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A limit theorem for occupation times of fractional Brownian motion

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Dedicated to Professor Shinzo Watanabe on the occasion of his 60th birthday

Abstract

Recently, N. Kôno gave a limit theorem for occupation times of fractional Brownian motion, which result generalizes the well-known Kallianpur–Robbins law for two-dimensional Brownian motion. This paper studies a functional limit theorem for Kôno's result. It is proved that, under a suitable normalization, the limiting process is the inverse of an extremal process.

Keywords: Fractional Brownian motion; Extremal process; Exponential distribution; Occupation times

1. Introduction

Let $\{B^{(2)}(t)\}_{t\geq 0}$ be a standard Brownian motion on the plane and let f be a bounded integrable function on $\mathbb{R}^2$. The following theorem due to Kallianpur–Robbins (1953) is well known.

Theorem A. *If* $\bar{f} := \int_{\mathbb{R}^2} f(x)\,\mathrm{d}x \neq 0$, *then*

$$\lim_{t\to\infty} P\left[\frac{2\pi}{\bar{f}\log t}\int_0^t f(B^{(2)}(u))\,\mathrm{d}u \leqslant x\right] = 1 - \mathrm{e}^{-x}, \quad x > 0.$$

An "invariance principle" for Theorem A was given by Kasahara–Kotani (1979). To explain the limiting process we first define the *canonical extremal process* $Y = \{Y(t)\}_{t\geq 0}$, which is defined to be a nondecreasing process such that $Y(0) = 0$ with the finite-dimensional marginal distributions

$$P[Y(t_1) \leqslant x_1, \ldots, Y(t_n) \leqslant x_n] = G(x_1)^{t_1} G(x_2)^{t_2 - t_1} \ldots G(x_n)^{t_n - t_{n-1}}$$

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for $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_n$, where $G(x) = e^{-1/x}$. One of the method of constructing Y is as follows. Let p be a Poisson point process on $(0, \infty) \times (\mathbb{R}\setminus\{0\})$ with intensity dx/x^2 (i.e., the mean measure of the counting process $N_p(dt, dx)$ is given by $dt\,dx/x^2$). It is easy to see that the maximal process of p is a canonical extremal process. Since such a point process p appears in the theory of Brownian excursions, Y may also be expressed using one-dimensional Brownian motion $B^{(1)}(t)$ as follows. Let $M = \{M(t)\}$ and $\ell = \{\ell(t)\}$ be the maximal process and the local time at 0 of $B^{(1)}(t)$, respectively, i.e.,

$$M(t) = \max_{0 \leqslant s \leqslant t} B^{(1)}(s), \qquad \ell(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t I_{[-\varepsilon,\varepsilon]}(B^{(1)}(s))\, ds.$$

Then

$$\{Y(t)\} \overset{\mathscr{L}}{=} \{M(\ell^{-1}(t))\}.$$

We refer to the textbook of Ikeda and Watanabe (1981) for the notation and fundamental results on point processes and Brownian excursions.

A functional limit theorem for Theorem A was given by Kasahara and Kotani (1979).

Theorem B. *Let* $\varphi(x) = xe^{2x}$. *Then,*

$$\frac{1}{\lambda} \int_0^{\varphi(\lambda t)} f(B^{(2)}(u))\, du \xrightarrow{\text{f.d.}} \frac{\bar{f}}{\pi} Z(t) \quad \text{as } \lambda \to \infty,$$

where $Z(t) = \ell(M^{-1}(t))$ *is the inverse process of a canonical extremal process* $Y(t)$.

Here, "$\xrightarrow{\text{f.d.}}$" denotes the weak convergence of all finite-dimensional distributions. See Remark 1.1 below for the reason why we use the above normalization due to D. Stroock (private communication) instead of

$$\frac{1}{\log \lambda} \int_0^{\lambda t} f(B^{(2)}(u))\, du.$$

We also remark that the assertion of Theorem B does not hold with respect to Skorohod's J_1-topology. If it does hold, then the limiting processes Z should necessarily be continuous, which is clearly a contradiction. In fact, we can claim the M_1-convergence, but we shall not go into details here. For an extension of Theorem B from the view point of Markov processes, see Kasahara (1982).

Recently, N. Kôno extended Theorem A for *fractional Brownian motions*: Let X^γ be a fractional Brownian motion with index $\gamma (0 < \gamma < 1)$. That is, X^γ is a real-valued centered Gaussian process such that

$$E[X^\gamma(t)X^\gamma(s)] = \tfrac{1}{2}\{t^{2\gamma} + s^{2\gamma} - |t-s|^{2\gamma}\}, \quad s, t \geqslant 0,$$

or, equivalently,

$$X^\gamma(0) = 0 \text{ and } E[(X^\gamma(t) - X^\gamma(s))^2] = |t-s|^{2\gamma}, \quad t, s \geqslant 0.$$

If $\gamma = \frac{1}{2}$, then X^γ is the usual Brownian motion. A d-dimensional fractional Brownian motion is defined to be an $\mathbb{R}^d$-valued Gaussian process

$$X^{\gamma,d}(t) = (X_1^\gamma(t), X_2^\gamma(t), \dots, X_d^\gamma(t)),$$

where $X_1^\gamma(t), X_2^\gamma, \dots$ are independent copies of X^γ. We shall consider only the recurrent case, i.e., $0 < \gamma d \leqslant 1$. If $0 < \gamma d < 1$, then the existence of jointly continuous local time $\ell_{\gamma,d}(t, x)$ is known and, therefore, it is easy to obtain a limit theorem for the occupation times: Let f be a bounded summable function on $\mathbb{R}^d$ and let $\bar{f} = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$ as before. We easily see by the self-similarity of the fractional Brownian motion that

$$\begin{aligned}\frac{1}{\lambda^{1-\gamma d}} \int_0^{\lambda t} f(X^{\gamma,d}(s))\,\mathrm{d}s &\overset{\mathscr{L}}{=} \lambda^{\gamma d} \int_{\mathbb{R}^d} f(\lambda^\gamma x)\ell_{\gamma,d}(t, x)\,\mathrm{d}x \\ &\xrightarrow{\mathscr{L}} \bar{f}\ell_{\gamma,d}(t, 0) \quad \text{as } \lambda \to \infty\end{aligned}$$

over the function space $C([0, \infty))$. (See Kasahara and Matsumoto (1996) for some remarks on the law of $\ell_{\gamma,d}(t, 0)$.) Therefore, the remaining case (i.e., $\gamma d = 1$) is the only interesting case and Kôno (1996) proved the following theorem, which extends Theorem A.

