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Yang–Mills fields and stochastic parallel transport in small geodesic balls ☆

Robert Otto Bauer

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

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Abstract

We develop a new method to obtain stochastic characterizations of Yang–Mills fields. Our main tool is the Itô-equation for the stochastic parallel transport. We estimate the drift terms in a small ball of radius ε and find that for a general connection the average rotation is of order ε^3 but that for a Yang–Mills connections the average rotation is of order ε^4. Using a Doob h-transform we give a new proof of the stochastic characterization of Yang–Mills fields by S. Stafford. Varying the starting point of the Brownian motion we obtain an unconditioned version of this result. By considering the horizontal Laplace equation we then apply our result to obtain a new analytic characterization of Yang–Mills fields.

MSC: 58G32; 60H30

Keywords: Stochastic parallel transport; Yang–Mills equations; Green function; Doob h-transform

1. Introduction

To date there are two stochastic characterizations of the Yang–Mills equations on a vector bundle (Bauer, 1998; Stafford, 1990). Both characterize connections that satisfy the Yang–Mills equations in terms of their associated stochastic parallel transport. In Bauer (1998), we showed that the "derivative" of the stochastic parallel transport under variations of the connections induced by gradient flows on the base manifold is a martingale if and only if the connection satisfies the Yang–Mills equations.

In Stafford (1990) the author studies the stochastic parallel transport along a Riemannian Brownian motion $x(t)$ until there exits a small geodesic ball of radius ε at a fixed point ζ. He found that for a general connection

$$E[v(\tau_\varepsilon) - \mathrm{Id} \mid x(\tau_\varepsilon) = \zeta] = \mathrm{O}(\varepsilon^3),$$

(where $v(\tau_\varepsilon)$ is the stochastic parallel transport from zero until the exit time τ_ε), and that a connection satisfies the Yang–Mills equations if and only if

$$E[v(\tau_\varepsilon) - \mathrm{Id} \mid x(\tau_\varepsilon) = \zeta] = \mathrm{O}(\varepsilon^4).$$

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E-mail address: rbauer@math.gatech.edu (R.O. Bauer).

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The proof is based on the Liao–Pinsky method (Liao and Pinsky, 1987) of first expanding the infinitesimal generator of the stochastic parallel transport, then solving a system of partial differential equations and evaluating the solutions at zero using a generalization of Pizetti's formula (Liao, 1988). To condition the Riemannian Brownian motion to exit the geodesic ball at a certain point the author uses a limiting procedure as in Liao and Pinsky (1987).

In this paper we develop a new method to prove this and other results. Our method is based on the Itô-integral equation for the stochastic parallel transport. We estimate the various terms using expansions of the metric and connection symbols in convenient coordinates and well-known bounds for the Green function and its derivatives. In Theorems 5.1 and 5.2 we use this method to characterize connections that satisfy the Yang–Mills equations. Theorem 5.2 provides a new proof of the result in Stafford (1990). In our proof, the conditioning of Brownian motion is incorporated from the beginning through a Doob h-transform.

Theorem 5.1 avoids the conditioning through variation of the starting point of the Brownian motion. Roughly stated, it says that if one starts a Brownian motion anywhere inside a ball of radius ε and evaluates the stochastic parallel transport at the exit time from the ball (at an unspecified exit point), then the average rotation is of order ε^4 if and only if the connection satisfies the Yang–Mills equations at the origin of the ball. This result lends itself naturally to a recasting in analytic terms. Corollary 5.3 shows that solutions u_ε of the horizontal Laplace equation $\Delta^{\mathrm{H}} u_\varepsilon = 0$ in a ball of radius ε with constant boundary value have variation of order ε^4 if and only if the connection is Yang–Mills at the origin.

To put our results into perspective they have to be compared to a characterization of Yang–Mills fields by physicists in the general context of the Penrose transform. Here, one studies extensions of holomorphic vector bundles over ambi-twistor space Q to formal neighborhoods of Q in $\mathbb{CP}^3 \times \mathbb{CP}^3$. In general, a vector bundle is given in terms of transition functions satisfying a cocycle condition. In particular, for each triple of open sets $U_\alpha, U_\beta, U_\gamma$ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, the transition functions $h_{\alpha\beta}, h_{\beta\gamma}, h_{\gamma\alpha}$ satisfy $h_{\alpha\beta} \circ h_{\beta\gamma} \circ h_{\gamma\alpha} = \mathrm{Id}$. It is shown in Isenberg and Yasskin (1979) and Witten (1978) that for a general connection A the corresponding holomorphic vector bundle over Q extends to the second formal neighborhood in $\mathbb{CP}^3 \times \mathbb{CP}^3$, meaning that

$$h_{\alpha\beta} \circ h_{\beta\gamma} \circ h_{\gamma\alpha} = \mathrm{Id} + \mathrm{O}(\varepsilon^3),$$

where ε is the distance of a point in $\mathbb{CP}^3 \times \mathbb{CP}^3$ to Q, and that the connection satisfies the Yang–Mills equations if and only if the vector bundle extends to the third formal neighborhood, i.e.

$$h_{\alpha\beta} \circ h_{\beta\gamma} \circ h_{\gamma\alpha} = \mathrm{Id} + \mathrm{O}(\varepsilon^4).$$

The striking similarity of the ambi-twistor representation to the results presented here will be explored in a forthcoming paper. The ambi-twistor representation is a generalization of twistor methods used in the representation of the much stronger self-dual Yang–Mills equations on $\mathbb{R}^4$. For self-dual fields, the representation lead to the explicit construction of all self-dual Yang–Mills fields of finite energy, the ADHM-construction (Atiyah et al., 1978). For the full Yang–Mills equations the results are much weaker and the characterizations, including the ones given here, have not led to the construction of Yang–Mills fields. Thus, our results are rather an illumination of the role the Yang–Mills equations play in the study of multiplicative functionals calculated along Brownian paths, i.e. stochastic parallel transports. In particular, for short time, the main contribution to the stochastic parallel transport comes from the Yang–Mills current. A detailed study of multiplicative operator functionals, which include stochastic parallel translation as a special case, can be found in Pinsky (1974).

