 \Psi_t^2} + \sum_{s=1}^t \exp \left[-\frac{4\alpha_{0,s} (\log x)^2}{(1 + 4\alpha_{0,s})^2} + 4\alpha_{0,s} \right]. \end{aligned} \quad (5.5)$$

Proof. We first show

$$\mathbb{E} \sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{W}}_t - \bar{\mathbf{p}}_t, h \rangle| \leq \text{Rad}(\mathcal{H}). \quad (5.6)$$

Since $\mathbb{E} \langle \bar{\mathbf{W}}_t, h \rangle = \langle \bar{\mathbf{p}}_t, h \rangle$, proceeding via the symmetrisation method [46, Lemma 26.2],

$$\mathbb{E} \sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{W}}_t - \bar{\mathbf{p}}_t, h \rangle| \leq \mathbb{E} \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n h_i (W_{i,t} - \tilde{W}_{i,t}),$$

where $\tilde{W}_{i,t}$ is an independent copy of $W_{i,t}$ for each $i = 1, \dots, n$. Now $W_{i,t} - \tilde{W}_{i,t} \stackrel{\mathcal{D}}{=} \sigma_i Z_i$ where σ_i are independent Rademacher random variables and Z_i are $\{0, 1\}$ -valued random variables with $\mathbb{P}(Z_i = 1) = 2p_{i,t}(1 - p_{i,t})$, independent of σ_i . So

$$\mathbb{E} \sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{W}}_t - \bar{\mathbf{p}}_t, h \rangle| \leq \mathbb{E} \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n h_i \sigma_i Z_i$$

The relation (5.6) follows upon conditioning on each Z_i ; indeed, for any binary vector $\mathbf{z} = (z_1, \dots, z_n) \in \{0, 1\}^n$,

$$\mathbb{E} \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n h_i \sigma_i z_i = \mathbb{E} \sup_{h \in \mathcal{H}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n h_i \sigma_i \mid \sigma_i : z_i = 1 \right] \leq \text{Rad}(\mathcal{H}).$$

As a consequence of the one-sided McDiarmid inequality (4.2),

$$\mathbb{P} \left(\sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{W}}_t - \bar{\mathbf{p}}_t, h \rangle| > \frac{1}{2} H t \Psi_t x + \text{Rad}(\mathcal{H}) \right) \leq \exp \left(-\frac{1}{2} n t^2 x^2 \Psi_t^2 \right). \quad (5.7)$$

Now let $J_t(\mathcal{H}) = \sup_{h \in \mathcal{H}} |\langle \bar{\mathbf{X}}_t - \bar{\mathbf{W}}_t, h \rangle|$, so that $\mathbb{E} J_t(\mathcal{H})^q \leq H^q \|\bar{\mathbf{J}}_t\|_q^q$. Lemma 5.3 provides a partial estimate for the moment-generating function of $J_t(\mathcal{H})$: for any $q \geq 1$, denoting $L_s = \log(4n^{-1} + 6\beta_s \sqrt{\pi} + 2\gamma_s) + \log(Ht)$ for each $s \leq t$,

$$\mathbb{E} e^{q \log J_t(\mathcal{H})} \leq \sum_{s=0}^{t-1} \exp(4q^2 \alpha_{s,t} + q L_s + q \log q).$$

Assuming that $\log x \geq 1 + 4\alpha_{0,t}$, choose $q = \log x(1 + 4\alpha_{0,t})^{-1}$, so, by the Chernoff approach,

$$\begin{aligned} & \mathbb{P} \left(\log J_t(\mathcal{H}) > \log x + \max_{s \leq t} L_s \right) \\ & \leq \mathbb{E} \exp \left(q \log J_t(\mathcal{H}) - q \max_{s \leq t} L_s - q \log x \right) \leq \sum_{s=1}^t \exp \left(-\frac{4\alpha_{0,s}(\log x)^2}{(1 + 4\alpha_{0,t})^2} \right). \end{aligned} \quad (5.8)$$

To extend to $0 \leq \log x < 1 + 4\alpha_{0,t}$, it suffices to add $4\alpha_{0,s}$ into each exponent, whereupon inequality (5.8) becomes the trivial bound. Together with (5.7), this implies Corollary 5.4. \square

We conclude this section with an error estimate for approximating $f(\mathbf{X}_t)$ for arbitrary three-times differentiable functions f , by the linear approximation (2.2). Aside from acting as a fundamental component of the proof of Proposition 6.1, it extends Theorem 2.1 to provide error estimates for the normal approximation to non-linear functions of \mathbf{X}_t .

Proposition 5.5. *There is a universal constant $0 < C \leq 72\pi$ such that, for any $f \in \mathcal{C}^3([0, 1]^n)$ and $t \geq 1$,*

$$\begin{aligned} & \mathbb{E} |f(\mathbf{X}_t) - f(\mathbf{p}_t) - \langle \zeta_t, \sqrt{n} \nabla f(\mathbf{p}_t) \rangle| \\ & \leq C \sqrt{1 + \log n} \left[1 + \sum_{s=0}^{t-1} (n^{-1} + n \psi_s^2) t e^{8\alpha_{s,t}} \right] \\ & \quad \left[n \max_{j,k=1,\dots,n} \|\partial_j \partial_k f\|_\infty + \sqrt{n} \max_{j=1,\dots,n} \sum_{k=1}^n \|\partial_j \partial_k^2 f\|_\infty \right]. \end{aligned}$$

