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Iterated random functions and slowly varying tails

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Abstract

Consider a sequence of i.i.d. random Lipschitz functions $\{\Psi_n\}_{n\geq 0}$. Using this sequence we can define a Markov chain via the recursive formula $R_{n+1} = \Psi_{n+1}(R_n)$. It is a well known fact that under some mild moment assumptions this Markov chain has a unique stationary distribution. We are interested in the tail behaviour of this distribution in the case when $\Psi_0(t) \approx A_0 t + B_0$. We will show that under subexponential assumptions on the random variable $\log^+(A_0 \vee B_0)$ the tail asymptotic in question can be described using the integrated tail function of $\log^+(A_0 \vee B_0)$. In particular we will obtain new results for the random difference equation $R_{n+1} = A_{n+1} R_n + B_{n+1}$.

MSC: 60H25; 60J10

Keywords: Stochastic recursions; Random difference equation; Stationary distribution; Subexponential distributions

1. Introduction

Consider a sequence of independent identically distributed (i.i.d.) random Lipschitz functions $\{\Psi_n\}_{n\geq 0}$, where $\Psi_n: \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$. Using this sequence we can define a Markov chain via the recursive formula

$$R_{n+1} = \Psi_{n+1}(R_n) \quad \text{for } n \geq 0, \tag{1.1}$$

where $R_0 \in \mathbb{R}$ is arbitrary but independent of the sequence $\{\Psi_n\}_{n\geq 0}$. Put $\Psi = \Psi_0$. We are interested in the existence and properties of the stationary distribution of the Markov chain

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$\{R_n\}_{n\geq 0}$, that is the solution of the stochastic fixed point equation

$$R \stackrel{d}{=} \Psi(R) \quad R \text{ independent of } \Psi, \tag{1.2}$$

where the distribution of random variable R is the stationary distribution of the Markov chain $\{R_n\}_{n\geq 0}$.

The main example, we have in mind, is the random difference equation, where Ψ is an affine transformation, that is $\Psi_n(t) = A_n t + B_n$ with $\{(A_n, B_n)\}_{n\geq 0}$ being an i.i.d. sequence of two-dimensional random vectors. Then the formula (1.1) can be written as

$$R_{n+1} = A_{n+1}R_n + B_{n+1} \quad \text{for } n \geq 0. \tag{1.3}$$

Put $(A, B) = (A_0, B_0)$. It is a well known fact that if

$$\mathbb{E}[\log|A|] < 0 \quad \text{and} \quad \mathbb{E}[\log^+|B|] < \infty,$$

then the Markov chain $\{R_n\}_{n\geq 0}$ given by (1.3) has a unique stationary distribution which can be represented as the distribution of the random variable

$$R = \sum_{n\geq 0} B_{n+1} \prod_{k=1}^{n} A_k, \tag{1.4}$$

for details see [28]. Random variables of this form can be found in analysis of probabilistic algorithms or financial mathematics, where R would be called a perpetuity. Such random variables occur also in number theory, combinatorics, as a solution to stochastic fixed point equation

$$R \stackrel{d}{=} AR + B \quad R \text{ independent of } (A, B), \tag{1.5}$$

atomic cascades, random environment branching processes, exponential functionals of Lévy processes, Additive Increase Multiplicative Decrease algorithms [17], COGARCH processes [22], and more. A variety of examples for possible applications of R can be found in [14,15,11].

From the application point of view, the key information is the behaviour of the tail of R, that is

$$\mathbb{P}[R > x] \quad \text{as } x \to \infty.$$

This problem was investigated by various authors, for example by Goldie and Grübel [14] and in a similar setting by Hitczenko and Wesołowski [18]. The first result says that if B is bounded, $\mathbb{P}[A \in [0, 1]] = 1$ and the distribution of A behaves like the uniform distribution in the neighbourhood of 1, then R given by (1.4) has thin tail, more precisely $\log \mathbb{P}[R \geq x] \sim -cx\log(x)$. Recall that for two positive functions $f(\cdot)$ and $g(\cdot)$, by $f(x) \sim g(x)$ we mean that $\lim_{x\to\infty} f(x)/g(x) = 1$. In this paper we are only interested in limits as $x \to \infty$, so from now we omit the specification of the limit.

