SPA: 2617 Model 1 pp. 1–25 (col. fig: NIL)

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications xx (xxxx) xxx–xxx

stochastic processes and their applications

www.elsevier.com/locate/spa

Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions

Christian Böinghoff*

Q1

Fachbereich Mathematik, Universität Frankfurt, Fach 187, D-60054 Frankfurt am Main, Germany

Received 11 November 2013; received in revised form 24 May 2014; accepted 24 May 2014

Abstract

In the present paper, we characterize the behavior of supercritical branching processes in random environment with linear fractional offspring distributions, conditioned on having small, but positive values at some large generation. As it has been noticed in previous works, there is a phase transition in the behavior of the process. Here, we examine the strongly and intermediately supercritical regimes The main result is a conditional limit theorem for the rescaled associated random walk in the intermediately case.

MSC: 60J80; 60K37; 92D25

Keywords: Branching process in random environment; Supercritical; Conditional limit theorem

1. Introduction

Branching processes in random environment (BPRE) are a stochastic model for the development of a population in discrete time. The model has first been introduced in [8,22]. In generalization to Galton–Watson processes, the reproductive success of all individuals of

* Tel.: +49 69 798 23409; fax: +49 69 798 28841.
E-mail addresses: Christian.Boeinghoff@gmx.de, boeinghoff@math.uni-frankfurt.de.

http://dx.doi.org/10.1016/j.spa.2014.05.009
0304-4149/

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

a generation is influenced by an environment which varies in an independent fashion from generation to generation.

As first noted in [1,14], there is a phase transition in the behavior of subcritical BPRE (see e.g. for an overview [13] and for detailed results [5,6,18,24–27,15]). Only recently, there has been interest in a phase transition in supercritical processes, conditioned on surviving and having small values at some large generation (see [9,10,21]). For the scaling limit of supercritical branching diffusions, a phase transition has been noted in [19].

In [9,19], the terminology of strongly, intermediately and weakly supercritical BPRE has been introduced in analogy to subcritical BPRE. In the present paper, we focus on the phase transition from strongly to intermediately supercriticality and characterize these regimes with limit results.

Let us formally introduce a branching process in random environment $(Z_n)_{n\in\mathbb{N}_0}$. For this, let Q be a random variable taking values in Δ, the space of all probability distributions on $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. An infinite sequence $\Pi = (Q_1, Q_2, \ldots)$ of i.i.d. copies of Q is called a *random environment*. By Q_n, we denote the (random) offspring distribution of an individual at generation $n - 1$.

Let Z_n be the number of individuals in generation n. Then Z_n is the sum of Z_{n-1} independent random variables with distribution Q_n. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is then called a *branching process in the random environment* Π, if Z_0 is independent of Π and given Π the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\big(Z_n \bigm| Z_{n-1} = z,\ \Pi = (q_1, q_2, \ldots)\big) \;=\; q_n^{*z} \tag{1.1}$$

for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, $z \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \Delta$, where q^{*z} is the z-fold convolution of the measure q.

By $\mathbb{P}$, we will denote the corresponding probability measure on the underlying probability space. We shorten $Q(\{y\}), q(\{y\})$ to $Q(y), q(y)$ and write

$$\mathbb{P}(\cdot \mid Z_0 = z) =: \mathbb{P}_z(\cdot).$$

For convenience, we write $\mathbb{P}(\cdot)$ instead of $\mathbb{P}_1(\cdot)$. Throughout the paper, we assume that the offspring distributions have the following form,

$$q\big(0\big) = a, \qquad q\big(k\big) = \frac{(1-a)(1-p)}{p}\ p^k, \quad \text{for } k \geq 1$$

where $a \in [0, 1)$ and $p \in (0, 1)$ are two random parameters. Note that $a = 1$ would imply that an individual becomes extinct with probability one. Thus we exclude this case. This class of offspring distributions is often called *linear fractional* as the generating functions have an explicit formula as a quotient of two linear functions. Also the concatenation of two linear fractional functions is again linear fractional and thus the generating function of Z_n can be calculated explicitly in this case. An important tool in the analysis of BPRE is the associated random walk $S = (S_n)_{n\geq 0}$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1},\ n \geq 1$ defined by

$$X_n := \log m(Q_n),$$

where

$$m(q) := \sum_{y=0}^{\infty} y\, q(y)$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

is the mean of the offspring distribution $q \in \Delta$. The expectation of Z_n can be expressed by S_n by

$$\mathbb{E}[Z_n \mid \Pi] = \prod_{k=1}^{n} m(Q_k) = \exp(S_n) \quad \mathbb{P}\text{-a.s.}$$

Averaging over the environment gives

$$\mathbb{E}[Z_n] = \mathbb{E}[m(Q)]^n. \tag{1.2}$$

A well-known estimate following from this by the Markov inequality is

$$\begin{aligned} \mathbb{P}(Z_n > 0 \mid \Pi) &= \min_{0 \le k \le n} \mathbb{P}(Z_k > 0 \mid \Pi) \\ &\le \min_{0 \le k \le n} \mathbb{E}[Z_k \mid \Pi] = \exp\Big(\min_{0 \le k \le n} S_k\Big) \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.3}$$

Here, we focus on supercritical BPRE, i.e. the case of $\mathbb{E}[X] > 0$.

As it has been described in [9] on the level of large deviations and for the most recent common ancestor, there is a phase transition in the supercritical regime. Our aim here is to describe the regimes of *strongly* and *intermediately* supercriticality more in detail. In these regimes, the event $\{Z_n = 1\}$ is typically realized in a favorable environment, i.e. conditioned on $\{Z_n = 1\}$, S_n will still be large.

Throughout the paper, we write $\{Z_\infty > 0\}$ for the event $\{Z_n > 0 \ \forall n \in \mathbb{N}\}$. The paper is organized as follows: In Section 2, we state our main results. Section 3 deals with special properties of linear fractional offspring distributions. Section 4 recalls some properties of conditioned random walks whereas our results are proved in Sections 5 and 6.

2. Results

2.1. *The strongly supercritical case*

In this part, we assume that

$$0 < \mathbb{E}\big[X e^{-X}\big] < \infty, \tag{2.1}$$

which we refer to as *strongly supercritical*. Note that this condition implies $\mathbb{E}[e^{-X}] < \infty$. First, we introduce a change of measure. Let $\phi : \Delta \to \mathbb{R}$ be a bounded and measurable function. Then the measure $\mathbf{P}$ is defined by

$$\mathbf{E}\big[\phi(Q)\big] := \gamma^{-1}\mathbb{E}\big[e^{-X}\phi(Q)\big], \tag{2.2}$$

where $\gamma := \mathbb{E}\big[e^{-X}\big]$. Under $\mathbf{P}$, S is still a random walk with positive drift as (2.1) implies

$$\mathbf{E}\big[X\big] = \gamma^{-1}\mathbb{E}\big[X e^{-X}\big] > 0.$$

Throughout this section, we assume that $\mathbf{E}\big[\log(1 - Q(0))\big] > -\infty$. As it is proved in [22][Theorem 3.1], this condition assures that $\mathbf{E}[X] > 0$ indeed implies that the process survives with a positive probability, i.e.

$$\lim_{n\to\infty} \mathbf{P}(Z_n > 0) = \mathbf{P}(Z_\infty > 0) > 0.$$

In the intermediately supercritical case, we require a slightly different condition. Our first result describes the asymptotics of having exactly one individual at some large generation.

Theorem 2.1.1. *Under (2.1), there is a constant $\vartheta > 0$ such that*

$$\mathbb{P}(Z_n = 1) \sim \vartheta\, \gamma^n.$$

The next theorem describes the distribution of Z_n, conditioned on Z_n being bounded.

Theorem 2.1.2. *Assume (2.1). As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \le k \le c$,*

$$\mathbb{P}(Z_n = k \mid 1 \le Z_n \le c) \to \frac{1}{c},$$

i.e. the limiting distribution is uniform on $\{1, \ldots, c\}$.

The fact that this limiting distribution is uniform seems rather linked to linear fractional offspring distributions whereas we suspect that Theorem 2.1.1 also holds for more general offspring distributions.

Our next theorem essentially says that conditioned on $Z_n = 1$, the process is of constant order at all times.

Theorem 2.1.3. *Under Assumption (2.1), there is a probability distribution $r = (r_z)_{z \in \mathbb{N}}$ such that for all $t \in (0, 1)$*

$$\lim_{n\to\infty} \mathbb{P}(Z_{\lfloor nt \rfloor} = z \mid Z_n = 1) = r_z.$$

Moreover, r_z does not depend on t and is given by

$$r_z = z\mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big].$$

Let us briefly explain the intuition behind these results. Given $S_n \approx 0$ and $\min_{0 \le k \le n} S_k \ge 0$, the process would have a large probability of surviving and being small at generation n (see [9]) and the event would be realized by *environmental stochasticity*. However, in the strongly supercritical case, such an environment has very small probability. Conditioned on $\{Z_n = 1\}$, S_n has still a positive drift and thus S_n will be large. Thus the event $\{Z_n = 1\}$ is here typically realized by *demographic stochasticity*. Growing within a favorable environment and then becoming small again would require an exponential number of independent subtrees becoming extinct. This has very small probability. Thus the conditioned process typically stays small at all generations as it is stated in Theorem 2.1.3.

This theorem also explains a result for the most recent common ancestor in [9][Corollary 2.3]. If the process is small at all times, then the most recent common ancestor of the population will be close to the final generations. We suspect that qualitatively, these results will also be true for more general offspring distributions.

2.2. *The intermediately supercritical case*

In this section, we assume that

$$\mathbb{E}\big[Xe^{-X}\big] = 0. \tag{2.3}$$

In [9], this regime has been characterized as *intermediately supercritical*. Here, we will prove conditional limit theorems describing this regime more in detail.

(2.3) suggests a change of measure. Recall the definition of $\mathbf{P}$ from (2.2). Due to Assumption (2.3), S is now a recurrent random walk under $\mathbf{P}$, i.e.

$$\mathbf{E}[X] = 0.$$

For our theorems, we require some regularity of the distribution of X.

Assumption 1. We assume that with respect to $\mathbf{P}$, X has finite variance, or more generally belongs to the domain of attraction of some strictly stable law s with index $\alpha \in (0, 2]$.