The paper is organized as follows. In Section 2 we introduce the differential geometric objects we will work with. Good references for this material are Roe (1993), Lawson (1985) and Bourguignon and Lawson (1981). Section 3 compares the Green function for a small Euclidean ball with the Green function for a geodesic ball on a Riemannian manifold (see Grüter and Widman, 1982; De Rham, 1946/1947, 1950). Section 4 establishes some probabilistic prerequisites. For more background on these topics we suggest Emery (1989), Ikeda and Watanabe (1989) and Norris (1992).

2. Geometric preliminaries

The differential objects we will be working with are a Riemannian manifold M with metric tensor $g(\cdot,\cdot)$, a vector bundle η over M with fiber $\eta_x \cong \mathbb{R}^n$ and compact structure group G. Denote the Lie algebra of G by $\mathfrak{g}$ and the adjoint and automorphism bundles by $\mathrm{Ad}\,\eta$ and $\mathrm{Aut}\,\eta$, respectively. Assume also η has a metric compatible with the action of G and an inclusion $G \subset \mathrm{SO}(n)$. Given any vector bundle ξ over M, we denote by $\Omega^p(\xi) \equiv \Gamma(\bigwedge^p T^*M \otimes \xi)$ the space of exterior differential p-forms with values in ξ.

Since our results are local in nature we will later assume that $M = U$ is a coordinate chart and a convenient local choice of gauge ρ: $\eta \,|\, U \cong U \times \mathbb{R}^n$ has been made. Then all bundles considered become cross products, ρ: $\mathrm{Aut}\,\eta \,|\, U \cong U \times G$ and ρ: $\mathrm{Ad}\,\eta \,|\, U \cong U \times \mathfrak{g}$.

The *gauge group* is $\mathfrak{G} = C^\infty(\mathrm{Aut}\,\eta)$, which in the trivial bundle case is $\mathfrak{G} \cong C^\infty(U, G)$. The choice of ρ introduces a flat covariant derivative given by $d = (\partial/\partial x^1, \dots, \partial/\partial x^d)$. Any covariant derivative (or connection) ∇ is then given by $\nabla = d + A = \{\partial/\partial x^i + A_i\}$, where $A_i(x) \in \mathfrak{g}$. One can think of A as a Lie algebra valued 1-form, or locally $A: U \to \mathbb{R}^d \otimes \mathfrak{g}$.

Gauge changes $s: U \to G$ act on $\nabla = d + A$ by

$$s^{-1} \circ \nabla \circ s = d + s^{-1}\,\mathrm{d}s + s^{-1} A s = d + \tilde{A}.$$

This means A and $\tilde{A} = s^{-1}\,\mathrm{d}s + s^{-1} A s$ represent the covariant derivative in different coordinates (or gauges).

For each connection ∇ on a vector bundle ξ there are operators d^∇: $\Omega^p(\xi) \to \Omega^{p+1}(\xi)$, $p \geqslant 0$, defined by

$$d^\nabla(\alpha \otimes \sigma) = \mathrm{d}\alpha \otimes \sigma + (-1)^p \alpha \otimes \nabla\sigma$$

for a real-valued differential p-form α and a section σ of ξ. This extends to general $\psi \in \Omega^p(\xi)$ by linearity. Note that $d^\nabla = \nabla$ on $\Omega^0(\xi)$.

Suppose now that ξ is furnished with an inner product preserved by ∇. Using the Riemannian metric we can define an inner product $\langle\,,\rangle$ in $\Omega^p(\xi)_x = \bigwedge^p T_x^*M \otimes \xi_x$.

Integrating this over M with respect to the Riemannian volume gives an inner product in $\Omega^p(\xi)$. We then define $\delta^\nabla\colon \Omega^{p+1}(\xi) \to \Omega^p(\xi)$, $p \geqslant 0$, to be the formal adjoint of the operator d^∇.

The *curvature* or *field* $F = F(\nabla)$ of a connection is defined by $F = d^\nabla \circ d^\nabla$. Locally, $F\colon U \to \mathbb{R}^d \wedge \mathbb{R}^d \otimes \mathfrak{g}$ is a Lie algebra-valued two-form, $F = \{F_{ij}\} = \{[\nabla_i, \nabla_j]\}$:

$$F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i + [A_i, A_j] \in \mathfrak{g}. \tag{2.1}$$

Under a gauge transformation $s \in \mathfrak{G}$ the curvature transforms by $F \to s^{-1} F s$.

The *Yang–Mills equations* are the Euler–Lagrange equations for the action integral

$$\|F\|^2 = \frac{1}{8\pi^2} \int_M \langle F \rangle^2 \, \mathrm{d}\mu_M = \frac{1}{8\pi^2} \int_U g^{ik} g^{jl} \langle F_{ij}, F_{kl} \rangle g \, \mathrm{d}x.$$

Here the second integral is in local coordinates, g^2 the determinant of the metric tensor (g_{ij}), (g^{ij}) the inverse matrix of (g_{ij}) and $\langle A, B \rangle = \operatorname{tr} AB^*$ is the trace inner product in $\mathfrak{g}$.