There is also the result of Kesten [20] and later on, in the same setting, of Goldie [13]. The essence of this result is that under Cramér's condition, that is if $\mathbb{E}[|A|^\alpha] = 1$ for some $\alpha > 0$ such that $\mathbb{E}[|B|^\alpha] < \infty$, the tail of R is regularly varying, i.e. $\mathbb{P}[R > x] \sim cx^{-\alpha}$ for some positive and finite constant c and R defined by (1.4).

Finally, the result of Grincevičius [16], which was later generalised by Grey [15], states that in the case of positive A if for some $\alpha > 0$ we have $\mathbb{E}[A^\alpha] < 1$ and $\mathbb{P}[B > x] \sim x^{-\alpha}L(x)$, where L is slowly varying (that is $L(cx) \sim L(x)$ for any positive c), then the tail of R is again regularly varying, in fact $\mathbb{P}[R > x] \sim cx^{-\alpha}L(x)$. Note that in this case the tail of perpetuity R exhibits the same rate of decay as the tail of the input, that is $\mathbb{P}[R > x] \sim c\mathbb{P}[B > x]$.

However, in the case when $\mathbb{P}[A > x]$ or $\mathbb{P}[B > x]$ is a slowly varying function of x, up to our knowledge, little is known about the behaviour of $\mathbb{P}[R > x]$ as $x \to \infty$. This is the problem we consider in the present paper.

The case of general fixed point equation (1.2) was studied by Goldie [13], where several particular forms of the transformation Ψ were treated. Later Mirek [24] found the tail asymptotic of the solution of (1.2) with Ψ being Lipschitz such that $\Psi(t) \approx \mathrm{Lip}(\Psi)t$, where $\mathrm{Lip}(\Psi)$ is the Lipschitz constant. The result says that if $\mathbb{E}[\log(\mathrm{Lip}(\Psi))] < 0$ and $\mathbb{E}[\mathrm{Lip}(\Psi)^\alpha] = 1$ for some $\alpha > 0$, then R solving (1.2) exhibits regularly varying tail $\mathbb{P}[|R| > x] \sim cx^\alpha$. Grey [15] also treated generalised fixed point equations (1.2) in the setting introduced by Grincevičius [16].

It turns out that the assumption $\mathbb{E}[\log(\mathrm{Lip}(\Psi))] < 0$ is necessary for the existence of the probabilistic solutions of (1.2). For the existence and asymptotic behaviour of the invariant measure of the Markov chain (1.1) in the critical case, that is $\mathbb{E}[\log(\mathrm{Lip}(\Psi))] = 0$, see [1,6,5,4].

This paper gives an answer to the question about asymptotic of $\mathbb{P}[R > x]$, where R solves (1.2), in the case of slowly varying input. Assuming that the Lipschitz function Ψ satisfies

$$At + B - D \le \Psi(t) \le At^+ + B^+ + D \quad \text{for } t \in \mathbb{R},$$

with $D > 0$ being relatively small and $A > 0$, we will show that under subexponential assumptions on the random variable $\log(A \vee B)$ one has

$$\mathbb{P}[R > x] \asymp \int_{\log(x)}^{\infty} \mathbb{P}[\log(A \vee B) > y]\mathrm{d}y.$$

Recall that for two positive functions $f(\cdot)$, $g(\cdot)$ by $f(x) \asymp g(x)$ we mean that $g(x) = O(f(x))$ and $f(x) = O(g(x))$. Furthermore, in our setting, the integral expression on the right hand side will be a slowly varying function of x. Moreover in several cases we will establish a precise tail asymptotic of R. In order to obtain full description of tail behaviour for the sequence $\{R_n\}_{n\ge 0}$ we will study finite time horizon. We will show that if distribution of $\log(A \vee B)$ is subexponential, then it holds true that

$$\mathbb{P}[R_n > x] \asymp n\mathbb{P}[A \vee B > x],$$

where $\{R_n\}_{n\ge 0}$ in given by (1.1).