As to the regularity of the offspring distribution, we require the following condition.

Assumption 2. We assume that there is an $\varepsilon > 0$ such that

$$\mathbf{E}\big[|\log(1 - Q(0))|^{\alpha+\varepsilon}\big] < \infty.$$

Our first theorem describes the asymptotics of the probability of having exactly one individual in some large generation n.

Theorem 2.2.1. *Assume (2.3). Then under Assumptions 1 and 2, there is a positive and finite constant θ such that as $n \to \infty$,*

$$\mathbb{P}(Z_n = 1) \sim \theta \mathbb{E}\big[e^{-X}\big]^n \mathbf{P}\big(\min_{0 \le k \le n} S_k \ge 0\big).$$

For a formula for θ, see Section 6. From [5][Lemma 2.1], it results that there is a slowly varying sequence $l(n)$ such that

$$\mathbb{P}(Z_n = 1) \sim \theta\, \gamma^n\, l(n)\, n^{-(1-\rho)},$$

where $\rho = s(\mathbb{R}^+)$ and s is the limiting stable law from Assumption 1.

Our next theorem describes the environment, conditioned on $\{Z_n = 1\}$. For this, define

$$S^n = \big(S^n_t\big)_{0 \le t \le 1} := \frac{l(n)}{n^{1/\alpha}} \big(S_{\lfloor nt \rfloor}\big)_{0 \le t \le 1}.$$

Then it holds that:

Theorem 2.2.2. *Assume (2.3). Under Assumptions 1 and 2,*

$$\mathcal{L}\big(S^n \mid Z_n = 1\big) \longrightarrow (L^+_t)_{t \in [0,1]},$$

where L^+ is the meander of a strictly stable Lévy process.

Essentially L^+ is a strictly stable Lévy process, conditioned on staying positive on (0, 1]. For more details see e.g. [5,16,17]. We can also specify the distribution of Z_n, given $1 \le Z_n \le c$ as in the strongly supercritical case.

Theorem 2.2.3. *Assume (2.3) and Assumptions 1 and 2. As $n \to \infty$ and for every $c \in \mathbb{N}$ and $1 \le k \le c$,*

$$\mathbb{P}(Z_n = k \mid 1 \le Z_n \le c) \to \frac{1}{c},$$

i.e. the limiting distribution is uniform on $\{1, \dots, c\}$.

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

For our next theorem, we require the successive global minima. Define the time of the first minimum up to generation n as

$$\tau_n := \min\big\{0 \le k \le n \mid S_k = \min\{0, S_1, \dots, S_n\}\big\}.$$

By

$$\tau_{k,n} := \min\big\{k \le j \le n \mid S_j = \min\{S_k, \dots, S_n\}\big\}, \tag{2.4}$$

we denote the time of the first minimum between generations k and n. Our next theorem proves that the conditioned BPRE is small in those minima.

Theorem 2.2.4. *Assume (2.3) and Assumptions 1 and 2. Let $t \in (0, 1)$. Then*

$$\lim_{n\to\infty} \mathbb{P}(Z_{\tau_{\lfloor nt \rfloor, n}} = z \mid Z_n = 1) = q(z),$$

where

$$q(z) := z\, \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big]$$

is a probability distribution on $\mathbb{N}$.

Let us briefly explain these results. Conditioned on $\{Z_n = 1\}$, S_n does not have a positive drift but converges – properly rescaled – to a Lévy meander. Thus the conditioned environment is less favorable than in the strongly supercritical case. In the minima of the Lévy meander, the process has to be small. Otherwise, many individuals would exhibit an environment favorable for growth making the event $\{Z_n = 1\}$ too improbable. In the excursions of S in between these minima, we expect (but did not prove here) that Z may grow and have exponentially large values. Thus *environmental* and *demographic* stochasticity have equal importance in this case.

3. The linear fractional offspring distribution

Remark. Throughout the paper, we only consider generalized geometric offspring distributions. Only for these distributions, an explicit formula for $\mathbb{P}(Z_n = 1 \mid \Pi)$ exists (except for some special related cases). As we condition on this probability, a useful formula is necessary in all our proofs. In the subcritical cases, one typically conditions on $\mathbb{P}(Z_n > 0 \mid \Pi)$. For this probability, a useful formula is known also for general offspring distributions (see e.g. [7]), involving the second factorial moments and not depending on the fine structure of the offspring distributions. Thus in contrast to the subcritical cases, we can only prove our results here for linear fractional offspring distributions. As in contrast to $\mathbb{P}(Z_n > 0 \mid \Pi)$, $\mathbb{P}(Z_n = 1 \mid \Pi)$ seems to depend on the fine structure of the offspring distributions (see [9]), generalizing the results seems to be difficult. However, the parallels to the subcritical case indicate that the theorems might be qualitatively true for more general offspring distributions (aside from the explicit limiting distributions proved here). □

Now, we present details on generalized geometric offspring distributions. Let ξ be a random variable on $\mathbb{N}_0$ with distribution q and let

$$q(0) = a, \qquad q(k) = \frac{(1-a)(1-p)}{p}\, p^k,$$

where $a \in [0, 1)$ and $p \in (0, 1)$. The expectation of ξ is

$$m_\xi := \sum_{k=1}^{\infty} k\, q(k) = \sum_{k=1}^{\infty} (1-a)(1-p)\, k\, p^{k-1} = (1-a)(1-p) \cdot \frac{1}{(1-p)^2} = \frac{1-a}{1-p}.$$

Inserting this, we get the formula

$$q(k) = (1-a)(1-p)p^{k-1} = \frac{1}{m_\xi}(1-a)^2 p^{k-1} = \frac{(1-q(0))^2}{m_\xi} p^{k-1},$$

i.e. uniformly in k,

$$0 \le m_\xi \cdot q(k) \le 1.$$

This fact will be used later. We can also rewrite the probability weights as

$$q(k) = q(1)p^{k-1}. \tag{3.1}$$

Let us now turn to the concatenation of linear fractional generating functions. As it is proved e.g. in [20][p. 156, Equation (6)], if the offspring distributions are linear fractional, then also Z_n, conditioned on Π has a linear fractional offspring distribution given by

$$\mathbb{P}(Z_n = z \mid \Pi) = \frac{e^{-S_n}}{\left(e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}\right)^2} \left(\frac{\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}} \right)^{z-1} \quad \text{a.s.},$$

where (see [20][p. 155]) $\eta_k := \eta_{Q_k}$ and

$$\eta_q = \frac{\sum_{j=0}^{\infty} j(j-1)q(j)}{2m_q^2} = \frac{\frac{(1-a)2p}{(1-p)^2}}{2m_q^2} = \frac{p}{1-a}$$

are the standardized second factorial moments of Q_k. Note that by summing over $z \in \mathbb{N}$, we get that a.s.

$$\mathbb{P}(Z_n > 0 \mid \Pi) = \frac{1}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}.$$

Defining

$$H_n := \frac{\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}} \tag{3.2}$$

we get the convenient formula

$$\mathbb{P}(Z_n = z \mid \Pi) = e^{-S_n} \mathbb{P}(Z_n > 0 \mid \Pi)^2 H_n^{z-1} \quad \text{a.s.} \tag{3.3}$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

In particular, we will use (as already proved in [9])

$$\mathbb{P}(Z_n = 1 \mid \Pi) = e^{-S_n}\, \mathbb{P}(Z_n > 0 \mid \Pi)^2 \quad \text{a.s.} \tag{3.4}$$

By (3.2), if $S_n \to \infty$ a.s. as $n \to \infty$, then also $H_n \to 1$ a.s.

For our proofs, we require the generating function of a distribution q,

$$f(s) := \sum_{k=0}^{\infty} s^k\, q(k), \quad s \in [0, 1].$$

Let f_k be the generating function of q_k and set

$$f_{0,n}(s) := \mathbb{E}\big[s^{Z_n} \mid Z_0 = 1\big].$$

As it is well-known, $f_{0,n}$ is the concatenation of the generating functions of each generation (see e.g. [5][Equation (3.2)]), i.e.

$$f_{0,n}(s) = f_1(f_2(\ldots(f_n(s))\ldots)).$$

More generally, let for $k \leq n$

$$f_{k,n}(s) := \mathbb{E}\big[s^{Z_n} \mid Z_k = 1\big]$$

resp.

$$f_{k,n}(s) = f_{k+1}(f_{k+2}(\ldots(f_n(s))\ldots)).$$

Thus we get the formula

$$\mathbb{P}(Z_n > 0 \mid \Pi) = 1 - f_{0,n}(0) \quad \text{a.s.} \tag{3.5}$$

Remark. In [5][Assumption B2], it is required that for some $\varepsilon > 0$

$$\mathbf{E}\Big[\big(\log^+(\eta)\big)^{\alpha+\varepsilon}\Big] < \infty,$$

where $\log^+(x) := \log(\max(x, 1))$. In our context, inserting the formula for η, this means

$$\mathbf{E}\Big[\Big(\log^+\Big(\frac{p}{1-a}\Big)\Big)^{\alpha+\varepsilon}\Big] < \infty.$$

As $p < 1$ and $\alpha \in (0, 2]$ and $a = Q(0)$, this condition is implied by

$$\mathbf{E}\Big[\big(\log(1 - Q(0))\big)^{\alpha+\varepsilon}\Big] < \infty.$$

4. Properties of random walks

Define for $n \geq 1$

$$L_n := \min\{S_1, \ldots, S_n\}, \qquad M_n := \max\{S_1, \ldots, S_n\}$$

and set $L_0 = M_0 = 0$. Recall the definition of τ_n,

$$\tau_n = \min\{0 \leq k \leq n \mid S_k = \min(0; L_n)\}.$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Next, we require the renewal function $u : \mathbb{R} \to \mathbb{R}$ defined by

$$u(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \le x, M_k < 0), \quad x \ge 0,$$
$$u(x) = 0, \quad x < 0.$$

For more details, see e.g. [5]. It is well-known that $u(0) = 1$. Let

$$\mathbf{P}_x\big((S_1, \dots, S_n) \in \cdot\big) = \mathbf{P}\big((S_1, \dots, S_n) \in \cdot \mid S_0 = x\big).$$

Here, we require that under Assumption 1, for every $x \ge 0$ and as $n \to \infty$

$$\mathbf{P}_x(L_n \ge -x) \sim u(x)\, n^{-(1-\rho)}\, l(n), \tag{4.1}$$

where the sequence $l(n)$ is slowly varying at infinity (see [5][Lemma 2.1]).