The Yang–Mills equations, or the Euler–Lagrange equations for the integral $\|F\|^2$, are $\delta^\nabla F = 0$. In coordinates on U this means explicitly,

$$(\delta^\nabla F)_k = -g^{ij} \left(\frac{\partial}{\partial x^i} F_{jk} + [A_i, F_{jk}] - \Gamma^l_{ij} F_{lk} - \Gamma^l_{ik} F_{jl} \right) = 0, \quad k = 1, \ldots, d,$$

where (Γ^l_{ij}) are the Christoffel symbols of the Levi–Civita connection on TM. We say ∇ is a *Yang–Mills connection* and $F = F(\nabla)$ is a *Yang–Mills field* if $\delta^\nabla F = 0$. If s lies in the gauge group $\mathfrak{G}$, then $\|s^{-1} F s\|^2 = \|F\|^2$. Therefore, the solutions of $\delta^\nabla F = 0$, as either Yang–Mills connections or Yang–Mills fields, are an invariant space under gauge transformation.

To do calculations near a point o in M we choose coordinates as follows. We use the exponential map exp to identify a small ball in $T_0 M$, centered at zero, with its image U under exp. Together with the identification $T_0 M \cong \mathbb{R}^d$ this provides a *normal* coordinate system at o. Over U we trivialize the tangent bundle $TM\,|_U \cong U \times \mathbb{R}^d$ using an orthonormal moving frame and the vector bundle $\xi\,|_U \cong U \times \mathbb{R}^n$ using a radial gauge. In these coordinates we have the well known (see, e.g. Stafford, 1990; Liao and Pinsky, 1987; Roe, 1993), expansions of the metric $\{g_{ij}\}$, the Christoffel symbols $\{\Gamma^i_{jk}\}$ and the connection symbols $\{A^\alpha_{i\beta}\}$;

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + \mathrm{O}(|x|^2), \\ \Gamma^i_{jk}(x) &= \tfrac{1}{2} R^i_{pjk}(0) x^p + \tfrac{1}{3} \partial_p R^i_{qjk}(0) x^p x^q + \mathrm{O}(|x|^3), \\ A^\alpha_{i\beta}(x) &= \tfrac{1}{2} F^\alpha_{pi\beta}(0) x^p + \tfrac{1}{3} \partial_p F^\alpha_{qi\beta}(0) x^p x^q + \mathrm{O}(|x|^3). \end{aligned} \tag{2.2}$$

By the symmetries of the curvature tensors R and F, we have $R^i_{jjk} = 0$, $F^\alpha_{jj\beta} = 0$. Consequently,

$$g^{ij}(x) \partial_i \Gamma^k_{jl}(x) = \tfrac{1}{3} \sum_{i=1}^d \partial_i R^k_{qil}(0) x^q + \mathrm{O}(|x|^2),$$

$$g^{ij}(x) \partial_i A^\alpha_{j\beta}(x) = \tfrac{1}{3} \sum_{i=1}^d \partial_i F^\alpha_{qi\beta}(0) x^q + \mathrm{O}(|x|^2). \tag{2.3}$$

Furthermore, at the origin o of a normal coordinate system in radial gauge, the Yang–Mills equations simplify as

$$\sum_{i=1}^{d} \frac{\partial}{\partial x^i} F_{ik}(\mathrm{o}) = 0, \quad k = 1, \ldots, d. \tag{2.4}$$

3. The Green function for a small geodesic ball

Let $B_\varepsilon \subset U$ be a small open ball of radius $\varepsilon > 0$ centered at o and write $S_\varepsilon = \partial B_\varepsilon$. In B_ε the Laplacian Δ on functions on M is given by

$$\Delta = g^{ij}(x)\left(\frac{\partial}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x)\frac{\partial}{\partial x^k}\right). \tag{3.1}$$

Denote $G_\varepsilon(x, y)$ the Green function for $\frac{1}{2}$ the Euclidean Laplacian $\Delta' = \sum_i \partial^2/(\partial x^i)^2$ on B_ε,

$$G_\varepsilon(x, y) = C_d\,|x - y|^{2-d} - C_d \varepsilon^{d-2}\,|y|^{2-d}\left|x - \frac{\varepsilon^2}{|y|^2} y\right|^{2-d} \tag{3.2}$$

with $C_d = \Gamma(d/2 - 1)/(2\pi^{d/2})$. Denote $G(x, y)$ the Green function for $1/2\Delta$ on B_ε. We have the following estimates in dimension $d \geqslant 3$ (Grüter and Widman, 1982, Theorem 3.3/3.4): For any $x, y \in B_\varepsilon$, setting $\delta(y) = \mathrm{dist}(y, \partial B_\varepsilon)$,

$$\begin{aligned}
&\text{(i)}\ G(x, y) \leqslant K\,|x - y|^{2-d},\\
&\text{(ii)}\ G(x, y) \leqslant K\delta(x)\,|x - y|^{1-d},\\
&\text{(iii)}\ \left|\operatorname*{grad}_x G(x, y)\right| \leqslant K\,|x - y|^{1-d},\\
&\text{(iv)}\ \left|\operatorname*{grad}_x \operatorname*{grad}_y G(x, y)\right| \leqslant K\,|x - y|^{-d}.
\end{aligned} \tag{3.3}$$

Furthermore, as follows from the Neumann series representation of $G(x, y)$ (De Rham, 1950, Section 17),

$$\begin{aligned}
&\text{(i)}\ |G(x, y) - G_\varepsilon(x, y)| \leqslant \mathrm{O}(|x - y|^{4-d}),\\
&\text{(ii)}\ \left|\operatorname*{grad}_x G(x, y) - \operatorname*{grad}_x G_\varepsilon(x, y)\right| \leqslant \mathrm{O}(|x - y|^{3-d}).
\end{aligned} \tag{3.4}$$

Remark 3.1. As is shown in De Rham (1946/1947), the difference between the Riemannian distance and the Euclidean distance is

$$(\mathrm{dist}(x, y))^2 - |x - y|^2 = \mathrm{O}(|x - y|^4).$$

Since in the calculations that will follow only the leading order terms are of importance it is not necessary (for the purpose of said calculations) to distinguish between Riemannian and Euclidean distance. Similarly, by (2.2), $g(x) = 1 + \mathrm{O}(|x|^2)$, and we will not distinguish between the Riemannian volume form $g(x)\,\mathrm{d}x$ and $\mathrm{d}x$.