Proof. Relying on the independent node approximation once again, the proof follows by comparing \mathbf{X}_t to \mathbf{W}_t , whence \mathbf{W}_t may be compared to \mathbf{p}_t using Lemma 4.2. Firstly,

$$\mathbb{E} \left| f(\mathbf{X}_t) - f(\mathbf{W}_t) - \sum_{j=1}^n \partial_j f(\mathbf{W}_t) (X_{j,t} - W_{j,t}) \right| \leq n^2 \max_{1 \leq j,k \leq n} \|\partial_j \partial_k f\|_\infty \|\bar{\mathbf{J}}_t\|_2^2,$$

which may be controlled by [Lemma 5.3](#). Focusing on the remaining cross-term, by two applications of Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n [\partial_j f(\mathbf{W}_t) - \partial_j f(\mathbf{p}_t)](X_{j,t} - W_{j,t}) \right| \\ \leq \left(\mathbb{E} \max_{j=1, \dots, n} [\partial_j f(\mathbf{W}_t) - \partial_j f(\mathbf{p}_t)]^2 \right)^{1/2} \cdot n \|\bar{J}_t\|_2, \end{aligned}$$

and consequently, due to [\(4.1\)](#), it suffices to estimate $\|\partial_j f(\mathbf{W}_t) - \partial_j f(\mathbf{p}_t)\|_\psi$. But, by [Theorem 4.1](#) and [\(4.5\)](#),

$$\|\partial_j f(\mathbf{W}_t) - \partial_j f(\mathbf{p}_t)\|_\psi \leq \sqrt{\frac{3n}{2}} \max_{k=1, \dots, n} \|\partial_j \partial_k f\|_\infty + \frac{1}{2} \sum_{k=1}^n \|\partial_j \partial_k^2 f\|_\infty,$$

and the result follows. \square

6. Stein's method for occupancy processes

We now turn to the problem of normal approximation, and proving [Theorem 2.1](#). Clearly, a direct application of the independent node coupling \mathbf{W}_t will not suffice. However, the conditional independence property of occupancy processes immediately implies a conditional central limit result: conditioning on $\mathbf{X}_t, \langle \zeta_{t+1}, h \rangle$ converges to a normal random variable for any $h \in \mathbb{R}^n$. An estimate of the convergence rate is given by the classical Berry–Esseen bound, for which some of the simplest proofs make use of Stein's method. To obtain the required *unconditional* estimate is more challenging, but fundamentally relies on this property.

The idea behind Stein's method is to estimate the difference between the expectations $\mathbb{E}g(X)$ and $\mathbb{E}g(Z)$ through a characterising operator (often called the *Stein operator*) \mathcal{A} which has the following property: if a random variable X satisfies $\mathbb{E}\mathcal{A}f(X) = 0$ for all f in an appropriate class of functions, then $X \stackrel{\mathcal{D}}{=} Z$. In the case of normal approximation, where $Z \sim \mathcal{N}(\mu, \sigma^2)$, the operator

$$\mathcal{A}f(x) = \sigma^2 f'(x) - (x - \mu)f(x)$$

suffices. Stein recognised that if, for some chosen function g , f_g solves the *Stein equation*

$$\mathcal{A}f_g(x) = g(x) - \mathbb{E}g(Z),$$

then, provided X is similar in law to Z , $\mathcal{A}f_g(X)$ should also have small expectation. Stein's method is often successful because bounding $|\mathbb{E}\mathcal{A}f(X)|$ over $|\mathbb{E}g(X) - \mathbb{E}g(Z)|$ is an appreciably simpler task in general.

We may divine the relationship between ζ_t and ξ_t by way of a series of one-step approximations. For any time $t \geq 1$, consider the s -step normal approximation $\zeta_t^{(s)}$ defined for $h \in \mathbb{R}^n$ by

$$\langle \zeta_t^{(s)}, h \rangle = \langle \zeta_{t-s}, D_{t-s,t}h \rangle + \sum_{r=t-s+1}^t \sigma_r [D_{r,t}h] \cdot z_r,$$

where each z_r is an independent standard normal random variable. We denote the special case $s = 1$ by $\tilde{\zeta}_t$ and remark that $\zeta_t^{(1)}$ and ξ_t are equal in distribution. By working with the L^q metric between distributions of random variables, we have at our disposal the following contraction

identity under translations by independent random variables. For any random variables X, Y , and Z , such that Z is independent of X and Y ,

$$\|\mathcal{L}(X + Z) - \mathcal{L}(Y + Z)\|_q \leq \|\mathcal{L}(X) - \mathcal{L}(Y)\|_q, \quad (6.1)$$

which is just a restatement of Young's inequality for convolutions. Together with the triangle inequality,

$$\begin{aligned} \|\mathcal{L}\langle \zeta_t, h \rangle - \mathcal{L}\langle \xi_t, h \rangle\|_q &\leq \sum_{s=0}^{t-1} \|\mathcal{L}\langle \zeta_t^{(s)}, h \rangle - \mathcal{L}\langle \zeta_t^{(s+1)}, h \rangle\|_q, \\ &\leq \sum_{s=1}^t \|\mathcal{L}\langle \zeta_s, D_{s,t}h \rangle - \mathcal{L}\langle \tilde{\zeta}_s, D_{s,t}h \rangle\|_q. \end{aligned}$$

This approach is also not too dissimilar to the Lindeberg argument; representing the process at time t as a sum of t individual steps, we have exchanged each component in this sum with its normal approximation in turn. Now, for any time t ,

$$\|D_t\|_\infty = \max_{1 \leq j \leq n} \sum_{i=1}^n \|\partial_j P_{i,t}\|_\infty \leq 1 + \alpha_t,$$

implying that $\|D_{s+1,t}h\|_\infty \leq \|h\|_\infty \exp(\alpha_{s,t})$, and hence [Theorem 2.1](#) follows from [Proposition 6.1](#). Here and throughout, C will denote a universal constant, but it will not necessarily be the same on each appearance. Furthermore, it will be assumed implicitly throughout that $\sigma_t[h] > 0$ for each $t \geq 1$.

Proposition 6.1. *There is a constant $0 < C < 700$ such that for any integer $t \geq 0$,*

$$\|\mathcal{L}\langle \zeta_{t+1}, h \rangle - \mathcal{L}\langle \tilde{\zeta}_{t+1}, h \rangle\|_q \leq C \sqrt{\frac{1 + \log n}{n}} \cdot \frac{\|h\|_\infty^{4-1/q} \kappa_t}{\sigma_{t+1}^{4-2/q}[h]}.$$

The strategy of proof is as follows: first, using the conditional independence property, we apply Stein's method under the appropriate conditional probability space to obtain an estimate for the one-step normal approximation. For this, let \mathbb{E}_t , \mathbb{P}_t , and Var_t denote expectation, probability, and variance conditional on \mathbf{X}_t . Let $h \in \mathbb{R}^n$ be arbitrary, and for any $f \in \mathcal{C}^1(\mathbb{R})$, define $\mathcal{A}_t f$ and $\tilde{\mathcal{A}}_t f$ as the Stein operators

$$\begin{aligned} \mathcal{A}_t f(x) &= \sigma_{t+1}^2[h] \cdot f'(x) - \{x - \langle \zeta_t, D_t h \rangle\} f(x), \\ \tilde{\mathcal{A}}_t f(x) &= \sigma_{t+1}^2[h] \cdot f'(x) - \{x - \mathbb{E}_t \langle \zeta_{t+1}, h \rangle\} f(x). \end{aligned}$$