Furthermore, we require the h-transform describing the random walk conditioned on never entering $(-\infty, 0)$ where we will denote the corresponding measure by $\mathbf{P}^+$. More precisely, for every oscillating random walk the renewal function u has the property

$$u(x) = \mathbf{E}\big[u(x + X); x + X \ge 0\big], \quad x \ge 0.$$

This martingale property allows to define the measure $\mathbf{P}^+$. The construction of this measure is described in detail in e.g. [5,16]. Here, we only briefly recall the definition.

Let $\mathcal{F}_n = \sigma\big(Q_1, \dots, Q_n, Z_0, \dots, Z_n\big)$ be the σ-algebra generated by the branching process and its environment up to generation n. Let R_n be a bounded random variable adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Then $\mathbf{P}^+$ is defined by

$$\mathbf{E}^+_x\big[R_n\big] = \frac{1}{u(x)}\mathbf{E}_x\big[R_n u(S_n); L_n \ge 0\big], \quad x \ge 0.$$

We will later use the following result from [5][Lemma 2.5].

Lemma 1. *Let U_n be a uniformly bounded sequence of random variables adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. If $U_n \to U_\infty$ $\mathbf{P}^+$-a.s. for some limiting random variable U_∞, then as $n \to \infty$, it holds that*

$$\mathbf{E}\big[U_n \mid L_n \ge 0\big] \to \mathbf{E}^+\big[U_\infty\big].$$

5. Proof of theorems in the strongly supercritical case

We assume throughout this section that $\mathbb{E}\big[Xe^{-X}\big] > 0$ and thus (using definition (2.2))

$$\mathbf{E}[X] > 0.$$

5.1. *Proof of Theorem 2.1.1*

Using (3.4) and the change of measure (2.2), we get that

$$\begin{aligned}\mathbb{P}(Z_n = 1) &= \mathbb{E}\big[e^{-S_n}\mathbb{P}(Z_n > 0 \mid \Pi)^2\big]\\ &= \gamma^n \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2\big].\end{aligned}$$

Under $\mathbf{P}$, Z is still a supercritical branching process in random environment with linear fractional offspring distributions. As it is proved in [22][Theorem 3.1], under the condition $\mathbf{E}\big[\log(1 - Q(0))\big] > -\infty$,

$$\lim_{n\to\infty} \mathbf{P}(Z_n > 0) = \mathbf{P}(Z_\infty > 0) > 0.$$

As the survival probability is monotonically decreasing, the limit above exists and we conclude

$$\mathbf{P}\big(\mathbb{P}(Z_\infty > 0 \mid \Pi) > 0\big) > 0.$$

By monotonicity, we may interchange the limit and the integration and get that

$$\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(Z_n = 1) = \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\big] =: \vartheta > 0.$$

This proves Theorem 2.1.1. □

5.2. *Proof of Theorem 2.1.2*

Using (3.3), we have a.s.

$$\mathbb{P}(Z_n = z \mid \Pi) = e^{-S_n}\mathbb{P}(Z_n > 0 \mid \Pi)^2 H_n^{z-1},$$

where

$$H_n = \frac{\sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1} e^{-S_k}}.$$

Using the change of measure (2.2) yields

$$\begin{aligned}\mathbb{P}(Z_n = z) &= \mathbb{E}\big[\mathbb{P}(Z_n = z \mid \Pi)\big] \\ &= \gamma^n \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2 H_n^{z-1}\big]. \qquad (5.1)\end{aligned}$$

Under $\mathbf{P}$, $S_n \to \infty$ a.s. and therefore $H_n \to 1$ a.s. As H_n is bounded by 1, we may use the dominated convergence theorem to conclude that

$$\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(Z_n = z) = \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\big] \qquad (5.2)$$

for every $z \in \mathbb{N}$. Inserting this yields for every $c \in \mathbb{N}$ and $1 \le z \le c$

$$\mathbb{P}(Z_n = z \mid 1 \le Z_n \le c) = \frac{\mathbb{P}(Z_n = z)}{\sum_{j=1}^{c} \mathbb{P}(Z_n = j)} = \frac{\gamma^{-n}\mathbb{P}(Z_n = z)}{\sum_{j=1}^{c} \gamma^{-n}\mathbb{P}(Z_n = j)}.$$

Taking the limit $n \to \infty$ and using (5.2) yields the theorem. □

5.3. *Proof of Theorem 2.1.3*

Let $z \in \mathbb{N}$ and $t \in (0, 1)$ be fixed. Then using the branching property, independence of the environment and (3.4) yields

$$\mathbb{P}(Z_{\lfloor nt\rfloor} = z, Z_n = 1) = \mathbb{E}\big[\mathbb{P}(Z_{\lfloor nt\rfloor} = z, Z_n = 1 \mid \Pi)\big] = \mathbb{P}(Z_{\lfloor nt\rfloor} = z)\mathbb{P}_z(Z_{n-\lfloor nt\rfloor} = 1).$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Conditioning on the environment and using (5.2), we get that as $n \to \infty$

$$\mathbb{P}(Z_{\lfloor nt \rfloor} = z) = \gamma^{\lfloor nt \rfloor}\big(\mathbf{E}\big[\mathbb{P}(Z_\infty > 0|\Pi)^2\big] + o(1)\big).$$

Starting from z-many individuals, $\{Z_n = 1\}$ implies that $z-1$ of the z subtrees must become extinct before time n. As conditioned on Π, all subtrees are independent and inserting (3.4), we get that

$$\begin{aligned}\mathbb{P}_z(Z_{n-\lfloor nt \rfloor} = 1) &= z\mathbb{E}\big[\mathbb{P}(Z_{n-\lfloor nt \rfloor} = 1 \mid \Pi)\mathbb{P}(Z_{n-\lfloor nt \rfloor} = 0 \mid \Pi)^{z-1}\big] \\ &= z\, \mathbb{E}\big[e^{-S_{n-\lfloor nt \rfloor}}\mathbb{P}(Z_{n-\lfloor nt \rfloor} > 0 \mid \Pi)^2\mathbb{P}(Z_{n-\lfloor nt \rfloor} = 0 \mid \Pi)^{z-1}\big] \\ &= z\, \gamma^{n-\lfloor nt \rfloor}\mathbf{E}\big[\mathbb{P}(Z_{n-\lfloor nt \rfloor} > 0 \mid \Pi)^2\mathbb{P}(Z_{n-\lfloor nt \rfloor} = 0 \mid \Pi)^{z-1}\big].\end{aligned}$$

By the dominated convergence theorem and monotonicity and as $\mathbf{P}(Z_\infty > 0) > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \mathbf{E}\big[\mathbb{P}(Z_{n-\lfloor nt \rfloor} > 0 \mid \Pi)^2\mathbb{P}(Z_{n-\lfloor nt \rfloor} = 0 \mid \Pi)^{z-1}\big] \\ &\quad = \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] > 0.\end{aligned}$$

Altogether, using Theorem 2.1.1, we get for $n \to \infty$

$$\begin{aligned}&\mathbb{P}(Z_{\lfloor nt \rfloor} = z \mid Z_n = 1) = \frac{\mathbb{P}(Z_{\lfloor nt \rfloor} = z, Z_n = 1)}{\mathbb{P}(Z_n = 1)} \\ &\quad = \frac{\mathbb{P}(Z_{\lfloor nt \rfloor} = z)\mathbb{P}_z(Z_{n-\lfloor nt \rfloor} = 1)}{\gamma^n(\theta + o(1))} \\ &\quad = \frac{\gamma^{\lfloor nt \rfloor}\big(\mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\big] + o(1)\big)z\gamma^{n-\lfloor nt \rfloor}\big(\mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] + o(1)\big)}{\gamma^n(\vartheta + o(1))}.\end{aligned}$$

Recall that $\vartheta = \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\big]$. Taking the limit $n \to \infty$ yields

$$\lim_{n\to\infty} \mathbb{P}(Z_{\lfloor nt \rfloor} = z \mid Z_n = 1) = z\, \mathbf{E}\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] =: r_z. \tag{5.3}$$

In particular, this distribution does not depend on t. Summing over $z \in \mathbb{N}$, as for $x \in [0, 1)$,

$$\sum_{z=0}^{\infty} z x^{z-1} = \frac{1}{(1-x)^2}$$

and $\mathbb{P}(Z_\infty = 0 \mid \Pi) = 1 - \mathbb{P}(Z_\infty > 0 \mid \Pi)$ we see that (5.3) is indeed a probability distribution on $\mathbb{N}$. □

6. Proof of theorems in the intermediately subcritical case

Define for $0 \le k \le n$

$$L_{k,n} = \min_{0\le j\le n-k} (S_{k+j} - S_k)\ . \tag{6.1}$$

Note that with this definition

$$\{\tau_n = k\} = \{\tau_k = k, L_{k,n} \ge 0\}. \tag{6.2}$$

Due to independence of the increments, $L_{k,n}$ and L_{n-k} have the same distribution.

First, we show that Z_n being small implies that with a high probability, the associated random walk attains its minimum at some early generation.