The main result gives description of tail asymptotic of the solution of (1.5) and also

$$R \overset{d}{=} AR^+ + B \quad R \text{ independent of } (A, B),$$

which is closely related to the ruin probability, for details see [9]. We can also obtain a description of the solutions to

$$R \overset{d}{=} A_1|R| + \sqrt{D + A_2R^2} \quad R \text{ independent of } (A_1, A_2, D),$$

where $\mathbb{P}[D > x] = o\left(\mathbb{P}\left[A_1 + \sqrt{A_2} > x\right]\right)$. This corresponds to an autoregressive process with ARCH(1) errors, which was described by Borkovec and Klüppelberg [3]. To find the behaviour of $\mathbb{P}[|R| > x]$ just take $\Psi(t) = |A_1t + \sqrt{D + A_2(t^+)^2}|$.

The paper is organised as follows: In Section 2 we will briefly recall basic definitions and properties of subexponential distributions, after that in Section 3 we will present a precise statement of the result followed by some remarks and sketch of the proof. Finally, in Section 4, we will give the full proof of the results.

2. Subexponential distributions

In this section we will recall well known notions from the theory of heavy-tailed distributions. Next we will quote a theorem about tail behaviour of a maxima of perturbed random walk, which will be particularly useful in the proof of the main result. Firstly, for a distribution F on $\mathbb{R}$ we define tail function $\overline{F}$ by the formula $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$.

Definition 2.1. A distribution F on $\mathbb{R}$ is called long-tailed if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$ and for any fixed $y \in \mathbb{R}$

$$\overline{F}(x+y) \sim \overline{F}(x).$$

We denote the class of long-tailed distributions by $\mathcal{L}$.

Notice that if $F \in \mathcal{L}$ then the function $x \mapsto \overline{F}(\log(x))$ is slowly varying as $x \to \infty$. Therefore one can use Potter's Theorem (see [2, Theorem 1.5.6]) to obtain the following corollary.

Corollary 2.2. *If $F \in \mathcal{L}$, then for any chosen $\Delta > 1$ and $\delta > 0$ there exists $X = X(\Delta, \delta)$ such that*

$$\frac{\overline{F}(x)}{\overline{F}(y)} \le \Delta e^{\delta|x-y|} \quad \text{for } x, y \ge X.$$

It turns out that class $\mathcal{L}$ is too big for our purposes. More precisely, we will need distributions satisfying some convolution properties. Recall that F^{*2} stands for the twofold convolution of the distribution F.

Definition 2.3. A distribution F on $\mathbb{R}$ is called subexponential if $F \in \mathcal{L}$ and

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x).$$

The class of subexponential distributions will be denoted by $\mathcal{S}$.

Note that if X_1 and X_2 are i.i.d. with distribution $F \in \mathcal{S}$, then by the definition above

$$\mathbb{P}[X_1 + X_2 > x] \sim 2\mathbb{P}[X_1 > x] \sim \mathbb{P}[X_1 \vee X_2 > x].$$

This is a type of phenomena that we want to use in the near future. We see that $\mathcal{S} \subset \mathcal{L}$ and it is a well known fact that this inclusion is proper. For examples of distributions in $\mathcal{L} \setminus \mathcal{S}$ see [10] or [26]. The following proposition is a well known fact which will be useful thought the proofs of the results. We follow the statement presented in [12].