We proceed by estimating $S_t f = \mathcal{A}_t f(\langle \zeta_{t+1}, h \rangle)$ through $\tilde{S}_t f = \tilde{\mathcal{A}}_t f(\langle \zeta_{t+1}, h \rangle)$. Our strategy for controlling $\tilde{S}_t f$ follows that of [\[13\]](#). As is customary with Stein's method, we first focus on the second term in these expressions. Indeed,

$$\mathbb{E}_t[\langle \zeta_{t+1}, h \rangle f(\langle \zeta_{t+1}, h \rangle)] = \frac{1}{\sqrt{n}} \sum_{i=1}^n h_i \{\mathbb{E}_t[X_{i,t+1} f(\langle \zeta_{t+1}, h \rangle)] - p_{i,t+1} \mathbb{E}_t f(\langle \zeta_{t+1}, h \rangle)\},$$

and hence, by conditioning on the event $X_{i,t+1} = 1$,

$$\mathbb{E}_t[X_{i,t+1} f(\langle \zeta_{t+1}, h \rangle)] = \mathbb{E}_t f \left[\langle \zeta_{t+1}, h \rangle + \frac{h_i}{\sqrt{n}} (1 - X_{i,t+1}) \right] P_{i,t}(\mathbf{X}_t).$$

Conditioning further on $X_{i,t+1} = 0$ reveals that

$$\begin{aligned} & \mathbb{E}_t[\{\langle \zeta_{t+1}, h \rangle - \mathbb{E}_t \langle \zeta_{t+1}, h \rangle\} f(\langle \zeta_{t+1}, h \rangle)] \\ &= \frac{1}{n} \sum_{i=1}^n h_i^2 P_{i,t}(X_t) [1 - P_{i,t}(X_t)] \mathbb{E}_t \int_0^1 f' \left[\langle \zeta_{t+1}, h \rangle + \frac{h_i}{\sqrt{n}} (u - X_{i,t+1}) \right] du. \end{aligned} \quad (6.2)$$

The method of bounding the integral term, and therefore $\mathbb{E}_t \tilde{S}_t f$, varies depending on q . Denoting by ϕ and Φ the density and distribution functions, respectively, of a standard normal random variable, we restate the fundamental lemma for Stein's method for normal approximation, linking bounds on the Stein equations with bounds for the Wasserstein and Kolmogorov metrics (the proof of which may be found in [12, Lemmas 2.3 & 2.4]).

Lemma 6.2 (Stein's Lemma). For any $\mu \in \mathbb{R}$, $\sigma > 0$, and $g \in C^1(\mathbb{R})$, defining f_g as the function satisfying

$$\sigma^2 f'_g(x) - (x - \mu) f_g(x) = g(x) - \frac{1}{\sigma} \int_{\mathbb{R}} g(x) \phi \left(\frac{x - \mu}{\sigma} \right) dx, \quad (6.3)$$

we have that $f_g \in C^2(\mathbb{R})$ and satisfies

$$\|f_g\|_{\infty} \leq 2\|g'\|_{\infty}, \quad \|f'_g\|_{\infty} \leq \frac{1}{\sigma} \|g'\|_{\infty}, \quad \|f''_g\|_{\infty} \leq \frac{2}{\sigma^2} \|g'\|_{\infty}.$$

Alternatively, for any $z \in \mathbb{R}$, defining f_z as the function satisfying

$$\sigma^2 f'_z(x) - (x - \mu) f_z(x) = \mathbb{1}_{x \leq z} - \Phi \left(\frac{x - \mu}{\sigma} \right),$$

we have that $f_z \in C^1(\mathbb{R})$ satisfying $\|f_z\|_{\infty} \leq \sigma^{-1}$, $\|f'_z\|_{\infty} \leq \sigma^{-1}$ and, for any $h \in \mathbb{R}$,

$$|(x + h - \mu) f_z(x + h) - (x - \mu) f_z(x)| \leq \frac{1}{\sigma} \left(\frac{|x - \mu|}{\sigma} + \frac{\sqrt{2\pi}}{4} \right) |h|.$$

The Wasserstein ($q = 1$) case is outlined in Section 6.1, while the more difficult Kolmogorov ($q = \infty$) case is treated in Section 6.2. The estimate for general q follows from these two cases by interpolation. Simultaneously, by utilising the independent node coupling from Section 5, the remaining dependence on X_t may be removed, and a bound on $\mathbb{E} S_t f$ obtained. Analogously to [20, Proposition 4.1], this involves two key estimates, one approximating $\mathbb{E}_t \langle \zeta_{t+1}, h \rangle$ by $\langle \zeta_t, D_t h \rangle$, and another for $\text{Var}_t \langle \zeta_{t+1}, h \rangle$ by $\sigma_{t+1}^2[h]$. The former is a direct consequence of Proposition 5.5:

$$\mathbb{E} |\mathbb{E}_t \langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle| \leq C \|h\|_{\infty} \kappa_t n^{-1/2} \sqrt{1 + \log n}, \quad (6.4)$$

with $C \leq 72\pi$. For the latter, by defining $V_t(\mathbf{x}) = n^{-1} \sum_{i=1}^n h_i^2 v_{i,t}(\mathbf{x})$ for $\mathbf{x} \in [0, 1]^n$, we have

$$\mathbb{E} |\sigma_{t+1}^2[h] - \text{Var}_t \langle \zeta_{t+1}, h \rangle| = \mathbb{E} |V_t(X_t) - V_t(\mathbf{p}_t)|.$$

Computing the derivatives of V_t reveals

$$\|DV_t\|_1 \leq n^{-1} \|h\|_{\infty}^2 (1 + \alpha_t), \quad \|DV_t\|_{2,1} \leq n^{-1/2} \|h\|_{\infty}^2 (1 + \alpha_t),$$

$$\|D^{(2)} V_t\|_{1,1} \leq \|h\|_{\infty}^2 (\gamma_t + 2\beta_t^2).$$

The desired estimate now follows by applying Theorem 5.1:

$$\mathbb{E} |\sigma_{t+1}^2[h] - \text{Var}_t \langle \zeta_{t+1}, h \rangle| \leq C \|h\|_{\infty}^2 \kappa_t n^{-1/2}, \quad (6.5)$$

with $C \leq 8\sqrt{\pi}$.