4. Probabilistic preliminaries

Let $(\mathscr{W}, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathscr{P})$ be a filtered probability space satisfying the usual conditions and let $b = \{b(t), t \geqslant 0\}$ be a Brownian motion on this probability space taking values in $\mathbb{R}^d$ with $\mathscr{P}(b(0) = 0) = 1$. In this paper we will denote Stratonovich differentials by ∂ and Itô differentials by d.

In a coordinate chart U the stochastic parallel transport $v = \{v(t), t \geqslant 0\}$ in η is defined by a system of stochastic differential equations (in the sense of Stratonovich)

$$\mathrm{d}x^i(t) = u^i_j(t)\partial b^j(t),$$

$$\mathrm{d}u^i_k(t) = -\Gamma^i_{lm}(x(t))u^m_k(t)u^l_j(t)\partial b^j(t),$$

$$\mathrm{d}v^\alpha_\beta(t) = -A^\alpha_{i\gamma}(x(t))v^\gamma_\beta(t)u^i_j(t)\partial b^j(t) \tag{4.1}$$

with the initial conditions $x(0) = x_0$, $u(0) = \mathrm{id}_{\mathbb{R}^d}$, and $v(0) = \mathrm{id}_{\mathbb{R}^n}$. In (4.1) $(A^\alpha_{i\beta})$ is the matrix representation of $A_i \in \mathfrak{g}$. A global solution is obtained by patching the solutions in the coordinate charts together. The process $x = \{x(t), t \geqslant 0\}$ thus obtained is called an M-valued (or Riemannian) Brownian motion and $u = \{u(t), t \geqslant 0\}$ its horizontal lift into the frame bundle of M. For details on this setup and existence and uniqueness results for system (4.1) see, e.g. Emery (1989), Norris (1992) and Ikeda and Watanabe (1989).

To obtain estimates on the stochastic parallel transport we need to know the Itô equation for $v(t)$. Using the usual Itô–Stratonovich conversion formula and the equality $\sum_{i=1}^d u^m_i(t)u^l_i(t) = g^{ml}(x(t))$, we find

$$\begin{aligned}\mathrm{d}v^\alpha_\beta(t) = {} & -A^\alpha_{i\gamma}(x(t))v^\gamma_\beta(t)u^i_j(t)\,\mathrm{d}b^j(t) \\ & - 1/2 g^{rl}(x(t))\partial_r A^\alpha_{l\gamma}(x(t))v^\gamma_\beta(t)\,\mathrm{d}t \\ & + 1/2 g^{rl}(x(t))A^\alpha_{l\gamma}(x(t))A^\gamma_{r\delta}(x(t))v^\delta_\beta(t)\,\mathrm{d}t \\ & + 1/2 g^{sr}(x(t))A^\alpha_{i\gamma}(x(t))v^\gamma_\beta(t)\Gamma^i_{rs}(x(t))\,\mathrm{d}t.\end{aligned} \tag{4.2}$$

Denote $\{x(t), t \geqslant 0\}$ an M-valued Brownian motion started at $x_0 \in B_\varepsilon$, and $\mathscr{P}^{x_0}$ its law, and set

$$\tau_\varepsilon = \inf\{t: x(t) \notin B_\varepsilon\},$$

the exit time of x from B_ε. Let $\underline{\varepsilon} \in S_\varepsilon$. For $x_0 = 0$ we construct a Riemannian Brownian motion conditioned to exit B_ε at $\underline{\varepsilon}$ as a Doob h-transform (see Pinsky, 1995). Set

$$h(x) = \frac{\partial}{\partial n_{\underline{\varepsilon}}} G(x, \underline{\varepsilon}), \quad x \in B_\varepsilon,$$

where $n_{\underline{\varepsilon}}$ is the inward unit normal vector at $\underline{\varepsilon}$. Similarly,

$$h_\varepsilon(x) = \frac{\partial}{\partial n_{\underline{\varepsilon}}} G_\varepsilon(x, \underline{\varepsilon}), \quad x \in B_\varepsilon.$$

Let $\mathscr{P}_\varepsilon = \mathscr{P}^{x_0}(.\,|\,x(\tau_\varepsilon)\,| = \underline{\varepsilon})$. Then, under $\mathscr{P}_\varepsilon$,

$$x(t) = w(t) + \int_0^t \operatorname{grad}\log h(x(s))\,\mathrm{d}s, \quad 0 \leqslant t < \tau_\varepsilon,$$

where $w = \{w(t), 0 \leqslant t < \tau_\varepsilon\}$ is a $\mathscr{P}_\varepsilon$-(Riemannian) Brownian motion. h is the Poisson kernel at $\underline{\varepsilon}$. By Grüter and Widman (1982, Lemma 3.1) combined with the Harnack inequality (Grigor'yan, 1995) we have

$$|\operatorname{grad}\log h(x)| \leqslant K\delta(x)^{-1}. \tag{4.3}$$

The exit distribution for Brownian motion in B_ε started at zero is known to have strictly positive density. Since the Poisson kernel (in our definition) gives the density of the exit distribution relative to non-normalized sphere measure (which assigns volume $c_d\varepsilon^{d-1}$ to S_ε), we have

$$h(0) \geqslant K\varepsilon^{1-d}. \tag{4.4}$$

Lemma 4.1. *Denote by E^{x_0} the expectation with respect to $\mathscr{P}^{x_0}$. Then*

$$E^{x_0}[\tau_\varepsilon^p] = \mathrm{O}(\varepsilon^{2p}), \quad p = 1, 2.$$

Lemma 4.2. *Denote by E_ε the expectation with respect to $\mathscr{P}_\varepsilon$. Then*

$$E_\varepsilon[\tau_\varepsilon^p] = \mathrm{O}(\varepsilon^{2p}), \quad p = 1, 2$$

and

$$E_\varepsilon\left[\left(\int_0^{\tau_\varepsilon} |\operatorname{grad}\log h(x(s))|\,\mathrm{d}s\right)^p\right] = \mathrm{O}(\varepsilon^p), \quad p = 1, 2.$$

The proof of Lemmas 4.1 and 4.2 will be given in the appendix.