Theorem C. *Let $d \geqslant 2$. Suppose $\gamma d = 1$ and let $f(x) \geqslant 0$ be a bounded integrable function on $\mathbb{R}^d$ such that $\bar{f} := \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \neq 0$. Then,*

$$\lim_{t\to\infty} P\left[\frac{1}{C \log t} \int_0^t f(X^{\gamma,d}(s))\,\mathrm{d}u \leqslant x\right] = 1 - \mathrm{e}^{-x}, \quad x > 0,$$

where

$$C = \frac{\bar{f}}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x. \tag{1.1}$$

The aim of the present paper is to obtain a functional limit theorem for Theorem C to extend Theorem B. Our main result is the following.

Theorem 1. *Under the assumptions of Theorem C, it holds*

$$A_\lambda(t) := \frac{1}{\lambda} \int_0^{\mathrm{e}^{\lambda t}} f(X^{\gamma,d}(u))\,\mathrm{d}u \xrightarrow{\text{f.d.}} CZ(t) \quad \text{as } \lambda \to \infty,$$

where C is as in (1.1) and Z is the inverse of a canonical extremal process as in Theorem B.

The assertion of Theorem 1 may be rewritten as follows:

$$\begin{aligned}&\lim_{\lambda\to\infty} P\left[\frac{1}{C} A_\lambda(t_1) \geqslant x_1, \dots, \frac{1}{C} A_\lambda(t_n) \geqslant x_n\right] \\ &\quad = \exp\left\{-\frac{x_1}{t_1} - \frac{x_2 - x_1}{t_2} - \cdots - \frac{x_n - x_{n-1}}{t_n}\right\}\end{aligned}$$

for $0 < t_1 \leqslant \cdots \leqslant t_n$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_n$.

For Markov processes it is proved by Kasahara (1982) that, in such cases where the occupation times converge in law to an exponential distribution under a suitable normalization, we have a limit theorem similar to Theorem 1 with the same limiting process Z as above. So the reader may think that Theorem 1 is natural, but in fact, it is amazing in some sense to the authors because in other cases (i.e., $0 < \gamma d < 1$) it is known that, as for the occupation time problems, fractional Brownian motions and Markov processes have distinct limiting distributions.

Remark 1.1. The reason why we consider the normalization of Theorem B is as follows. Let $Z^*(t)$ be any limiting process of

$$\frac{1}{\log \lambda}\int_0^{\lambda t} f(B^{(2)}(u))\,\mathrm{d}u.$$

Then $Z^*(t)$ is degenerate in the sense that it does not depend on t (i.e., $Z^*(t) = Z^*(0+)$, $t \geqslant 0$): Since $E[\int_0^t f(B^{(2)}(u))]\,\mathrm{d}u$ is asymptotically equal to $C \log t$ as $t \to \infty$, we obtain

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{1}{\log \lambda}\left\{E\left[\int_0^{\lambda t} f(B^{(2)}(u))\,\mathrm{d}u\right] - E\left[\int_0^{\lambda} f(B^{(2)}(u))\,\mathrm{d}u\right]\right\}\\
&\quad = \lim_{\lambda\to\infty} \frac{C}{\log \lambda}\{\log(\lambda t) - \log \lambda\} = 0.
\end{aligned}$$

Since $\int_0^{\lambda t} f(B^{(2)}(u))\,\mathrm{d}u$ is monotone in t if $f(x) \geqslant 0$, this implies $Z^*(t) = Z^*(0+)$, $t \geqslant 0$.

Remark 1.2. The assumption that f is nonnegative is not essential and can be removed. It is, however, crucial that $C \neq 0$. Otherwise, the limiting process degenerates and we need another normalization.

Remark 1.3. The condition that $\gamma d = 1$ can be weakened: The assertion of Theorem 1 is still valid for d-dimensional fractional Brownian motion with each independent component has scaling parameter $0 < \gamma_j < 1$ satisfying $\gamma_1 + \cdots + \gamma_d = 1$.

2. Proof of Theorem 1

For the proof of Theorem 1 we adopt the moment method of Darling–Kac (1957), Bingham (1971) and Kasahara (1982). We first introduce a linear transformation on $C_b[0, \infty)$ the space of all real-valued bounded continuous functions defined on $[0, \infty)$.

Definition 2.1. For every $s > 0$ and for every $p \in C_b[0, \infty)$ we define $U_s p \in C_b[0, \infty)$ as follows:

$$U_s p(t) = \int_t^{\infty} \mathrm{e}^{-s\xi} p(\xi)\,\mathrm{d}\xi + t\mathrm{e}^{-st} p(t), \quad t \geqslant 0. \tag{2.1}$$

Let us consider some examples. Let $\mathbf{1}$ denote the function which is identically equal to 1. (i.e., $\mathbf{1}(t) \equiv 1$.) Then,

$$U_s\mathbf{1}(t) = \frac{1}{s}\,\mathrm{e}^{-st} + \mathrm{e}^{-st}t,$$

$$U_{s_1}U_{s_2}\mathbf{1}(t) = \left\{\frac{1}{s_2(s_1+s_2)} + \frac{t}{s_1+s_2} + \frac{1}{(s_1+s_2)^2}\right\}\mathrm{e}^{-(s_1+s_2)t}.$$

As a special case, we have

$$U_{s_1}\mathbf{1}(0) = \frac{1}{s_1}, \qquad U_{s_1}U_{s_2}\mathbf{1}(0) = \frac{1}{s_2(s_1+s_2)} + \frac{1}{(s_1+s_2)^2}. \tag{2.2}$$

Without loss of generality we may and do assume that $C = 1$. The following is the key lemma.

