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Interface in a one-dimensional Ising spin system

Thierry Bodineau*

ENS 45 rue d'Ulm 75005 Paris and CNRS 1321 University of Paris 7, France

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Abstract

In this paper we will be studying the interface in a one-dimensional Ising spin system with a ferromagnetic Kac potential $\gamma J(\gamma|r|)$. Below the critical temperature, when γ tends to 0, two distinct thermodynamic phases with different magnetizations appear. We will see that the local magnetization converges to one of these two values. On intervals of length γ^{-k} the local magnetization will stay almost constant, but on longer intervals interfaces take place between different phases. We prove first a large deviation principle and apply Friedlin and Wentzell theory to estimate the position where the first interface appears.

Keywords: Gibbs fields; Interfaces; Large deviations

1. Introduction

Kac et al. (1963) introduces a family of potentials which are the functions $\gamma \exp(-\gamma|r|)$ depending on the scaling parameter γ. Lebowitz and Penrose (1966) have used an extension of the Kac potentials to treat the liquid–vapor transition. In their paper, the Kac potentials model, as the parameter γ tends to 0, the long-range attractive part of the intermolecular forces.

Following the approach of Cassandro et al. (1993), we study the phenomenon of metastability in a one-dimensional Ising spin system with a ferromagnetic Kac potential. However, we use here different methods of proof and we generalize their results to more general single spin distributions.

One of the problems in the study of thermodynamical systems is to understand thoroughly the phenomenon of metastability. Many attempts have been made in this direction with dynamical systems in $\mathbb{R}^d$ driven by a vector field $b(x) = -\nabla a(x)$ where a is a double-well potential (see Galves et al., 1987; Mathieu, 1994). We get the same kind of results but our model does not allow us to use the well known methods of the stochastic differential equations (see Day, 1983). In particular we had to prove a large deviation principle and to get some decorrelation properties. In Cassandro et al.

* E-mail: bodineau@dmi.ens.fr.

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(1984), the reader will also find two examples of stochastic processes showing metastable behavior: the Curie–Weiss model and the contact process of Harris.

For their study Cassandro et al. (1993) used different spatial scales. To simplify, we have only kept the scaling parameter γ and we use weak neighborhoods at the macroscopic scale. We consider the one-dimensional interacting spin system at the limit as γ goes to 0. The interaction between any two given spins becomes smaller but the range increases to infinity. Instead of studying the discrete system, we map the discrete configurations to obtain block spins. Former works with block spins by Eisele and Ellis (1983) and by Comets (1987) in dynamics were done on finite regions, whereas in this paper, we are interested in infinite regions. Working with an infinite volume is one of the major problems because we know only the conditional probabilities of the Gibbs measure on bounded intervals and we do not have an accurate expression of the Gibbs measure as in the case of the torus. However, as in the mean field model the local magnetization tends to equilibrium values, below the critical temperature. These values, denoted $\pm m_\beta$, are completely insensitive to the details of J if we assume that J satisfies a normalization condition. Some other phenomena appear with the infinite volume; in particular we get a metastability property.

First we establish a large deviation principle when γ tends to 0. We change the proof of Cassandro et al. (1993) and use heavily some of the properties of the one-dimensional Gibbs measure. Some geometrical remarks make our method technically simpler and allow us to work on block spins without considering the discrete system. As a direct consequence of the large deviation estimates, we prove that the profiles are rigid on intervals of length γ^{-k} as the temperature decreases. The trajectories, however, are no longer constant when observed on much longer spatial intervals.

In the second part we are interested in studying the average distance where the local magnetization changes from one equilibrium value to the other. To get an intuitive picture of what happens we can see the local magnetization as a continuous process which oscillates about an equilibrium value denoted as m_β. After an exponential distance the process will perform an abrupt transition to the other equilibrium. The smaller the value of γ, the further is the position where the first transition occurs. By replacing time with space, we make an analogy with dynamical systems. The metastability property could be interpreted in terms of exit time from a neighborhood of a stable equilibrium position (see Freidlin and Wentzell, 1983). The action functional found in the first part will play the role of the double-well potential introduced by Galves et al. (1987). In this article the minima of the rate functional are the symmetric thermodynamic magnetizations $\pm m_\beta$ which have already been calculated by Eisele and Ellis (1983). The Markov property of the Gibbs measure, which is peculiar to one dimension, enables us to adapt the Friedlin–Wentzell proof. This specific property linked to the one-dimensional case does not allow us to generalize to higher dimensions.

In Section 2 we describe the model and state some general results which will be used in subsequent sections. In Section 3 we prove a large deviation principle and in Section 4 we study the position where the first interface appears.

2. The model

In this section we introduce some notation and recall well known definitions in equilibrium statistical mechanics.

2.1. Description of the system

We study a one-dimensional spin system with values in $[-1, 1]$. Let S_i be the value of the spin at i. A spin configuration is a sequence $S = \{S_i; i \in \mathbb{Z}\}$ which belongs to the phase space called $\Omega = [-1, 1]^{\mathbb{Z}}$. We are interested only in average values like the local magnetization, so we are led to define a continuous version of the system with block spins in order to use a weak topology which naturally introduces local averages.

Definition 2.1. We denote by E the space of magnetic profiles. The set E is a subset of $\mathscr{L}^{\infty}(\mathrm{d}r, \mathbb{R})$, the set of the bounded measurable functions.

$$E = \{\sigma \in \mathscr{L}^{\infty}(\mathrm{d}r, \mathbb{R}) | \|\sigma\|_{\infty} \leqslant 1\}.$$

Definition 2.2. We consider the $\mathscr{L}^2_{\mathrm{loc}}$ weak topology τ on E which satisfies

$$m_n \to_{\tau} m \;\Leftrightarrow\; \forall\, L \in \mathbb{R}, \lim_{n \to 0} m_n|_{[-L,L]} \text{ weakly in } \mathscr{L}^2(\mathrm{d}r, [-L, L]).$$

Definition 2.3. Let κ be the function from Ω to E which maps the configuration S to the piecewise constant function σ_γ in E defined by

$$\text{for all } x \text{ in } \mathbb{R}, \quad \sigma_\gamma(x) = S_{[x/\gamma]}. \tag{2.1}$$

2.2. The interactions

Let J be a continuous function on $\mathbb{R}$ and we assume that J satisfies the following properties:

$$J(r) = J(-r), \tag{2.2}$$

$$\text{for } r \text{ in } [-1, 1]^{\mathrm{c}}, \quad J(r) = 0, \tag{2.3}$$

$$\text{for } |r| < 1, \quad J(r) > 0, \tag{2.4}$$

$$\int J(r)\,\mathrm{d}r = 1. \tag{2.5}$$

Definition 2.4. A family of Kac potentials is a family of functions J_γ depending on the scaling parameter γ. These functions are defined in terms of J by the rule:

$$\text{for all } r \text{ in } \mathbb{R}, \quad J_\gamma(r) = \gamma J(\gamma r). \tag{2.6}$$

Furthermore, we assume for the sake of simplicity that the parameter γ takes values in the set $\{2^{-n} | n \in \mathbb{N}\}$; by slight modification of the proof this condition can be dropped.

When γ is fixed, the particles are in interaction under the potential J_γ. If Δ is a finite subset of $\mathbb{Z}$, the energy of the configuration $S_\Delta = \{S_i;\ i \in \Delta\}$ given the external condition $\xi = \{\xi_i;\ i \in \Delta^c\}$ is

$$H_\gamma^\Delta(S_\Delta | \xi) = -\frac{1}{2} \sum_{\substack{i \neq j \\ i,j \in \Delta}} J_\gamma(i-j) S_i S_j - \sum_{\substack{i \in \Delta \\ j \in \Delta_c}} J_\gamma(i-j) S_i \xi_j. \tag{2.7}$$

According to physical considerations we know that the most likely configurations are the ones with the lowest energy. The conditions imposed on J imply that the interaction is symmetric (2.2), has a finite range (2.3) and is ferromagnetic (2.4). Being ferromagnetic means that on the average the spins are oriented in the same direction in order to minimize the energy of the configuration. We note that the neigborhood interacts with the configuration $S_\Delta = \{S_i;\ i \in \Delta\}$ only on the frontier

$$\delta\Delta = \{i \in \Delta^c | \exists j \in \Delta,\ J_\gamma(i-j) > 0\}.$$

Let p be a symmetric measure on $\mathbb{R}$ with bounded support, which we take to be $[-1, 1]$ for simplicity. We will impose additional hypotheses on p later. For each γ we define at the temperature $1/\beta$ a Gibbs measure on Ω.