5. Probabilistic characterization of Yang–Mills fields

Theorem 5.1. *The field F is Yang–Mills at* o *if and only if*

$$\sup_{x_0 \in B_\varepsilon} E_{x_0}[\,|v(\tau_\varepsilon) - \mathrm{Id}|\,] = \mathrm{O}(\varepsilon^4).$$

Proof. By (4.2) we have

$$\begin{aligned} v(\tau_\varepsilon) - \mathrm{Id} = &- \int_0^{\tau_\varepsilon} A(x(t))\langle u(t)\,\mathrm{d}b(t)\rangle v(t) \\ &- 1/2\int_0^{\tau_\varepsilon} g^{ki}(x(t))\partial_k A_i(x(t))v(t)\,\mathrm{d}t \\ &+ 1/2\int_0^{\tau_\varepsilon} g^{ki}(x(t))A_i(x(t))A_k(x(t))v(t)\,\mathrm{d}t \\ &+ 1/2\int_0^{\tau_\varepsilon} g^{ml}(x(t))\Gamma^i_{lm}(x(t))A_i(x(t))v(t)\,\mathrm{d}t. \end{aligned} \tag{5.1}$$

Since the integrand of the martingale term is bounded and $E_{x_0}[\tau_\varepsilon] < \infty$, we have

$$E_{x_0}\left[\int_0^{\tau_\varepsilon} A(x(t))\langle u(t)\,\mathrm{d}b(t)\rangle v(t)\right] = 0.$$

By (2.2) A and Γ are $\mathrm{O}(\varepsilon)$ in B_ε. Using this and Lemma 4.1 it follows that the expectation of the third and fourth term on the right-hand side in (5.1) is $\mathrm{O}(\varepsilon^4)$. From (2.3) it follows that the expected value of the second term is

$$-1/6E\left[\int_0^{\tau_\varepsilon}\sum_{i=1}^{d}\partial_i F_{qi}(0)x^q(t)v(t)\,\mathrm{d}t\right] + \mathrm{O}(\varepsilon^4),$$

where we suppressed the x_0 subscript in the expectation. Expand $v(t) = \mathrm{Id} + v(t) - \mathrm{Id}$. Then

$$E\left[\int_0^{\tau_\varepsilon}\left|\sum_i \partial_i F_{qi}(0)x^q(t)(v(t)-\mathrm{Id})\right|\,\mathrm{d}t\right] \leqslant c\varepsilon E\left[\int_0^{\tau_\varepsilon} |\,v(t)-\mathrm{Id}\,|\,\mathrm{d}t\right].$$

Denote the martingale term and the drift term of $v(t)-\mathrm{Id}$ by $M(t)$ and $D(t)$, respectively. Then

$$E\left[\int_0^{\tau_\varepsilon} |\,D(t)\,|\,\mathrm{d}t\right] \leqslant cE\left[\int_0^{\tau_\varepsilon} t\,\mathrm{d}t\right] \leqslant cE[\tau_\varepsilon^2] = \mathrm{O}(\varepsilon^4)$$

and writing $M_{\tau_\varepsilon}^* = \sup_{t\leqslant\tau_\varepsilon}|\,M(t)\,|$, by Burkholder's inequality for $p=2$,

$$\begin{aligned}E\left[\int_0^{\tau_\varepsilon} |\,M(t)\,|\,\mathrm{d}t\right] &\leqslant E[M_{\tau_\varepsilon}^*\tau_\varepsilon] \leqslant (E[(M_{\tau_\varepsilon}^*)^2])^{1/2}(E[\tau_\varepsilon^2])^{1/2}\\ &\leqslant c(E[\tau_\varepsilon])^{1/2}(E[\tau_\varepsilon^2])^{1/2} = \mathrm{O}(\varepsilon^3).\end{aligned}$$

Leaving aside all terms of order $\mathrm{O}(\varepsilon^4)$ we are left with

$$-\,1/6\sum_i \partial_i F_{qi}(0)E\left[\int_0^{\tau_\varepsilon} x^q(t)\,\mathrm{d}t\right]. \tag{5.2}$$

This last expectation is

$$\int_{B_\varepsilon} G(x,x_0)x^q\,\mathrm{d}x.$$

By (3.4),

$$\int_{B_\varepsilon} G(x,x_0)x^q\,\mathrm{d}x = \int_{B_\varepsilon} G_\varepsilon(x,x_0)x^q\,\mathrm{d}x + \mathrm{O}(\varepsilon^5).$$

Suppose now that $x_0 = (y,0,\ldots,0)$. Then $\int_{B_\varepsilon} G_\varepsilon(x,x_0)x^q\,\mathrm{d}x = 0$ for $q \neq 1$. From the scaling properties of the Euclidean Green function we obtain

$$\int_{B_\varepsilon} G_\varepsilon(x,x_0)x^1\,\mathrm{d}x = \varepsilon^3\int_{B_1} G_1(x,x_0)x^1\,\mathrm{d}x.$$