Lemma 2.1. *Let* $\varphi(x) = \mathrm{e}^x - 1$ *and for every* $s_1, s_2, \ldots, s_n > 0$ *define*

$$\begin{aligned}&\phi_n^{(\lambda)}(s_1, \ldots, s_n)\\ &= \int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} E\left[\prod_{j=1}^{n} f(X^{\gamma,d}(\varphi(\lambda t_j)))\varphi'(\lambda t_j)\right]\mathrm{d}t_1\cdots\mathrm{d}t_n.\end{aligned} \tag{2.3}$$

Then,

$$\lim_{\lambda\to\infty} \phi_n^{(\lambda)}(s_1, \ldots, s_n) = U_{s_1}U_{s_2}\ldots U_{s_n}\mathbf{1}(0).$$

The reason why we set $\varphi(x)$ as above is that this function is asymptotically equal to e^x and satisfies $\varphi(0) = 0$, which condition is technically convenient. But in fact $\varphi(x)$ may be any other smooth function such that $\varphi^{-1}(x)$ is asymptotically equal to $\log x$ (e.g. $\varphi(x) = \mathrm{e}^x, x\mathrm{e}^x$, etc; cf. Theorem B). We postpone the proof of Lemma 2.1 until the next section and we shall prove Theorem 1. The idea is due to Bingham (1971). Lemma 2.1 implies that

$$\begin{aligned}&\lim_{\lambda\to\infty}\int_0^\infty\cdots\int_0^\infty \mathrm{e}^{-\Sigma s_j t_j} E\left[\prod_{j=1}^{n} f(X^{\gamma,d}(\varphi(\lambda t_j)))\varphi'(\lambda t_j)\right]\mathrm{d}t_1\cdots\mathrm{d}t_n\\ &= \Phi_n(s_1, \ldots, s_n),\end{aligned} \tag{2.4}$$

where

$$\Phi_n(s_1, \ldots, s_n) = \sum_\pi U_{s_{\pi(1)}}U_{s_{\pi(2)}}\cdots U_{s_{\pi(n)}}\mathbf{1}(0).$$

Here, π runs over all permutations of $\{1, \ldots, n\}$ and, for example, by (2.2)

$$\Phi_1(s_1) = \frac{1}{s_1}, \qquad \Phi_2(s_1, s_2) = \frac{1}{s_1 s_2} + \frac{2}{(s_1+s_2)^2}.$$

Integrating by parts, (2.4) may be rewritten as

$$\lim_{\lambda\to\infty} s_1\ldots s_n\int_0^\infty\cdots\int_0^\infty \mathrm{e}^{-\Sigma s_j t_j} E[A_\lambda^*(t_1)\ldots A_\lambda^*(t_n)]\,\mathrm{d}t_1\cdots\mathrm{d}t_n = \Phi_n(s_1, \ldots, s_n),$$

where

$$A_\lambda^*(t) := \frac{1}{\lambda}\int_0^{\varphi(\lambda t)} f(X^{\gamma,d}(u))\,\mathrm{d}u.$$

Notice that $A_\lambda^*(t)$ and $A_\lambda(t)$ have the same limiting process. In order to find the relationship between the right-hand side Φ_n and the process Z, we use the idea of 'invariance principle': Let us consider the special case of two-dimensional Brownian motion, in which case we have, by Theorem B, $A_\lambda \xrightarrow{\text{f.d.}} Z$. Therefore, by a routine argument we see that the limiting function $\Phi_n(s_1, \dots, s_n)$ is in fact the Laplace transform of $E[Z(t_1) \dots Z(t_n)]$; for every $s_j > 0$ and $n \geqslant 1$,

$$\Phi(s_1, \dots, s_n) = s_1 \dots s_n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\Sigma s_j t_j} E[Z(t_1) \dots Z(t_n)]\,\mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

So (2.4) can be rewritten as

$$\lim_{\lambda\to\infty} s_1 \dots s_n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\Sigma s_j t_j} E[A_\lambda(t_1) \dots A_\lambda(t_n)]\,\mathrm{d}t_1 \cdots \mathrm{d}t_n$$
$$= s_1 \dots s_n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\Sigma s_j t_j} E[Z(t_1) \dots Z(t_n)]\,\mathrm{d}t_1 \cdots \mathrm{d}t_n$$

which implies

$$\lim_{\lambda\to\infty} E[A_\lambda(t_1) \dots A_\lambda(t_n)] = E[Z(t_1) \dots Z(t_n)]$$

for every $t_1, \dots, t_n \geqslant 0$ ($n \geqslant 1$). Here it should be noticed that the right-hand side is continuous in $(t_1, \dots, t_n)$. Now keeping in mind that repetition of $\{t_j\}$ is allowed (e.g., $E[A_\lambda(t)^2]$ may be rewritten as $E[A_\lambda(t)A_\lambda(t)]$), this implies

$$\lim_{\lambda\to\infty} E[A_\lambda(t_1)^{m_1} \dots A_\lambda(t_n)^{m_n}] = E[Z(t_1)^{m_1} \dots Z(t_n)^{m_n}]$$

for every $t_j \geqslant 0$, $m_j \geqslant 1$ and $n \geqslant 1$. This proves the assertion of Theorem 1 since it is easy to see that the limiting law is characterized by moments. □

The proof of Lemma 2.1 will be given in Section 3 and in the rest of this section we shall explain the idea.

