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A representation formula for transition probability densities of diffusions and applications

Zhongmin Qian[a,b], Weian Zheng[c,d,*,1]

[a] *Université Paul-Sabatier, 118, route de Narbonne, 31062 Toulouse, France*
[b] *CNRS, France*
[c] *East China Normal University, Shanghai, China*
[d] *Department of Mathematics, University of California, 103 MSTB, Irvine, CA 92697, USA*

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Abstract

We establish a representation formula for the transition probability density of a diffusion perturbed by a vector field, which takes a form of Cameron–Martin's formula for pinned diffusions. As an application, by carefully estimating the mixed moments of a Gaussian process, we deduce explicit, strong lower and upper estimates for the transition probability function of Brownian motion with drift of linear growth.

Keywords: Heat kernel estimates; Diffusion

1. Introduction

In the paper (Qian et al., 2003) by the present authors in collaboration with F. Russo, a comparison theorem (see below Theorem 2.4) for the density function of a Brownian motion with drift in terms of the transition function of the diffusion with two-valued drift has been established. By computing the transition function for a model diffusion, sharp bounds for the transition function of one-dimensional Brownian motion with bounded drift has been established in Qian and Zheng (2002). The goal of this paper is to extend the above results to the case with unbounded drift, which is important in many applications. We obtain a representation formula for the transition function

* Corresponding author. Department of Mathematics, University of California, 103 MSTB, Irvine, CA 92697, USA. Tel.: +1-9498245379; fax: +1-9498247993.

E-mail addresses: qian@math.ups-tlse.fr (Z. Qian), wzheng@uci.edu (W. Zheng).

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of a diffusion perturbed by a vector field, which takes a form of Cameron–Martin's formula for pinned diffusions. As consequences we deduce a very precise estimate for the density function of Brownian motion with a drift bounded by linear growth.

Although there are many excellent results on the estimates of the transition density functions of diffusions with generators in divergence form (see Aronson, 1967; Davies, 1989 and the literature therein), the explicit estimates for Brownian motion with *drift of order not faster than linear growth* presented in this paper (see Theorems 3.1 and 3.2 below) are new and useful to the applications of the statistical inference of diffusion processes with stochastic volatility used in the mathematical finance.

The paper is organized as the following. In Section 2, several folklore facts about conditional diffusion processes are recalled. We then deduce an integral representation for the transition probability density of a diffusion perturbed by a vector field, which is the main tool we will use to establish the lower and upper bounds. Indeed, the representation theorem of this type is applied to a more general setting, and is very useful formula in obtaining information about the density functions perturbed by some drift. Therefore it has interest by its own. In order to prove Theorems 3.1 and 3.2 we need several technical estimates about the mixed moments of the linear diffusion, which will be done in Section 3.

2. A representation formula

In this section we deduce our first result, a presentation formula (Theorem 2.4) for the transition function of a diffusion perturbed by a vector field. We begin with some remarks about pinned diffusions or conditional diffusions. The materials presented here belong to the tool box about Markov processes, though we could not find a reference which address these issues. The reader may regard them as a set of folklore facts, which are known to experts in stochastic analysis.

2.1. Conditional diffusions

Let $(X_t, \mathbb{P}^x)$ be a (time homogenous) diffusion process with its natural filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and state space M (for example a complete Riemannian manifold such as $\mathbb{R}^n$). Suppose its transition probability function $P_t(x, \mathrm{d}y)$ possesses a positive, continuous density function $p(x,t,y)$ for all $t>0$, with respect to a σ-finite measure μ on M (in many applications it will be a weighted Riemann–Lebesgue measure on M).

For $T>0$ and a point $y\in M$, define a non-homogenous transition density function

$$H_{T,y}(s,z;t,w) = \frac{p(z,t-s,w)\,p(w,T-t,y)}{p(z,T-s,y)} \tag{2.1}$$

for all $0\leqslant s<t<T$, and a transition probability function

$$Q_{s,t}^{T,y} f(z) \triangleq \int_M f(w) H_{T,y}(s,z;t,w)\mu(\mathrm{d}w) \quad \text{for } 0\leqslant s<t<T. \tag{2.2}$$

We will omit indices T, y if no confusion may arise. In most part of this section, both $T>0$ and $y\in M$ are fixed. Then for any $0\leqslant s<t<T$, and functions f and g we have

$$\mathbb{P}^x\{Q_{s,t}^{T,X_T}f(X_s)g(X_T)|\mathscr{F}_s\}=\mathbb{P}^x\{f(X_t)g(X_T)|\mathscr{F}_s\}, \tag{2.3}$$

and thus formally (by taking $g=\delta_y$), for every $0\leqslant s<t<T$,

$$\mathbb{P}^x\{Q_{s,t}f(X_s)\delta_y(X_T)|\mathscr{F}_s\}=\mathbb{P}^x\{f(X_t)\delta_y(X_T)|\mathscr{F}_s\} \tag{2.4}$$

or equivalently

$$\mathbb{P}^x\{(f(X_t)|\mathscr{F}_s)|X_T=y\}=\mathbb{P}^x\{Q_{s,t}f(X_s)|X_T=y\},$$

which explains that the process $(X_t)_{t<T}$ possesses the Markov property with transition function $Q_{s,t}$ under "the conditional probability" $\mathbb{P}^x\{\cdot|X_T=y\}$.

For given $T>0$ and two points x and y in M, define

$$N_t=\frac{p(X_t,T-t,y)}{p(x,T,y)}\quad\text{for all } t<T. \tag{2.5}$$

Then $(N_t)_{t<T}$ is a non-negative martingale under the probability $\mathbb{P}^x$, therefore define a probability $\mathbb{P}_T^{x,y}$ on the σ-algebra $\sigma\{\mathscr{F}_t: t<T\}$ by

$$\left.\frac{\mathrm{d}\mathbb{P}_T^{x,y}}{\mathrm{d}\mathbb{P}^x}\right|_{\mathscr{F}_t}=\frac{p(X_t,T-t,y)}{p(x,T,y)}\quad\text{for all } t<T \tag{2.6}$$

called the conditional probability of (X_t) such that $X_0=x$ and $X_T=y$. Since (X_t) is continuous, $\sigma\{\mathscr{F}_t : t<T\}$ equals $\mathscr{F}_T$. Note that $\mathbb{P}_T^{x,y}$ may be not absolutely continuous with respect to $\mathbb{P}^x$ on $\mathscr{F}_T$. In the literature, $\mathbb{P}_T^{x,y}$ is also denoted by $\mathbb{P}^x(\cdot|X_T=y)$ or $\mathbb{P}(\cdot|X_0=x, X_T=y)$.

Lemma 2.1. *The continuous process $(X_t, t\leqslant T)$ under the probability $\mathbb{P}_T^{x,y}$ is a diffusion process with transition density function $H(s,z;t,w)$.*

The diffusion process $(X_t, t\leqslant T; \mathbb{P}_T^{x,y})$ is called a pinned diffusion, a diffusion conditioned on $X_0=x$ and $X_T=y$, or a diffusion bridge from x to y with running time T.