Definition 2.5. For each finite subset Δ in $\mathbb{Z}$ we introduce a probability measure on $[-1, 1]^\Delta$

$$\tilde{\mu}_{\beta,\gamma}^\Delta(S_\Delta | \xi) = \frac{\exp(-\beta H_\gamma^\Delta(S_\Delta | \xi)) \prod_{i \in \Delta} p(\mathrm{d}S_i)}{Z_{\beta,\gamma}^\Delta(\xi)}, \tag{2.8}$$

where $Z_{\beta,\gamma}^\Delta(\xi)$ is the normalization factor

$$Z_{\beta,\gamma}^\Delta(\xi) = \int \exp(-\beta H_\gamma^\Delta(S_\Delta \xi)) \prod_{i \in \Delta} p(\mathrm{d}S_i).$$

The measure $\tilde{\mu}_{\beta,\gamma}^\Delta(S_\Delta | \xi)$ is called the Gibbs distribution in Δ with boundary condition $\xi = \{\xi_i | i \in \Delta^c\}$. In our case, there is a unique measure $\tilde{\mu}_{\beta,\gamma}$ on Ω which is defined by the conditional probabilities above.

Eq. (2.8) above are called the Dobrushin–Landford–Ruelle equations denoted by DLR (see Georgii, 1988). We insist on dimension one and on finite range interactions, so that there would be a unique solution $\tilde{\mu}_{\beta,\gamma}$ to the DLR equations for all positive γ. Furthermore, the Gibbs measure has the Markov property that we will use later. Let us define $\mu_{\beta,\gamma}$ as the image of $\tilde{\mu}_{\beta,\gamma}$ under the mapping κ.

The inverse temperature β modifies the strength of the interactions. The larger the value of β, the more the spins interact with their neighborhoods. As β passes above the critical value denoted by β_c, two distinct thermodynamical phases appear and the local magnetization is close to two distincts values $\pm m_\beta$. Before going on to the large deviation principle, we will describe the additional conditions needed on p and we will compute β_c and m_β.

2.3. *Hypotheses on the measure p*

We recall that p is a symmetric measure on $\mathbb{R}$ with support $[-1, 1]$. We assume that the log of the Laplace transform of p

$$\Lambda(\alpha) = \ln\left(\int \exp(x\alpha)p(\mathrm{d}x)\right)$$

is finite and strictly convex on $\mathbb{R}$. We assume in addition that

$$\text{for all } t \geqslant 0, \text{ the third derivative } \Lambda'''(t) \leqslant 0. \tag{2.9}$$

This condition is connected to some usual properties of physical systems and is satisfied for any distribution p in the GHS class. We refer the reader to Ellis for a precise definition (1985) and discussion (1976). The Bernoulli measure $p = \frac{1}{2}(\delta_1 + \delta_{-1})$ which defines the case of spins taking values $\{-1, 1\}$ belongs to GHS. We generalize the paper of Cassendro et al. (1993) which deals only with the Bernoulli measure.

We assume (2.9) in order that the system has only two stable equilibria, though the interface problem is reduced to transition from one to another with a symmetry property. Without (2.9) it may happen that more stable equilibria exist, and the characteristics of the interface will depend on those equilibria which are apart. Eisele and Ellis (1988) have prove more general conditions to get only one second-order phase transition and no other phase transition.

The Legendre–Fenchel transform of Λ is denoted Λ^*. The function f, defined by

$$\text{for all } x \text{ in } \mathbb{R}, \quad f(x) = \Lambda^*(x) - \frac{\beta}{2}x^2, \tag{2.10}$$

plays an important role in the qualitative behavior of the system.

Lemma 2.1. *The critical value of the system is*

$$\beta_c = \frac{1}{\int x^2 p(\mathrm{d}x)}.$$

More precisely, f has a unique minimum $x = 0$ for β smaller than β_c and for each β above β_c, the function f admits two symmetric minima denoted $\pm m_\beta$. Furthermore, we have

$$\exists c > 0, \quad \forall m \in \mathbb{R}, \quad |f(m) - f(m_\beta)| \geqslant c(|m| - m_\beta)^2. \tag{2.11}$$

The inequality (2.11) will be relevant in Section 4 to prove the large deviation inequality for open sets.

Proof. By condition (2.9) we see that Λ' is concave on $\mathbb{R}^+$. This implies that the inverse function $\Lambda^{*\prime}$ is convex on the interval $[0, 1[$.

So for all $\beta > \Lambda^{*\prime\prime}(0) = 1/\int x^2 p(\mathrm{d}x)$ there is only one positive solution m_β of the equation $f'(m) = 0$. Using the convexity of $\Lambda^{*\prime}$, we get also

$$f''(m_\beta) > 0.$$

This completes the proof of the lemma. $\square$

In this paper we are only interested in the case of β above β_c.

3. A large deviation principle: preliminary results

The purpose of this section is to prove a large deviation principle for independent identically distributed (i.i.d.) variables, as well as regularity properties of some rate function which will be used in the next section to obtain large deviation estimates on the measures $\mu_{\beta,\gamma}$.

3.1. The i.i.d. case

Let I be a bounded interval of $\mathbb{R}$. We use the weak topology on $\mathscr{L}^2(\mathrm{d}r, I)$ and denote by $\langle .,. \rangle$ the duality brackets.

Definition 3.1. Let E_I be the set of measurable functions with values in $[-1, 1]$, that is

$$E_I = \{\sigma|_I | \sigma \in E\}. \tag{3.1}$$

We give E_I the restriction of the weak topology on $\mathscr{L}^2(\mathrm{d}r, I)$.

Let ρ_γ^I be the image on E_I of the product probability measure $\otimes_{\mathbb{Z}} P$ under the mapping $(S_i)_{i\in\mathbb{Z}} \to \sigma_\gamma(\cdot)1_I$, with σ_γ defined in (2.1).

Theorem 3.1. *The measure ρ_γ^I obeys a large deviation principle on E_I with rate function*

$$E_I \to [0, +\infty[$$

$$\sigma \to \int_I \Lambda^*(\sigma(r))\,\mathrm{d}r.$$

Proof. We suppose without loss of generality that I is the interval $[-n, n]$, where n is any integer. This statement derives from a more general theorem which requires some hypotheses that we are going to check. First we prove a result which describes the asymptotics of the Laplace transform of ρ_γ^I.

Lemma 3.1. *For all f in $\mathscr{L}^2(\mathrm{d}r, I)$, we have*

$$\lim_{\gamma\to 0} \gamma \ln\left[\rho_\gamma^I\left(\exp\left(\frac{\langle \sigma 1_I, f\rangle}{\gamma}\right)\right)\right] = \int_I \Lambda(f(r))\,\mathrm{d}r.$$

Proof. Let $(X_i)_{i\in N}$ be an i.i.d. sequence with law p, we see that

$$\rho_\gamma^I\left(\exp\left(\frac{\langle \sigma 1_I, f\rangle}{\gamma}\right)\right) = E\left(\exp\left(\sum_{i=-[n/\gamma]}^{[n/\gamma]-1} \frac{X_i \int_{i\gamma}^{(i+1)\gamma} f(s)\,\mathrm{d}s}{\gamma}\right)\right),$$

since (X_i) are independent, we have

$$\rho_\gamma^I\left(\exp\left(\frac{\langle \sigma 1_I, f\rangle}{\gamma}\right)\right) = \prod_{i=-[n/\gamma]}^{[n/\gamma]-1} \left(\exp\left(\Lambda\left(\frac{\int_{i\gamma}^{(i+1)\gamma} f(s)\,\mathrm{d}s}{\gamma}\right)\right)\right),$$

We define the function f_γ by

$$f_\gamma = \sum_{i=-[n/\gamma]}^{[n/\gamma]-1} 1_{[i\gamma,(i+1)\gamma[}\frac{1}{\gamma}\int_{i\gamma}^{(i+1)\gamma} f(s)\,\mathrm{d}s.$$

We know that f_γ converges strongly in $\mathscr{L}^1(\mathrm{d}r, I)$ to f as γ tends to 0. As Λ is Lipschitz continuous, we get for all f in $\mathscr{L}^2(\mathrm{d}r, I)$

$$\int_I \Lambda(f_\gamma) \to \int_I \Lambda(f).$$

The lemma follows. □

The compactness of E_I ensures the tightness of the family $\{\rho_\gamma^I\}$. Furthermore, we check that $f \to \int_I \Lambda^*(f)\,\mathrm{d}s$ is the Legendre transform of $f \to \int_I \Lambda(f)$. Due to the assumption that Λ^* is strictly convex, we see that $f \to \int_I \Lambda^*(f(s))\,\mathrm{d}s$ is a strictly convex functional. Since the previous hypotheses hold, by applying Theorem 1.1 of Baldi (1988), we obtain the large deviation principle claimed above. □

3.2. *Study of the action functional*

From now on we fix $\beta > \beta_c$.