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Lemma 2. *For every $\varepsilon > 0$ and $z \in \mathbb{N}$ there is an $m = m(\varepsilon) \in \mathbb{N}$ such that*

$$\mathbb{P}(Z_n = z, \tau_n > m) < \varepsilon \gamma^n \mathbf{P}(L_n \geq 0)$$

Proof. Decomposing at the minimum and using (3.3) with $H_n \leq 1$ yields

$$\begin{aligned}\mathbb{P}(Z_n = z, \tau_n > m) &= \sum_{k=m+1}^{n} \mathbb{E}\big[\mathbb{P}(Z_n = z \mid \Pi); \tau_n = k\big] \\ &\leq \sum_{k=m+1}^{n} \mathbb{E}\big[\mathbb{P}(Z_n = 1 \mid \Pi); \tau_n = k\big] \\ &= \sum_{k=m+1}^{n} \mathbb{E}\big[e^{-S_n}\mathbb{P}(Z_n > 0 \mid \Pi)^2; \tau_n = k\big].\end{aligned}$$

Next using the standard estimate (1.3) and (6.2) and the change of measure, we get that

$$\begin{aligned}\mathbb{P}(Z_n = z, \tau_n > m) &\leq \sum_{k=m+1}^{n} \mathbb{E}\big[e^{-S_n + 2L_n}; \tau_k = k, L_{k,n} \geq 0\big] \\ &= \gamma^n \sum_{k=m+1}^{n} \mathbf{E}\big[e^{L_n}; \tau_k = k, L_{k,n} \geq 0\big] \\ &= \gamma^n \sum_{k=m+1}^{n} \mathbf{E}\big[e^{S_k}; \tau_k = k\big]\mathbf{P}(L_{n-k} \geq 0).\end{aligned}$$

Using [5][Lemma 2.2] with $u(x) = e^{-x}$, we get for $m \in \mathbb{N}$ large enough

$$\sum_{k=m+1}^{n} \mathbf{E}\big[e^{S_k}; \tau_k = k\big]\mathbf{P}(L_{n-k} \geq 0) < \varepsilon\, \mathbf{P}(L_n \geq 0).$$

This yields the claim. □

In particular, we have proved that for every $\varepsilon > 0$, if m is large enough

$$\limsup_{n\to\infty} \mathbf{P}(L_n \geq 0)^{-1} \sum_{k=m+1}^{n} \mathbf{E}\big[\mathbb{P}(Z_k > 0 \mid \Pi)^2; \tau_n = k\big] < \varepsilon. \tag{6.3}$$

6.1. *Proof of Theorem 2.2.1*

In the following proofs, we will require the shift of the environment. Let for $\Pi = (Q_1, Q_2, \ldots)$ and $k \in \mathbb{N}$ denote the environment shifted by k generations by

$$\theta_k \circ \Pi = (Q_{k+1}, Q_{k+2}, \ldots).$$

The main idea of the following lemma is to use results for critical BPRE, i.e. [5][Lemma 4.1].

Lemma 3. *Under Assumption 1, there is a positive and finite constant θ such that as $n \to \infty$*

$$\mathbf{E}[\mathbb{P}(Z_n > 0 \mid \Pi)^2] \sim \theta\, \mathbf{P}(L_n \geq 0).$$

Proof. Following [5], we will first define a suitable sequence of uniformly bounded random variables which satisfy the conditions of [5][Lemma 4.1]. In the second step, we will apply [5][Lemma 4.1] and get the result of Lemma 3 after a short calculation.

To avoid any confusion, we adopt the notation from [5][Lemma 4.1]. Let $I_{Z_n>0}$ be the indicator function of the event $\{Z_n > 0\}$. Then for $n \in \mathbb{N}$

$$V_n := \mathbb{P}(Z_n > 0 \mid \Pi) \cdot I_{Z_n>0} \quad \text{a.s.},$$

forms a uniformly bounded sequence of random variables $(V_n)_{n\in\mathbb{N}}$ adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Next noting that $Z_n > 0$ implies $Z_k > 0$ and using (3.5), we get random variables adapted to $(\mathcal{F}_k)_{k\in\mathbb{N}}$

$$\begin{aligned} V_{k,n} &:= \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi) \cdot I_{Z_n>0};\, Z_k > 0 \mid L_{k,n} \geq 0, \mathcal{F}_k\big] \\ &= \mathbf{E}\big[\big(1 - f_{0,k}\big(f_{k,n}(0)\big)\big) \cdot I_{Z_n>0} \mid L_{k,n} \geq 0, \mathcal{F}_k\big] \quad \text{a.s.} \end{aligned}$$

Both $1 - f_{k,n}(0) = \mathbb{P}(Z_n > 0 \mid \Pi, Z_k = 1)$ a.s. and $I_{Z_n>0}$ (given Z_k) are bounded, nonnegative and nonincreasing in n and thus converge $\mathbf{P}^+$-a.s. More precisely, given $\mathcal{F}_k$,

$$I_{Z_n>0} \overset{n\to\infty}{\longrightarrow} I_{Z_\infty>0} \quad \mathbf{P}^+ - \text{a.s.}$$

where $\mathbb{P}^+$ acts on $\theta_k \circ \Pi$. Moreover, given $\mathcal{F}_k$

$$\mathbb{P}(Z_n > 0 \mid \Pi) = 1 - f_{0,k}\big(f_{k+1,n}(0)\big) \overset{n\to\infty}{\longrightarrow} 1 - f_{0,k}\Big(P_\infty^k\Big) \quad \mathbf{P}^+ - \text{a.s.}$$

where we defined

$$P_\infty^k := \mathbb{P}(Z_\infty = 0 \mid \theta_k \circ \Pi) \quad \text{a.s.} \tag{6.4}$$

Thus the conditions of Lemma 1 are met and as $n \to \infty$

$$V_{k,n} \to V_\infty(Z_k, f_{0,k}) \quad \text{a.s.},$$

where for $z \in \mathbb{N}_0$ and $g \in \mathcal{C}_b\big([0, 1]\big)$

$$\begin{aligned} V_\infty(z, g) &= \mathbf{E}_z^+\big[\big(1 - g\big(P_\infty\big)\big) \cdot I_{Z_\infty>0}\big] \\ &= \mathbf{E}^+\big[\big(1 - g\big(P_\infty\big)\big) \cdot \mathbb{E}_z[I_{Z_\infty>0} \mid \Pi]\big] \\ &= \mathbf{E}^+\big[\big(1 - g\big(P_\infty\big)\big) \cdot \mathbb{P}_z(Z_\infty > 0 \mid \Pi)\big]. \end{aligned}$$

Next, recall that by [5][Lemma 2.1], for fixed k and as $n \to \infty$, $\mathbf{P}(L_n \geq 0) \sim \mathbf{P}(L_{k,n} \geq 0)$. Consequently,

$$\begin{aligned} &\mathbf{E}[V_n;\, Z_k > 0, L_{k,n} \geq 0 \mid \mathcal{F}_k] \\ &\quad = \mathbf{P}(L_{k,n} \geq 0)\mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi) \cdot I_{Z_n>0};\, Z_k > 0 \mid L_{k,n} \geq 0, \mathcal{F}_k\big] \\ &\quad = \mathbf{P}(L_n \geq 0)\big(V_\infty(Z_k, f_{0,k}) + o(1)\big) \quad \text{a.s.} \end{aligned}$$

and the conditions of [5][Lemma 4.1] are met for $m = 0$. Thus by [5][Lemma 4.1], we get that

$$\mathbf{E}[V_n;\, Z_{\tau_n} > 0] = \mathbf{P}(L_n \geq 0)\Big(\sum_{k=0}^{\infty} \mathbf{E}[V_\infty(Z_k, f_{0,k});\, \tau_k = k] + o(1)\Big). \tag{6.5}$$

Noting that $\{Z_n > 0\}$ implies $\{Z_{\tau_n} > 0\}$ and conditioning on the environment yields

$$\begin{aligned} \mathbf{E}[V_n;\, Z_{\tau_n} > 0] &= \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi) \cdot I_{Z_n>0};\, Z_{\tau_n} > 0\big] \\ &= \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi) \cdot I_{Z_n>0}\big] = \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2\big]. \end{aligned}$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Inserting this into (6.5), we get that

$$\mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2\big] = \mathbf{E}[V_n; Z_{\tau_n} > 0] = \big(\theta + o(1)\big)\mathbf{P}(L_n \geq 0),$$

where

$$\theta := \sum_{k=0}^{\infty} \mathbf{E}[V_\infty(Z_k, f_{0,k}); \tau_k = k]. \tag{6.6}$$

Clearly, $V_\infty(Z_k, f_{0,k}) \leq \mathbf{P}^+_{Z_k}(Z_\infty > 0)$, and thus by [5][Equation (4.10)], the sum on the right-hand side is convergent. As it is proved in [5][Proposition 3.1], $P_\infty = \mathbb{P}_z(Z_\infty = 0 \mid \Pi) < 1$ $\mathbf{P}^+$-a.s. for all $z \geq 1$. As for $s < 1$ and $f_{0,k}(0) < 1$

$$f_{0,k}(s) < f_{0,k}(1) = 1$$

this proves $\theta > 0$. □

Proof of Theorem 2.2.1. Using the change of measure and the explicit formula for $\mathbb{P}(Z_n = 1 \mid \Pi)$ in the case of linear fractional offspring distributions, we get that

$$\mathbb{P}(Z_n = 1) = \mathbb{E}\big[e^{-S_n}\mathbb{P}(Z_n > 0 \mid \Pi)^2\big] = \gamma^n\, \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2\big]. \tag{6.7}$$

The theorem now results from Lemma 3. □

There is another representation of θ. Using $\mathbb{P}_z(Z_\infty > 0 \mid \Pi) = 1 - \mathbb{P}_1(Z_\infty = 0 \mid \Pi)^z$ yields

$$\begin{aligned} V_\infty(z, g) &= \mathbf{E}^+\big[\big(1 - g\big(P_\infty\big)\big)\mathbb{P}_z(Z_\infty > 0 \mid \Pi)\big] \\ &= \mathbf{E}^+\big[\big(1 - g\big(P_\infty\big)\big)\big(1 - (P_\infty)^z\big)\big]. \end{aligned}$$

Taking into account the definition of generating functions and applying Fubini's theorem to interchange the expectations (note that $\mathbf{E}^+$ only acts on the shifted environment $\theta_k \circ \Pi$, i.e. P^k_∞ whereas $\mathbf{E}$ only acts on Z_k and $f_{0,k}$), we get the following representation of θ.

$$\begin{aligned} \mathbf{E}[V_\infty(Z_k, f_{0,k}); \tau_k = k] &= \mathbf{E}\Big[\mathbf{E}^+\Big[(1 - f_{0,k}(P^k_\infty))\mathbf{E}^+\big[(1 - (P^k_\infty)^{Z_k}) \mid \theta_k \circ \Pi\big]\Big]; \tau_k = k\Big] \\ &= \mathbf{E}\Big[\mathbf{E}^+\Big[\big(1 - f_{0,k}(P^k_\infty)\big)\big(1 - f_{0,k}(P^k_\infty)\big)\Big]; \tau_k = k\Big] \\ &= \mathbf{E}\big[\mathbf{E}^+\big[(1 - f_{0,k}(P^k_\infty))^2\big]; \tau_k = k\big]. \end{aligned} \tag{6.8}$$