2.2. Cameron–Martin's formula for pinned diffusions

In this subsection we consider an L_b-diffusion process $(X_t, \mathbb{P}^x)$, where L_b is an elliptic differential operator of second order. L_b may be in non-divergence form

$$L_b=\frac{1}{2}\sum_{i,j}g^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b^i(x)\frac{\partial}{\partial x_i}, \tag{2.7}$$

in this case the symmetric matrix (g^{ij}) is uniformly continuous, or L_b may be in divergence form

$$L_b=\frac{1}{2}\frac{1}{q(x)}\sum_{i,j}\frac{\partial}{\partial x_i}\left(q(x)g^{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_i b^i(x)\frac{\partial}{\partial x_i}, \tag{2.8}$$

where coefficients (g^{ij}), positive function $q(x)$ and $(b^i(x))$ are Borel measurable. In both the cases (g^{ij}) is supposed to be positive definite. Let $(M_t)_{t\geqslant 0}$ be the martingale part of (X_t). Then

$$\langle M^i, M^j\rangle_t = \int_0^t g^{ij}(X_s)\,\mathrm{d}s.$$

Let $p_b(x,t,y)$ denote the transition density function of the L_b-diffusion with respect to the Lebesgue measure. By Itô's formula we have the following (Lyons and Zheng, 1990; Zheng, 1995).

Lemma 2.2. *The conditional diffusion process* $(X_t, t\leqslant T, \mathbb{P}_T^{x,y})$ *possesses infinitesimal generator*

$$\begin{aligned} A &= L_b + \nabla_z^g \log p(z, T-t, y)\nabla^g \\ &= L_b + \sum_{i,j} g^{ij}(z)\,\frac{\partial}{\partial z_i}\log p(z,T-t,y)\,\frac{\partial}{\partial z_j}. \end{aligned}$$

Let

$$c(x) = \sum_i c^i(x)\,\frac{\partial}{\partial x_i}$$

be a measurable vector field on $\mathbb{R}^n$ which is of at most linear growth, such that

$$\mathbb{P}^x \exp\left(\frac{1}{2}\int_0^T |c|_g^2(X_s)\,\mathrm{d}s\right) < \infty.$$

Define a probability measure $\mathbb{Q}^x$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathscr{F}_t} = \exp\left[\int_0^t \langle c(X_s), \mathrm{d}M_s\rangle_g - \frac{1}{2}\int_0^t |c|_g^2(X_s)\,\mathrm{d}s\right].$$

The lower index g indicates that both the norm and the inner product are computed in term of the metric (g_{ij}) (which is the inverse of (g^{ij})). Thus

$$\langle c(X_s), \mathrm{d}M_s\rangle_g = \sum_{i,j} g_{ij}(X_s)c^i(X_s)\,\mathrm{d}M_s^j$$

and

$$|c|_g^2(X_s) = \sum_{i,j} g_{ij}(X_s)c^i(X_s)c^j(X_s).$$

By the Cameron–Martin formula, $(X_t, \mathscr{F}_t, \mathbb{Q}^x)$ is an $L+c$-diffusion process, with transition density $p_{b+c}(x,t,y)$.

Lemma 2.3. *Suppose* $p_b(x,t,y)$ *and* $p_{b+c}(x,t,y)$ *are continuous, and suppose the function*

$$y \to \mathbb{P}_T^{x,y}\left(\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T |c|^2(X_t)\,\mathrm{d}t}\right)$$

is continuous, then

$$\frac{p_{b+c}(x,T,y)}{p_b(x,T,y)} = \mathbb{P}_T^{x,y}\exp\left\{\int_0^T \langle c(X_t), \mathrm{d}M_t\rangle_g - \frac{1}{2}\int_0^T |c|_g^2(X_t)\,\mathrm{d}t\right\}. \tag{2.9}$$

Proof. For every $0<\varepsilon<T$ and any bounded, continuous function φ we have

$$\begin{aligned}
&\mathbb{P}_T^{x,y}\left(\varphi(X_{T-\varepsilon})\mathrm{e}^{\int_0^{T-\varepsilon}\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^{T-\varepsilon}|c|^2(X_t)\,\mathrm{d}t}\right)\\
&\quad = \frac{1}{p_b(x,T,y)}\,\mathbb{P}^x\left(p_b(X_{T-\varepsilon},\varepsilon,y)\varphi(X_{T-\varepsilon})\mathrm{e}^{\int_0^{T-\varepsilon}\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^{T-\varepsilon}|c|^2(X_t)\,\mathrm{d}t}\right)
\end{aligned}$$

then multiplying by $p_b(x,T,y)$ both sides and integrating in y over $\mathbb{R}^n$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} p_b(x,T,y)\mathbb{P}_T^{x,y}\left(\varphi(X_{T-\varepsilon})\mathrm{e}^{\int_0^{T-\varepsilon}\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^{T-\varepsilon}|c|^2(X_t)\,\mathrm{d}t}\right)\mathrm{d}y\\
&\quad = \mathbb{P}^x\left(\varphi(X_{T-\varepsilon})\mathrm{e}^{\int_0^{T-\varepsilon}\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^{T-\varepsilon}|c|^2(X_t)\,\mathrm{d}t}\right).
\end{aligned}$$

From Lyons and Zheng (1990), the bounded variational part of $\{M_t\}_t$ under $\mathbb{P}_T^{x,y}$ satisfies

$$\mathbb{P}_T^{x,y}\left\{\exp\left\{k\int_0^T\left|\sum_j g^{ij}(X_t)\,\frac{\partial}{\partial x_j}\log p(X_t,T-t,y)\right|\,\mathrm{d}t\right\}\right\}<\infty$$

for any bounded constant k. Since c is locally bounded and at most of linear growth, we can let $\varepsilon\to 0$ and obtain thus

$$\begin{aligned}
&\int_{\mathbb{R}^d} p_b(x,T,y)\mathbb{P}_T^{x,y}\left(\varphi(X_T)\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T|c|^2(X_t)\,\mathrm{d}t}\right)\mathrm{d}y\\
&\quad = \mathbb{P}^x\left(\varphi(X_T)\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T|c|^2(X_t)\,\mathrm{d}t}\right),
\end{aligned}$$

where the exchanges of limits with integrals are justified under our conditions. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} p_{b+c}(x,T,y)\varphi(y)\,\mathrm{d}y\\
&\quad = \mathbb{Q}^x(\varphi(X_T))\\
&\quad = \mathbb{P}^x\left(\varphi(X_T)\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T|c|^2(X_t)\,\mathrm{d}t}\right)\\
&\quad = \int_{\mathbb{R}^d} p_b(x,T,y)\mathbb{P}_T^{x,y}\left(\varphi(X_T)\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T|c|^2(X_t)\,\mathrm{d}t}\right)\mathrm{d}y\\
&\quad = \int_{\mathbb{R}^d} \varphi(y)\,p_b(x,T,y)\mathbb{P}_T^{x,y}\left(\mathrm{e}^{\int_0^T\langle c(X_t),\mathrm{d}M_t\rangle - \frac{1}{2}\int_0^T|c|^2(X_t)\,\mathrm{d}t}\right)\mathrm{d}y.
\end{aligned}$$