We define on E the functional

$$\mathscr{F}(\sigma) = \int_{\mathbb{R}} [f(\sigma(r)) - f(m_\beta)]\,\mathrm{d}r + \frac{\beta}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} J(r-r')(\sigma(r)-\sigma(r'))^2\,\mathrm{d}r\,\mathrm{d}r'. \tag{3.2}$$

We note that $\mathscr{F}$ is the generalization to the infinite model of the rate functional found by Eisele and Ellis (1983) in the case of the torus. To simplify the calculation we define $\mathscr{F}_\Delta$, when Δ is a bounded interval containing 0, by

$$\mathscr{F}_\Delta(\sigma) = \int_\Delta \Lambda^*(\sigma) - \frac{\beta}{2}\langle J * \sigma 1_\Delta, \sigma 1_\Delta\rangle. \tag{3.3}$$

Let $\sigma|_\Delta \otimes \xi$ be the extension of the profile σ by the profile ξ outside Δ.

Lemma 3.2. *For any pair $\lambda = (\lambda^+, \lambda^-)$ in $\{-1, 1\}^2$, we introduce the function*

$$\chi_\lambda = \lambda^+ m_\beta 1_{\mathbb{R}^+} + \lambda^- m_\beta 1_{\mathbb{R}^{-*}} \tag{3.4}$$

Since σ is any profile in E, we get

$$\mathscr{F}(\sigma|_\Delta \otimes \chi_\lambda) = \mathscr{F}_{\Delta+\delta\Delta}(\sigma|_\Delta \otimes \chi_\lambda) - \mathscr{F}_{\Delta+\delta\Delta}(m_\beta). \tag{3.5}$$

This proof is a straightforward computation and is left to the reader.

Theorem 3.2. *$\mathscr{F}$ is a lower semicontinuous function on E.*

Proof. Let I be a bounded interval. We denote by $\delta I + I$ the set

$$\{x \in \mathbb{R} | \exists y \in I \ |x - y| \leqslant 1\}.$$

We introduce another functional

$$\mathscr{G}_I(\sigma) = \int_I A^*(\sigma) + \frac{\beta}{2}\int_I \mathrm{d}r \int_{I^c} \mathrm{d}r'\, \sigma^2(r) J(r-r')$$
$$-\frac{\beta}{2}\langle J * \sigma 1_I, \sigma 1_I\rangle - \beta\langle J * \sigma 1_I, \sigma 1_{I^c}\rangle - \int_I f(m_\beta)\,\mathrm{d}r, \tag{3.6}$$

which satisfies

$$\mathscr{F}(\sigma) = \mathscr{G}_I(\sigma) + \int_{I^c} [f(\sigma(r)) - f(m_\beta)]\,\mathrm{d}r + \frac{\beta}{4}\int_{I^c}\int_{I^c} J(r-r')(\sigma(r) - \sigma(r'))^2\,\mathrm{d}r\,\mathrm{d}r'.$$

By the equation $\mathscr{F}(\sigma) = \sup_I(\mathscr{G}_I(\sigma))$, we see that the lower semicontinuity of $\mathscr{F}$ could be deduced from the one of $\mathscr{G}_I$. Hence, the proof will be complete once we show the lower semicontinuity of all the terms which compose $\mathscr{G}_I$. The support of $\mathscr{G}_I$ is included in $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$ so we just have to prove the result for the weak topology of $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$.

Let h be the function defined by

$$\text{for all } \sigma \text{ in } E, \quad h(\sigma) = \langle J * \sigma 1_I, \sigma 1_I\rangle.$$

We will prove the continuity of h for the weak topology of $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$.

We note that $\mathscr{H}_I = \{J * \sigma 1_I | \sigma \in E\}$ is a subset in $\mathscr{C}(I + \delta I)$, bounded for the norm $\|\cdot\|_\infty$. Then, as I is compact, J is continuous and each σ in E is uniformly bounded, we check that $\mathscr{H}_I$ is uniformly equicontinuous. From this and from the Ascoli theorem, there exists, for each ε positive, a finite set of continuous functions $\{g_i\}_{i \leqslant N}$ which satisfy the condition

$$\mathscr{H}_I \subset \bigcup_{i \leqslant N} \{f | \forall x \in I + \delta I, \quad |f(x) - g_i(x)| < \varepsilon\}.$$

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of profiles converging to σ in $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$. Because

$$|h(\sigma_n) - h(\sigma)| \leqslant |\langle J * \sigma_n 1_I, (\sigma_n - \sigma)1_I\rangle| + |\langle J * \sigma 1_I, (\sigma_n - \sigma)1_I\rangle|,$$

we have

$$|h(\sigma_n) - h(\sigma)| \leqslant \max_{i \leqslant N} [|\langle g_i, (\sigma_n - \sigma)1_I \rangle| + |\langle J * \sigma 1_I, (\sigma_n - \sigma)1_I \rangle| + \varepsilon].$$

Therefore,

$$\limsup_{n \to \infty} |h(\sigma_n) - h(\sigma)| \leqslant \varepsilon.$$

So h is continuous for the topology τ. An analogous argument can be used to show that the function $\sigma \to \langle J * \sigma 1_I, \sigma 1_{I^c} \rangle$ is also continuous.

We define the function $\tilde{h}$ by

$$\text{for } \sigma \text{ in } E, \quad \tilde{h}(\sigma) = \int_I \mathrm{d}r \int_{I^c} \mathrm{d}r\, J(r - r')\sigma(r)^2.$$

The function $\tilde{h}$ is convex, continuous for the strong topology of $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$. This implies the lower semicontinuity for the weak topology of $\mathscr{L}^2(\mathrm{d}r, I + \delta I)$. Finally, noticing that $\sigma \to \int_I \Lambda^*(\sigma(r))\,\mathrm{d}r$ is the Legendre–Fenchel transform of $f \to \int_I \Lambda(f(r))\,\mathrm{d}r$, we check the lower semicontinuity of $\mathscr{G}_I$. $\square$

Using the previous notation we introduce the sets $\mathscr{G}_I^\varepsilon$

Definition 3.2. For each ε positive and for each bounded interval I, we define the set $\mathscr{G}_I^\varepsilon$ by

$$\mathscr{G}_I^\varepsilon = \{1_I g_i\}_{i \leqslant N} \tag{3.7}$$

4. A large deviation principle for $\mu_{\beta,\gamma}$

In this section we shall use the previous results to obtain large deviation estimates for $\mu_{\beta,\gamma}$.

4.1. Notation

Definition 4.1. Let V_ε be the weak neighborhood of 0 in E defined by

$$f \in V_\varepsilon \Leftrightarrow \forall g \in \mathscr{G}_{[0,1]}^\varepsilon, \quad |\langle g, f \rangle| < \varepsilon, \tag{4.1}$$

with $\mathscr{G}_{[0,1]}^\varepsilon$ as in the Definition 3.2.

Definition 4.2. We denote by T the translation operator on E

$$\text{for } L \text{ in } \mathbb{R}, \text{ for } \sigma \text{ in } E, \quad T_L(\sigma) = \sigma(\,.\,-L). \tag{4.2}$$

Definition 4.3. We introduce the closed set $D^{l^+,l^-}_{\lambda^+,\lambda^-}(\varepsilon)$ which contains the profiles close to $\lambda^+ m_\beta$ around the location l^+ and close to $\lambda^- m_\beta$ around the location $-l^-$

$$\forall(\lambda^+,\lambda^-)\in\{-1,1\}^2,\ \forall(l^+,l^-)\in\mathbb{N}^2,$$
$$D^{l^+,l^-}_{\lambda^+,\lambda^-}(\varepsilon)=(T_{(l^+)}\bar{V}_\varepsilon+\chi_\lambda)\cap(T_{(l^++1)}\bar{V}_\varepsilon+\chi_\lambda)\cap(T_{(-l^--1)}\bar{V}_\varepsilon+\chi_\lambda)$$
$$\cap(T_{(-l^--2)}\bar{V}_\varepsilon+\chi_\lambda),$$

where $\bar{V}_\varepsilon$ is the closure of V_ε and χ_λ is defined in (3.4).