6.2. *Proof of Theorem 2.2.2*

Lemma 4. *For every $\varepsilon > 0$ there is an $m = m(\varepsilon) \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \mathbf{P}(L_n \geq 0)^{-1}\mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2\big|\big] < \varepsilon.$$

Proof. We decompose according to τ_n and let $0 \leq l \leq n$. Then

$$\begin{aligned} &\mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2\big|\big] \\ &\quad\leq \sum_{k=0}^{n} \mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2\big|; \tau_n = k\big] \end{aligned}$$

$$= \sum_{k=0}^{l} \mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2\big|;\, \tau_n = k\big]$$
$$+ \sum_{k=l+1}^{n} \mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2\big|;\, \tau_n = k\big]. \tag{6.9}$$

As to the second term, using the fact that $\mathbb{P}(Z_n > 0 \mid \Pi)$ is a.s. decreasing in n, we get that

$$\sum_{k=l+1}^{n} \mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2\big|;\, \tau_n = k\big]$$
$$\leq \sum_{k=l+1}^{n} \mathbf{E}\big[\mathbb{P}(Z_k > 0 \mid \Pi)^2;\, \tau_n = k\big].$$

Using (6.3), if l is chosen large enough,

$$\sum_{k=l+1}^{n} \mathbf{E}\big[\mathbb{P}(Z_k > 0 \mid \Pi)^2;\, \tau_n = k\big] \leq \frac{\varepsilon}{2}\, \mathbf{P}(L_n \geq 0). \tag{6.10}$$

For the first term in (6.9), we condition on the environment up to generation k and get that

$$\sum_{k=0}^{l} \mathbf{E}[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2\big|;\, \tau_n = k] \tag{6.11}$$
$$\leq \sum_{k=0}^{l} \mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2\big|;\, \tau_k = k,\, L_{k,n} \geq 0\big]$$
$$= \sum_{k=0}^{l} \mathbf{E}\big[\mathbf{E}\big[\big|\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2\big|;\, L_{k,n} \geq 0 \mid \mathcal{F}_k\big];\, \tau_k = k\big]$$
$$= \sum_{k=0}^{l} \mathbf{E}[\psi(f_{0,k}, m, n-k);\, \tau_k = k]\mathbf{P}(L_{n-k} \geq 0), \tag{6.12}$$

where for $g \in \mathcal{C}_b\big([0,1]\big)$, the space of bounded and continuous functions on $[0, 1]$, we define

$$\psi(g, m, n) := \mathbf{E}\Big[\Big|\big(1 - g\big(f_{0,n}(0)\big)\big)^2 - \big(1 - g\big(f_{0,m}(0)\big)\big)^2\Big| \mid L_n \geq 0\Big].$$

Note that in (6.12), the expectation in ψ acts on the shifted environment $\theta_k \circ \Pi$.
As $n \to \infty$, $f_{0,n}(0) \to \mathbb{P}(Z_\infty = 0 \mid \Pi)$ a.s. Using Lemma 1 then yields

$$\lim_{n\to\infty} \psi(g, m, n) = \mathbf{E}^+\Big[\Big|\big(1 - g\big(P_\infty\big)\big)^2 - \big(1 - g\big(f_{0,m}(0)\big)\big)^2\Big|\Big].$$

As $m \to \infty$, $f_{0,m}(0) \to P_\infty$ a.s. and thus by the dominated convergence theorem, the above term tends to zero as $m \to \infty$. Applying this and $\mathbf{P}(L_{n-k} \geq 0) \sim \mathbf{P}(L_n \geq 0)$, we get that

$$\sum_{k=0}^{l} \big|\mathbf{E}[\mathbb{P}(Z_n > 0 \mid \Pi)^2 - \mathbb{P}(Z_{\tau_n+m} > 0 \mid \Pi)^2;\, \tau_n = k]\big|$$
$$\leq \sum_{k=0}^{l} \mathbf{E}[\psi(f_{0,k}, m, n-k);\, \tau_k = k]\, \mathbf{P}(L_n \geq 0) \leq \frac{\varepsilon}{2}\mathbf{P}(L_n \geq 0) \tag{6.13}$$

if m is large enough. This completes the proof. □

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Proof of Theorem 2.2.2. The proof follows the same lines as the proof of [5][Theorem 1.6]. Let ϕ be a bounded and continuous function on $D[0,1]$, the space of càdlàg functions on $[0,1]$. Recall that

$$S^n = \left(S^n_t\right)_{0\le t\le 1} = \frac{l(n)}{n^{1/\alpha}}\left(S_{\lfloor nt\rfloor}\right)_{0\le t\le 1}.$$

Then considering the change of measure to $\mathbf{P}$, (3.4) and Theorem 2.2.1, as $n\to\infty$

$$\mathbb{E}[\phi(S^n)\mid Z_n=1] = \frac{\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_n>0\mid \Pi)^2\big]}{\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]} \sim \frac{1}{\theta}\frac{\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_n>0\mid \Pi)^2\big]}{\mathbf{P}(L_n\ge 0)}.$$

Thus it is enough to prove that as $n\to\infty$

$$\Big|\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_n>0\mid \Pi)^2\big]-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]\Big| = o\big(\mathbf{P}(L_n\ge 0)\big).$$

By the triangle inequality and Lemma 4, for every $\varepsilon>0$, if $m\in\mathbb{N}$ is large enough

$$\begin{aligned}
&\Big|\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_n>0\mid \Pi)^2\big]-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]\Big|\\
&\quad\le \Big|\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{\tau_n+m}>0\mid \Pi)^2\big]-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]\Big|\\
&\qquad+\sup|\phi|\mathbf{E}\Big[\big|\mathbb{P}(Z_n>0\mid \Pi)^2\big]-\mathbb{P}(Z_{\tau_n+m}>0\mid \Pi)^2\big|\Big]\\
&\quad= \Big|\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{\tau_n+m}>0\mid \Pi)^2\big]-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]\Big|+\varepsilon\mathbf{P}(L_n\ge 0).
\end{aligned}$$

Next, we are going to prove that the first term can be bounded by $\varepsilon\mathbf{P}(L_n\ge 0)$ if m is large enough. We decompose according to the time of the minimum. By (6.3), for every $\varepsilon>0$ and for $l\in\mathbb{N}$ large enough, and as $\mathbb{P}(Z_n>0\mid \Pi)$ is non-increasing in n,

$$\begin{aligned}
&\Big|\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{\tau_n+m}>0\mid \Pi)^2\big]-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2\big]\Big|\\
&\quad\le \Big|\sum_{k=0}^{l}\Big(\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{k+m}>0\mid \Pi)^2;\tau_n=k\big]\\
&\qquad-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2;\tau_n=k\big]\Big)\Big|\\
&\qquad+2\sup|\phi|\mathbf{E}\big[\mathbb{P}(Z_{k+m}>0\mid \Pi)^2;\tau_n>l\big]\\
&\quad\le \Big|\sum_{k=0}^{l}\Big(\mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{k+m}>0\mid \Pi)^2;\tau_n=k\big]\\
&\qquad-\mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n>0\mid \Pi)^2;\tau_n=k\big]\Big)\Big|\\
&\qquad+\varepsilon\mathbf{P}(L_n\ge 0).
\end{aligned}$$

Next we decompose the process S^n according to generation k, i.e. let for $0\le k\le n$

$$S^{k,n}_t = n^{-1/\alpha}l(n)\ S_{\lfloor nt\rfloor\wedge k}$$

and

$$\overline{S}^{k,n}_t = n^{-1/\alpha}l(n)\ (S_{\lfloor nt\rfloor}-S_{\lfloor nt\rfloor\wedge k}).$$

Thus

$$S^n = S^{k,n}+\overline{S}^{k,n}.$$

Recall the definition of $L_{k,n} = \min_{0\le j\le n-k}(S_{k+j} - S_k)$ from (6.1). Next, note that

$$\begin{aligned}\sum_{k=0}^{l} &\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n})\mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2; \tau_n = k\big] \\ &= \sum_{k=0}^{l} \mathbf{E}\Big[\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,n} \ge 0 \mid \mathcal{F}_{k+m}\big]\mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2; \tau_k = k\Big].\end{aligned} \tag{6.14}$$

Conditioning on the environment yields

$$\begin{aligned}&\mathbf{E}\Big[\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,n} \ge 0 \mid \mathcal{F}_{k+m}\big]\mathbb{P}(Z_{k+m} > 0|\Pi)\cdot I_{Z_{k+m}>0}; \tau_k = k\Big] \\ &\quad = \mathbf{E}\Big[\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,n} \ge 0 \mid \mathcal{F}_{k+m}\big]\mathbb{P}(Z_{k+m} > 0|\Pi)^2; \tau_k = k\Big].\end{aligned} \tag{6.15}$$

Set for $w \in D[0,1]$ and $x \ge 0$

$$\psi(w,x) := \mathbf{E}\big[\phi(w + \overline{S}^{k+m,n}); L_{k+m,n} \ge -x\big].$$

Note that (see [5][Proof of Theorem 1.5])

$$\{L_{k,n} \ge 0\} = \{L_{k,k+m} \ge 0\} \cap \{L_{k+m,n} \ge -(S_{k+m} - S_k)\}.$$

Thus, we may rewrite

$$\begin{aligned}&\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,n} \ge 0 \mid \mathcal{F}_{k+m}\big] \\ &\quad = \mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,k+m} \ge 0, L_{k+m,n} \ge -(S_{k+m} - S_k) \mid \mathcal{F}_{k+m}\big] \\ &\quad = \psi(S^{k+m,n}, S_{k+m} - S_k)\, I_{L_{k,k+m}\ge 0} \quad \text{a.s.}\end{aligned}$$

Using [5][Lemma 2.3] and [5][Lemma 2.1] yields (where the expectation in ψ is taken with respect to the shifted environment $\theta_{k+m} \circ \Pi$)

$$\begin{aligned}\psi(w,x) &= \mathbf{E}\big[\phi(w + \overline{S}^{k+m,n}); L_{k+m,n} \ge -x\big] \\ &= \mathbf{P}(L_{k+m,n} \ge -x)\big(\mathbf{E}\big[\phi(w + L^+)\big] + o(1)\big) \\ &= u(x)\,\mathbf{P}(L_n \ge 0)\big(\mathbf{E}\big[\phi(w + L^+)\big] + o(1)\big).\end{aligned} \tag{6.16}$$