The conclusion follows from the fact that

$$y \to \mathbb{P}_T^{x,y}\left(e^{\int_0^T \langle c(X_t), dM_t\rangle - \frac{1}{2}\int_0^T |c|^2(X_t)\,dt}\right)$$

is continuous. □

Theorem 2.4. *Same conditions as in Lemma* 2.3. *Let*

$$U_t = \exp\left(\int_0^t \langle c(X_s), dM_s\rangle_g - \frac{1}{2}\int_0^t |c(X_s)|_g^2\,ds\right)$$

which is a martingale up to time T. Then

$$p_{b+c}(x,T,y) = p_b(x,T,y) + \int_0^T \mathbb{P}^x\{U_t\langle c(X_t), \nabla_x^g p_b(X_t, T-t, y)\rangle_g\}\,dt. \quad (2.10)$$

Proof. By Lemma 2.3

$$\frac{p_{b+c}(x,T,y)}{p_b(x,T,y)} = \mathbb{P}_T^{x,y}(U_T).$$

On the other hand,

$$\left.\frac{d\mathbb{P}_T^{x,y}}{d\mathbb{P}^x}\right|_{\mathscr{F}_t} = \frac{p_b(X_t, T-t, y)}{p_b(x,T,y)}, \quad \forall t < T,$$

of which the right-hand side will be denoted by N_t for $t < T$. By Girsanov's theorem

$$\tilde{U}_t = U_t - \int_0^t \frac{1}{N_s}\,d\langle U, N\rangle_s$$

is a martingale under $\mathbb{P}_T^{x,y}$ for $t < T$. While as (U_t) is the exponential martingale of $\int_0^t \langle c(X_s), dM_s\rangle_g$ so that

$$U_t = 1 + \int_0^t U_s\langle c(X_s), dM_s\rangle.$$

On the other hand,

$$\langle U, N\rangle_t = \int_0^t U_s N_s\langle c(X_s), \nabla_x^g \log p_b(X_s, T-s, y)\rangle_g\,ds,$$

so that

$$\begin{aligned}\frac{p_{b+c}(x,T,y)}{p_b(x,T,y)} &= \mathbb{P}_T^{x,y}(U_T) = \mathbb{P}_T^{x,y}\left(\tilde{U}_T + \int_0^T \frac{1}{N_t}\,d\langle U,N\rangle_t\right)\\ &= 1 + \mathbb{P}_T^{x,y}\left(\int_0^T U_t\langle c(X_t), \nabla_x^g \log p_b(X_t, T-t, y)\rangle_g\,dt\right)\\ &= 1 + \int_0^T \mathbb{P}_T^{x,y}\{U_t\langle c(X_t), \nabla_x^g \log p_b(X_t, T-t, y)\rangle_g\}\,dt. \quad (2.11)\end{aligned}$$

Since U_t is $\mathscr{F}_t$-measurable,

$$\begin{aligned}
&\mathbb{P}_T^{x,y}\{U_t\langle c(X_t), \nabla_x^g \log p_b(X_t, T-t, y)\rangle_g\} \\
&\quad = \mathbb{P}^x \left(\frac{p_b(X_t, T-t, y)}{p_b(x, T, y)} U_t \langle c(X_t), \nabla_x^g \log p_b(X_t, T-t, y)\rangle_g \right) \\
&\quad = \frac{1}{p_b(x, T, y)} \mathbb{P}^x\{U_t\langle c(X_t), \nabla_x^g p_b(X_t, T-t, y)\rangle_g\}
\end{aligned}$$

and therefore; together with Eq. (2.11);

$$\frac{p_{b+c}(x, T, y)}{p_b(x, T, y)} = 1 + \frac{1}{p_b(x, T, y)} \int_0^T \mathbb{P}^x\{U_t\langle c(X_t), \nabla_x^g p_b(X_t, T-t, y)\rangle_g\}\, \mathrm{d}t \quad (2.12)$$

which in turn yields the claim of the theorem. □

Remark 2.5. The conditions in Lemma 2.3 hold in many practical situations. For example when (g_{ij}) is elliptic, q is uniformly bounded from zero and above, and the additional vector field b has at most linear growth. In the manifold case with smooth metric (g_{ij}), Lemma 2.3 is applicable if

$$L = \frac{1}{2} \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right) + \sum_i b^i \frac{\partial}{\partial x_i},$$

the manifold (M, g_{ij}) is stochastically complete. In all these cases, it may be verified that U_T and $\langle U, N\rangle_T$ both are integrable with respect to the conditional distribution $\mathbb{P}_T^{x,y}$ under our assumptions.

It is clear all the results still hold for a vector field c dependent on t. As a direct consequence we have the following

Corollary 2.6. *Fix $y \in M$. We have the following conclusions*:

1. *If $\langle c(\cdot), \nabla_x \log p_b(\cdot, t, y)\rangle \geqslant 0$ for all t and x, then*
$$p_{b+c}(x, t, y) \geqslant p_b(x, t, y) \quad \textit{for all } (t, x).$$
2. *If $\langle c(\cdot), \nabla_x \log p_b(\cdot, t, y)\rangle \leqslant 0$ for all t and x, then*
$$p_{b+c}(x, t, y) \leqslant p_b(x, t, y) \quad \textit{for all } (t, x).$$

The usefulness of this corollary of course depends on the knowledge of the comparison transition density $p_b(x, t, y)$. Required information in applying Corollary 2.6 may be obtained for some class of diffusion processes. Let us consider a simple case, that is we consider L_b-diffusions in $\mathbb{R}^n$ with

$$L_b = \tfrac{1}{2}\Delta + b,$$

where Δ is the Laplacian in $\mathbb{R}^n$ and b is a vector field. The L_b-diffusion is called a Brownian motion with drift. Let $y \in \mathbb{R}^n$ be a fixed point, let α, β be two constants. Let

$$\tilde{b}(x) = \beta \nabla |x-y| - \alpha \nabla |x-y|^2$$

(we note that $\nabla|x-y| = \mathrm{sgn}(x-y)$ in one-dimensional case), which is at most of linear growth. Then for all $x \neq y$, $\nabla_x p_{\tilde{b}}(x,t,y)$ exists. From the symmetry, according to Qian et al. (2003) there is a function $f(t,r)$, $(t \geqslant 0, r > 0)$ non-increasing in r such that $p_{\tilde{b}}(x,t,y) = f(t,|x-y|)$. Thus

$$\langle \nabla_x \log p_{\tilde{b}}(x,t,y), \nabla|x-y| \rangle \leqslant 0.$$

Thus when $g^{ij}(.) = \delta_{ij}$ we have

Theorem 2.7 (Qian et al., 2003). *Let*

$$\tilde{b}(x) = \beta\nabla_x|x-y| - \alpha\nabla_x|x-y|^2$$

and let $y \in \mathbb{R}^n$. *Consider Brownian motion with drift* b (*b is measurable and of at most linear growth*). *Then*

1. *if* $\langle b(x) - \tilde{b}(x), \nabla_x|x-y| \rangle \geqslant 0$ *for all* (x,t) *such that* $|x-y| > 0$, *then*

$$p_b(x,t,y) \leqslant p_{\tilde{b}}(x,t,y) \quad \text{for all } (x,t),$$

2. *if* $\langle b(x) - \tilde{b}(x), \nabla_x|x-y| \rangle \leqslant 0$ *for all* (x,t) *such that* $|x-y| > 0$, *then*

$$p_b(x,t,y) \geqslant p_{\tilde{b}}(x,t,y) \quad \text{for all } (x,t).$$

Indeed, this theorem follows from representation (2.10) applying to $c = b - \tilde{b}$ and $L_{\tilde{b}}$-diffusion.