Proposition 2.4. *Suppose that $F \in \mathcal{S}$. Let $G_1, \ldots, G_n$ be distributions such that $\overline{G}_i(x) \sim c_i \overline{F}(x)$ for some constants $c_i \ge 0$, $i = 1, \ldots, n$. Then*

$$\overline{G_1 * \cdots * G_n}(x) \sim (c_1 + \cdots + c_n)\overline{F}(x).$$

If $c_1 + \cdots + c_n > 0$, then $G_1 * \cdots * G_n \in \mathcal{S}$.

The following theorem by Palmowski and Zwart [25] is crucial for our future purposes. The result itself deals with i.i.d. sequence $\{(X_n, Y_n)\}_{n\ge 0}$, where X_n are i.i.d. increments of negatively driven stochastic process and Y_n being maxima of this process taken at some renewal epochs. Nevertheless the result and the proof presented in [25] remains valid for arbitrary i.i.d. sequence $\{(X_n, Y_n)\}_{n\ge 0}$.

Theorem 2.5. *Let $\{(X_n, Y_n)\}_{n\ge 0}$ be a sequence of i.i.d. two-dimensional random vectors such that $\mathbb{E}[X_1] < 0$ and $\mathbb{E}[X_1 \vee Y_1] < \infty$. Assume that distribution on $\mathbb{R}_+$ given by the tail function*

$$x \mapsto 1 \wedge \int_x^\infty \mathbb{P}[X_1 \vee Y_1 > y]\mathrm{d}y$$

is subexponential. Then

$$\mathbb{P}\left[\sup_{n\ge 0}\left\{Y_{n+1} + \sum_{j=1}^{n} X_j\right\} > x\right] \sim -\frac{1}{\mathbb{E}[X_1]}\int_x^\infty \mathbb{P}[X_1 \vee Y_1 > y]\mathrm{d}y.$$

The $\mathbb{R}_+$ in the theorem above and for the rest of the paper stands for $[0, +\infty)$. For conditions on F guaranteeing subexponentiality of distribution given by the tail function $x \mapsto 1 \wedge \int_x^\infty \overline{F}(y)\mathrm{d}y$ see [21].

3. Main result

In this section we will give a precise statement of the main result of the paper followed by some remarks and idea behind the proof.

3.1. Statement

Recall that we consider a Markov chain $\{R_n\}_{n\ge 0}$ given by (1.1), where for each $n \in \mathbb{N}$ the function $\Psi_n : \mathbb{R} \to \mathbb{R}$ satisfies

$$A_n t + B_n - D_n \le \Psi_n(t) \le A_n t^+ + B_n^+ + D_n \quad \text{for } t \in \mathbb{R} \tag{3.1}$$

and some random variables A_n, B_n and D_n. We are assuming that $\{(\Psi_n, A_n, B_n, D_n)\}_{n\ge 0}$ are i.i.d., where Ψ_n are Lipschitz functions with

$$\mathrm{Lip}(\Psi_n) = \sup_{t_1\ne t_2}\left|\frac{\Psi_n(t_1) - \Psi_n(t_2)}{t_1 - t_2}\right|. \tag{3.2}$$

Put $(\Psi, A, B, D) = (\Psi_0, A_0, B_0, D_0)$. From now our standing assumptions will be

$$\begin{aligned} &A, D \ge 0 \quad \text{a.s.}, \ \mathbb{E}[\log(A)] > -\infty, \qquad \mathbb{E}[\log(\mathrm{Lip}(\Psi))] < 0,\\ &\qquad \mathbb{E}[\log^+ |B \pm D|] < \infty. \end{aligned} \tag{3.3}$$

Recall that $\log^+(x) = \log(x \vee 1)$. Note that (3.1) implies

$$A_n \le \mathrm{Lip}(\Psi_n)$$

and hence $\mathbb{E}[\log(A)] < 0$. For infinite time horizon, that is the case of the stationary distribution, we will also need to assume

$$\mathbb{E}[\log^+(A \vee B)^{1+\gamma}] < \infty \quad \text{for some } \gamma > 0. \tag{3.4}$$