Also note that for fixed k and m, $S^{k+m,n}$ converges uniformly to 0 as $n \to \infty$ a.s. and that ϕ is continuous and bounded. Using this and (6.16), we get that a.s.

$$\begin{aligned}&\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n}); L_{k,n} \ge 0 \mid \mathcal{F}_{k+m}\big] \\ &\quad = u(S_{k+m} - S_k)\mathbf{P}(L_n \ge 0) I_{L_{k,k+m}\ge 0}\big(\mathbf{E}[\phi(L^+)] + o(1)\big).\end{aligned}$$

Inserting this into (6.14) and using (6.15) yields

$$\begin{aligned}\sum_{k=0}^{l} &\mathbf{E}\big[\phi(S^{k+m,n} + \overline{S}^{k+m,n})\mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2; \tau_n = k\big] \\ &= \big(\mathbf{E}\big[\phi(L^+)\big] + o(1)\big)\mathbf{P}(L_n \ge 0) \\ &\quad \times \sum_{k=0}^{l} \mathbf{E}\Big[u(S_{k+m} - S_k)\mathbb{P}(Z_{k+m} > 0 \mid \Pi)\cdot I_{Z_{k+m}>0}; \tau_k = k, L_{k,k+m} \ge 0\Big].\end{aligned}$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Finally, by the definition of $\mathbf{P}^+$, we have (recall $u(0) = 1$)

$$\mathbf{E}\Big[u(S_{k+m} - S_k)\mathbb{P}(Z_{k+m} > 0 \mid \Pi) \cdot I_{Z_{k+m}>0};\ \tau_k = k,\ L_{k,k+m} \geq 0\Big]$$
$$= \mathbf{E}\Big[\mathbf{E}^+\Big[\mathbb{P}(Z_{k+m} > 0 \mid \Pi) \cdot I_{Z_{k+m}>0} \mid \mathcal{F}_k\Big];\ \tau_k = k\Big]$$
$$\stackrel{m\to\infty}{\longrightarrow} \mathbf{E}\big[V_\infty(Z_k,\ f_{0,k});\ \tau_k = k\big],$$

where $V_\infty(Z_k,\ f_{0,k})$ is defined in Lemma 3. Inserting all this yields

$$\sum_{k=0}^{l} \mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2;\ \tau_n = k\big]$$
$$= \big(\mathbf{E}\big[\phi(L^+)\big] + o(1)\big)\mathbf{P}(L_n \geq 0) \sum_{k=0}^{l}\Big(\mathbf{E}\big[V_\infty(Z_k,\ f_{0,k});\ \tau_k = k\big] + o(1)\Big).$$

On the other hand, by Lemmas 3 and 4,

$$\mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2;\ \tau_n = k\big] = \mathbf{P}(L_n \geq 0) \sum_{k=0}^{l}\Big(\mathbf{E}\big[V_\infty(Z_k,\ f_{0,k});\ \tau_k = k\big] + o(1)\Big).$$

Thus for every $\varepsilon > 0$ if l and m are large enough,

$$\Big|\sum_{k=0}^{l} \mathbf{E}\big[\phi(S^n)\mathbb{P}(Z_{k+m} > 0 \mid \Pi)^2;\ \tau_n = k\big]$$
$$- \mathbf{E}\big[\phi(L^+)\big]\mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2;\ \tau_n = k\big]\Big| \leq \varepsilon \mathbf{P}(L_n \geq 0).$$

This proves the theorem. □

6.3. *Proof of Theorems 2.2.3 and 2.2.4*

The following lemma describes the probability that the process has some value, conditioned on a favorable environment.

Lemma 5. *For every $z, k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \mathbf{E}\big[e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \mid L_n \geq 0\big] = z\mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big],$$

where the limit does not depend on k.

Proof. We will prove the lemma by induction with respect to z.

For $z = 1$, the explicit formula for the probability (3.3) yields

$$\lim_{n\to\infty} \mathbf{E}\big[e^{S_n}\mathbb{P}(Z_n = k \mid \Pi) \mid L_n \geq 0\big] = \lim_{n\to\infty} \mathbf{E}\big[\mathbb{P}(Z_n > 0 \mid \Pi)^2 H_n^{k-1} \mid L_n \geq 0\big].$$

Recall that

$$H_n = \frac{\sum_{k=0}^{n-1} \eta_{k+1}e^{-S_k}}{e^{-S_n} + \sum_{k=0}^{n-1} \eta_{k+1}e^{-S_k}}.$$

Under $\mathbf{P}^+$, $S_n \to \infty$ a.s. and thus $e^{-S_n} \to 0$ a.s. Consequently $H_n \to 1$ $\mathbf{P}^+$-a.s. and as $n \to \infty$

$$e^{S_n}\mathbb{P}(Z_n = k \mid \Pi) \to \mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \quad \mathbf{P}^+ - \text{a.s.}$$

Using this together with Lemma 1 and $H_n \le 1$ yields for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbf{E}\big[e^{S_n}\mathbb{P}(Z_n = k \mid \Pi) \mid L_n \ge 0\big] = \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\big],$$

which completes the proof for $z = 1$.

Let us now assume that for $z \in \mathbb{N}$

$$e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \to z\,\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1} \quad \mathbf{P}^+ - \text{a.s.}$$

and thus

$$\lim_{n\to\infty} \mathbf{E}\big[e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \mid L_n \ge 0\big] = z\,\mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big].$$

Then starting from $z+1$-many individuals, Z_n is the sum of $z+1$-many independent and identically distributed random variables. Thus a.s.

$$\begin{aligned} e^{S_n}\mathbb{P}_{z+1}(Z_n = k \mid \Pi) &= e^{S_n}\sum_{j=0}^{k}\mathbb{P}(Z_n = j \mid \Pi)\mathbb{P}_z(Z_n = k - j \mid \Pi) \\ &= \mathbb{P}(Z_n = 0 \mid \Pi)e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) + \sum_{j=1}^{k-1}\mathbb{P}(Z_n = j \mid \Pi)e^{S_n}\mathbb{P}_z(Z_n = k - j \mid \Pi) \\ &\quad + e^{S_n}\mathbb{P}(Z_n = k \mid \Pi)\mathbb{P}_z(Z_n = 0 \mid \Pi). \end{aligned}$$

For the first summand, by assumption of the induction

$$e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \xrightarrow{n\to\infty} z\,\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0)^{z-1}$$

$\mathbf{P}^+$-a.s. for every $k > 0$. Under $\mathbf{P}^+$, $\mathbb{P}(Z_n = 0 \mid \Pi) \to \mathbb{P}(Z_\infty = 0 \mid \Pi)$. Thus as $n \to \infty$, $\mathbf{P}^+$ a.s.

$$\mathbb{P}(Z_n = 0 \mid \Pi)e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \to z\,\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^z.$$

As to the second part, again $e^{S_n}\mathbb{P}_z(Z_n = k - j \mid \Pi)$ converges $\mathbf{P}^+$ a.s. for every $k - j > 0$. Note that $e^{S_n}\mathbb{P}(Z_n = j \mid \Pi)$ converges $\mathbf{P}^+$-a.s. and thus, as $S_n \to \infty$ $\mathbf{P}^+$-a.s., we have $\mathbb{P}(Z_n = j \mid \Pi) \to 0$ $\mathbf{P}^+$ a.s. Consequently, as there are only finitely many summands,

$$\sum_{j=1}^{k-1}\mathbb{P}(Z_n = j \mid \Pi)e^{S_n}\mathbb{P}_z(Z_n = k - j \mid \Pi) \xrightarrow{n\to\infty} 0 \quad \mathbf{P}^+ - \text{a.s.}$$

For the last summand, note that

$$\mathbb{P}_z(Z_n = 0 \mid \Pi) = \mathbb{P}(Z_n = 0 \mid \Pi)^z \xrightarrow{n\to\infty} \mathbb{P}(Z_\infty = 0 \mid \Pi)^z$$

and consequently, as $n \to \infty$, $\mathbf{P}^+$-a.s.

$$e^{S_n}\mathbb{P}(Z_n = k \mid \Pi)\mathbb{P}_z(Z_n = 0 \mid \Pi) \to \mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_n = 0 \mid \Pi)^z.$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Putting this together and applying Lemma 1, we get that

$$\lim_{n\to\infty} \mathbf{E}\big[e^{S_n}\mathbb{P}_{z+1}(Z_n = k \mid \Pi) \mid L_n \geq 0\big]$$
$$= z\,\mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^z\big] + \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^z\big]$$
$$= (z+1)\,\mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^z\big].$$

This ends up the induction. □

Proof of Theorem 2.2.3. Fix $c \in \mathbb{N}$. Then for $1 \leq k \leq c$

$$\mathbb{P}(Z_n = k \mid 1 \leq Z_n \leq c) = \frac{\mathbb{P}(Z_n = k)}{\sum_{j=1}^{c} \mathbb{P}(Z_n = j)} ends.$$

Next, using the change of measure and the decomposition according to the global minimum, for every $0 \leq m \leq n$,

$$\mathbb{P}(Z_n = k) = \gamma^n \sum_{i=0}^{m} \mathbf{E}\big[e^{S_n}\mathbb{P}(Z_n = k \mid \Pi); \tau_i = i, L_{i,n} \geq 0\big] + \mathbb{P}(Z_n = k, \tau_n > m).$$

Let $\varepsilon > 0$. By Lemma 2 and Theorem 2.2.1, the second term can be bounded by $\varepsilon\mathbb{P}(Z_n = 1)$ for m large enough and as $n \to \infty$. Examining the first term, we get that

$$\gamma^n \sum_{i=0}^{m} \mathbf{E}\big[e^{S_n}\mathbb{P}(Z_n = k \mid \Pi); \tau_i = i, L_{i,n} \geq 0\big]$$
$$= \gamma^n \sum_{i=0}^{m} \mathbf{E}\big[e^{S_i}e^{S_n - S_i}\mathbb{P}(Z_n = k \mid \Pi); \tau_i = i, L_{i,n} \geq 0\big]$$
$$= \gamma^n \sum_{i=0}^{m} \mathbf{E}\big[e^{S_i}\psi(Z_i, n-i); \tau_i = i\big]\mathbf{P}(L_{n-i} \geq 0),$$

where

$$\psi(z, n) = \mathbf{E}\big[e^{S_n}\mathbb{P}_z(Z_n = k \mid \Pi) \mid L_n \geq 0\big].$$

The expectation in ψ is taken with respect to the shifted environment $\theta_i \circ \Pi$. Note that as $n \to \infty$, using Lemma 5,

$$\psi(z, n) \to z\,\mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2\mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big].$$

This term does not depend on k as $n \to \infty$. Thus for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbb{P}(Z_n = 1) / \lim_{n\to\infty} \mathbb{P}(Z_n = k) = 1.$$

Theorem 2.2.3 immediately results from this. □

Our next result concerns the time of the prospective global minima. For convenience, we shorten $\tau_{\lfloor nt \rfloor}, Z_{\lfloor nt \rfloor}, S_{\lfloor nt \rfloor}$ to $\tau_{nt}, Z_{nt}, S_{nt}$, i.e. we drop $\lfloor\cdot\rfloor$ in the indices.