Consider the following two vector fields on the Euclidean space $\mathbb{R}^n$: $b(x) = \sum_i b^i(x)\, \partial/\partial x^i$ with

$$b^i(x) = \gamma_i \,\mathrm{sgn}(x_i - y_i) - 2\alpha(x_i - y_i)$$

and the standard one used in Theorem 2.7

$$\begin{aligned}\tilde{b}(x) &= \beta\nabla_x|x-y| - \alpha\nabla_x|x-y|^2 \\ &= \beta\,\frac{x-y}{|x-y|} - 2\alpha(x-y),\end{aligned}$$

where α, β and γ_i are constants. It is easy to see that

$$\begin{aligned}\langle b(x) - \tilde{b}(x), \nabla_x|x-y| \rangle &= \sum_{j=1}^{n} \left\{ \gamma \,\mathrm{sgn}(x_j - y_j) - \beta\,\frac{x_j - y_j}{|x-y|} \right\} \frac{x_j - y_j}{|x-y|} \\ &= \frac{\gamma}{|x-y|} \sum_{j=1}^{n} |x_j - y_j| - \beta\end{aligned}$$

which, together with the following elementary inequality

$$|x-y| \leqslant \sum_{i=1}^{n} |x_i - y_i| \leqslant \sqrt{n}|x-y|,$$

implies the following:

Lemma 2.8. *Under the above notations, we have*

(1) *if* $\beta \geqslant 0$, $\gamma = \beta$ (*or if* $\beta \leqslant 0$, $\gamma = \beta/\sqrt{n}$), *and for any* $\alpha \in \mathbb{R}$,

$$\langle b(x) - \tilde{b}(x), \nabla_x |x - y| \rangle \geqslant 0 \quad \text{for all } x \neq y,$$

(2) *if* $\beta \geqslant 0$, $\gamma = \beta/\sqrt{n}$ (*or if* $\beta \leqslant 0$, $\gamma = \beta$), *and for any* $\alpha \in \mathbb{R}$,

$$\langle b(x) - \tilde{b}(x), \nabla_x |x - y| \rangle \leqslant 0 \quad \text{for all } x \neq y.$$

Let $p_{k_1,k_2,y}(x,t,z)$ denote the transition density function of the following diffusion:

$$\mathrm{d}X_t^i = \mathrm{d}W_t^i + (k_1 \operatorname{sgn}(X_t^i - y_i) - k_2(X_t^i - y_i))\,\mathrm{d}t \quad (i = 1, \ldots, n)$$

and let $\tilde{p}_{\beta,\alpha,y}(x,t,z)$ be that of the diffusion

$$\mathrm{d}X_t = \mathrm{d}W_t + (\beta \nabla_x |X_t - y| - \alpha \nabla_x |X_t - y|^2)\,\mathrm{d}t.$$

Thus $p_{k_1,k_2,y}(x,t,z) = p_b(x,t,z)$ with $b(x) = \sum_i b^i(x)\partial/\partial x^i$ where

$$b^i(x) = k_1 \operatorname{sgn}(x_i - y_i) - k_2(x_i - y_i).$$

We would like to mention that if the dimension $n = 1$, then $\tilde{p}_{\beta,\alpha,y}(x,t,z)$ coincides with $p_{\beta,2\alpha,y}(x,t,z)$, where the factor 2 comes from the fact that

$$\alpha \nabla_x |x - y|^2 = 2\alpha(x - y).$$

As applying Theorem 2.7 to the vector field b we thus have

Corollary 2.9. *Let* $\alpha, \beta \in \mathbb{R}$ *be two constants.*

(1) *If* $\beta \geqslant 0$ *then*

$$p_{\beta,2\alpha,y}(x,t,y) \leqslant \tilde{p}_{\beta,\alpha,y}(x,t,y) \quad \text{for all } (x,t)$$

and

$$p_{\beta/\sqrt{n},2\alpha,y}(x,t,y) \geqslant \tilde{p}_{\beta,\alpha,y}(x,t,y) \quad \text{for all } (x,t).$$

(2) *If* $\beta \leqslant 0$ *then*

$$p_{\beta/\sqrt{n},2\alpha,y}(x,t,y) \leqslant \tilde{p}_{\beta,\alpha,y}(x,t,y) \quad \text{for all } (x,t)$$

and

$$p_{\beta,2\alpha,y}(x,t,y) \geqslant \tilde{p}_{\beta,\alpha,y}(x,t,y) \quad \text{for all } (x,t).$$

The Comparison Theorem 2.7 together with Corollary 2.9 allow us to deduce explicit, sharp upper and lower bounds for the transition function of a Brownian motion with a general drift of linear growth. As announced, these bounds are established through carefully estimating the density function $p_{\beta,\alpha,y}(x,t,y)$.

A simple application of the Comparison Theorem 2.7 we establish

Lemma 2.10. *Let* $b(x)=\sum_i b^i(x)\partial/\partial x_i$ *be a vector field on* $\mathbb{R}^n$, *and* $y\in\mathbb{R}^n$. *Let*

$$\tilde{b}(x)=\beta\nabla_x|x-y|-\alpha\nabla_x|x-y|^2.$$

(1) *If for all* $i=1,\dots,n$ *and* x

$$|b^i(x)|\leqslant\varepsilon_i+\delta_i|x-y|,$$

where $\varepsilon=(\varepsilon_i)$, $\delta=(\delta_i)$, $\varepsilon_i\geqslant 0$ *and* $\delta_i\geqslant 0$, *then*

$$\langle b(x)-\tilde{b}(x),\nabla_x|x-y|\rangle\leqslant 0$$

with $\beta=|\varepsilon|$ *and* $\alpha=-|\delta|/2$.

(2) *If for all* $i=1,\dots,n$ *and* x

$$|b^i(x)|\leqslant\varepsilon_i+\delta_i|x-y|,$$

then

$$\langle b(x)-\tilde{b}(x),\nabla_x|x-y|\rangle\geqslant 0$$

with $\beta=-|\varepsilon|$ *and* $\alpha=|\delta|/2$.