Remark 6.3. The $\log n$ term in Proposition 6.1 arises only from Eq. (6.4), and so may be removed, provided one can derive an $\mathcal{O}(n^{-1/2})$ bound for this term. For example, consider an occupancy process with

$$S_{i,t}(\mathbf{x}) = f_i \left(n^{-1} \sum_{j=1}^n s_{ij} x_j \right), \quad C_{i,t}(\mathbf{x}) = g_i \left(n^{-1} \sum_{j=1}^n c_{ij} x_j \right), \quad (6.6)$$

where $f_i, g_i \in \mathcal{C}^2([0, \infty))$ and $s_{ij}, c_{ij} \geq 0$, with $s_{ii} = c_{ii} = 0$. This particular process was studied in [4]. Denote $\mathbf{s}_i = (s_{i1}, \dots, s_{in})$ and similarly for \mathbf{c}_i . In this instance, the Hessian matrices of $S_{i,t}, C_{i,t}$ conveniently factorise into a sum of rank-one real-valued matrices scaled by real-valued functions, and for any $j, k \neq i$,

$$\partial_j \partial_k P_{i,t}(\mathbf{x}) = \frac{x_i s_{ij} s_{ik}}{n^2} f_i'' \left(\frac{1}{n} \sum_{l=1}^n s_{il} x_l \right) + \frac{(1-x_i) c_{ij} c_{ik}}{n^2} g_i'' \left(\frac{1}{n} \sum_{l=1}^n c_{il} x_l \right).$$

So by Taylor's Theorem,

$$\mathbb{E} |\mathbb{E}_t \langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle| \leq \frac{\|h\|_\infty}{n^{3/2}} \sum_{i=1}^n \|f_i''\|_\infty \mathbb{E} \langle \zeta_t, \mathbf{s}_i \rangle^2 + \|g_i''\|_\infty \mathbb{E} \langle \zeta_t, \mathbf{c}_i \rangle^2,$$

where here $\|f_i''\|_\infty$ is understood as the supremum of f_i over the convex hull of $\{n^{-1} \sum_{j=1}^n s_{ij}\}_{i=1}^n$, and likewise for $\|g_i''\|_\infty$. Observing that

$$n^{-1} \sum_{i=1}^n \|f_i''\|_\infty \|\mathbf{s}_i\|_\infty^2 + \|g_i''\|_\infty \|\mathbf{c}_i\|_\infty^2 \leq 2n\Gamma_t,$$

with the help of Theorem 5.1 it can be shown that

$$\mathbb{E} |\mathbb{E}_t \langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle| \leq C \|h\|_\infty \kappa_t n^{-1/2}, \quad (6.7)$$

with $C \leq 16\sqrt{\pi}$.

6.1. The Wasserstein metric

Following the arguments in [13], let $g \in \mathcal{C}^1(\mathbb{R})$ be arbitrary, and take f_g to be the solution to $\mathcal{A}_t f_g(x) = g(x) - \mathbb{E}_t g(\langle \tilde{\zeta}_{t+1}, h \rangle)$. Since $f_g \in \mathcal{C}^2(\mathbb{R})$, there exists a random variable $Y_{i,t}$ such that

$$\int_0^1 f'_g \left[\langle \zeta_{t+1}, h \rangle + \frac{h_i}{\sqrt{n}}(u - X_{i,t+1}) \right] du = f'_g(\langle \zeta_{t+1}, h \rangle) + \frac{h_i(1 - 2X_{i,t+1})}{2\sqrt{n}} f''_g(Y_{i,t}).$$

Thus, $|\mathbb{E}_t \tilde{S}_t f| \leq T_1 + T_2$, where

$$T_1 = |\sigma_{t+1}^2[h] - \text{Var}_t \langle \zeta_{t+1}, h \rangle| \mathbb{E}_t f'_g(\langle \zeta_{t+1}, h \rangle),$$

$$T_2 = \frac{1}{2n^{3/2}} \sum_{i=1}^n |h_i|^3 P_{i,t}(\mathbf{X}_t) [1 - P_{i,t}(\mathbf{X}_t)] \mathbb{E}_t (1 - 2X_{i,t+1}) f''_g(Y_{i,t}).$$

Using (6.5) to bound T_1 and Lemma 6.2 to bound the derivatives of f_g ,

$$\mathbb{E} |T_1| \leq \frac{C \|g'\|_\infty \|h\|_\infty^2 \kappa_t}{\sigma_{t+1}[h] \sqrt{n}}, \quad |T_2| \leq \frac{\|g'\|_\infty \|h\|_\infty^3}{\sigma_{t+1}^2[h] \sqrt{n}}.$$

Since $\sigma_{t+1}[h] \leq \|h\|_\infty$, it follows that

$$|\mathbb{E}\tilde{S}_t f_g| \leq \mathbb{E}|\tilde{S}_t f_g| \leq \frac{C\|g'\|_\infty\|h\|_\infty^3\kappa_t}{\sigma_{t+1}^2[h]\sqrt{n}},$$

from which (6.4) implies

$$|\mathbb{E}g(\langle\zeta_{t+1}, h\rangle) - \mathbb{E}g(\langle\tilde{\zeta}_{t+1}, h\rangle)| \leq C\sqrt{\frac{1+\log n}{n}} \cdot \frac{\|g'\|_\infty\|h\|_\infty^3\kappa_t}{\sigma_{t+1}^2[h]}, \quad (6.8)$$

which is exactly Proposition 6.1 with $q = 1$. Passing the upper bounds on universal constants through these arguments reveals that $C \leq 72\pi + 8\sqrt{\pi} + 1 < 242$.