4.2. Large deviation inequality for closed sets

Let us state the large deviation inequality for closed sets.

Theorem 4.1. *Let F be a set in E closed for the topology τ. We get*

$$\limsup_{\gamma\to 0}\gamma\ln\mu_{\beta,\gamma}(F)\leqslant -\inf_F\mathscr{F}(\sigma). \tag{4.3}$$

Proof. We divide this proof into 5 steps.

Step 1: We assume that F is a closed cylinder set with basis $I=[-a,a]$, i.e.

$$\sigma\in F\ \Leftrightarrow\ \forall\sigma'\in E,\ \sigma|_I\otimes\sigma'\in F.$$

We fix λ^+,λ^- and l^+,l^- (with $l^+>a$ and $l^->a$) and ε positive. Let D_ε be the shorthand of $D^{l^+,l^-}_{\lambda^+,\lambda^-}(\varepsilon)$ and $\Delta=[-l^-,l^+]$. Our first goal will be to prove the inequality

$$\limsup_{\gamma\to0}\gamma\ln\mu_{\beta,\gamma}(F\cap D_\varepsilon)$$
$$\leqslant-\inf_{F\cap D_\varepsilon}\mathscr{F}(\sigma)+\left[\limsup_{\gamma\to0}\gamma\ln\mu_{\beta,\gamma}(D_\varepsilon)+\inf_E\mathscr{F}(\sigma|_\Delta\otimes\chi_\lambda)\right]+O(\varepsilon). \tag{4.4}$$

We notice that with a small error we can write a continuous version of the energy H^Δ_γ introduced in (2.7)

$$H^\Delta_\gamma(\sigma)=-\frac{1}{2\gamma}\langle J*\sigma 1_\Delta,\sigma 1_\Delta+2\sigma 1_{\Delta^c}\rangle+\frac{1}{\gamma}|\Delta|^2O(\gamma), \tag{4.5}$$

where the error $O(\gamma)$ is a function which tends to 0 as γ goes to 0. Because of the definition of V_ε (4.1), we get

$$\forall m\in D_\varepsilon,\ \forall\sigma\in E,\quad|\langle J*\sigma 1_\Delta,2(m-\chi_\lambda)1_{\delta\Delta}\rangle|\leqslant O(\varepsilon), \tag{4.6}$$

where $O(\varepsilon)$ tends to 0 as ε goes to 0.

Using the continuous version of the energy (4.5), we have

$$\mu_{\beta,\gamma}(F\cap D_\varepsilon)$$
$$=\int d\mu_{\beta,\gamma}(m)1_{D_\varepsilon}\rho^\Delta_\gamma\left(\frac{1_F}{Z_\Delta(m)}\exp\left(\frac{\beta}{2\gamma}\left(\langle J*\sigma 1_\Delta,\sigma 1_\Delta+2m1_{\delta\Delta}\rangle+\frac{|\Delta|^2O(\gamma)}{\gamma}\right)\right)\right).$$

Therefore, by applying (4.6) we modify the external conditions with a slight error and we see that

$$\mu_{\beta,\gamma}(F \cap D_\varepsilon)$$
$$= \mu_{\beta,\gamma}(D_\varepsilon)\rho_\gamma^\Delta\left(\frac{1_F}{Z_\Delta(\chi_\lambda)}\exp\left(\frac{\beta}{2\gamma}\left(\langle J * \sigma 1_\Delta, \sigma 1_\Delta + 2\chi_\lambda 1_{\delta\Delta}\rangle + \frac{|\Delta|^2 O(\gamma) + O(\varepsilon)}{\gamma}\right)\right)\right).$$

Combining Theorem 3.1 and Varadhan's theorem, we obtain

$$\limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(F \cap D_\varepsilon))$$
$$\leqslant \limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(D_\varepsilon)) + \sup_F\left(\frac{\beta}{2}\langle J * \sigma 1_\Delta, \sigma 1_\Delta + 2\chi_\lambda 1_{\delta\Delta}\rangle - \int_\Delta \Lambda^*(\sigma)\right)$$
$$- \sup_E\left(\frac{\beta}{2}\langle J * \sigma 1_\Delta, \sigma 1_\Delta + 2\chi_\lambda 1_{\delta\Delta}\rangle - \int_\Delta \Lambda^*(\sigma)\right) + O(\varepsilon).$$

As a consequence of the equation below

$$\frac{\beta}{2}\langle J * \sigma 1_\Delta, \sigma 1_\Delta + 2\chi_\lambda 1_{\delta\Delta}\rangle - \int \Lambda^*(\sigma)$$
$$= -\mathscr{F}_{\Delta+\delta\Delta}(\sigma|_\Delta \otimes \chi_\lambda) + \int_{\delta\Delta} \Lambda^*(m_\beta) - \frac{\beta}{2}\langle J * \chi_\lambda 1_\Delta, \chi_\lambda 1_\Delta\rangle,$$

we get

$$\limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(F \cap D_\varepsilon))$$
$$\leqslant \limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(D_\varepsilon)) - \inf_F(\mathscr{F}_{\Delta+\delta\Delta}(\sigma|_\Delta \otimes \chi_\lambda) - \mathscr{F}_{\Delta+\delta\Delta}(m_\beta))$$
$$+ \sup_E(\mathscr{F}_{\Delta+\delta\Delta}(\sigma|_\Delta \otimes \chi_\lambda) - \mathscr{F}_{\Delta+\delta\Delta}(m_\beta)) + O(\varepsilon).$$

By applying Lemma 3.2, we see that

$$\limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(F \cap D_\varepsilon))$$
$$\leqslant \limsup_{\gamma\to 0} \gamma \ln(\mu_{\beta,\gamma}(D_\varepsilon)) - \inf_F \mathscr{F}(\sigma|_\Delta \otimes \chi_\lambda) + \sup_E \mathscr{F}(\sigma|_\Delta \otimes \chi_\lambda) + O(\varepsilon).$$

Since σ is in F, then $\sigma|_\Delta \otimes \chi_\lambda$ is in $F \cap D_\varepsilon$ and we have

$$\inf_F(\mathscr{F}(\sigma|_\Delta \otimes \chi_\lambda)) \geqslant \inf_{F \cap D_\varepsilon}(\mathscr{F}(\sigma)).$$

The inequality (4.4) now follows immediately.

Step 2: In the previous step, we have reduced the proof to the computation of $\mu_{\beta,\gamma}(D_\varepsilon)$. Two cases remain:

Case 1: $\lambda^+ = \lambda^-$. Noticing that for any pair (λ^-, λ^+) such that $\lambda^- = \lambda^+$, we have

$$\inf_E \mathscr{F}(\sigma|_\Delta \otimes \chi_\lambda) \leqslant \mathscr{F}(m_\beta) = 0,$$

$$\limsup_{\gamma \to 0} \gamma \ln \mu_{\beta,\gamma}(D_\varepsilon) \leqslant 0,$$

and using the inequality (4.4), we check that

$$\limsup_{\gamma \to 0} \gamma \ln \mu_{\beta,\gamma}(F \cap D_\varepsilon) \leqslant -\inf_{F \cap D_\varepsilon} \mathscr{F}(\sigma) + O(\varepsilon). \tag{4.7}$$

Case 2: $\lambda^+ = -\lambda^-$. We must now handle the case in which $\mu_{\beta,\gamma}(D_\varepsilon)$ is the probability of leaping from one equilibrium value to another between the positions l^- and l^+. One of the main problem due to the infinite volume is that we had to compute directly the probability $\mu_{\beta,\gamma}(D_\varepsilon)$. We cannot iterate the conditioning argument. To deal with this difficulty we have used a symmetry property of the Gibbs measure. Our argument is completely different than that of Cassandro et al. (1993).

Lemma 4.1.

$$\limsup_{\gamma \to 0} \gamma \ln \mu_{\beta,\gamma}(D_\varepsilon) \leqslant -\inf_{D_\varepsilon} \mathscr{F}(\sigma) + O(\varepsilon). \tag{4.8}$$

Proof. We deal with the case $\lambda^+ = -\lambda^- = 1$. The other case is obtained by symmetry.