Proof. Indeed, suppose

$$|b^i(x)|\leqslant\varepsilon_i+\delta_i|x-y|,$$

then, for any $x\neq y$,

$$\begin{aligned}
&\langle b(x)-\tilde{b}(x),\nabla_x|x-y|\rangle\\
&\quad=\sum_{i=1}^n\left\{b^i(x)-\beta\frac{x_i-y_i}{|x-y|}+2\alpha(x_i-y_i)\right\}\frac{x_i-y_i}{|x-y|}\\
&\quad=\sum_{i=1}^n b^i(x)\frac{x_i-y_i}{|x-y|}-\beta+2\alpha|x-y|\\
&\quad\leqslant\frac{1}{|x-y|}\sum_{i=1}^n\varepsilon_i|x_i-y_i|+\sum_{i=1}^n\delta_i|x_i-y_i|-\beta+2\alpha|x-y|\\
&\quad\leqslant|\varepsilon|-\beta+(|\delta|+2\alpha)|x-y|
\end{aligned}$$

upon setting $\beta=|\varepsilon|$ and $\alpha=-|\delta|/2$ we have

$$\langle b(x)-\tilde{b}(x),\nabla_x|x-y|\rangle\leqslant 0$$

which proves the first claim. Similarly for $y\neq x$

$$\langle b(x)-\tilde{b}(x),\nabla_x|x-y|\rangle=\sum_{i=1}^n b^i(x)\frac{x_i-y_i}{|x-y|}-\beta+2\alpha|x-y|$$

$$\geqslant -\frac{1}{|x-y|}\sum_{i=1}^{n}\varepsilon_i|x_i-y_i|$$

$$-\sum_{i=1}^{n}\delta_i|x_i-y_i|-\beta+2\alpha|x-y|$$

$$\geqslant (-|\varepsilon|-\beta)-(|\delta|-2\alpha)|x-y|$$

which yields the second conclusion. □

By Theorem 2.7, Corollary 2.9 and Lemma 2.10 we thus establish the following:

Proposition 2.11. *Let $b(x)=\sum_i b^i(x)\partial/\partial x_i$ be a vector field on $\mathbb{R}^n$, and let c_1 and c_2 be two non-negative constants, $y\in\mathbb{R}^n$. If for all x and $i=1,\dots,n$*

$$|b^i(x)|\leqslant \varepsilon_i+\delta_i|x-y|$$

for some $\varepsilon_i\geqslant 0$, $\delta_i\geqslant 0$, then

$$p_{c_1,-c_2,y}(x,t,y)\leqslant p_b(x,t,y)\leqslant p_{-c_1,c_2,y}(x,t,y)$$

for all (x,t), where

$$c_1=\sqrt{\sum_{i=1}^{n}\varepsilon_i^2}\quad \textit{and}\quad c_2=\sqrt{\sum_{i=1}^{n}\delta_i^2}.$$

The nice feature of the last inequality is that the bound function $p_{\beta,\alpha,y}(x,t,z)$ in this estimate has a product form, more precisely

$$p_{\beta,\alpha,y}(x,t,z)=\prod_{j=1}^{n}p_{\beta,\alpha,y_j}(x_j,t,z_j)$$

in terms of the standard coordinate system $x=(x_1,\dots,x_n)$.

3. Brownian motion with drift in $\mathbb{R}^n$

In this section we establish precise estimates for $p_{\beta,\alpha,y}(x,t,y)$ which allow us, by Proposition 2.11 to establish strong estimates for $p_b(x,t,y)$, where b is a vector field of at most linear growth.

3.1. Bounds for diffusion with drift of linear growth

Let, for every $q>0$, $t\geqslant 0$,

$$a_{1,q}=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|x|^q e^{-x^2/2}\,dx,\qquad \sigma(t,c)=\sqrt{\frac{1-e^{-2ct}}{2c}}$$

and set $\sigma(t,0)=\lim_{c\to 0}\sigma(t,c)=\sqrt{t}$.

As usual $p_b(x,t,y)$ is the transition density function of the Brownian motion with drift b. We prove the following two theorems. For lower bounds we have

Theorem 3.1. *Fix $y\in\mathbb{R}^n$. Suppose that $b=(b_i)_i$ is a vector field with at most linear growth, and*

$$|b^i(x)| \leqslant \varepsilon_i + \delta_i|x-y|, \quad \forall x\in\mathbb{R}^n$$

for some $\varepsilon_i \geqslant 0$, $\delta_i \geqslant 0$. Define

$$c_1 = \sqrt{\sum_{i=1}^n \varepsilon_i^2} \quad \text{and} \quad c_2 = \sqrt{\sum_{i=1}^n \delta_i^2}.$$

Then

$$p_b(x,t,y) \geqslant \prod_{i=1}^n \left\{ h_{-c_2}(x_i-y_i,t,0) \times \exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}} + |x_i-y_i|\right) - \frac{c_1^2}{2}t\right]\right\} \tag{3.1}$$

for all $t>0$ and x,y, where $h_{-c_2}(x_i-y_i,t,0)$ is given by

$$h_\alpha(x_i,t,z_i) = \frac{1}{\sqrt{2\pi}\sigma(t,\alpha)}\exp\left(-\frac{|z_i-\mathrm{e}^{-\alpha t}x_i|^2}{2\sigma(t,\alpha)^2}\right). \tag{3.2}$$

The upper bound is given in the following

Theorem 3.2. *Under the same assumption in Theorem 3.1, then for every $q>1$ we have*

$$p_b(x,t,y) \leqslant \prod_{i=1}^n \left\{ h_{c_2}(x_i-y_i,t,0) + \frac{c_1}{\sqrt{2\pi}\sigma(t,c_2)}\left(\zeta_q\sqrt{\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}} + \rho_q|x_i-y_i|\right) \times \exp\left(-\frac{\mathrm{e}^{-2c_2t}|x_i-y_i|^2}{2q\sigma(t,c_2)^2} + \frac{c_1^2}{2(q-1)}t\right)\right\}, \tag{3.3}$$

where

$$\zeta_q = q2^{2-1/q}\sqrt[q]{a_{1,q}}, \qquad \rho_q = q2^{2-1/q}.$$

Indeed, we will prove that $p_{c_1,-c_2,y}(x,t,y)$ has the right-hand side of (3.1) as its lower bound, $p_{-c_1,c_2,y}(x,t,y)$ has the right-hand side of (3.3) as its upper bound, then Proposition 2.11 thus lead to the estimates in Theorems 3.1 and 3.2. Since both the bounds in (3.1) and (3.3) and $p_{\pm c_1,\mp c_2,y}(x,t,y)$ have product forms, therefore we only need to prove these estimates for one-dimensional case. Thus we only consider the case $n=1$ in what follows.

To see the above lower bound is sharp, we first notice that this lower bound coincides with the transition density function $h_{-c_2}(x-y,t,0)$ of the linear diffusion with drift $c_2(\cdot - y)$ if $c_1 = 0$. Moreover, if $c_2 = 0$, since

$$\lim_{\alpha\to 0} h_\alpha(x-y,t,0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right),$$

and thus if the vector field satisfies $|b| \leqslant c_1$, Theorem 3.1 implies in this case that

$$p_b(x,t,y) \geqslant \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right) \exp\left[-2c_1\left(a_{1,1}\sqrt{t} + |x-y|\right) - \frac{c_1^2}{2}t\right]$$

which is very close to the best possible we could hope under the bound of $|b|$, since it has the exact leading Gaussian term $\left(1/\sqrt{2\pi t}\right)\exp(-|x-y|^2/2t)$ and the correct leading term for large t. Indeed, the best estimate one could achieve is the following: if $|b| \leqslant c_1$ then (which is proved in Qian et al., 2003, see also Gradinaru et al., 2001; Karatzas and Shreve, 1988)

$$p_b(x,t,y) \geqslant \frac{1}{t\sqrt{2\pi t}} \int_{|x-y|}^{\infty} z \exp\left(-\frac{|z+c_1 t|^2}{2t}\right) \mathrm{d}z.$$

To see the sharpness of our upper bound in Theorem 3.2, we have the same remark as the previous lower bound. First when $c_1 = 0$, then the upper bound is reduced to $h_{c_2}(x-y,t,0)$ which is the best possible one can hope when the drift b has linear growth. On the other hand if $c_2 = 0$, that is b is bounded with bound c_1, then the upper bound for this case reads as, for every $q > 1$,

$$\begin{aligned} p_b(x,t,y) \leqslant{}& \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right) \\ &+ \frac{c_1}{\sqrt{2\pi t}}\left(\zeta_q\sqrt{t} + \rho_q|x-y|\right) \exp\left(-\frac{|x-y|^2}{2qt} + \frac{c_1^2}{2(q-1)}t\right) \end{aligned}$$

which is a very strong estimate for $p_b(x,t,y)$ in terms of the bound of b.