6.2. The Kolmogorov metric

Our arguments follow closely those of [12, Theorem 3.4]. Let $z \in \mathbb{R}$ be arbitrary, and take f_z to be the solution to $\mathcal{A}_t f_z(x) = \mathbb{1}\{x \leq z\} - \mathbb{P}_t(\langle\tilde{\zeta}_{t+1}, h\rangle \leq z)$. By rearranging the Stein equation for f'_z and inserting into (6.2), we obtain

$$\begin{aligned} \sum_{i=1}^n v_{i,t} \left[\int_0^1 \mathbb{P}_t \left(\langle\zeta_{t+1}, h\rangle + \frac{h_i}{\sqrt{n}}(u - X_{i,t+1}) \leq z \right) du - \mathbb{P}_t(\langle\tilde{\zeta}_{t+1}, h\rangle \leq z) \right] \\ = \sigma_{t+1}^2[h]T_1 + T_2 - \sum_{i=1}^n v_{i,t} \int_0^1 \mathbb{E}_t I_{i,t}(u) du \end{aligned} \quad (6.9)$$

where $v_{i,t} = n^{-1}h_i^2 v_{i,t}(X_t)$, and T_1 , T_2 , $I_{i,t}(u)$ for $u \in [0, 1]$ are given by

$$T_1 = [\langle\zeta_t, D_t h\rangle - \mathbb{E}_t \langle\zeta_{t+1}, h\rangle] \mathbb{E}_t f_z(\langle\zeta_{t+1}, h\rangle)$$

$$T_2 = [\sigma_{t+1}^2[h] - \text{Var}_t \langle\zeta_{t+1}, h\rangle] \mathbb{E}_t [\{\langle\zeta_{t+1}, h\rangle - \langle\zeta_t, D_t h\rangle\} f_z(\langle\zeta_{t+1}, h\rangle)]$$

$$\begin{aligned} I_{i,t}(u) = & \left\{ \langle\zeta_{t+1}, h\rangle + \frac{h_i}{\sqrt{n}}(u - X_{i,t+1}) - \langle\zeta_t, D_t h\rangle \right\} \\ & \times f_z \left(\langle\zeta_{t+1}, h\rangle + \frac{h_i}{\sqrt{n}}(u - X_{i,t+1}) \right) \\ & - \{\langle\zeta_{t+1}, h\rangle - \langle\zeta_t, D_t h\rangle\} f_z(\langle\zeta_{t+1}, h\rangle). \end{aligned}$$

Applying Lemma 6.2 to $I_{i,t}$ implies

$$\mathbb{E}_t |I_{i,t}(u)| \leq \frac{1}{\sigma_{t+1}[h]} \left(\frac{|\langle\zeta_{t+1}, h\rangle - \langle\zeta_t, D_t h\rangle|}{\sigma_{t+1}[h]} + \frac{\sqrt{2\pi}}{4} \right) \frac{\|h\|_\infty}{\sqrt{n}}.$$

Now, since $\|f_z\|_\infty$ is bounded uniformly in z , the right-hand side of (6.9) may be bounded in magnitude independently of z . For the moment, let M denote such a bound. Then, since

$$\begin{aligned} \mathbb{P}_t \left(\langle\zeta_{t+1}, h\rangle + \frac{h_i}{\sqrt{n}}u \leq z + \frac{\|h\|_\infty}{\sqrt{n}} \right) & \geq \mathbb{P}_t(\langle\zeta_{t+1}, h\rangle \leq z), \\ \left| \mathbb{P}_t \left(\langle\tilde{\zeta}_{t+1}, h\rangle \leq z + \frac{\|h\|_\infty}{\sqrt{n}} \right) - \mathbb{P}_t(\langle\tilde{\zeta}_{t+1}, h\rangle \leq z) \right| & \leq \frac{\|h\|_\infty}{\sigma_{t+1}[h]\sqrt{2\pi n}}, \end{aligned}$$

substituting $z + n^{-1/2}\|h\|_\infty$ for z in (6.9) gives

$$\text{Var}_t \langle\zeta_{t+1}, h\rangle [\mathbb{P}_t(\langle\zeta_{t+1}, h\rangle \leq z) - \mathbb{P}_t(\langle\tilde{\zeta}_{t+1}, h\rangle \leq z)] \leq M + \frac{\|h\|_\infty^3}{\sigma_{t+1}[h]\sqrt{2\pi n}}.$$

By performing a similar procedure for the lower bound,

$$\sigma_{t+1}^2[h]|\mathbb{P}_t(\langle \zeta_{t+1}, h \rangle \leq z) - \mathbb{P}_t(\langle \tilde{\zeta}_{t+1}, h \rangle \leq z)| \leq M + \frac{2\|h\|_\infty^3}{\sigma_{t+1}[h]\sqrt{2\pi n}} + T_3, \quad (6.10)$$

where

$$T_3 = |\sigma_{t+1}^2[h] - \text{Var}_t\langle \zeta_{t+1}, h \rangle| |\mathbb{P}_t(\langle \zeta_{t+1}, h \rangle \leq z) - \mathbb{P}_t(\langle \tilde{\zeta}_{t+1}, h \rangle \leq z)|.$$

Inequalities (6.4) and (6.5) together with the estimate for $\|f_z\|_\infty$ in Lemma 6.2 immediately imply

$$\mathbb{E}|T_1| \leq \frac{C\|h\|_\infty \kappa_t}{\sigma_{t+1}[h]\sqrt{n}}, \quad \mathbb{E}|T_3| \leq \frac{C\|h\|_\infty^2 \kappa_t}{\sqrt{n}}.$$

Once again, since $\text{Var}_t\langle \zeta_{t+1}, h \rangle \leq \|h\|_\infty^2$,

$$\mathbb{E}_t|\langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle| \leq \|h\|_\infty + |\mathbb{E}_t\langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle|,$$

and by liberal use of (6.4) and (6.5) with Lemma 6.2,

$$\mathbb{E}|T_2| \leq C\sqrt{\frac{1+\log n}{n}} \cdot \frac{\|h\|_\infty^3 \kappa_t}{\sigma_{t+1}[h]}, \quad \mathbb{E}|I_{i,t}(u)| \leq \frac{C\|h\|_\infty^2 \kappa_t}{\sigma_{t+1}[h]\sqrt{n}}.$$

Altogether, combining these estimates with (6.9) and (6.10) gives

$$\|\mathcal{L}\langle \zeta_{t+1}, h \rangle - \mathcal{L}\langle \tilde{\zeta}_{t+1}, h \rangle\|_\infty \leq C\sqrt{\frac{1+\log n}{n}} \cdot \frac{\|h\|_\infty^4 \kappa_t}{\sigma_{t+1}^4[h]},$$

which is Proposition 6.1 with $q = \infty$. Passing the upper bounds on universal constants through these arguments reveals that $C \leq 216\pi + 9\sqrt{\pi} + 1 < 700$.