Let D be $T_{l^- + 3/2} D_\varepsilon$. As $\mu_{\beta,\gamma}$ is shift invariant, we note that D and D_ε have the same probability. We define s, a symmetry mapping on E, by

$$\text{for } \sigma \text{ in } E, \text{ for all } r \text{ in } \mathbb{R}, \quad s(\sigma)(r) = \sigma(-r). \tag{4.9}$$

We notice that this symmetry leaves the conditional probabilities invariant:

$$\mu_{\beta,\gamma}(\sigma | \mathscr{B}_{[-n,n]^c})(\omega) = \mu_{\beta,\gamma}(s(\sigma) | \mathscr{B}_{[-n,n]^c})(s(\omega)). \tag{4.10}$$

By the uniqueness of the one-dimensional Gibbs measure we check that

$$s \circ \mu_{\beta,\gamma} = \mu_{\beta,\gamma}. \tag{4.11}$$

We are thus led to introduce a symmetrical set H, deduced from D, in order to apply (4.7):

$$H = s(D) \cap D. \tag{4.12}$$

Let $L = l^+ + l^- + 3$. We partition H into 3 sets belonging respectively to the σ-fields $\mathscr{B}_{[1/2, L+1/2]}$, $\mathscr{B}_{[-1/2, 1/2]}$, $\mathscr{B}_{[-L-1/2, -1/2]}$

$$F_1 = \{\sigma \in E \mid \exists \sigma' \in H,\ \sigma|_{[1/2, L+1/2]} = \sigma'\}, \tag{4.13}$$

$$C = \{\sigma \in E \mid \exists \sigma' \in H,\ \sigma|_{[-1/2, 1/2]} = \sigma'\}, \tag{4.14}$$

$$F_2 = \{\sigma \in E \mid \exists \sigma' \in H,\ \sigma|_{[-L-1/2, -1/2]} = \sigma'\}. \tag{4.15}$$

The probability $\mu_{\beta,\gamma}^{[1/2,k]}(F_1|\sigma,\sigma')$ with boundary conditions σ on $]-\infty,1/2[$ and σ' on $]k,\infty[$ was introduced in Definition 2.5. When σ is in C the following approximation holds $\mu_{\beta,\gamma}$-a.s.

$$\forall\, k > L + \tfrac{1}{2},\quad \mu_{\beta,\gamma}^{[1/2,k]}(F_1|\sigma,\sigma') = \mu_{\beta,\gamma}^{[1/2,k]}(F_1|-m_\beta,\sigma')\exp(O(\varepsilon)/\gamma), \tag{4.16}$$

where $O(\varepsilon)$ does not depend on γ, L, σ'. We insist on the fact that the equation above is a formal notation, it makes sense only with continuous versions of H_γ (4.5).

We introduce the formal notation

$$\mu_{\beta,\gamma}(F_1|-m_\beta) = \lim_{k\to\infty}\mu_{\beta,\gamma}^{[1/2,k]}(F_1|-m_\beta,\sigma'),$$

where σ' is any block spins in E.

So, for each profile σ in C, by applying (4.16) we get

$$\mu_{\beta,\gamma} - \text{a.s.},\quad \mu_{\beta,\gamma}(F_1|\mathscr{B}_{[-1/2,1/2]})(\sigma) = \exp\left(\frac{O(\varepsilon)}{\gamma}\right)\mu_{\beta,\gamma}(F_1|-m_\beta). \tag{4.17}$$

After these preliminaries we turn to the computation of $\mu_{\beta,\gamma}(H)$

$$\mu_{\beta,\gamma}(H) = \mu_{\beta,\gamma}(C\,\mu_{\beta,\gamma}(F_1\cap F_2|\mathscr{B}_{[-1/2,1/2]})).$$

A one-dimensional Gibbs field with finite range interactions is a Markov chain, so the conditional law depends only on the values of $\sigma|_{[-1/2,1/2]}$. The sets F_1 and F_2 are independent conditionally to $\mathscr{B}_{[-1/2,1/2]}$; it follows

$$\mu_{\beta,\gamma}(H) = \mu_{\beta,\gamma}(1_C\,\mu_{\beta,\gamma}(F_1|\mathscr{B}_{[-1/2,1/2]})\mu_{\beta,\gamma}(F_2|\mathscr{B}_{[-1/2,1/2]})). \tag{4.18}$$

Then noticing that

$$\mu_{\beta,\gamma}(F_2|\mathscr{B}_{[-1/2,1/2]})(\sigma) = \mu_{\beta,\gamma}(s(F_1)|\mathscr{B}_{[-1/2,1/2]})(\sigma), \tag{4.19}$$

and using (4.10), we get

$$\mu_{\beta,\gamma}(F_2|\mathscr{B}_{[-1/2,1/2]})(\sigma) = \mu_{\beta,\gamma}(F_1|\mathscr{B}_{[-1/2,1/2]})(s(\sigma)). \tag{4.20}$$

Combining the preceding with Eq. (4.18), this implies

$$\mu_{\beta,\gamma}(H) = \mu_{\beta,\gamma}[1_C\,\mu_{\beta,\gamma}(F_1|\mathscr{B}_{[-1/2,1/2]})(\sigma)\mu_{\beta,r}(F_1|\mathscr{B}_{[-1/2,1/2]})(s(\sigma))]. \tag{4.21}$$

If $\bar{V}_\varepsilon$ is properly chosen we note that when σ is in $\bar{V}_\varepsilon$ then $s(\sigma)$ will also belong to this set. Therefore, by Eq. (4.17), we see that

$$\mu_{\beta,\gamma}(H) = \mu_{\beta,\gamma}(1_C)\mu_{\beta,\gamma}(F_1|-m_\beta)^2\exp(O(\varepsilon)/\gamma),$$

we have

$$\mu_{\beta,\gamma}(H) \geqslant \mu_{\beta,\gamma}(1_C\mu_{\beta,\gamma}(F_1|-m_\beta))^2\exp(O(\varepsilon)/\gamma),$$

and finally

$$\mu_{\beta,\gamma}(H) \geqslant \mu_{\beta,\gamma}(1_C\mu_{\beta,\gamma}(F_1|\mathscr{B}_{[-1/2,1/2]}))^2\exp(O(\varepsilon)/\gamma).$$

By again applying the Markov property, we get

$$\mu_{\beta,\gamma}(H) \geqslant \mu_{\beta,\gamma}(CF_1)^2 \exp(O(\varepsilon)/\gamma). \tag{4.22}$$

We use now for the set H the large deviation estimate obtained in (4.7) and by (4.22) we see that

$$\limsup_{\gamma\to 0} 2\gamma \ln \mu_{\beta,\gamma}(CF_1) \leqslant -\inf_H \mathscr{F}(\sigma) + O(\varepsilon).$$

From Definition (4.12) of H in terms of D, it is clear that

$$2\inf_D \mathscr{F}(\sigma) \leqslant O(\varepsilon) + \inf_H \mathscr{F}(\sigma).$$

This completes the lemma. $\square$

By Lemma 4.1 to inequality (4.4), we see that

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(D_\varepsilon \cap F) \leqslant -\inf_{D_\varepsilon\cap F} \mathscr{F}(\sigma) + O(\varepsilon). \tag{4.23}$$

Step 3: We shall now estimate the probability of the subset of F which contains the profiles which are not close to an equilibrium value. We define

$$\text{for } L>0, \quad W_L = \left[\bigcup_{\substack{a\leqslant l^+\leqslant L\\ a\leqslant l^-\leqslant L}} \bigcup_{\lambda^+\lambda^-} D_{l^+,l^-}^{\lambda^+,\lambda^-}(\varepsilon)\right]^{c}. \tag{4.24}$$

In this step, we will prove the existence of two constants C_1 and C_2

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(W_L) \leqslant -LC_1 + C_2. \tag{4.25}$$

Lemma 4.2.

$$\exists c>0 \textit{ such that for all } \sigma \textit{ in } W_L,\ \mathscr{F}(\sigma) \geqslant c(L-a). \tag{4.26}$$

Proof. By the lower semicontinuity of $\mathscr{G}_I$ (3.6) we get

$$\forall l \in [-L,-a]\cup[a,L], \qquad \inf_{\sigma\in(T_l V_\varepsilon \pm m_\beta)^c} \mathscr{G}_{[l,l+1]}(\sigma) > c > 0,$$

so, by iterating the procedure with different l, we prove the lemma. $\square$

We denote $\Lambda = [-L, L]$ and with an error proportional to $\exp(\alpha/\gamma)$ we fix arbitrarily the external conditions outside Λ equal to m_β. We get

$$\mu_{\beta,\gamma}(W_L) = \rho_\gamma^\Lambda\left(\frac{1_{W_L}}{Z(m_\beta)}\exp\left(\frac{\beta}{2\gamma}\langle J*\sigma 1_\Lambda, \sigma 1_\Lambda + 2m_\beta 1_{\delta\Lambda}\rangle + \frac{|\Lambda|^2 O(\gamma)+\alpha\beta}{\gamma}\right)\right).$$

By Eq. (4.2.5), we see that

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(W_L\cap F) \leqslant \alpha'\beta - \inf_{W_L} \mathscr{F}(m).$$

The statement (4.25) will derive from Lemma 4.2 applied to the equation above.