It follows immediately from the above lower and upper bounds the following Varadhan's asymptotic for small time t:

$$\lim_{t\to 0} 2t \log p(x,t,y) \to -|x-y|^2.$$

3.2. Several technical facts

As we have mentioned, we will prove Theorems 3.1 and 3.2 by establishing an estimate for $p_{-c_1,c_2,y}(x,t,y)$ from above, and a bound for $p_{c_1,-c_2,y}(x,t,y)$ from below, and we only need to consider the case that $n = 1$.

An exact formulas are known (but only for one-dimensional case, Gradinaru et al., 2001; Karatzas and Shreve, 1988) for $p_{\beta,0,y}(x,t,y)$ and $p_{0,\alpha,y}(x,t,y)$, and they are given by

$$p_{\beta,0,y}(x,t,y)=\frac{1}{t\sqrt{2\pi t}}\int_{|x-y|}^{\infty} z\exp\left(-\frac{|z-\beta t|^2}{2t}\right)\,\mathrm{d}z \tag{3.4}$$

and

$$p_{0,\alpha,y}(x,t,y)=\sqrt{\frac{\alpha}{\pi(1-\mathrm{e}^{-2\alpha t})}}\exp\left(-\frac{\alpha\mathrm{e}^{-2\alpha t}|x-y|^2}{(1-\mathrm{e}^{-2\alpha t})}\right).$$

However we know no closed formula for the transition probability density $p_{\beta,\alpha,y}(x,t,z)$, and indeed the main goal of the present section is to develope some methods of estimating $p_{\beta,\alpha,y}(x,t,z)$ which we believe will be useful in treating other problems.

For simplicity denote by $h_\alpha(x,t,z)$ the transition probability density of the linear diffusion process

$$\mathrm{d}\xi_t=-\alpha\xi_t\,\mathrm{d}t+\mathrm{d}B_t,\qquad \xi_0=x. \tag{3.5}$$

The unique strong solution to Eq. (3.5) is well known and is given by (see for example Ikeda and Watanabe, 1981, (8.2))

$$\xi_t=\mathrm{e}^{-\alpha t}x+\int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}B_s. \tag{3.6}$$

Thus by a simple computation we can see that $h_\alpha(x,t,z)$ is given by (3.2).

Obviously

$$p_{0,\alpha,y}(x,t,y)=h_\alpha(x-y,t,0).$$

Let us give estimates about the transition density function $p_{c_1,-c_2,y}(x,t,y)$ from below, and $p_{-c_1,c_2,y}(x,t,y)$ from above, where c_1, c_2 are two non-negative constants. The proof of Theorems 3.1 and 3.2 follow these estimate easily.

Given $\alpha\in\mathbb{R}$, consider the following Gaussian diffusion $(\xi_t,\mathscr{F}_t,\mathbb{P}^x)$:

$$\mathrm{d}\xi_t=-\alpha\xi_t\,\mathrm{d}t+\mathrm{d}B_t,\qquad \xi_0=x, \tag{3.7}$$

of which the transition probability density is known as Gaussian function, denoted by $h_\alpha(x,t,z)$. Then by using the Cameron–Martin formula we may add a two-valued drift $\beta\,\mathrm{sgn}(\cdot)$. Thus the transition density function $p_{\beta,\alpha,y}(x,t,y)$ can be expressed in terms of $h_\alpha(x,t,z)$ and the conditional distribution of the Gaussian diffusion ξ_s such that $\xi_0=x$ and $\xi_t=y$. Our results will then follow careful estimates about these conditional probabilities.

In fact under the probability measure $\mathbb{Q}^x$ defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathscr{F}_t}=\exp\left(\beta\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s-\frac{\beta^2}{2}t\right)$$

for all t, the process (ξ_t) is a weak solution to our model process

$$\mathrm{d}X_t=\beta\,\mathrm{sgn}(X_t)\,\mathrm{d}t-\alpha X_t\,\mathrm{d}t+\mathrm{d}B_t,\qquad X_0=x, \tag{3.8}$$

and thus, by the Cameron–Martin formula

$$\frac{p_{\beta,\alpha}(x,t,0)}{h_\alpha(x,t,0)} = \mathbb{P}^{x,0} \exp\left(\beta \int_0^t \operatorname{sgn}(\xi_s)\, \mathrm{d}B_s - \frac{\beta^2}{2} t\right), \tag{3.9}$$

where $\mathbb{P}^{x,0}$ denotes the conditional distribution $\mathbb{P}^x(\cdot|\xi_t = 0)$. Let

$$M_s = \exp\left(\beta \int_0^s \operatorname{sgn}(\xi_u)\, \mathrm{d}B_u - \frac{\beta^2}{2} s\right).$$

Since

$$\nabla_x h_\alpha(x,t,0) = -\frac{2\alpha \mathrm{e}^{-2\alpha t}}{1-\mathrm{e}^{-2\alpha t}} x h_\alpha(x,t,0), \tag{3.10}$$

by Theorem 2.4

$$p_{\beta,\alpha}(x,t,0) = h_\alpha(x,t,0) - \frac{2\beta|\alpha|^{3/2}}{\sqrt{\pi}} \int_0^t \frac{\mathrm{e}^{-2\alpha(t-s)}}{|1-\mathrm{e}^{-2\alpha(t-s)}|^{3/2}} \mathbb{P}^x \times \left(M_s |\xi_s| \mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}} \right) \mathrm{d}s. \tag{3.11}$$

Lemma 3.3. *For $q \geqslant 1$, and $s < t$ we have*

$$\mathbb{P}^x\left(|\xi_s|^q \mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}} \right) = \sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}} A_s^{q+1} \mathrm{e}^{-\frac{\alpha q \mathrm{e}^{-2\alpha t}}{(q-1)\mathrm{e}^{-2\alpha(t-s)}+1-q\mathrm{e}^{-2\alpha t}} x^2} \times \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \left| z + \frac{2\alpha \mathrm{e}^{-\alpha s} A_s}{1-\mathrm{e}^{-2\alpha s}} x \right|^q \mathrm{e}^{-z^2/2}\, \mathrm{d}z,$$

where

$$A_s = \sqrt{\frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha} \frac{1-\mathrm{e}^{-2\alpha(t-s)}}{q\mathrm{e}^{-2\alpha(t-s)}(1-\mathrm{e}^{-2\alpha s}) + (1-\mathrm{e}^{-2\alpha(t-s)})}} = \sqrt{\frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha} \frac{1-\mathrm{e}^{-2\alpha(t-s)}}{(q-1)\mathrm{e}^{-2\alpha(t-s)} + (1-q\mathrm{e}^{-2\alpha t})}}.$$