Definition 2.2. Throughout the paper we set $t_0 = 0$ and let $0 < t_1 < \cdots < t_n$. We denote by $C_n(t_1, \dots, t_n)$ the covariance matrix of

$$(X^\gamma(t_1) - X^\gamma(t_0), \dots, X^\gamma(t_n) - X^\gamma(t_{n-1})),$$

and for every $\lambda > 0$ we define

$$C_n^\lambda(t_1, \dots, t_n) = C_n(\varphi(\lambda t_1), \dots, \varphi(\lambda t_n)).$$

So $C_n^\lambda(t_1, \dots, t_n)$ is the covariance matrix of

$$(X^\gamma(\varphi(\lambda t_1)) - X^\gamma(\varphi(\lambda t_0)), \dots, X^\gamma(\varphi(\lambda t_n)) - X^\gamma(\varphi(\lambda t_{n-1}))),$$

and the diagonal elements are

$$C_n^\lambda(t_1, \dots, t_n)_{jj} = (\Delta\varphi(\lambda t_j))^{2\gamma}, \quad j = 1, \dots, n$$

where

$$\Delta\varphi(\lambda t_j) = \varphi(\lambda t_j) - \varphi(\lambda t_{j-1}), \quad j = 1, \dots, n.$$

Now writing down the Gaussian kernel it is not difficult to see that the right-hand side of (2.3) is asymptotically equal to

$$\int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{(\det C_n^\lambda(t_1, \dots, t_n))^{d/2}} \mathrm{d}t_1 \dots \mathrm{d}t_n \tag{2.5}$$

since $\sqrt{2\pi}^{-d}\int f(x)\,\mathrm{d}x = 1$ by assumption. The difficulty in proving Lemma 2.1 comes not only from the complicated $\det C_n^\lambda(t_1, \dots, t_n)$ but also from the fact that

$$\begin{aligned}\lim_{\lambda\to\infty} &\int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{(\det C_n^\lambda(t_1, \dots, t_n))^{d/2}} \mathrm{d}t_1 \dots \mathrm{d}t_n \\ &\neq \int\cdots\int_{0<t_1<\cdots<t_n} \lim_{\lambda\to\infty} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{(\det C_n^\lambda(t_1, \dots, t_n))^{d/2}} \mathrm{d}t_1 \dots \mathrm{d}t_n.\end{aligned} \tag{2.6}$$

However, in the Markovian case (i.e., $\gamma = \frac{1}{2}$, $C_n^\lambda(t_1, \dots, t_n)$ is diagonal and so

$$\det C_n^\lambda(t_1, \dots, t_n) = \prod_{j=1}^n (\Delta\varphi(\lambda t_j))^{2\gamma} \tag{2.7}$$

and hence

$$\begin{aligned}&\int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{(\det C_n^\lambda(t_1, \dots, t_n))^{d/2}} \mathrm{d}t_1 \dots \mathrm{d}t_n. \\ &= \int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{\Delta\varphi(\lambda t_1)\dots\Delta\varphi(\lambda t_n)} \mathrm{d}t_1 \dots \mathrm{d}t_n,\end{aligned} \tag{2.8}$$

since $\gamma d = 1$. Therefore, the assertion of Lemma 2.1 may be reduced to

$$\begin{aligned}\lim_{\lambda\to\infty} &\int\cdots\int_{0<t_1<\cdots<t_n} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\dots\varphi'(\lambda t_n)}{\Delta\varphi(\lambda t_1)\dots\Delta\varphi(\lambda t_n)} \mathrm{d}t_1 \dots \mathrm{d}t_n \\ &= U_{s_1}U_{s_2}\dots U_{s_n}\mathbf{1}(0).\end{aligned} \tag{2.9}$$

If we define

$$U_s^\lambda p(t) = \int_t^\infty \mathrm{e}^{-s\xi} \frac{\varphi'(\lambda\xi)}{\varphi(\lambda\xi) - \varphi(\lambda t)} p(\xi)\,\mathrm{d}\xi, \tag{2.10}$$

then (2.9) can be rewritten as

$$\lim_{\lambda\to\infty} U_{s_1}^\lambda \dots U_{s_n}^\lambda \mathbf{1}(0) = U_{s_1}U_{s_2}\dots U_{s_n}\mathbf{1}(0).$$

Thus, for the proof of Lemma 2.1 it suffices to show that

$$\lim_{\lambda\to\infty} U_s^\lambda p(t) = U_s p(t), \quad p \in C_b([0, \infty)). \tag{2.11}$$

This idea is already given in Kasahara (1982), but in the present case the difficulty is that we have to treat non-Markovian cases, where (2.7) does *not* hold. Nevertheless, we shall show that, thanks to a special situation coming from the crucial condition $\gamma d = 1$, an analogue of (2.8) does hold and the problem may be reduced to (2.9) as in the Markovian case and hence the problem can be reduced to (2.11). We emphasize here that an analogue does not hold unless $\gamma d = 1$.

Of course, in the above we only explained the idea and did not mention about technical conditions. For example, precisely speaking, (2.5) and (2.10) diverge. So the domain of integration in (2.5) should be restricted to the set $\{(t_1, \dots, t_n) : \Delta\varphi(\lambda t_j) \geqslant 1 \ (j = 1, \dots, n)\}$ and also (2.10) should be modified a little (see Definition 3.1 in Section 3):

$$U_s^\lambda p(t) = \int_t^\infty e^{-s\xi} \frac{\varphi'(\lambda\xi)}{\varphi(\lambda\xi) - \varphi(\lambda t)} I(\varphi(\lambda\xi) - \varphi(\lambda t) \geqslant 1) p(\xi)\, d\xi. \tag{2.12}$$

3. Proof of Lemma 2.1

We first define an operator $U_s^{\lambda,M}$ which generalize the operator U_s^λ we defined in (2.12). ($U_s^\lambda = U_s^{\lambda,1}$). As we shall see later, $U_s^{\lambda,M}$ converges in some sense to U_s which we defined in Definition 2.1.