6.3. A central limit theorem

It now only remains to prove Corollary 2.2, once again making use of the one-step approximations $\tilde{\zeta}_t$. Unfortunately, Proposition 6.1 is insufficient as $\sigma_t^{-2}[h]$ is potentially unbounded. Instead, by utilising Proposition 6.1 in conjunction with a crude bound, the dependence on $\sigma_t[h]$ may be removed at the expense of a suboptimal exponent in n . For any function $g \in \mathcal{C}^1(\mathbb{R})$,

$$\begin{aligned} |\mathbb{E}g(\langle \zeta_{t+1}, h \rangle) - \mathbb{E}g(\langle \tilde{\zeta}_{t+1}, h \rangle)| &\leq \|g'\|_\infty \{|\mathbb{E}\langle \zeta_{t+1}, h \rangle - \mathbb{E}\langle \tilde{\zeta}_{t+1}, h \rangle| + \\ &\quad \mathbb{E}|\mathbb{E}_t\langle \zeta_{t+1}, h \rangle - \langle \zeta_t, D_t h \rangle| + \sigma_{t+1}[h]\}, \end{aligned}$$

and, since

$$\mathbb{E}|\langle \zeta_{t+1}, h \rangle - \mathbb{E}_t\langle \zeta_{t+1}, h \rangle| \leq \mathbb{E}|\sigma_{t+1}^2[h] - \text{Var}_t\langle \zeta_{t+1}, h \rangle|^{1/2} + \sigma_{t+1}[h],$$

it follows from (6.5) that

$$|\mathbb{E}g(\langle \zeta_{t+1}, h \rangle) - \mathbb{E}g(\langle \tilde{\zeta}_{t+1}, h \rangle)| \leq C\|g'\|_\infty (\sigma_{t+1}[h] + \|h\|_\infty \kappa_t n^{-1/4} \sqrt{1+\log n}).$$

By this argument, and proceeding as in the proof of Proposition 6.1, conditioning instead on X_1, \dots, X_t , we obtain Proposition 6.4.

Proposition 6.4. *There is a universal constant $C > 0$ such that, for any $t \geq 0$, any function $g \in \mathcal{C}^1(\mathbb{R})$, and any $h_1, \dots, h_{t+1} \in \mathbb{R}^n$,*

$$\left| \mathbb{E}g \left(\sum_{s=1}^t \langle \zeta_s, h_s \rangle + \langle \zeta_{t+1}, h_{t+1} \rangle \right) - \mathbb{E}g \left(\sum_{s=1}^t \langle \zeta_s, h_s \rangle + \langle \tilde{\zeta}_{t+1}, h_{t+1} \rangle \right) \right| \leq C \|g'\|_\infty \|h_{t+1}\|_\infty \kappa_t \cdot n^{-1/6} \sqrt{1 + \log n}.$$

Through the Cramér–Wold device, to prove [Corollary 2.2](#), we prove instead the stronger result that, for any collection of sequences $\{h_1^{(n)}, \dots, h_t^{(n)}\}_{n=1}^\infty$ with each $h_s^{(n)} \in \mathbb{R}^n$ and $\|h_s^{(n)}\|_\infty \leq H$ for some $H > 0$,

$$\lim_{n \rightarrow \infty} \left\| \mathcal{L} \left[\sum_{s=1}^t \langle \zeta_s^{(n)}, h_s^{(n)} \rangle \right] - \mathcal{L} \left[\sum_{s=1}^t \langle \xi_s^{(n)}, h_s^{(n)} \rangle \right] \right\|_1 = 0. \quad (6.11)$$

Proceeding by induction on the case $t = 1$ which holds by [Proposition 6.4](#), assume (6.11) holds for some $t \geq 1$. For each n , let $h_{t+1}^{(n)} \in \mathbb{R}^n$ be such that $\|h_{t+1}^{(n)}\|_\infty \leq H$. Immediately, from [Proposition 6.4](#),

$$\lim_{n \rightarrow \infty} \left\| \mathcal{L} \left[\sum_{s=1}^{t+1} \langle \zeta_s^{(n)}, h_s^{(n)} \rangle \right] - \mathcal{L} \left[\sum_{s=1}^t \langle \zeta_s^{(n)}, h_s^{(n)} \rangle + \langle \tilde{\zeta}_t^{(n)}, h_{t+1}^{(n)} \rangle \right] \right\|_1 = 0,$$

and furthermore, from the induction hypothesis,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \mathcal{L} \left[\sum_{s=1}^{t-1} \langle \zeta_s^{(n)}, h_s^{(n)} \rangle + \langle \zeta_t^{(n)}, h_t^{(n)} + D_t^{(n)} h_{t+1}^{(n)} \rangle \right] \right. \\ \left. - \mathcal{L} \left[\sum_{s=1}^{t-1} \langle \xi_s^{(n)}, h_s^{(n)} \rangle + \langle \xi_t^{(n)}, h_t^{(n)} + D_t^{(n)} h_{t+1}^{(n)} \rangle \right] \right\|_1 = 0. \end{aligned}$$

But now the contraction identity [\(6.1\)](#) tells us that

$$\lim_{n \rightarrow \infty} \left\| \mathcal{L} \left[\sum_{s=1}^t \langle \zeta_s^{(n)}, h_s^{(n)} \rangle + \langle \tilde{\zeta}_t^{(n)}, h_s^{(n)} \rangle \right] - \mathcal{L} \left[\sum_{s=1}^{t+1} \langle \xi_s^{(n)}, h_s^{(n)} \rangle \right] \right\|_1 = 0.$$

The variance of $\sum_{s=1}^{t+1} \langle \xi_s^{(n)}, h_s^{(n)} \rangle$ converges to the quantity given by [Corollary 2.2](#), and hence the result holds for any $t \geq 1$.