Using (3.2) and a change of variables, we obtain

$$\int_{B_1} G_1(x,x_0)x^1\,\mathrm{d}x = C_d\,\mathrm{vol}(S^{d-2})\int_0^1\int_0^{\pi/2} t^d\cos^{d-2}\alpha\sin\alpha\; f(y,t,\alpha)\,\mathrm{d}t\,\mathrm{d}\alpha,$$

where

$$f(y,t,\alpha) = (y^2+t^2-2yt\sin\alpha)^{1-d/2} - (1+y^2t^2-2yt\sin\alpha)^{1-d/2}$$
$$-(y^2+t^2+2yt\sin\alpha)^{1-d/2} + (1+y^2t^2+2yt\sin\alpha)^{1-d/2}. \qquad (5.3)$$

We are done if we can show that this integral is non-zero for $y \neq 0$. Assume $y > 0$ (the case $y < 0$, except for a sign, being completely analogous). Then, for $\alpha > 0$, $f(y,t,\alpha)$ is of the form

$$\frac{1}{x^m} - \frac{1}{(x+a)^m} - \frac{1}{y^m} + \frac{1}{(y+a)^m}$$

with $y > x$, $a > 0$ and $m > 0$. But this equals

$$\frac{1}{x^m}\left(1 - \left(\frac{1}{1+a/x}\right)^m\right) - \frac{1}{y^m}\left(1 - \left(\frac{1}{1+a/y}\right)^m\right) > 0.$$

The result now follows from (2.4) and (5.2). □

We now use the above method to give a new proof of the probabilistic characterization of Yang–Mills fields in Stafford (1990).

Theorem 5.2. *The field F is Yang–Mills at* o *if and only if*

$$E_o[v(\tau_\varepsilon) - \mathrm{Id} \mid x(\tau_\varepsilon) = \underline{\varepsilon}] = \mathrm{O}(\varepsilon^4) \quad \textit{for all } \underline{\varepsilon} \in S_\varepsilon.$$

Proof. Choose normal coordinates so that $\underline{\varepsilon} = (\varepsilon, 0, \ldots, 0)$. Under the law $\mathscr{P}_\varepsilon$ the Itô decomposition of $v(t)$ contains an additional term

$$v(\tau_\varepsilon) - \mathrm{Id} = -\int_0^{\tau_\varepsilon} A(x(t))\langle u(t)\,\mathrm{d}b(t)\rangle v(t)$$
$$- \int_0^{\tau_\varepsilon} A(x(t))v(t)\mathrm{grad}\log h(x(t))\,\mathrm{d}t$$
$$-1/2\int_0^{\tau_\varepsilon} g^{ki}(x(t))\partial_k A_i(x(t))v(t)\,\mathrm{d}t$$
$$+1/2\int_0^{\tau_\varepsilon} g^{ki}(x(t))A_i(x(t))A_k(x(t))v(t)\,\mathrm{d}t$$
$$+1/2\int_0^{\tau_\varepsilon} g^{ml}(x(t))\Gamma^i_{lm}(x(t))A_i(x(t))v(t)\,\mathrm{d}t. \qquad (5.4)$$

As in the proof of Theorem 5.1 it follows from Lemma 4.2 that the expectation of the first term on the right-hand side vanishes, the expectation of the fourth and fifth term is $\mathrm{O}(\varepsilon^4)$, and the expected value of the third term is

$$-1/6E_\varepsilon\left[\int_0^{\tau_\varepsilon} \sum_{i=1}^{d} \partial_i F_{qi}(0)x^q(t)v(t)\,\mathrm{d}t\right] + \mathrm{O}(\varepsilon^4).$$

Expand $v(t) = \mathrm{Id} + v(t) - \mathrm{Id}$. Then

$$E_\varepsilon\left[\int_0^{\tau_\varepsilon}\left|\sum_i \partial_i F_{qi}(0)x^q(t)(v(t)-\mathrm{Id})\right|\,\mathrm{d}t\right] \leqslant c\varepsilon E_\varepsilon\left[\int_0^{\tau_\varepsilon} |\,v(t)-\mathrm{Id}\,|\,\mathrm{d}t\right].$$

The four terms of the Itô decomposition of $v(t) - \mathrm{Id}$ that already appeared in (5.1) can be dealt with as in the proof of Theorem 5.1, using Lemma 4.2. The additional term is bounded by

$$\begin{aligned}
&c\varepsilon^2 E_\varepsilon\left[\int_0^{\tau_\varepsilon}\int_0^s |\operatorname{grad}\log h(x(t))|\,\mathrm{d}t\,\mathrm{d}s\right]\\
&\quad\leqslant c\varepsilon^2 E_\varepsilon\left[\int_0^{\tau_\varepsilon} |\operatorname{grad}\log h(x(t))|\,\mathrm{d}t\,\tau_\varepsilon\right]\\
&\quad\leqslant c\varepsilon^2 E_\varepsilon\left[\left(\int_0^{\tau_\varepsilon} |\operatorname{grad}\log h(x(t))|\,\mathrm{d}t\right)^2\right]^{1/2} E_\varepsilon[\tau_\varepsilon^2]^{1/2} = \mathrm{O}(\varepsilon^5),
\end{aligned}$$

where we used Hölder's inequality and Lemma 4.2. Thus, the expectation of the third term is

$$-1/6\sum_i \partial_i F_{qi}(0) E_\varepsilon\left[\int_0^{\tau_\varepsilon} x^q(t)\,\mathrm{d}t\right] + \mathrm{O}(\varepsilon^4). \tag{5.5}$$

For the expectation of the second term in (5.4) one proceeds by expanding $v(t) = \mathrm{Id} + v(t) - \mathrm{Id}$ inside the integrand. The part containing $v(t) - \mathrm{Id}$ can be shown to be $\mathrm{O}(\varepsilon^4)$ much the same as in the proof of Theorem 5.1, using Lemma 4.2 instead of Lemma 4.1. Hence, the second term's contribution is

$$-E_\varepsilon\left[\int_0^{\tau_\varepsilon} A(x(t))\operatorname{grad}\log h(x(t))\,\mathrm{d}t\right] + \mathrm{O}(\varepsilon^4). \tag{5.6}$$