Definition 3.1. For every $s > 0$, $M \geqslant 1$ and for every $p \in C_b[0, \infty)$ we define $U_s^{\lambda,M} p \in C_b[0, \infty)$ by

$$U_s^{\lambda,M} p(t) = \int_t^\infty e^{-s\xi} p(\xi) \frac{\varphi'(\lambda\xi)}{\varphi(\lambda\xi) - \varphi(\lambda t)} I(\varphi(\lambda\xi) - \varphi(\lambda t) \geqslant M)\, d\xi, \quad \lambda > 0. \tag{3.1}$$

Notice that (3.1) may be rewritten as follows:

$$U_s^{\lambda,M} p(t) = \int_{\xi_0}^\infty e^{-s\xi} p(\xi) \frac{\partial}{\partial\xi} T_\lambda(\xi; t)\, d\xi, \tag{3.2}$$

where

$$T_\lambda(\xi; t) := \frac{1}{\lambda} \log(\varphi(\lambda\xi) - \varphi(\lambda t)) = \frac{1}{\lambda} \log(e^{\lambda\xi} - e^{\lambda t})$$

and

$$\xi_0 = \xi_0(\lambda, t; M) := \frac{1}{\lambda} \log(e^{\lambda t} + M). \tag{3.3}$$

We shall also use the convention that $T_\lambda(\xi; t) = 0$ when $\xi < \xi_0$. We remark that

$$0 \leqslant \frac{1}{\lambda} \log M = T_\lambda(\xi_0; t) \leqslant T_\lambda(\xi; t) \leqslant \xi, \quad \xi \in [\xi_0, \infty), \tag{3.4}$$

$$\lim_{\lambda \to \infty} \xi_0 = t, \tag{3.5}$$

and

$$\lim_{\lambda \to \infty} T_\lambda(\xi; t) = \xi, \quad \xi > t. \tag{3.6}$$

Furthermore, it holds that

$$\sup_{t \geq 0} \int_t^\infty e^{-s\xi} |T_\lambda(\xi; t) - \xi| \, d\xi \to 0 \quad \text{as } \lambda \to \infty. \tag{3.7}$$

Indeed, the convergence for each fixed t follows from (3.6) combined with (3.4), and by the monotonicity in t of $\int_t^\infty e^{-s\xi} T_\lambda(\xi; t)\, d\xi$, we see that the convergence is uniform for $t \geq 0$. The following lemma is what we mentioned in (2.11).

Lemma 3.1. *For every $M \geq 1$, $s > 0$ and $p \in C_b[0, \infty)$,*

(i) $\| U_s p \|_\infty \leq (1/s) \| p \|_\infty$,

(ii) $| U_s^{\lambda, M} p(t) | \leq \| p \|_\infty s \int_t^\infty e^{-s\xi} \xi \, d\xi \quad (t \geq 0, \lambda > 0)$

(iii) $\| U_s^{\lambda, M} p \|_\infty \leq (1/s) \| p \|_\infty \quad (\lambda > 0)$

(iv) $\lim_{\lambda \to \infty} \| U_s^{\lambda, M} p(t) - U_s p(t) \|_\infty = 0$.

Proof. (i)

$$\begin{aligned} |U_s p(t)| &\leq \| p \|_\infty \left(\int_t^\infty e^{-s\xi} \, d\xi + e^{-st} t \right) \\ &\leq \| p \|_\infty \times \int_0^\infty e^{-s\xi} \, d\xi \leq \frac{1}{s} \| p \|_\infty. \end{aligned}$$

(ii) Integrating by parts, we see from (3.2) that

$$\begin{aligned} |U_s^{\lambda, M} p(t)| &\leq \| p \|_\infty U_s^{\lambda, M} \mathbf{1}(t) \\ &= \| p \|_\infty \left(e^{-s\xi} T_\lambda(\xi; t) |_{\xi = \xi_0}^\infty + s \int_{\xi_0}^\infty e^{-s\xi} T_\lambda(\xi; t) \, d\xi \right) \\ &\leq 0 + \| p \|_\infty \times s \int_t^\infty e^{-s\xi} \xi \, d\xi. \end{aligned}$$

Here we have used $T_\lambda(\xi; t) \leq \xi$ (see (3.4)) and $\xi_0 > t$.

(iii) is an easy consequence of (ii).

(iv) In view of (ii), it is a routine work to see that it suffices to show the assertion only for p having compact support, and furthermore, by (i) and (iii), it suffices to prove assuming that p is a smooth function with compact support. Let ξ_0 be as in (3.3). Then,

$$\begin{aligned} U_s^{\lambda, M} p(t) &= e^{-s\xi} p(\xi) T_\lambda(\xi; t) |_{\xi = \xi_0}^\infty - \int_{\xi_0}^\infty (e^{-s\xi} p(\xi))' T_\lambda(\xi; t) \, d\xi \\ &= - e^{-s\xi_0} p(\xi_0) \frac{1}{\lambda} \log M - \int_t^\infty (e^{-s\xi} p(\xi))' T_\lambda(\xi; t) \, d\xi \end{aligned}$$

(with the convention that $T_\lambda(\xi;\, t) = 0$ when $\xi < \xi_0$) and

$$U_s p(t) = -\int_t^\infty (\mathrm{e}^{-s\xi} p(\xi))' \xi \,\mathrm{d}\xi.$$

Therefore,

$$\begin{aligned} &|U_s^{\lambda, M} p(t) - U_s p(t)| \\ &\leqslant \mathrm{e}^{-s\xi_0} |p(\xi_0)| \frac{1}{\lambda} \log M + \left| \int_t^\infty (\mathrm{e}^{-s\xi} p(\xi))' (T_\lambda(\xi;\, t) - \xi) \,\mathrm{d}\xi \right| \\ &\leqslant \|p\|_\infty \frac{1}{\lambda} \log M + (s\|p\|_\infty + \|p'\|_\infty) \int_t^\infty \mathrm{e}^{-s\xi} |T_\lambda(\xi;\, t) - \xi| \,\mathrm{d}\xi, \end{aligned}$$

which converges uniformly to 0 as $\lambda \to \infty$ by (3.7). □

Lemma 3.2. *For every* $M \geqslant 1$, $s_1, s_2, \ldots, s_n > 0$ *and* $p \in C_b[0, \infty)$,

(i) $\| U_{s_1}^{\lambda, M} U_{s_2}^{\lambda, M} \ldots U_{s_n}^{\lambda, M} p \|_\infty \leqslant (1/s_1 s_2 \ldots s_n) \|p\|_\infty$.