Proof. It follows a direct computation. In fact, since ξ_s has the Gaussian distribution $N(a, \sigma^2)$ where

$$a = \mathrm{e}^{-\alpha s} x, \quad \sigma^2 = \frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha}$$

we therefore have

$$\mathbb{P}^x\left(|\xi_s|^q \mathrm{e}^{\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}} \right) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} |z|^q \mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}z|^2}{1-\mathrm{e}^{-2\alpha(t-s)}} - \frac{|z-\alpha|^2}{2\sigma^2}}\, \mathrm{d}z = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} |z|^q \mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}z|^2}{1-\mathrm{e}^{-2\alpha(t-s)}} - \frac{\alpha|z-\mathrm{e}^{-\alpha s}x|^2}{1-\mathrm{e}^{-2\alpha s}}}\, \mathrm{d}z.$$

Changing the variable

$$\sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}\,\frac{q\mathrm{e}^{-2\alpha(t-s)}(1-\mathrm{e}^{-2\alpha s})+(1-\mathrm{e}^{-2\alpha(t-s)})}{1-\mathrm{e}^{-2\alpha(t-s)}}}\,z=u,$$

the last integral may be written as

$$\mathbb{P}^x\left(|\xi_s|^q\mathrm{e}^{\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)=\sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}}\,A_s^{q+1}\,\mathrm{e}^{-\frac{\alpha\mathrm{e}^{-2\alpha s}}{1-\mathrm{e}^{-2\alpha s}}x^2+\frac{2\alpha^2\mathrm{e}^{-2\alpha s}}{(1-\mathrm{e}^{-2\alpha s})^2}A_s^2x^2}$$

$$\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|z+\frac{2\alpha\mathrm{e}^{-\alpha s}A_s}{1-\mathrm{e}^{-2\alpha s}}x\right|^q\mathrm{e}^{-z^2/2}\,\mathrm{d}z,$$

and the lemma follows immediately. □

Lemma 3.4. *Let* $\alpha=-c_2<0$. *Then*

$$\mathbb{P}^x\left(|\xi_s|\mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)\leqslant\frac{\mathrm{e}^{2c_2(t-s)}-1}{\mathrm{e}^{2c_2t}-1}\left(a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}}+|x|\right)$$

$$\times\exp\left(-\frac{c_2\mathrm{e}^{2c_2t}}{\mathrm{e}^{2c_2t}-1}x^2\right).$$

Proof. Indeed, by the previous lemma we have

$$\mathbb{P}^x\left(|\xi_s|\mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)=\frac{\mathrm{e}^{2c_2(t-s)}-1}{\mathrm{e}^{2c_2t}-1}\exp\left(-\frac{c_2\mathrm{e}^{2c_2t}}{\mathrm{e}^{2c_2t}-1}x^2\right)$$

$$\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|z\sqrt{\frac{\mathrm{e}^{2c_2s}-1}{2c_2}}+\sqrt{\frac{\mathrm{e}^{2c_2t}-\mathrm{e}^{2c_2s}}{\mathrm{e}^{2c_2t}-1}}\,x\right|\mathrm{e}^{-z^2/2}\,\mathrm{d}z,$$

then the inequality follows from the elementary inequality: for all $s\leqslant t$,

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|z\sqrt{\frac{\mathrm{e}^{2c_2s}-1}{2c_2}}+\sqrt{\frac{\mathrm{e}^{2c_2t}-\mathrm{e}^{2c_2s}}{\mathrm{e}^{2c_2t}-1}}x\right|\mathrm{e}^{-z^2/2}\,\mathrm{d}z$$

$$\leqslant a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}}+|x|.\qquad\square$$

Lemma 3.5. *Let* $\alpha=c_2\geqslant 0$, *and let* $(\xi_s,\mathbb{P}^x)$ *be the Gaussian diffusion* (3.7). *Then for every* $q\geqslant 1$ *we have*

$$\mathbb{P}^x\left(|\xi_s|^q\mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)\leqslant 2^{q-1}\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}\right)^{q/2}+|x|^q\right)$$

$$\times\left(\frac{1-\mathrm{e}^{-2c_2(t-s)}}{1-\mathrm{e}^{-2c_2t}}\right)^{(q+1)/2}\exp\left(-\frac{c_2\mathrm{e}^{-2c_2t}}{1-\mathrm{e}^{-2c_2t}}x^2\right).$$

Proof. By Lemma 3.3

$$\mathbb{P}^x\left(|\xi_s|^q \mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)$$

$$= \left(\frac{1-\mathrm{e}^{-2\alpha(t-s)}}{(q-1)\mathrm{e}^{-2\alpha(t-s)}+(1-q\mathrm{e}^{-2\alpha t})}\right)^{(q+1)/2} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{\alpha q \mathrm{e}^{-2\alpha t}}{(q-1)\mathrm{e}^{-2\alpha(t-s)}+1-q\mathrm{e}^{-2\alpha t}}x^2}$$

$$\times \int_{\mathbb{R}} \left| z\sqrt{\frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha}} + \sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}}\,\mathrm{e}^{-\alpha s}A_s x\right|^q \mathrm{e}^{-z^2/2}\,\mathrm{d}z,$$

since

$$\sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}}\,\mathrm{e}^{-\alpha s}A_s = \mathrm{e}^{-\alpha s}\sqrt{\frac{(1-\mathrm{e}^{-2\alpha(t-s)})}{q\mathrm{e}^{-2\alpha(t-s)}(1-\mathrm{e}^{-2\alpha s})+(1-\mathrm{e}^{-2\alpha(t-s)})}} \leqslant 1,$$

we have

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \left| z\sqrt{\frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha}} + \sqrt{\frac{2\alpha}{1-\mathrm{e}^{-2\alpha s}}}\mathrm{e}^{-\alpha s}A_s x\right|^q \mathrm{e}^{-z^2/2}\,\mathrm{d}z$$

$$\leqslant 2^{q-1}\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2\alpha t}}{2\alpha}\right)^{q/2} + |x|^q\right).$$

It is easy to see that

$$-\frac{\alpha q\mathrm{e}^{-2\alpha t}}{(q-1)\mathrm{e}^{-2\alpha(t-s)}+1-q\mathrm{e}^{-2\alpha t}} \leqslant -\frac{\alpha \mathrm{e}^{-2\alpha t}}{1-\mathrm{e}^{-2\alpha t}}$$

and

$$\left(\frac{1-\mathrm{e}^{-2\alpha(t-s)}}{(q-1)\mathrm{e}^{-2\alpha(t-s)}+(1-q\mathrm{e}^{-2\alpha t})}\right)^{(q+1)/2} \leqslant \left(\frac{1-\mathrm{e}^{-2\alpha(t-s)}}{1-\mathrm{e}^{-2\alpha t}}\right)^{(q+1)/2}.$$

Collecting these estimates together we thus obtain

$$\mathbb{P}^x\left(|\xi_s|^q \mathrm{e}^{-\frac{\alpha q|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right) \leqslant \left(\frac{1-\mathrm{e}^{-2c_2(t-s)}}{1-\mathrm{e}^{-2c_2 t}}\right)^{(q+1)/2} \exp\left(-\frac{c_2\mathrm{e}^{-2c_2 t}}{1-\mathrm{e}^{-2c_2 t}}x^2\right) 2^{q-1}$$

$$\times\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2 t}}{2c_2}\right)^{q/2} + |x|^q\right). \qquad \square$$

3.3. Lower bound for $p_{c_1,-c_2}$ and upper bound for p_{-c_1,c_2}

As mentioned, it is sufficient to consider the case where the dimension $n = 1$.