Step 4: We now collect all the previous bounds to compute $\mu_{\beta,\gamma}(F)$. We fix L and ε. Because,

$$\mu_{\beta,\gamma}(F) \leqslant \mu_{\beta,\gamma}(W_L) + \mu_{\beta,\gamma}(F \cap W_L^c),$$

we have

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(F) \leqslant \limsup_{\gamma\to 0} \gamma \ln 2\max(\mu_{\beta,\gamma}(W_L),\ \mu_{\beta,\gamma}(W_L^c \cap F)),$$

and therefore,

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(F) \leqslant \max\left(C_1 - LC_2, \limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(F \cap W_L^c)\right). \tag{4.27}$$

Noticing that,

$$\mu_{\beta,\gamma}(F \cap W_L^c) \leqslant 4(L-a)^2 \max_{\lambda^+,\lambda^-} \max_{\substack{a \leqslant l^+ \leqslant L \\ a \leqslant l^- \leqslant L}} (\mu_{\beta,\gamma}(F \cap D_{\lambda^+,\lambda^-}^{l^+,l^-})),$$

and using the inequalities (4.7) and (4.23) of the second step, we can check, as L goes to infinity, that

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(F) \leqslant -\inf_F \mathscr{F}(\sigma) + O(\varepsilon). \tag{4.28}$$

As ε tends to 0, we arrive at the required result.

Step 5: We must now remove the restriction on the sets and prove Theorem 4.1 for any closed set F. Let

$$F_R = \{\sigma \,|\, \exists\, \sigma' \in F,\ \sigma|_{[-R,R]} = \sigma'|_{[-R,R]}\}.$$

The set F_R is a closed cylinder, so we can use the large deviation estimate (4.28). Noticing that F is included in F_R, we see that

$$\limsup_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(F) \leqslant -\sup_R\left(\inf_{F_R} \mathscr{F}(\sigma)\right).$$

We assume that $\sup_R \inf_{F_R} \mathscr{F}(\sigma) = \alpha < \infty$; otherwise, the statement is obvious. Let σ_n be a profile chosen in F_n in order that the sequence $\{\sigma_n\}$ satisfies

$$\lim_{n\to\infty} \mathscr{F}(\sigma_n) = \alpha.$$

By compactness of E, we can extract a subsequence converging for the topology τ to a profile σ in F. Using the lower semicontinuity of $\mathscr{F}$ (Theorem 3.2) we get

$$\alpha \geqslant \mathscr{F}(\sigma) \geqslant \inf_F \mathscr{F}.$$

The statement (4.3) follows. $\square$

4.3. Large deviation inequality for open sets

In this section, we shall exploit the same methods as the ones developed in Section 4.2. We will prove a large deviation inequality for open sets.

Theorem 4.2. *Let O be an open set for the topology τ. We get*

$$\liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant -\inf_{O} \mathscr{F}(\sigma). \tag{4.29}$$

Proof

Step 1: Let σ be any profile in E. We recall that χ_λ was introduced in (3.4). First we assume that O is a neighborhood of the profile $\sigma|_{[-R,R]} \otimes \chi_\lambda$, where R is positive and λ is in $\{-1, 1\}^2$. We denote by σ_R the profile $\sigma|_{[-R,R]} \otimes \chi_\lambda$. We can suppose without any restriction that O is a cylinder set of basis $\Delta = [-R, R]$. We fix ε positive and we denote, in analogy with Definition 4.3, by $\mathring{D}$ the interior of $D_{R,R}^{\lambda^+,\lambda^-}(\varepsilon)$. We will prove

$$\liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant -\mathscr{F}(\sigma_R) + \left[\liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(\mathring{D}) + \inf_{E} \mathscr{F}(\sigma' \otimes \chi_\lambda)\right] + O(\varepsilon). \tag{4.30}$$

We have

$$\mu_{\beta,\gamma}(O) \geqslant \mu_{\beta,\gamma}\left[1_{\mathring{D}}(m)\rho_\gamma^\Delta\left(\frac{1_O}{Z_\Delta(m)}\exp\left(\frac{\beta}{2\gamma}\left(\langle J * \sigma' 1_\Delta, \sigma' 1_\Delta + 2m 1_{\delta\Delta}\rangle + \frac{1}{\gamma}|\Delta|^2 O(\gamma)\right)\right)\right)\right],$$

so by changing the external conditions, we get

$$\mu_{\beta,\gamma}(O) \geqslant \mu_{\beta,\gamma}\left[1_{\mathring{D}}\rho_\gamma^\Delta\left(\frac{1_O}{Z_\Delta(\chi_\lambda)}\exp\left(\frac{\beta}{2\gamma}\left(\langle J * \sigma' 1_\Delta, \sigma' 1_\Delta + 2\chi_\lambda 1_{\delta\Delta}\rangle + \frac{|\Delta|^2 O(\gamma) + O(\varepsilon)}{\gamma}\right)\right)\right)\right].$$

This will allow us to apply Theorem 3.1

$$\liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant \liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(\mathring{D}) - \inf_{\sigma'\in O} \mathscr{F}(\sigma'|_\Delta \otimes \chi_\lambda) + \inf_{\sigma'\in E} \mathscr{F}(\sigma'|_\Delta \otimes \chi_\lambda) + O(\varepsilon). \tag{4.31}$$

Then noticing that σ_R is in O and using (4.31), we obtain finally (4.30).

Step 2: To check

$$\liminf_{\gamma\to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant -\mathscr{F}(\sigma_R), \tag{4.32}$$

we need to estimate $\mu_{\beta,\gamma}(\mathring{D})$ for the different values of $\{\lambda^+, \lambda^-\}$.

Case 1: $\lambda^+ = \lambda^-$. The profile m_β (respectively $-m_\beta$) belongs to $\mathring{D}$ so by Theorem 4.1 we get

$$\lim_{\gamma \to 0} \mu_{\beta,\gamma}(\mathring{D}) \geqslant \tfrac{1}{4}.$$

When combined with equality (4.30), we see that

$$\liminf_{\gamma \to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant -\mathscr{F}(\sigma_R) + O(\varepsilon). \tag{4.33}$$

By taking the limit as ε tends to 0, we prove the statement (4.32).

Case 2: $\lambda^+ = -\lambda^-$. The derivation of (4.32) differs in no way from the proof given in Section 4.2 (Step 2), thus we will not repeat the same argument here.

Step 3: Hence the proof will be complete once we show (4.29) for any open set O in E. Let O be an open set and σ be a profile in O. We suppose that $\mathscr{F}(\sigma)$ is finite. Thus, by condition (2.11), we can choose λ such that $\sigma - \chi_\lambda$ belongs to $\mathscr{L}^2(\mathrm{d}r, \mathbb{R})$. Therefore, we check that

$$\lim_{R \to 0} \mathscr{F}(\sigma_R) = \mathscr{F}(\sigma).$$

Since R is sufficiently large, σ_R belongs to O. So we can apply the previous results for O_R a neighborhood of σ_R included in O and we get as R goes to infinity

$$\text{for } \sigma \text{ in } O, \quad \liminf_{\gamma \to 0} \gamma \ln \mu_{\beta,\gamma}(O) \geqslant -\mathscr{F}(\sigma). \tag{4.34}$$

Theorem 4.2 follows. □

Before going on to the study of the profiles we will state an immediate consequence of the large deviation principle.