The expectations in (5.5) and (5.6) are

$$\int_{B_\varepsilon} x^q G(0,x)h(x)/h(0)\,\mathrm{d}x$$

and

$$T = \int_{B_\varepsilon} A(x)\operatorname{grad}\log h(x)G(0,x)h(x)/h(0)\,\mathrm{d}x,$$

respectively. We integrate by parts in (T), using $G(0,.) = 0$ on S_ε. This gives

$$\begin{aligned}
T = &-\sum_i \int_{B_\varepsilon} \partial_i A_i(x)G(0,x)h(x)/h(0)\,\mathrm{d}x\\
&-\sum_i \int_{B_\varepsilon} A_i(x)\partial_i G(0,x)h(x)/h(0)\,\mathrm{d}x.
\end{aligned} \tag{5.7}$$

By (2.3), the first of these integrals equals

$$-\frac{1}{3}\sum_i \partial_i F_{qi}(0)\int_{B_\varepsilon} x^q G(0,x)h(x)/h(0)\,\mathrm{d}x + \mathrm{O}(\varepsilon^4). \tag{5.8}$$

For the second, using (2.2), we get

$$-\frac{1}{2}\sum_i F_{pi}(0)\int_{B_\varepsilon} x^p \partial_i G(0,x)h(x)/h(0)\,\mathrm{d}x + \mathrm{O}(\varepsilon^4).$$

It follows from (3.4) that $|h(x) - h_\varepsilon(x)| = \mathrm{O}(\delta(x)^{3-d})$ and we can replace h, G by h_ε, G_ε, incurring an error $\mathrm{O}(\varepsilon^4)$, giving us

$$-\frac{1}{2}\sum_i F_{pi}(0)\int_{B_\varepsilon} x^p \partial_i G_\varepsilon(0,x)h_\varepsilon(x)/h(0)\,\mathrm{d}x + \mathrm{O}(\varepsilon^4).$$

Since h_ε and $\partial_i G(0,.)$ are even in the component x^p unless $p=1$, the integral vanishes unless $p=1$. Now, $G_\varepsilon(0,x) = f(|x|^2)$ and so $\partial_i G_\varepsilon(0,x) = f'(|x|^2)2x^i$. Again, for $i \neq 1$, h_ε, $f'(|\,.\,|^2)$ are even in x^i, and the integral vanishes. But for $i = p = 1$ we have $F_{ip} = 0$. Combining (5.5) and (5.8) we find that the total expectation is

$$\frac{1}{6}\sum_i \partial_i F_{qi}(0)\int_{B_\varepsilon} x^q G(0,x)h(x)/h(0)\,\mathrm{d}x + \mathrm{O}(\varepsilon^4).$$

By the same steps as above we get

$$E_\varepsilon[v(\tau_\varepsilon - \mathrm{Id})] = \frac{1}{6}\sum_i \partial_i F_{1i}(0)\int_{B_\varepsilon} x^1 G_\varepsilon(0,x)h_\varepsilon(x)/h(0)\,\mathrm{d}x + \mathrm{O}(\varepsilon^4). \tag{5.9}$$

The scaling properties of h_ε, G_ε and (4.4) imply that the integral is $\mathrm{O}(\varepsilon^3)$ and it is straightforward to check that it is strictly positive (for $x^1 > 0$, $h_\varepsilon(x^1,x^2,\ldots,x^d) > h_\varepsilon(-x^1,x^2,\ldots,x^d)$).

It is now clear that if F satisfies the Yang–Mills equations at o, (2.4), then $E_\varepsilon[v(\tau_\varepsilon) - \mathrm{Id}] = \mathrm{O}(\varepsilon^4)$. Conversely, if $E_\varepsilon[v(\tau_\varepsilon) - \mathrm{Id}]$ is $\mathrm{O}(\varepsilon^4)$, then $\sum_j \partial_j F_{j1}(0) = 0$, which is the Yang–Mills "equation", (2.4), for $k=1$. If we replace $\underline{\varepsilon}$ by $\underline{\varepsilon}' = (0,\ldots,0,\varepsilon,0,\ldots,0)$, where ε is the kth coordinate, we obtain the same result for any k. In fact, expansion (5.9) still holds if we replace $\underline{\varepsilon}$ by $\underline{\varepsilon}'$ and $F_{j1}(0)$ by $F_{jk}(0)$. □

We next give an analytic reformulation of Theorem 5.1. Denote Δ^{H} the horizontal Laplacian on η, whose local form is

$$\Delta^{\mathrm{H}} = g^{jk}(\nabla_j\nabla_k - \Gamma^i_{jk}\nabla_i).$$

Note that $1/2\Delta^{\mathrm{H}}$ is the infinitesimal generator of $\{v(t), t \geqslant 0\}$. It is well known (Hsu, 1987) that for a fixed vector $V \in \mathbb{R}^n$ the section $u_\varepsilon : \overline{B_\varepsilon} \to \mathbb{R}^n$ defined by $u_\varepsilon(x) = E^x[v(\tau_\varepsilon)^{-1}V]$ satisfies

$$\Delta^{\mathrm{H}} u_\varepsilon(x) = 0 \quad \text{for } x \in B_\varepsilon,$$

$$u_\varepsilon|_{\partial B_\varepsilon} = V. \tag{5.10}$$

In fact, for ε small enough, say $\varepsilon < \varepsilon_1$, u_ε is the unique solution of (5.10).

Corollary 5.3. *Denote* u_ε, *for* $0 < \varepsilon < \varepsilon_1$, *a family of solutions to the horizontal Laplace equation* (5.10). *Then the following are equivalent*,
(a) $\sup_{x,y \in B_\varepsilon} |u_\varepsilon(x) - u_\varepsilon(y)| = \mathrm{O}(\varepsilon^4)$,
(b) *the connection* A *solves the Yang–Mills equations at* o.