(ii) $\lim_{\lambda \to \infty} U_{s_1}^{\lambda, M} U_{s_2}^{\lambda, M} \ldots U_{s_n}^{\lambda, M} p(t) = U_{s_1} U_{s_2} \ldots U_{s_n} p(t), \quad t \geqslant 0.$

Proof. The assertion is an easy consequence of Lemma 3.1. □

The reader will probably find the following arguments tedious. But the authors would like to stress that the difficulty comes from (2.6), which is due to the singularity of the integration around the boundary.

Definition 3.2. Let $\Delta t_j = t_j - t_{j-1}$ and $t_0 = 0$ as before. For every $n = 1, 2, \ldots$ and $M \geqslant 1$, $a > 1$, we define

(1) $G(n;\, M) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ \Delta t_j \geqslant M,\ j = 1, \ldots, n\}$.
(2) $H_{jk}(n;\, M, a) = \{(t_1, \ldots, t_n) \in G(n;\, M);\ 1/a < \Delta t_j / \Delta t_k < a\}$.
(3) $G(n;\, M, a) = G(n;\, M) \setminus \bigcup_{j \neq k} H_{jk}(n;\, M, a)$.
(4) $G^\lambda(n;\, M) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ (\varphi(\lambda t_1), \ldots, \varphi(\lambda t_n)) \in G(n;\, M)\}$.
(5) $H_{jk}^\lambda(n;\, M, a) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ (\varphi(\lambda t_1), \ldots, \varphi(\lambda t_n)) \in H_{jk}(n;\, M, a)\}$.
(6) $G^\lambda(n;\, M, a) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n;\ (\varphi(\lambda t_1), \ldots, \varphi(\lambda t_n)) \in G(n;\, M, a)\}$.

Lemma 3.3. *For every* $M \geqslant 1$,

$$\lim_{\lambda \to \infty} \int \cdots \int_{G^\lambda(n;\, M)} \mathrm{e}^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1) \ldots \varphi'(\lambda t_n)}{\Delta\varphi(\lambda t_1) \ldots \Delta\varphi(\lambda t_n)} \,\mathrm{d}t_1 \ldots \mathrm{d}t_n = U_{s_1} U_{s_2} \ldots U_{s_n} \mathbf{1}(0). \tag{3.8}$$

Proof. Since the left-hand side may be expressed as $\lim_{\lambda \to \infty} U_{s_1}^{\lambda, M} \ldots U_{s_n}^{\lambda, M} \mathbf{1}(0)$, the assertion follows immediately from Lemma 3.2(ii) by setting $t = 0$. □

Lemma 3.4. *Let* $1 \leqslant j, k \leqslant n, j \neq k$. *Then, for every* $M \geqslant 1$ *and* $a > 1$,

(i)

$$\lim_{\lambda\to\infty} \int \cdots \int_{H_{jk}^{\lambda}(n;M,a)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_n)}{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_n)} dt_1 \ldots dt_n = 0.$$

(ii)

$$\lim_{\lambda\to\infty} \int \cdots \int_{H_{jk}^{\lambda}(n;M,a)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_n)}{(\det C_n^{\lambda}(t_1,\ldots,t_n))^{d/2}} dt_1 \ldots dt_n = 0.$$

Proof. Let $j < k$. By Lemma 3.1(iii),

$$\begin{aligned}
&\int \cdots \int_{H_{jk}^{\lambda}(n;M,a)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_n)}{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_n)} dt_1 \ldots dt_n \\
&= \int \cdots \int_{H_{jk}^{\lambda}(k;M,a)} e^{-\Sigma_{j\leqslant k} s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_k)}{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_k)} U_{s_{k+1}}^{\lambda,M} \ldots U_{s_n}^{\lambda,M}\mathbf{1}(t_k)\, dt_1 \ldots dt_k \\
&\leqslant \frac{1}{s_{k+1}\ldots s_n} \int \cdots \int_{H_{jk}^{\lambda}(k;M,a)} e^{-\Sigma_{j\leqslant k} s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_k)}{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_k)} dt_1 \ldots dt_k \\
&\leqslant \frac{1}{s_{k+1}\ldots s_n} \int \cdots \int_{G^{\lambda}(k-1;M)} e^{-\Sigma_{j\leqslant k-1} s_j t_j} \frac{\varphi'(\lambda t_1)\ldots\varphi'(\lambda t_{k-1})}{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_{k-1})} dt_1 \ldots dt_{k-1} \frac{2}{\lambda}\log a \\
&\leqslant \frac{s_k}{s_1\ldots s_n}\frac{2}{\lambda}\log a \to 0 \quad (\lambda\to\infty),
\end{aligned}$$

which proves (i). From Lemma 3.3 of Csörgő et al. (1995) (see also Goldman, 1984) we have

$$\det C_n(t_1, t_2, \ldots, t_n) \geqslant 2^{-n}\{\Delta t_1 \ldots \Delta t_n\}^{2\gamma} \tag{3.9}$$

and, hence,

$$\det C_n^{\lambda}(t_1, \ldots, t_n) \geqslant 2^{-n}\{\Delta\varphi(\lambda t_1)\ldots\Delta\varphi(\lambda t_n)\}^{2\gamma}.$$

Since $\gamma d = 1$, (ii) follows from (i). □

The following lemma is essentially due to Kôno (1996) but for the convenience of the reader we shall give the proof.