4.4. *The average shape of the profiles on small regions*

Proposition 4.1. *Let V be a neighborhood of* 0 *sufficiently small and any sequence* (a_γ) *with*

$$\gamma \ln a_\gamma \to 0,$$

$$a_\gamma \to \infty.$$

We introduce the open sets

$$\lambda = \pm 1, \qquad A_\gamma^\lambda = \bigcap_{\substack{-a_\gamma < l < a_\gamma \\ l \in \mathbb{Z}}} (T_l V + \lambda m_\beta),$$

and we get

$$\lambda = \pm 1, \quad \lim_{\gamma \to 0} \mu_{\beta,\gamma}(A_\gamma^\lambda) = \tfrac{1}{2}. \tag{4.35}$$

This proposition shows that on intervals of length a_γ a profile has a strong probability to stay close to one of the equilibrium states. The reader should take note that this result tells us nothing about the profiles behavior on regions larger than a_γ.

Proof. We denote $(A_\gamma^1 \cup A_\gamma^{-1})^c$ by B_γ. As the system is symmetric, it suffices to prove

$$\lim_{\gamma \to 0} \mu_{\beta,\gamma}(B_\gamma) = 0. \tag{4.36}$$

We distinguish two subsets of B_γ

$$B_\gamma^1 = \{\sigma \mid \exists\, l \in \mathbb{Z},\ |l| < a_\gamma,\ \sigma \notin T_l V \pm m_\beta\}$$

$$B_\gamma^2 = \{\sigma \mid \exists\, \lambda,\ \exists\, l \in \mathbb{Z},\ |l| < a_\gamma,\ \sigma \in T_l V + \lambda m_\beta \text{ and } \sigma \in T_{(l+1)} V - \lambda m_\beta\}.$$

Noticing that

$$\mu_{\beta,\gamma}(B_\gamma^1) \leqslant 2a_\gamma \mu_{\beta,\gamma}((V + m_\beta)^c \cap (V - m_\beta)^c),$$

and using Lemma 4.2 we have

$$\exists\, c > 0,\ \exists\, \gamma_0 > 0 \text{ such that } \forall\, \gamma < \gamma_0,\ \mu_{\beta,\gamma}(B_\gamma^1) \leqslant 2a_\gamma \exp(-c/\gamma).$$

Each element σ in B_γ^2 satisfies

$$\exists\, l,\ |l| < a_\gamma,\ \exists\, r \in [l, l+1] \text{ such that } \sigma \notin T_r V \pm m_\beta,$$

so an analogous argument will imply

$$\exists\, c > 0,\ \exists\, \gamma_0 > 0 \text{ such that } \forall\, \gamma < \gamma_0,\ \mu_{\beta,\gamma}(B_\gamma^2) \leqslant 2a_\gamma \exp(-c/\gamma).$$

Collecting all the previous bounds we get (4.36) from which the proposition follows.

5. Study of the profiles

In the previous section (Proposition 4.1) we proved that the more likely configurations are locally close to an equilibrium state. We shall use the large deviation principle to obtain more information on the shape of the profiles and to prove a metastability property due to the infinite volume. We generalize the local study of Eisele and Ellis and we prove that a profile will keep a local magnetization close to the value $\pm m_\beta$ on a distance of the order of $\exp(\Phi/\gamma)$ before jumping to the opposite stable value. We make an analogy with dynamical systems and exploit Friedlin–Wentzell theory to estimate the position where the first interface appears.

5.1. Notation

Let V be a weak neighborhood of 0 which is a cylinder with basis $[0, 1]$. We are led to introduce $\mathscr{L}_V^\gamma$, the function from E into $\gamma\mathbb{Z} = \{n\gamma \mid n \in \mathbb{Z}\}$ which associates to each profile of E the position of the first interface after 0. More explicitly $\mathscr{L}_V^\gamma$ is defined by

Definition 5.1.

$$\mathscr{L}_V^\gamma(\sigma) = \inf\{l \in \gamma\mathbb{Z} \mid \exists\, l' \in \gamma\mathbb{Z}, l' > 0, l > l', \exists\, \lambda, \sigma \in (T_{l'}V + \lambda m_\beta) \cap (T_l V - \lambda m_\beta)\}.$$

At position l' the profile enters for the first time in a neighborhood of the state λm_β and $\mathscr{L}_V^\gamma(\sigma)$ is the first position after l' where the profile hits a neighborhood of $-\lambda m_\beta$.

Definition 5.2. Let A be

$$A = \left\{\sigma \mid \int_{\mathbb{R}^+} (\sigma(r) - m_\beta)^2\, \mathrm{d}r < \infty, \int_{\mathbb{R}^-} (\sigma(r) + m_\beta)^2\, \mathrm{d}r < \infty\right\}, \tag{5.1}$$

and define Φ to be

$$\Phi = \inf_A \mathscr{F}(\sigma). \tag{5.2}$$

We insist on the fact that the profiles in A leap at least one time. The constant Φ could be interpreted as the cost of a leap.

5.2. Evaluation of $\mathscr{L}_V^\gamma$

Before going on to estimate $\mathscr{L}_V^\gamma$ we need to prove an extension of the large deviation principle.

Theorem 5.1. *Let O be an open cylinder with compact basis in $\mathbb{R}^+$. We suppose in addition that O is symmetric, i.e.*

$$O = \{-\sigma \mid \sigma \in O\}. \tag{5.3}$$

We get for all ε positive

$\exists\, R > 0, \exists\, \gamma_0 > 0$ *such that* $\forall\, \gamma < \gamma_0$,

$$\mu_{\beta,\gamma} - a.s., \quad \mu_{\beta,\gamma}(T_R O \mid \mathscr{B}_{\mathbb{R}^-})(\omega) \geqslant \exp\left(-\frac{\inf_O \mathscr{F}(\sigma) + \varepsilon}{\gamma}\right). \tag{5.4}$$

It is interesting to note that the assumption (5.3) on O is relevant; in fact, a mixing property for general sets will never hold on finite distance.

Proof. We fix ε positive. Let ω and ω' be two block spins, so the probability $\mu_{\beta,\gamma}$ conditional on ω or ω' makes sense. Since ω' is in the set $\omega + T_{-1}V_\varepsilon$ (with V_ε as in Definition 4.1), by using the same kind of argument as in (4.16), we check for any cylinder set H of basis included in $\mathbb{R}^+$

$$\text{for all } \gamma < \gamma_0, \quad \mu_{\beta,\gamma}(H \mid \mathscr{B}_{\mathbb{R}^-})(\omega') = \mu_{\beta,\gamma}(H \mid \mathscr{B}_{\mathbb{R}^-})(\omega) \exp(\mathrm{O}(\varepsilon)/\gamma). \tag{5.5}$$

For any profile ω in E, we define G_ω by $\omega + T_{(-1)}V_\varepsilon$. We note that $E = \bigcup_{\omega\in E} G_\omega$. Thus, by the compactness of E, one has the existence of a finite family $(G_{\omega_i})_{i\leqslant N}$ which covers E. We denote G_{ω_i} by G_i. Furthermore, the sets G_i are regular, i.e. they satisfy the property

$$\inf_{G_i} \mathscr{F} = \inf_{\bar{G}_i} \mathscr{F} \tag{5.6}$$

where $\bar{G}_i$ is the closure of G_i.

Let G_i be given, we will prove that

$$\liminf_{\gamma\to 0} \gamma \ln \inf_{\substack{\omega\in G_i \\ \mu_{\beta,\gamma}-\text{a.s.}}} [\mu_{\beta,\gamma}(T_R O|\mathscr{B}_{\mathbb{R}^-})(\omega)] \geqslant -\inf_O \mathscr{F}(\sigma) + \varepsilon, \tag{5.7}$$

where the constant R will be fixed later.

Noticing that

$$\mu_{\beta,\gamma}(G_i \cap T_R O) = \int \mathrm{d}\mu_{\beta,\gamma}(\sigma) 1_{G_i}(\sigma)\mu_{\beta,\gamma}(T_R O|\mathscr{B}_{\mathbb{R}^-})(\sigma),$$

by Eq. (5.5), we have $\mu_{\beta,\gamma}$-a.s. for ω in G_i

$$\mu_{\beta,\gamma}(G_i \cap T_R O) = \mu_{\beta,\gamma}(G_i)\mu_{\beta,\gamma}(T_R O|\mathscr{B}_{\mathbb{R}^-})(\omega)\exp(O(\varepsilon)/\gamma), \tag{5.8}$$

as G_i is regular (5.6), by applying the large deviation property we get from (5.8)

$$\liminf_{\gamma\to 0} \gamma \ln \inf_{\substack{\omega\in G_i \\ \mu_{\beta,\gamma}-\text{a.s.}}} [\mu_{\beta,\gamma}(T_R O|\mathscr{B}_{\mathbb{R}^-})(\omega)] \geqslant -\inf_{G_i\cap T_R O} \mathscr{F}(\sigma) + \inf_{G_i} \mathscr{F}(\sigma) + O(\varepsilon). \tag{5.9}$$

One is led to prove that for a suitable constant R, $\inf_{G_i\cap T_R O} \mathscr{F}$ is almost equal to $\inf_{G_i} \mathscr{F} + \inf_{T_R O} \mathscr{F}$.