Proof. By the remarks above, for $0 < \varepsilon < \varepsilon_1$, we have $u_\varepsilon(x) = E^x[v(\tau_\varepsilon)^{-1}V]$. The result now follows from Theorem 5.1. □

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Appendix. Proof of Lemmata

Our proofs follow the exposition in Chung and Zhao (1995).

Lemma A.1 (Lemma 4.1 in the main part). *Denote by E^{x_0} the expectation with respect to $\mathscr{P}^{x_0}$. Then*

$$E^{x_0}[\tau_\varepsilon^p] \leqslant K\varepsilon^{2p}, \quad p = 1, 2.$$

Proof. First,

$$E^{x_0}[\tau_\varepsilon] = \int_{B_\varepsilon} G(x_0, y)\,\mathrm{d}y \leqslant K\varepsilon^2$$

by (3.3), (i). Second, using the Markov property,

$$\begin{aligned} \frac{1}{2}E^{x_0}[\tau_\varepsilon^2] &= E^{x_0}\left[\int_0^{\tau_\varepsilon} \tau_\varepsilon \circ \theta_t\,\mathrm{d}t\right] \\ &= \int_0^\infty E^{x_0}\{t < \tau_\varepsilon;\ \tau_\varepsilon \circ \theta_t\}\,\mathrm{d}t \\ &= \int_0^\infty E^{x_0}\{t < \tau_\varepsilon;\ E^{x(t)}[\tau_\varepsilon]\}\,\mathrm{d}t \end{aligned} \tag{A.1}$$

and this equals

$$\int_0^\infty E^{x_0}\left\{t < \tau_\varepsilon; \int_{B_\varepsilon} G(x(t), y)\,\mathrm{d}y\right\}\,\mathrm{d}t = \int_{B_\varepsilon \times B_\varepsilon} G(x_0, x)G(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

By (3.3), (i), this is bounded by

$$\int_{B_\varepsilon \times B_\varepsilon} c|x - x_0|^{2-d}|x - y|^{2-d}\,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon^2 \int_{B_\varepsilon} c'|x_0 - y|^{2-d}\,\mathrm{d}y \leqslant K\varepsilon^4. \quad \square \tag{A.2}$$

Lemma A.2 (Lemma 4.2 in the main part). *Denote by E_ε the expectation with respect to $\mathscr{P}_\varepsilon$. Then*

$$E_\varepsilon[\tau_\varepsilon^p] = \mathrm{O}(\varepsilon^{2p}), \quad p = 1, 2$$

and

$$E_\varepsilon\left[\left(\int_0^{\tau_\varepsilon} |\operatorname{grad}\log h(x(s))|\,\mathrm{d}s\right)^p\right] = \mathrm{O}(\varepsilon^p), \quad p = 1, 2.$$

Proof. We only prove the most difficult part. The other statements of the Lemma then are easy exercises. Set

$$a(t) = \int_0^t |\operatorname{grad}\log h(x(s))|\,\mathrm{d}s.$$

Then, using the Markov property,

$$\frac{1}{2}E_\varepsilon\left[\left(\int_0^{\tau_\varepsilon} |\operatorname{grad}\log h(x(s))|\,\mathrm{d}s\right)^2\right]$$

$$= E_\varepsilon\left[\int_0^{\tau_\varepsilon} a(\tau_\varepsilon)\circ\theta_t\,|\operatorname{grad}\log h(x(t))|\,\mathrm{d}t\right.$$

$$= \int_0^\infty E_\varepsilon\{t < \tau_\varepsilon;\; E_\varepsilon^{x(t)}[a(\tau_\varepsilon)]\,|\operatorname{grad}\log h(x(t))|\,\}\,\mathrm{d}t. \tag{A.3}$$

But

$$E_\varepsilon^{x(t)}[a(\tau_\varepsilon)] = \int_{B_\varepsilon} G(x(t),y)\frac{h(y)}{h(x(t))}\,|\operatorname{grad}\log h(y)|\,\mathrm{d}y.$$

Thus, (A.3) equals

$$\int_{B_\varepsilon\times B_\varepsilon} G(0,x)\frac{h(x)}{h(0)}\,|\operatorname{grad}\log h(x)|\,G(x,y)\frac{h(y)}{h(x)}\,|\operatorname{grad}\log h(y)|\,\mathrm{d}x\,\mathrm{d}y.$$

The y-integral is

$$\int_{B_\varepsilon} G(x,y)\operatorname{grad} h(y)\,\mathrm{d}y.$$

Near x the integrand is bounded by $K\delta(x)^{-d}\,|x-y|^{2-d}$, see (3.3) (i), (iv). Integration over a ball at x with radius $\delta(x)/2$ gives $c\delta(x)^{2-d}$. Outside of this ball the integrand is bounded by $K\delta(x)^{1-d}\delta(y)\delta(y)^{-d}$ (see (3.3), (ii), (iv)). Integration over B_ε gives $c'\delta(x)^{1-d}\varepsilon$. Thus, (A.3) is bounded by

$$c''\int_{B_\varepsilon}\frac{G(0,x)}{h(0)}\,|\operatorname{grad}\log h(x)|\,(\delta(x)^{2-d}+\varepsilon\delta(x)^{1-d})\,\mathrm{d}x.$$

In $B_{\varepsilon/2}$ bound the integrand by $c'''\,|x|^{2-d}$, using (4.3), (4.4) and (3.3)(i). The integral over this ball then is bounded by $c^{(4)}\varepsilon^2$. Outside of this ball bound the integrand by $c^{(5)}(\delta(x)^{2-d}+\varepsilon\delta(x)^{1-d})$, using (4.3), (4.4) and (3.3)(ii). The integral over B_ε now is bounded by $c^{(6)}\varepsilon^2$. This proves the lemma. □

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