Lemma 3.5. *For every* $M \geqslant 1$,

$$\lim_{a\to\infty} \sup_{G(n;M,a)} \left| \frac{\det C_n(t_1, t_2, \ldots, t_n)}{(\Delta t_1 \ldots \Delta t_n)^{2\gamma}} - 1 \right| = 0.$$

Proof. For $0 < t_1 < \cdots < t_n$ let $\hat{C}_n(t_1, t_2, \ldots, t_n)$ denote the correlation matrix of $\{X^{\gamma}(t_j) - X^{\gamma}(t_{j-1})\}_{j=1}^{n}$, i.e., $\hat{C}_n(t_1, t_2, \ldots, t_n)$ is the $n \times n$ matrix with elements

$$r_{ij} = \frac{|t_{i-1} - t_j|^{2\gamma} + |t_i - t_{j-1}|^{2\gamma} - |t_i - t_j|^{2\gamma} - |t_{i-1} - t_{j-1}|^{2\gamma}}{2(t_i - t_{i-1})^{\gamma}(t_j - t_{j-1})^{\gamma}}.$$

Notice that

$$\det \hat{C}_n(t_1, t_2, \dots, t_n) = \frac{\det C_n(t_1, t_2, \dots, t_n)}{\{\Delta t_1 \dots \Delta t_n\}^{2\gamma}}. \tag{3.10}$$

As is pointed out by Kôno (1996) we easily have

$$|r_{ij}| \leqslant \left(\frac{\Delta t_i \wedge \Delta t_j}{\Delta t_i \vee \Delta t_j}\right)^{\gamma}, \quad i \neq j.$$

(Here we used the fact that $0 < \gamma < \frac{1}{2}$ which follows from $d \geqslant 2$ and $\gamma d = 1$). Combining this with the definition of $G(n; M, a)$ we have

$$\lim_{a \to \infty} \sup_{G(n; M, a)} |r_{ij}| = 0, \quad i \neq j.$$

Since $r_{11} = r_{22} = \cdots = 1$, this implies

$$\lim_{a \to \infty} \sup_{G(n; M, a)} |\det \hat{C}_n(t_1, t_2, \dots, t_n) - 1| = 0. \tag{3.11}$$

The assertion of the lemma follows from (3.10) and (3.11). □

Lemma 3.6. *For every $n \geqslant 1$ and $M \geqslant 1$,*

$$\lim_{\lambda \to \infty} \int \cdots \int_{G^{\lambda}(n; M)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)}{(\det C_n^{\lambda}(t_1, \dots, t_n))^{d/2}} \, dt_1 \dots dt_n$$

$$= U_{s_1} U_{s_2} \dots U_{s_n} \mathbf{1}(0). \tag{3.12}$$

Proof. By Lemma 3.4(ii), it holds that for every $a > 1$,

$$\lim_{\lambda \to \infty} \int \cdots \int_{G^{\lambda}(n; M)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)}{(\det C_n^{\lambda}(t_1, \dots, t_n))^{d/2}} \, dt_1 \dots dt_n$$

$$= \lim_{\lambda \to \infty} \int \cdots \int_{G^{\lambda}(n; M, a)} e^{-\Sigma s_j t_j} \frac{\varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)}{(\det C_n^{\lambda}(t_1, \dots, t_n))^{d/2}} \, dt_1 \dots dt_n$$

provided that the right-hand side exists. Combining this with Lemmas 3.3 and 3.5 we easily have the assertion. □

Lemma 3.7. *Let $G(n; M)$ and $C_n(t_1, t_2, \dots, t_n)$ be as before. Then,*

$$\lim_{M \to \infty} \sup_{G(n; M)} \sup_{x \in \mathbb{R}^n, |x| \leqslant 1} (C_n(t_1, t_2, \dots, t_n)^{-1} x, x) = 0.$$

Proof. Let $\hat{C}_n(t_1, t_2, \dots, t_n)$ be as in the proof of Lemma 3.5. By (3.9) it holds for $0 < t_1 < \cdots < t_n$ that

$$\det \hat{C}_n(t_1, t_2, \dots, t_n) = \frac{\det C_n(t_1, t_2, \dots, t_n)}{(\Delta t_1 \dots \Delta t_n)^{2\gamma}} \geqslant \frac{1}{2^n}.$$

Now let $0 < \rho_1 \leqslant \cdots \leqslant \rho_n$ be eigenvalues of $\hat{C}_n(t_1, t_2, \dots, t_n)$. Since

$$\rho_k \leqslant \operatorname{trace} \hat{C}_n(t_1, t_2, \dots, t_n) = n, \quad k = 1, \dots, n,$$

we see that the minimal eigenvalue is uniformly positive,

$$\rho_1 = \frac{\det \hat{C}_n(t_1, t_2, \dots, t_n)}{\rho_2 \dots \rho_n} \geqslant \frac{1}{2^n \rho_2 \dots \rho_n} \geqslant \frac{1}{2^n n^{n-1}}.$$

Therefore, for $(t_1, \dots, t_n) \in G(n; M)$, letting $y = (x_1/(\Delta t_1)^\gamma, \dots, x_n/(\Delta t_n)^\gamma)$ we have

$$\begin{aligned}(C_n(t_1, t_2, \dots, t_n)^{-1}x, x) &= (\hat{C}_n(t_1, t_2, \dots, t_n)^{-1}y, y)\\ &\leqslant 2^n n^{n-1} \sum_{k=1}^{n} \frac{x_k^2}{(\Delta t_k)^{2\gamma}} \leqslant \frac{2^n n^{n-1}}{M^{2\gamma}} |x|^2,\end{aligned}$$

which proves the assertion. □

Lemma 3.8. *For every $s_1, s_2, \dots, s_n > 0$ and $M \geqslant 1$ define*

$$\phi_n^{(\lambda)}(s_1, \dots, s_n; M) = \int \cdots \int_{G^\lambda(n;M)} e^{-\Sigma s_j t_j} E\left[\prod_{j=1}^{n} f(X^{\gamma,d}(\varphi(\lambda t_j)))\varphi'(\lambda t_j)\right] dt_1 \dots dt_n.$$

Then,

(i) $\limsup_{\lambda \to \infty} \phi_n^{(\lambda)}(s_1, \dots, s_n; M) \leqslant U_{s_1} U_{s_2} \dots U_{s_n} \mathbf{1}(0)$, $M \geqslant 1$,

(ii) $\lim_{M \to \infty} \limsup_{\lambda \to \infty} |\phi_n^{(\lambda)}(s_1, \dots, s_n; M) - U_{s_1} U_{s_2} \dots U_{s_n} \mathbf{1}(0)| = 0$.