Let σ_i be an element of G_i which satisfies the propery

$$\mathscr{F}(\sigma_i) \leqslant \inf_{G_i} \mathscr{F} + \tfrac{\varepsilon}{2},$$

$$\sigma_i(r) = \lambda m_\beta \quad \text{for } r > d,$$

where d is a constant and λ is an element in $\{-1, 1\}$ depending on σ_i.

For any λ the symmetry of O enables us to find σ in O which has the following property:

$$\mathscr{F}(\sigma) \leqslant \inf_O \mathscr{F} + \varepsilon/2,$$

$$\sigma(r) = \lambda m_\beta \quad \text{for } r < -d.$$

Let R be $2d + 1$, then the profile $\sigma_i \otimes m_\beta \otimes T_R\sigma$ belongs to $G_i \cap T_R O$ and we get

$$\inf_{G_i} \mathscr{F} - \inf_{G_i\cap T_R O} \mathscr{F} \geqslant -\mathscr{F}(\sigma) - \frac{\varepsilon}{2} \geqslant -\inf_O \mathscr{F} - \varepsilon.$$

This implies (5.7) for any set G_i. Therefore, noticing that the family $(G_i)_{i \leqslant N}$ covers E and using (5.7) for all i, it can be checked that the statement (5.4) holds for suitable constants γ_0 and R. □

We have made all the preparation necessary to estimate $\mathscr{L}_V^\gamma$.

Theorem 5.2. *Let V be a sufficiently small symmetric neighborhood of* 0, *for each positive ε we get*

$$\lim_{\gamma \to 0} \mu_{\beta,\gamma}\left(\exp\left(\frac{\Phi - \varepsilon}{\gamma}\right) \leqslant \mathscr{L}_V^\gamma \leqslant \exp\left(\frac{\Phi + \varepsilon}{\gamma}\right)\right) = 1, \tag{5.10}$$

where Φ was introduced in Definition 5.2.

Proof.

Step 1: First, we will prove

$$\lim_{\gamma \to 0} \mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma \leqslant \exp\left(\frac{\Phi + \varepsilon}{\gamma}\right)\right) = 1. \tag{5.11}$$

We fix a positive ε. Due to the definition of Φ (5.2), there exists an integer N and a profile σ in E which satisfy the following properties:

$$\begin{aligned} &\mathscr{F}(\sigma) \leqslant \Phi + \frac{\varepsilon}{4}, \\ &\sigma(r) = -m_\beta \quad \text{for } r \leqslant 1, \\ &\sigma(r) = m_\beta \quad \text{for } r \geqslant N. \end{aligned} \tag{5.12}$$

We introduce $O = O(\sigma) \cup O(-\sigma)$, where $O(\sigma)$ is a neighborhood of σ defined by

$$O(\sigma) = (V - \sigma) \cap (T_N V + \sigma).$$

Hence, by the symmetry of O and Theorem 5.1, we get for a suitable integer R

$$\exists\, \gamma_0 \text{ such that } \forall\, \gamma < \gamma_0,$$

$$\mu_{\beta,\gamma}\text{-a.s.,} \quad \mu_{\beta,\gamma}(O_R | \mathscr{B}_{\mathbb{R}^-})(\omega) \geqslant \exp\left(-\frac{\inf_O \mathscr{F}(\sigma) + \varepsilon/4}{\gamma}\right) \tag{5.13}$$

where $O_R = T_R O$. Without any restriction, we can assume that O is a cylinder of basis $[0, N+1]$ and we set $R' = R + N + 1$. Before getting into the details, it may be helpful to make a couple of remarks. Eq. (5.13) tells us that the probability of leaping from an equilibrium value to another on the interval $[0, R']$ is always greater than a constant c_γ. So if we iterate (5.13) on the intervals $([iR', (i+1)R'])_{i \leqslant n}$, we see that the probability of leaping on $[0, nR']$ will increase more than nc_γ. One is led to predict that $\mathscr{L}_V^\gamma$ ought to be of the order of $1/c_\gamma$.

We note that

$$\left\{\mathscr{L}_V^\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right\} \subset \bigcap_{n=0}^{l^*} (T_{R'n}O_R)^c, \tag{5.14}$$

where l^* is the integer part of $(1/R')\exp((\Phi+\varepsilon)/\gamma)$.

Because of the Markov property of the Gibbs measure, we have

$$\mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right) \leqslant \mu_{\beta,\gamma}\left[\bigcap_{n=0}^{l^*-1} (T_{R'n}O_R)^c \mu_{\beta,\gamma}((T_{R'l^*}O_R)^c|\mathscr{B}_{]-\infty,R'l^*]})\right].$$

Combining the shift invariance of $\mu_{\beta,\gamma}$ and Eq. (5.13) proves

$$\mu_{\beta,\gamma}\text{-a.s.},\quad \mu_{\beta,\gamma}(T_{R'l^*}O_R|\mathscr{B}_{]-\infty,R'l^*]})(\omega) \geqslant \exp\left(-\frac{\Phi+\varepsilon/2}{\gamma}\right),$$

and therefore,

$$\mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right) \leqslant \mu_{\beta,\gamma}\left[\bigcap_{n=0}^{l^*-1} (T_{R'n}O_R)^c\right]\left(1-\exp\left(-\frac{\Phi+\varepsilon/2}{\gamma}\right)\right).$$

So we can iterate the procedure to get

$$\mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right) \leqslant \left[1-\exp\left(-\frac{\Phi+\varepsilon/2}{\gamma}\right)\right]^{(1/R')\exp((\Phi+\varepsilon)/\gamma)-1},$$

and more precisely

$$\mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma > \exp\left(\frac{\Phi+\varepsilon}{\gamma}\right)\right) \leqslant c\exp\left(-\frac{1}{R'}\exp\left(\frac{\varepsilon}{2\gamma}\right)\right).$$

By taking the limit as γ goes to 0, we then derive (5.11).

Step 2: Theorem 5.2 will be complete once we show that

$$\lim_{\gamma\to 0} \mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma \geqslant \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\right) = 1. \tag{5.15}$$

We introduce the length $\mathscr{L}_V^{\gamma'}$ defined by

$$\mathscr{L}_V^{\gamma'}(\sigma) = \sup\{l \in \gamma\mathbb{Z} | \mathscr{L}_V^\gamma > l \geqslant 0,\ \exists\lambda,\ \sigma \in (T_l V + \lambda m_\beta) \cap (T_{\mathscr{L}_V^\gamma} V - \lambda m_\beta)\}.$$

We denote by $\mathscr{M}_V$ the set $\{\mathscr{L}_V^{\gamma'} = 0,\ \mathscr{L}_V^\gamma < \infty\}$.

Let σ be any profile in the closure of $\mathscr{M}_V$. If σ is in $\mathscr{M}_V$, we see that $\mathscr{F}(\sigma) \geqslant \Phi$. If $\mathscr{L}_V^\gamma(\sigma)$ is infinite then σ is not in a neighborhood of an equilibrium value, this implies

$$\mathscr{F}(\sigma) \geqslant \Phi.$$

So we get for the closure of $\mathscr{M}_V$ that

$$\inf_{\bar{\mathscr{M}}_V} \mathscr{F} = \Phi.$$

Noticing that

$$\mu_{\beta,\gamma}\left(\mathscr{L}_V^\gamma \leqslant \exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\right) \leqslant \sum_{l=0}^{[\exp((\Phi-\varepsilon)/\gamma)\gamma^{-1}]} \mu_{\beta,\gamma}(T_{l\gamma}\mathscr{M}_V),$$

and applying the large deviation principle for the closure of $\mathscr{M}_V$, we have

$$\sum_{l=0}^{[\exp((\Phi-\varepsilon)/\gamma)\gamma^{-1}]} \mu_{\beta,\gamma}(T_{l\gamma}\mathscr{M}_V) \leqslant \gamma^{-1}\exp\left(\frac{\Phi-\varepsilon}{\gamma}\right)\exp\left(\frac{-\Phi+\varepsilon/2}{\gamma}\right).$$

As γ goes to 0, the statement (5.15) follows. □

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