7. Long-term behaviour

An advantage to working with approximations of stochastic processes is that it is generally easier to study the stationary and long-term behaviour of the approximation than it is the original process. For example, we can identify a critical phase in the time-homogeneous case, where $P_{i,t} = P_i$, $i = 1, \dots, n$, does not depend on t . While Brouwer’s Fixed Point Theorem guarantees that there is at least one fixed point, most occupancy processes have a trivial fixed point anyway (for example, corresponding to extinction). Under sufficiently strict assumptions, it is possible to identify precisely globally stable equilibria of the deterministic system. We extend partial relations to vectors by pointwise comparison, and say that $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ and $\mathbf{x} \neq \mathbf{y}$. Defining $J_0 = \lim_{\mathbf{x} \rightarrow 0^+} DP(\mathbf{x})$ where $P = (P_1, \dots, P_n)$, and letting $r(J_0)$ denote the spectral radius of the matrix J_0 , the results of [\[47\]](#) are interpreted within our context as follows.

Theorem 7.1 ([47, Theorem 2.2]). Suppose that, for every $\mathbf{x} > \mathbf{0}$ and $i, j = 1, \dots, n$, $\partial_j P_i(\mathbf{x}) > 0$, and that $DP(\mathbf{x}) \geq DP(\mathbf{y})$ for all $\mathbf{0} < \mathbf{x} < \mathbf{y}$. Assume also that $P_i(\mathbf{1}) \neq 1$ for some $i = 1, \dots, n$. Then, the limit $\mathbf{p}_\infty := \lim_{t \rightarrow \infty} \mathbf{p}_t$ exists and is independent of $\mathbf{p}_0 \neq \mathbf{0}$. Furthermore, $\mathbf{p}_\infty = \mathbf{0}$ if and only if $\mathbf{0}$ is a fixed point of P and $r(J_0) \leq 1$.

If a globally stable equilibrium does indeed exist, it is natural to consider the limit random variable \mathbf{Z}_∞ . Fortunately, as an autoregressive process, conditions for ergodicity of \mathbf{Z}_t are well-known [42]. Let $V_t = \text{diag}(p_{i,t}(1 - p_{i,t}))_{i=1}^n$ and $\Sigma_t = \text{Cov}(\mathbf{Z}_t)$, and observe that Σ_t satisfies

$$\Sigma_{t+1} = DP(\mathbf{p}_t) \Sigma_t DP(\mathbf{p}_t)^\top + V_t.$$

Let $V_\infty := \lim_{t \rightarrow \infty} V_t$ and $J_\infty = DP(\mathbf{p}_\infty)$. If Σ_t converges to some matrix Q as $t \rightarrow \infty$, then it must satisfy the discrete Lyapunov equation

$$Q = J_\infty Q J_\infty^\top + V_\infty. \quad (7.1)$$

If such a Q exists, then, by vectorising (7.1), we see that it must also satisfy

$$\text{vec } Q = (I - J_\infty \otimes J_\infty)^{-1} \text{vec } V_\infty. \quad (7.2)$$

There are three immediate consequences of (7.2): (i) the solution to (7.1) is necessarily unique, (ii) a solution exists if and only if every pair of eigenvalues λ_1, λ_2 of J_∞ satisfies $\lambda_1 \lambda_2 \neq 1$, and, (iii) $Q = 0$ if and only if $\mathbf{p}_\infty \in \{\mathbf{0}, \mathbf{1}\}$. Altogether, we have

Proposition 7.2. Assume that $\mathbf{p}_\infty = \lim_{t \rightarrow \infty} \mathbf{p}_t$ exists. Then, $\Sigma_\infty := \lim_{t \rightarrow \infty} \Sigma_t$ exists if and only if $r(J_\infty) < 1$, in which case Σ_∞ is the solution to (7.1). Furthermore, when Σ_∞ exists, if $\mathbf{p}_\infty = \mathbf{0}$, then $\mathbf{Z}_t \xrightarrow{\mathbb{P}} \mathbf{0}$ as $t \rightarrow \infty$; otherwise

$$\mathbf{Z}_t \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{p}_\infty, \Sigma_\infty) \quad \text{as } t \rightarrow \infty. \quad (7.3)$$

Assuming the conditions of Proposition 7.2, while each $\mathbf{Z}_t^{(n)}$ is geometrically ergodic, to prove Corollary 2.3, we will show that the rate of convergence of $\langle \xi_t^{(n)}, h \rangle$ to $\langle \xi_\infty^{(n)}, h \rangle$ is uniform in n . This is accomplished in Lemma 7.3.

Lemma 7.3. Suppose that $\sup_n \|\mathbf{J}_\infty^{(n)}\|_1 < 1$, $\Gamma^{(n)} = \mathcal{O}(n^{-1})$, and $\sup_n \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. For any $h \in \ell^\infty$, there is a $\rho < 1$, and $C > 0$, independent of n, t , such that, for every $t \geq 1$ and $n \in \mathbb{N}$,

$$|\langle \bar{\mathbf{p}}_t^{(n)} - \bar{\mathbf{p}}_\infty^{(n)}, h \rangle| \leq C \|h\|_\infty \rho^{2t} \quad (7.4)$$

$$\|\mathcal{L}\langle \xi_t^{(n)}, h \rangle - \mathcal{L}\langle \xi_\infty^{(n)}, h \rangle\|_1 \leq C \|h\|_\infty t \rho^t. \quad (7.5)$$

Proof. For each $n \in \mathbb{N}$, representing the process $\mathbf{Z}_t^{(n)}$ by the recursion

$$\mathbf{Z}_{t+1}^{(n)} - \mathbf{p}_{t+1}^{(n)} = D_t(\mathbf{Z}_t^{(n)} - \mathbf{p}_t^{(n)}) + V_t^{1/2} \mathbf{z}_t, \quad (7.6)$$

where $\mathbf{z}_t \stackrel{\text{iid}}{\sim} \mathcal{N}(0, I)$ for each $t = 0, 1, \dots$, we introduce a coupled process $\tilde{\mathbf{Z}}_t^{(n)}$ satisfying $\tilde{\mathbf{Z}}_0^{(n)} = \mathbf{Z}_0^{(n)}$ and

$$\tilde{\mathbf{Z}}_{t+1}^{(n)} - \mathbf{p}_\infty^{(n)} = J_\infty^\top (\tilde{\mathbf{Z}}_t^{(n)} - \mathbf{p}_\infty^{(n)}) + V_\infty^{1/2} \mathbf{z}_t. \quad (7.7)$$