Lemma 6. *For every $\varepsilon > 0$ and $t \in (0, 1)$, there is an $m \in \mathbb{N}$ such that*

$$\mathbf{P}(\tau_{nt,n} > \lfloor nt \rfloor + m \mid L_n \geq 0) \leq \varepsilon.$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

Proof. The main idea is to apply [5][Lemma 2.2]. First note that decomposing at time $\lfloor nt\rfloor$ and by independence,

$$\begin{aligned}&\mathbf{P}(\tau_{nt,n} > \lfloor nt\rfloor + m, L_n \geq 0)\\&\quad = \int_0^\infty \mathbf{P}(\tau_{n-nt} > m, L_{n-nt} \geq -x)\mathbf{P}(S_{nt} \in dx, L_{nt} \geq 0)\, dx.\end{aligned}$$

Next, we can rewrite

$$\begin{aligned}\mathbf{P}(\tau_{n-nt} > m, L_{n-nt} \geq -x) &= \sum_{k=m+1}^{n-\lfloor nt\rfloor} \mathbf{P}(\tau_k = k, S_k \geq -x)\mathbf{P}(L_{n-nt-k} \geq 0)\\&= \sum_{k=m+1}^{n-\lfloor nt\rfloor} \mathbf{E}\big[u(-S_k); \tau_k = k\big]\mathbf{P}(L_{n-nt-k} \geq 0),\end{aligned}$$

where $u(y) := 1_{y\leq x}$. Obviously, u is nonnegative, nonincreasing in y and $\int_0^\infty u(y)dy = x < \infty$. Thus all conditions of [5][Lemma 2.2] are met. Applying this lemma yields for every x, every $\varepsilon > 0$ and if m is large enough

$$\sum_{k=m+1}^{n-\lfloor nt\rfloor} \mathbf{E}\big[u(-S_k); \tau_k = k\big]\mathbf{P}(L_{n-nt-k} \geq 0) \leq \varepsilon\mathbf{P}(L_{n-nt} \geq 0).$$

Thus we get that

$$\begin{aligned}\mathbf{P}(\tau_{nt,n} > \lfloor nt\rfloor + m, L_n \geq 0) &\leq \varepsilon \int_0^\infty \mathbf{P}(S_{nt} \in dx, L_{nt} \geq 0)\mathbf{P}(L_{n-nt} \geq 0)\, dx\\&= \varepsilon\mathbf{P}(L_{nt} \geq 0)\mathbf{P}(L_{n-nt} \geq 0) \leq \varepsilon\mathbf{P}(L_n \geq 0).\end{aligned}$$

For the last step, note that $\{L_{nt} \geq 0\} \cap \{L_{nt,n} \geq 0\}$ implies $\{L_n \geq 0\}$. Thus, as the two terms in the first event are independent, we get that

$$\mathbf{P}(L_{nt} \geq 0)\mathbf{P}(L_{n-nt} \geq 0) \leq \mathbf{P}(L_n \geq 0). \qquad \square$$

Proof of Theorem 2.2.4. Let $0 \leq m \leq n$ and $t \in (0, 1)$. Decomposing according to the global minimum yields

$$\begin{aligned}\mathbb{P}\big(Z_{\tau_{nt,n}} = z, Z_n = 1\big) &= \sum_{k=0}^{m} \mathbb{P}\big(Z_{\tau_{nt,n}} = z, Z_n = 1, \tau_n = k\big)\\&\quad + \mathbb{P}\big(Z_{\tau_{nt,n}} = z, Z_n = 1, \tau_n > m\big), \qquad (6.17)\end{aligned}$$

By Lemma 2 and Theorem 2.2.1, for every $\varepsilon > 0$ and m large enough,

$$\limsup_{n\to\infty} \mathbb{P}(Z_n = 1)^{-1}\mathbb{P}\big(Z_{\tau_{nt,n}} = z, Z_n = 1, \tau_n > m\big) \leq \varepsilon.$$

Thus we only have to consider the first sum on the right-hand side in (6.17). Let $m < \ell < n - \lfloor nt\rfloor$ be specified later. Then we get that

$$\begin{aligned}&\sum_{k=0}^{m} \mathbb{P}\big(Z_{\tau_{nt,n}} = z, Z_n = 1, \tau_n = k\big)\\&\quad = \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{n-\lfloor nt\rfloor} \mathbf{E}\big[e^{S_n}\mathbb{P}(Z_{nt+j} = z, Z_n = 1 \mid \Pi);\end{aligned}$$

$$
\begin{aligned}
&\quad \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} = \lfloor nt \rfloor + j\big] \\
&= \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{\ell} \mathbf{E}\big[e^{S_{nt+j}} e^{S_n - S_{nt+j}} \mathbb{P}(Z_{nt+j} = z, Z_n = 1 \mid \Pi); \\
&\quad \times \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} = \lfloor nt \rfloor + j\big] + \gamma^n \sum_{k=0}^{m} \sum_{j=\ell+1}^{n-\lfloor nt \rfloor} \mathbf{E} \\
&\quad \times \big[e^{S_n} \mathbb{P}(Z_{nt+j} = z, Z_n = 1 \mid \Pi); \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} = \lfloor nt \rfloor + j\big] \\
&=: s_1 + s_2.
\end{aligned} \tag{6.18}
$$

In view of $\mathbb{P}(Z_{nt+j} = z, Z_n = 1 \mid \Pi) \leq \mathbb{P}(Z_n = 1 \mid \Pi) \leq e^{-S_n}$ a.s. (see this special property of generalized geometric offspring in (3.3)), the second summand can be bounded by

$$
\begin{aligned}
s_2 &\leq \gamma^n \sum_{k=0}^{m} \sum_{j=\ell+1}^{n-\lfloor nt \rfloor} \mathbf{P}\big(\tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} = \lfloor nt \rfloor + j\big) \\
&= \gamma^n \sum_{k=0}^{m} \mathbf{P}\big(\tau_k = k\big) \mathbf{P}(L_{n-k} \geq 0) \mathbf{P}(\tau_{nt-k,n-k} > \lfloor nt \rfloor + \ell - k \mid L_{n-k} \geq 0\big).
\end{aligned}
$$

Using Lemma 6 yields for every fixed $m \in \mathbb{N}$ and $\varepsilon > 0$, if $\ell = \ell(m, \varepsilon)$ is large enough,

$$
\begin{aligned}
s_2 &\leq \frac{\varepsilon}{\sum_{k=0}^{m} \mathbf{P}\big(\tau_k = k\big)} \gamma^n \sum_{k=0}^{m} \mathbf{P}\big(\tau_k = k\big) \mathbf{P}(L_{n-k} \geq 0) \\
&\leq \frac{\varepsilon}{\sum_{k=0}^{m} \mathbf{P}\big(\tau_k = k\big)} \gamma^n \mathbf{P}(L_{n-m} \geq 0) \sum_{k=0}^{m} \mathbf{P}(\tau_k = k) \\
&\leq \varepsilon\, \gamma^n \mathbf{P}(L_{n-m} \geq 0).
\end{aligned}
$$

In the second step, we have used the fact that $\mathbf{P}(L_n \geq 0)$ is decreasing in n.
As $\mathbf{P}(L_{n-m} \geq 0) \sim \mathbf{P}(L_n \geq 0)$ as $n \to \infty$ and using Theorem 2.2.1, we get that

$$
s_2 \leq \varepsilon \mathbb{P}(Z_n = 1).
$$

Thus s_2 may be neglected as $n \to \infty$ and then $m \to \infty$.
Let us turn to the term s_1. First note that the event $\{\tau_{nt,n} = j\}$ can be written as

$$
\{\tau_{nt,n} = j\} = \{\tau_{nt,j} = j\} \cap \{L_{j,n} \geq 0\}.
$$

Moreover,

$$
\{\tau_{nt,n} = j, L_n \geq 0\} = \{\tau_{nt,j} = j, L_j \geq 0\} \cap \{L_{j,n} \geq 0\},
$$

where both events are independent. Conditioning on $\mathcal{F}_{nt+j}$, we get that

$$
\begin{aligned}
s_1 = \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{\ell} &\mathbf{E}\big[e^{S_{nt+j}} \mathbb{P}(Z_{nt+j} = z \mid \Pi); \tau_k = k, L_{k,nt+j} \geq 0, \tau_{nt,nt+j} = \lfloor nt \rfloor + j\big] \\
&\cdot \mathbf{E}\big[e^{S_{n-nt-j}} \mathbb{P}_z(Z_{n-nt-j} = 1 \mid \Pi); L_{n-nt-j} \geq 0\big]
\end{aligned}
$$

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

$$= \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{\ell} \mathbf{E}\big[e^{S_{nt+j}} \mathbb{P}(Z_{nt+j} = z \mid \Pi); \tau_k = k, L_{k,nt+j} \geq 0, \tau_{nt,nt+j} = \lfloor nt \rfloor + j\big]$$

$$\cdot \mathbf{E}\big[e^{S_{n-nt-j}} \mathbb{P}_z(Z_{n-nt-j} = 1 \mid \Pi) | L_{n-nt-j} \geq 0\big] \mathbf{P}(L_{n-nt-j} \geq 0). \tag{6.19}$$

Next, we use the explicit formula for the probability in (3.3) and get that a.s.