Lower bound estimate: We again may assume $y=0$. In this section $\beta=c_1>0$ and $\alpha=-c_2\leqslant 0$. Applying Jensen's inequality to Eq. (3.9) we have

$$\frac{p_{\beta,\alpha}(x,t,0)}{h_\alpha(x,t,0)}\geqslant \exp\left[\beta\mathbb{P}^{x,0}\left(\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s\right)-\frac{\beta^2}{2}t\right]. \tag{3.12}$$

However under $\mathbb{P}^{x,0}$ the semimartingale $\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s$ possesses the following decomposition:

$$\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s=N_t+\int_0^t \langle \mathrm{sgn}(\xi_s),\nabla_x\log h_\alpha(\xi_s,t-s,0)\rangle\,\mathrm{d}s,$$

where N_t is a martingale under $\mathbb{P}^{x,0}$. Thus by (3.10),

$$\begin{aligned}
&\mathbb{P}^{x,0}\left(\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s\right)\\
&\quad=\int_0^t \mathbb{P}^{x,0}(\langle \mathrm{sgn}(\xi_s),\nabla_x\log h_\alpha(\xi_s,t-s,0)\rangle)\,\mathrm{d}s\\
&\quad=-\int_0^t \frac{2\alpha\mathrm{e}^{-2\alpha(t-s)}}{1-\mathrm{e}^{-2\alpha(t-s)}}(\mathbb{P}^{x,0}|\xi_s|)\,\mathrm{d}s\\
&\quad=-\frac{1}{h_\alpha(x,t,0)}\int_0^t \frac{2\alpha\mathrm{e}^{-2\alpha(t-s)}}{1-\mathrm{e}^{-2\alpha(t-s)}}\mathbb{P}^{x}(h_\alpha(\xi_s,t-s,0)|\xi_s|)\,\mathrm{d}s\\
&\quad=-\frac{1}{\sqrt{\pi}}\frac{1}{h_\alpha(x,t,0)}\int_0^t \frac{2|\alpha|^{3/2}\mathrm{e}^{-2\alpha(t-s)}}{|1-\mathrm{e}^{-2\alpha(t-s)}|^{3/2}}\mathbb{P}^{x}\left(|\xi_s|\mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)\,\mathrm{d}s.
\end{aligned}$$

By Lemma 3.4, as $\alpha=-c_2<0$,

$$\begin{aligned}
\mathbb{P}^{x,0}\left(\int_0^t \mathrm{sgn}(\xi_s)\,\mathrm{d}B_s\right)&\geqslant -\frac{1}{\sqrt{\pi}}\left(a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}}+|x|\right)\frac{1}{\mathrm{e}^{2c_2t}-1}\\
&\quad\times\frac{\exp\left(-\frac{c_2\mathrm{e}^{2c_2t}}{\mathrm{e}^{2c_2t}-1}x^2\right)}{h_\alpha(x,t,0)}\int_0^t \frac{2c_2^{3/2}\mathrm{e}^{2c_2(t-s)}}{\sqrt{\mathrm{e}^{2c_2(t-s)}-1}}\,\mathrm{d}s\\
&=-2\left(a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}}+|x|\right).
\end{aligned}$$

Therefore

$$\frac{p_{c_1,-c_2}(x,t,0)}{h_{-c_2}(x,t,0)}\geqslant \exp\left[-2c_1\left(a_{1,1}\sqrt{\frac{\mathrm{e}^{2c_2t}-1}{2c_2}}+|x|\right)-\frac{c_1^2}{2}\,t\right].$$

Upper bound estimate: The upper bound follows from the representation formula, Lemma 3.5 and the Hölder inequality. Indeed, by the Hölder inequality and Lemma 3.5, for any $q>1$ of which p is the conjugate exponent,

$$\mathbb{P}^x\left(M_s|\xi_s|\mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right) \leqslant \mathrm{e}^{\frac{(p-1)c_1^2}{2}t}\left(\mathbb{P}^x\left(|\xi_s|^q\mathrm{e}^{-\frac{q\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)\right)^{1/q}$$

$$\leqslant 2^{1-1/q}\left(\frac{1-\mathrm{e}^{-2c_2(t-s)}}{1-\mathrm{e}^{-2c_2t}}\right)^{1/2+1/2q}\mathrm{e}^{\frac{(p-1)c_1^2}{2}t-\frac{c_2\mathrm{e}^{-2c_2t}}{q(1-\mathrm{e}^{-2c_2t})}x^2}$$

$$\times\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}\right)^{q/2}+|x|^q\right)^{1/q}.$$

After integrating in s we deduce that

$$\int_0^t\frac{\mathrm{e}^{-2\alpha(t-s)}}{|1-\mathrm{e}^{-2\alpha(t-s)}|^{3/2}}\mathbb{P}^x\left(M_s|\xi_s|\mathrm{e}^{-\frac{\alpha|\mathrm{e}^{-\alpha(t-s)}\xi_s|^2}{1-\mathrm{e}^{-2\alpha(t-s)}}}\right)\mathrm{d}s$$

$$\leqslant 2^{1-1/q}\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}\right)^{q/2}+|x|^q\right)^{1/q}\mathrm{e}^{\frac{(p-1)c_1^2}{2}t-\frac{c_2\mathrm{e}^{-2c_2t}}{q(1-\mathrm{e}^{-2c_2t})}x^2}$$

$$\times\int_0^t\frac{\mathrm{e}^{-2\alpha(t-s)}}{|1-\mathrm{e}^{-2\alpha(t-s)}|^{3/2}}\left(\frac{1-\mathrm{e}^{-2c_2(t-s)}}{1-\mathrm{e}^{-2c_2t}}\right)^{1/2+1/2q}\mathrm{d}s$$

$$=2^{1-1/q}\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}\right)^{q/2}+|x|^q\right)^{1/q}\mathrm{e}^{\frac{(p-1)c_1^2}{2}t-\frac{c_2\mathrm{e}^{-2c_2t}}{q(1-\mathrm{e}^{-2c_2t})}x^2}\frac{q}{\alpha}\left(\frac{1}{1-\mathrm{e}^{-2c_2t}}\right)^{1/2}.$$

Thus by Eq. (3.11)

$$p_{-c_1,c_2}(x,t,0)$$

$$\leqslant h_{c_2}(x,t,0)+\frac{2^{2-1/q}qc_1}{\sqrt{\pi}}\left(\frac{c_2}{1-\mathrm{e}^{-2c_2t}}\right)^{1/2}\left(a_{1,q}\left(\frac{1-\mathrm{e}^{-2c_2t}}{2c_2}\right)^{q/2}+|x|^q\right)^{1/q}$$

$$\times\exp\left(-\frac{c_2\mathrm{e}^{-2c_2t}}{q(1-\mathrm{e}^{-2c_2t})}x^2+\frac{(p-1)c_1^2}{2}t\right)$$

which yields the upper bound.

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