Additionally, by Proposition 7.2, $\mathbf{Z}_t^{(n)}$ has a limit as $t \rightarrow \infty$ for each $n \in \mathbb{N}$, which we shall denote by $\mathbf{Z}_\infty^{(n)}$ with $\text{Cov}(\mathbf{Z}_\infty^{(n)}) = \Sigma_\infty^{(n)}$. Denoting $\tilde{\xi}_t^{(n)} = n^{-1/2}(\tilde{\mathbf{Z}}_t^{(n)} - \mathbf{p}_t^{(n)})$ and

$\xi_\infty^{(n)} = n^{-1/2}(\mathbf{Z}_\infty^{(n)} - \mathbf{p}_\infty^{(n)})$, it suffices to bound $\|\mathcal{L}\langle \xi_t^{(n)}, h \rangle - \mathcal{L}\langle \tilde{\xi}_t^{(n)}, h \rangle\|_1$ and $\|\mathcal{L}\langle \tilde{\xi}_t^{(n)}, h \rangle - \mathcal{L}\langle \xi_\infty^{(n)}, h \rangle\|_1$. For the former, observe that

$$\begin{aligned}\|\mathbf{J}_\infty^{(n)} - \mathbf{D}_t^{(n)}\|_1 &= \max_{j=1, \dots, n} \sum_{i=1}^n |\partial_j P_i^{(n)}(\mathbf{p}_t^{(n)}) - \partial_j P_i^{(n)}(\mathbf{p}_\infty^{(n)})| \\ &\leq \Gamma^{(n)} \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1,\end{aligned}$$

and hence, for any $t \geq 0$,

$$\|\mathbf{p}_{t+1}^{(n)} - \mathbf{p}_\infty^{(n)}\|_1 \leq \|\mathbf{J}_\infty^{(n)}\|_1 \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1 + \Gamma^{(n)} \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1^2.$$

Letting $\rho^2 := \frac{1}{2}(1 + \sup_n \|\mathbf{J}_\infty^{(n)}\|_1) < 1$, by assumption, there exists some T such that, for every $t \geq T$ and $n \in \mathbb{N}$, $\|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1 \leq (1 - \rho^2)/\Gamma^{(n)}$. Therefore, $\|\mathbf{p}_{t+1}^{(n)} - \mathbf{p}_\infty^{(n)}\|_1 \leq \rho^2 \|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1$ for all $t \geq T$, and hence, for every $n \in \mathbb{N}$,

$$\|\mathbf{p}_t^{(n)} - \mathbf{p}_\infty^{(n)}\|_1 \leq Cn\rho^{2t}, \quad \text{for all } t \geq 0, \quad (7.8)$$

implying (7.4). Similarly, $\|\mathbf{D}_t^{(n)}\|_1 \leq \|\mathbf{J}_\infty^{(n)}\|_1 + \|\mathbf{J}_\infty^{(n)} - \mathbf{D}_t^{(n)}\|_1 \leq \rho^2$ for all $t \geq T$, and so, for all $0 \leq s < t$,

$$\|\mathbf{D}_{s,t}^{(n)}\|_\infty \leq C\rho^{2(t-s)}, \quad \|\tilde{\mathbf{D}}_{s,t}^{(n)}\|_\infty \leq \rho^{2(t-s)}. \quad (7.9)$$

Additionally, according to (7.1),

$$\|\Sigma_\infty^{(n)}\|_1 \leq \frac{1}{1 - \rho^2} \max_i p_{i,\infty}^{(n)} (1 - p_{i,\infty}^{(n)}) \leq \frac{1}{4(1 - \rho^2)}. \quad (7.10)$$

From the recursion formulae (7.6) and (7.7) and the identity $(a - b)^2 \leq |a^2 - b^2|$ for $a, b \geq 0$,

$$\text{Var}[\langle \xi_t^{(n)} - \tilde{\xi}_t^{(n)}, h \rangle] \leq \frac{1}{n} \sum_{s=1}^t \|\mathbf{D}_{s,t}^{(n)}\|_\infty^2 \|h\|_\infty^2 \|\mathbf{p}_\infty^{(n)} - \mathbf{p}_s^{(n)}\|_1.$$

Thus, from Jensen's inequality and inequalities (7.8) and (7.9),

$$\|\mathcal{L}\langle \xi_t^{(n)}, h \rangle - \mathcal{L}\langle \tilde{\xi}_t^{(n)}, h \rangle\|_1 \leq C\|h\|_\infty t \rho^t,$$

as required. Finally, since $\langle \tilde{\xi}_t^{(n)}, h \rangle$ and $\langle \xi_\infty^{(n)}, h \rangle$ are normally distributed,

$$\begin{aligned}\|\mathcal{L}\langle \tilde{\xi}_t^{(n)}, h \rangle - \mathcal{L}\langle \xi_\infty^{(n)}, h \rangle\|_1^2 &\leq |\text{Var}\langle \tilde{\xi}_t^{(n)}, h \rangle - \text{Var}\langle \xi_\infty^{(n)}, h \rangle| \\ &\leq \|\mathbf{J}_\infty^{(n)}\|_1^{2t} \|h\|_\infty^2 \|\Sigma_\infty^{(n)}\|_1,\end{aligned}$$

and so (7.10) implies that

$$\|\mathcal{L}\langle \tilde{\xi}_t^{(n)}, h \rangle - \mathcal{L}\langle \xi_\infty^{(n)}, h \rangle\|_1 \leq \|h\|_\infty \frac{\rho^t}{1 - \rho},$$

and hence (7.5). \square

The result of Lemma 7.3 implies that for any sequence of times $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$, $\|\mathcal{L}\langle \xi_{\tau_n}^{(n)}, h \rangle - \mathcal{L}\langle \xi_\infty^{(n)}, h \rangle\|_1 \rightarrow 0$ as $n \rightarrow \infty$. Additionally, if $\tau_n \leq \frac{1}{17\alpha}\epsilon \log n$ for $\epsilon > 0$, then Proposition 6.4 implies that

$$\|\mathcal{L}\langle \xi_{\tau_n}^{(n)}, h \rangle - \mathcal{L}\langle \xi_{\tau_n}^{(n)}, h \rangle\|_1 = \mathcal{O}(1)\|h\|_\infty n^{-1/6+\epsilon}(1 + \log n)^{3/2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Together, these two facts imply Corollary 2.3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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