$$e^{S_{nt+j}} \mathbb{P}(Z_{nt+j} = z \mid \Pi) = \mathbb{P}(Z_{nt+j} > 0 \mid \Pi)^2 H_{nt+j}^{z-1}.$$

Using Lemma 5 for $k = 1$, as $n \to \infty$

$$\begin{aligned} &\mathbf{E}\big[e^{S_{n-nt-j}} \mathbb{P}_z(Z_{n-nt-j} = 1 \mid \Pi) \mid L_{n-nt-j} \geq 0\big] \\ &\quad = z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] + o(1). \end{aligned}$$

Inserting this into (6.19) yields

$$\begin{aligned} s_1 &= \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{\ell} \mathbf{E}\big[\mathbb{P}(Z_{nt+j} > 0 \mid \Pi)^2 H_{nt+j}^{z-1}; \tau_k = k, L_{k,nt+j} \geq 0, \tau_{nt,nt+j} = \lfloor nt \rfloor + j\big] \\ &\quad \cdot \Big(z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] + o(1)\Big) \mathbf{P}(L_{n-nt-j} \geq 0) \\ &= \gamma^n \sum_{k=0}^{m} \sum_{j=0}^{\ell} \mathbf{E}\big[\mathbb{P}(Z_{nt+j} > 0 \mid \Pi)^2 H_{nt+j}^{z-1}; \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} = \lfloor nt \rfloor + j\big] \\ &\quad \cdot \Big(z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] + o(1)\Big) \\ &= \gamma^n \sum_{k=0}^{m} \mathbf{E}\big[\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)^2 H_{\tau_{nt,n}}^{z-1}; \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} \leq \lfloor nt \rfloor + \ell\big] \\ &\quad \cdot \Big(z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big] + o(1)\Big). \end{aligned}$$

Note that always $\tau_{nt,n} \geq \tau_n$. In the first factor, we can condition on $\{L_{k,n} \geq 0\}$ and get that

$$\begin{aligned} &\mathbf{E}\big[\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)^2 H_{\tau_{nt,n}}^{z-1}; \tau_k = k, L_{k,n} \geq 0, \tau_{nt,n} \leq \lfloor nt \rfloor + \ell\big] \\ &\quad = \mathbf{E}\Big[\mathbf{E}\big[\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)^2 H_{\tau_{nt,n}}^{z-1}, \tau_{nt,n} \leq \lfloor nt \rfloor \\ &\qquad + \ell \mid L_{k,n} \geq 0, \mathcal{F}_k\big]; \tau_k = k\Big] \mathbf{P}(L_{n-k} \geq 0) \\ &\quad \sim \mathbf{P}(L_n \geq 0) \mathbf{E}\Big[\mathbf{E}\big[\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)^2 H_{\tau_{nt,n}}^{z-1}; \tau_{nt,n} \leq \lfloor nt \rfloor + \ell \mid L_{k,n} \geq 0, \mathcal{F}_k\big]; \tau_k = k\Big]. \end{aligned}$$

By [5][Proof of Lemma 2.7], $\tau_{nt,n} \to \infty$ $\mathbf{P}^+$-a.s. as $n \to \infty$. Thus, $\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)$ converges a.s. with respect to $\mathbf{P}^+$. Moreover, as under $\mathbf{P}^+$, $S_n \to \infty$ and $S_{\tau_{nt,n}} \to \infty$ as $n \to \infty$, we get $H_{\tau_{nt,n}}^{z-1} \to 1$ $\mathbf{P}^+$-a.s. Using this, Lemmas 1 and 6, we get a.s. as $n \to \infty$

$$\mathbf{E}\big[\mathbb{P}(Z_{\tau_{nt,n}} > 0 \mid \Pi)^2 H_{\tau_{nt,n}}^{z-1}; \tau_{nt,n} \leq \lfloor nt \rfloor + \ell \mid L_{k,n} \geq 0, \mathcal{F}_k\big] \to \mathbf{E}^+\big[(1 - f_{0,k}(P_\infty^k))^2\big],$$

where P_∞^k has been defined in (6.4) and the expectation only acts on the shifted environment $\theta_k \circ \Pi$. We can now formulate the limit. As s_2 and thus the corresponding probability on the event $\{\tau_{nt,n} > nt + m\}$ can be neglected as $n \to \infty$ and if $m \to \infty$, we get that

$$\begin{aligned}
&\mathbb{P}(Z_{\tau_{nt,n}} = z, Z_n = 1) \\
&\quad \sim z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big]\gamma^n \mathbf{P}(L_n \geq 0) \\
&\qquad \times \sum_{k=0}^{\infty} \mathbf{E}\big[\mathbf{E}^+\big[(1 - f_{0,k}(P_\infty^k))^2\big]; \tau_k = k\big].
\end{aligned}$$

Together with Theorem 2.2.1 and the formula for θ in (6.8), we get that as $n \to \infty$

$$\lim_{n\to\infty} \mathbb{P}\big(Z_{\tau_{nt,n}} = z \mid Z_n = 1\big) = z\, \mathbf{E}^+\big[\mathbb{P}(Z_\infty > 0 \mid \Pi)^2 \mathbb{P}(Z_\infty = 0 \mid \Pi)^{z-1}\big].$$

As proved in Theorem 2.1.3, this is indeed a probability distribution on $\mathbb{N}$. □

Uncited references

Q2

[2], [3], [4], [11], [12] and [23].

Acknowledgment

The author thanks Vincent Bansaye from Ecole Polytechnique, Palaiseau for fruitful discussions on this topic. The author also would like to thank the anonymous referees for their useful comments and improvements.

References

[1] V.I. Afanasyev, Limit theorems for a conditional random walk and some applications. Diss. Cand. Sci., Moscow, MSU, 1980.

[2] V.I. Afanasyev, Limit theorems for an intermediately subcritical and a strongly subcritical branching process in a random environment, Discrete Math. Appl. 11 (2001) 105–131.

[3] V.I. Afanasyev, Ch. Böinghoff, G. Kersting, V.A. Vatutin, Limit theorems for weakly subcritical branching processes in random environment, J. Theoret. Probab. (2010).

[4] V.I. Afanasyev, Ch. Böinghoff, G. Kersting, V.A. Vatutin, Limit theorems for intermediately subcritical branching
Q3 processes in random environment, Ann. I.H. Poincaré (B) (2012) http://dx.doi.org/10.1214/13-AIHP538. in press.

[5] V.I. Afanasyev, J. Geiger, G. Kersting, V.A. Vatutin, Criticality for branching processes in random environment, Ann. Probab. 33 (2005) 645–673.

[6] V.I. Afanasyev, J. Geiger, G. Kersting, V.A. Vatutin, Functional limit theorems for strongly subcritical branching processes in random environment, Stochastic Process. Appl. 115 (2005) 1658–1676.

[7] A. Agresti, On the extinction times of varying and random environment branching processes, J. Appl. Probab. 12 (1975) 39–46.

[8] K.B. Athreya, S. Karlin, On branching processes with random environments: I, II, Ann. Math. Stat. 42 (1971) 1499–1520, 1843–1858.

[9] V. Bansaye, C. Böinghoff, Small values for supercritical branching processes in random environment, Ann. I.H. Poincaré (B) (2012) in press.

[10] V. Bansaye, C. Böinghoff, Lower large deviations for supercritical branching processes in random environment, Proc. Steklov Inst. Math. 282 (2013) 15–34.

[11] J. Bertoin, Lévy Processes, Cambridge University Press, Cambridge, 1996.

[12] J. Bertoin, R.A. Doney, On conditioning a random walk to stay non-negative, Ann. Probab. 22 (1994) 2152–2167.

[13] M. Birkner, J. Geiger, G. Kersting, Branching processes in random environment—a view on critical and subcritical cases. Proceedings of the DFG-Schwerpunktprogramm, in: Interacting Stochastic Systems of High Complexity, Springer, Berlin, 2005, pp. 265–291.

[14] F.M. Dekking, On the survival probability of a branching process in a finite state i.i.d. environment, Stochastic Process. Appl. 27 (1988) 151–157.

[15] C. Böinghoff, G. Kersting, Simulations and a conditional limit theorem for intermediately subcritical branching processes in random environment, Proc. Steklov Inst. Math. 282 (2013) 45–61.

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009

[16] R.A. Doney, Conditional limit theorems for asymptotically stable random walks, Z. Wahrscheinlichkeitstheor. 1
Verwandte Geb. 70 (1985) 351–360.

[17] R. Durrett, Condiotend limit theorems for some null recurrent Markov processes, Ann. Probab. 6 (1978) 798–828. 2

[18] J. Geiger, G. Kersting, V.A. Vatutin, Limit theorems for subcritical branching processes in random environment, 3
Ann. Inst. Henry Poincare (B) 39 (2003) 593–620.

[19] M. Hutzenthaler, Supercritical branching diffusions in random environment, Electron. Commun. Probab. 16 (2011) 4
781–791.

[20] M.V. Kozlov, On large deviations of branching processes in a random environment: geometric distribution of 5
descendants, Discrete Math. Appl. 16 (2006) 155–174.

[21] M. Nakashima, Lower deviations of branching processes in random environment with geometrical offspring 6
distributions, Stochastic Process. Appl. 123 (2013) 3560–3587.

[22] W.L. Smith, W.E. Wilkinson, On branching processes in random environments, Ann. Math. Stat. 40 (1969) 7
814–827.

[23] H. Tanaka, Time reversal of random walks in one dimension, Tokyo J. Math. 12 (1989) 159–174. 8

[24] V.A. Vatutin, A limit theorem for an intermediate subcritical branching process in a random environment, Theory 9
Probab. Appl. 48 (2004) 481–492.

[25] V.A. Vatutin, E.E. Dyakonova, Galton–Watson branching processes in random environment. I: limit theorems, 10
Theory Probab. Appl. 48 (2004) 314–336.

[26] V.A. Vatutin, E.E. Dyakonova, Galton–Watson branching processes in random environment. II: finite-dimensional 11
distributions, Theory Probab. Appl. 49 (2005) 275–308.

[27] V.A. Vatutin, E.E. Dyakonova, Branching processes in random environment and the bottlenecks in the evolution of 12
populations, Theory Probab. Appl. 51 (2007) 189–210.

Please cite this article in press as: C. Böinghoff, Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions, Stochastic Processes and their Applications (2014), http://dx.doi.org/10.1016/j.spa.2014.05.009