Proof. Let $g^\lambda(t_1, \dots, t_n; x)$ $(x \in \mathbb{R}^n)$ denote the density function of

$$(X^\gamma(\varphi(\lambda t_1)) - X^\gamma(\varphi(\lambda t_0)), \dots, X^\gamma(\varphi(\lambda t_n)) - X^\gamma(\varphi(\lambda t_{n-1})))$$

i.e.,

$$g^\lambda(t_1, \dots, t_n; x) = \frac{1}{(2\pi)^{n/2} \sqrt{\det C_n^\lambda(t_1, \dots, t_n)}} \exp\{-\tfrac{1}{2}(C_n^\lambda(t_1, \dots, t_n)^{-1}x, x)\}.$$

By Lemma 3.7 we have

$$\lim_{M \to \infty} \sup_{G^\lambda(n;M)} \left| \frac{g^\lambda(t_1, \dots, t_n; x)}{g^\lambda(t_1, \dots, t_n; 0)} - 1 \right| = 0, \tag{3.13}$$

the convergence being uniform for $x \in \mathbb{R}^n$ on every compact set. The density function of

$$(X^{\gamma,d}(\varphi(\lambda t_1)) - X^{\gamma,d}(\varphi(\lambda t_0)), \dots, X^{\gamma,d}(\varphi(\lambda t_n)) - X^{\gamma,d}(\varphi(\lambda t_{n-1})))$$

is given by

$$g_d^\lambda(t_1, \dots, t_n; x^1, \dots, x^d) = \prod_{k=1}^{d} g^\lambda(t_1, \dots, t_n; x^k) \quad (x^1, \dots, x^d) \in (\mathbb{R}^n)^d.$$

Therefore, by (3.13)

$$\lim_{M \to \infty} \sup_{G^\lambda(n;M)} \left| \frac{g_d^\lambda(t_1, \dots, t_n; x^1, \dots, x^d)}{g_d^\lambda(t_1, \dots, t_n; 0, \dots, 0)} - 1 \right| = 0, \tag{3.14}$$

the convergence being uniform for $(x^1, \dots, x^d) \in (\mathbb{R}^n)^d$ on every compact set, and it holds

$$\phi_n^{(\lambda)}(s_1, \dots, s_n; M) = \int \cdots \int_{G^\lambda(n;M)} e^{-\sum s_j t_j} \, dt_1 \dots dt_n$$

$$\times \int \cdots \int_{(\mathbb{R}^d)^n} g_d^\lambda(t_1, \dots, t_n; x^1, \dots, x^d) \varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)$$

$$\times f(x_1) f(x_1 + x_2) \dots f(x_1 + \cdots + x_n) \, dx_1 \dots dx_n,$$

where $x_i = (x_i^1, \dots, x_i^d)$ $(i = 1, \dots, n)$ when $x^j = (x_1^j, \dots, x_n^j)$ $(j = 1, \dots, d)$. Now by (3.14) it suffices to show the assertion (ii) of the lemma replacing $\phi_n^{(\lambda)}$ by

$$\tilde{\phi}_n^{(\lambda)}(s_1, \dots, s_n; M) = \int \cdots \int_{G^\lambda(n;M)} e^{-\sum s_j t_j} \, dt_1 \dots dt_n$$

$$\times \int \cdots \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} g_d^\lambda(t_1, \dots, t_n; 0, \dots, 0) \varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)$$

$$\times f(x_1) f(x_1 + x_2) \dots f(x_1 + \cdots + x_n) \, dx_1 \dots dx_n,$$

$$= C^n \int \cdots \int_{G^\lambda(n;M)} e^{-\sum s_j t_j} \frac{\varphi'(\lambda t_1) \dots \varphi'(\lambda t_n)}{(\det C_n^\lambda(t_1 \dots t_n))^{d/2}} \, dt_1 \dots dt_n,$$

where C is as before (see (1.1)) and recall that we assumed that $C = 1$. Therefore, the assertion (ii) follows from Lemma 3.6. In a similar way, we can show (i) using the obvious inequality

$$\phi_n^{(\lambda)}(s_1, \dots, s_n; M) \leqslant \tilde{\phi}_n^{(\lambda)}(s_1, \dots, s_n; M).$$

Lemma 3.9. *For every $M \geqslant 1$,*

$$\lim_{\lambda \to \infty} |\phi_n^{(\lambda)}(s_1, \dots, s_n) - \phi_n^{(\lambda)}(s_1, \dots, s_n; M)| = 0.$$

Proof.

$$|\phi_n^{(\lambda)}(s_1, \dots, s_n) - \phi_n^{(\lambda)}(s_1, \dots, s_n; M)|$$

$$\leqslant \sum_{k=1}^n \int \cdots \int_{\substack{0 < t_1 < \cdots < t_n \\ \Delta\varphi(\lambda t_k) \leqslant M}} e^{-\sum s_j t_j} E\left[\prod_{j=1}^n f(X^{\gamma,d}(\varphi(\lambda t_j))) \varphi'(\lambda t_j) \right] dt_1 \dots dt_n$$

$$\leqslant \|f\|_\infty \sum_{k=1}^n \int \cdots \int_{\substack{0 < t_1 < \cdots < t_n \\ \Delta\varphi(\lambda t_k) \leqslant M}} e^{-\sum_{j \neq k} s_j t_j}$$

$$\times E\left[\prod_{j \neq k}^n f(X^{\gamma,d}(\varphi(\lambda t_j))) \varphi'(\lambda t_j) \right] \varphi'(\lambda t_k) \, dt_1 \dots dt_n$$

$$= \|f\|_\infty \frac{M}{\lambda} \sum_{k=1}^n \phi_{n-1}^{(\lambda)}(s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_n).$$

Here we understand that $\phi_{n-1}^{(\lambda)} = 1$ when $n = 1$. Therefore, by mathematical induction, we have the assertion from Lemma 3.8(i). $\square$

We are now ready to prove Lemma 2.1: The assertion follows from Lemmas 3.8(ii) and 3.9.

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