In order to ensure that the stationary distribution has right-unbounded support we will need to assume the following tail behaviour

$$\begin{aligned} &\mathbb{P}[A \vee (B \pm D) > x] \sim \mathbb{P}[A \vee B > x],\\ &\qquad \mathbb{P}[A > x, B - D \le -x] = o(\mathbb{P}[A \vee B > x]). \end{aligned} \tag{3.5}$$

Define a probability distribution F_I on $\mathbb{R}_+$ via its tail function $\overline{F}_I$ which is given by

$$\overline{F}_I(x) = 1 \wedge \int_x^\infty \mathbb{P}[\log(A \vee B) > y]\mathrm{d}y. \tag{3.6}$$

Theorem 3.1. *Assume that conditions (3.1), (3.3), (3.4) and (3.5) are satisfied and that F_I defined by (3.6) is subexponential. Then the Markov chain $\{R_n\}_{n\geq 0}$ given by (1.1) converges in distribution to a unique stationary distribution which is a unique solution of (1.2). Furthermore*

$$-\frac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]} \leq \liminf_{x\to\infty} \frac{\mathbb{P}[R > x]}{\overline{F}_I(\log(x))} \leq \limsup_{x\to\infty} \frac{\mathbb{P}[R > x]}{\overline{F}_I(\log(x))} \leq -\frac{1}{\mathbb{E}[\log(A)]}. \tag{3.7}$$

In particular, if $B - D > 0$ a.s., then

$$\mathbb{P}[R > x] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_{\log(x)}^\infty \mathbb{P}[\log(A \vee B) > y]\mathrm{d}y. \tag{3.8}$$

Moreover, if

- $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$ *then*
$$\mathbb{P}[R > x] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_{\log(x)}^\infty \mathbb{P}[\log^+(B) > y]\mathrm{d}y,$$
- $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$ *then*
$$\mathbb{P}[R > x] \sim -\frac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]} \int_{\log(x)}^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y.$$

Since in last two cases of the above theorem we obtain $\mathbb{P}[R > x] \sim c\overline{F}_I(\log(x))$ with $F_I \in \mathcal{S} \subseteq \mathcal{L}$ and some constant c we see that in each case the distribution of R exhibits slowly varying tail.

Remark 3.2. From the proof of Theorem 3.1 one can see that in order to establish the lower bound in (3.7) one only uses the fact that the distribution of the random variable $A \vee B$ has a slowly varying tail. Precisely, assume (3.1), (3.3), (3.5) and that the function $x \mapsto \mathbb{P}[A \vee B > x]$ is slowly varying. Then

$$-\frac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]} \leq \liminf_{x\to\infty} \frac{\mathbb{P}[R > x]}{\overline{F}_I(\log(x))}$$

where the function $\overline{F}_I$ is given by (3.6). Since when $F_I \in \mathcal{L}$ it is true that $\mathbb{P}[A \vee B > x] = \mathbb{P}[\log(A \vee B) > \log(x)] = o(\overline{F}_I(\log(x)))$ and we can also conclude that $\mathbb{P}[A \vee B > x] = o(\mathbb{P}[R > x])$.

In order to obtain an extensive description of the asymptotic properties of the Markov chain $\{R_n\}_{n\geq 0}$ given by (1.1) we will also investigate the tail behaviour of random variables R_n for finite n. Put

$$\overline{F}(x) = \mathbb{P}\left[\log(A \vee B) > x\right]. \tag{3.9}$$

It turns out that in case of finite time horizon one can obtain result analogous to Theorem 3.1.

Theorem 3.3. *Assume (3.1), (3.3), (3.5), $0 \le n < \infty$ and that F defined by (3.9) is subexponential. Assume additionally that*

$$\mathbb{P}[R_0 > x] \sim w\mathbb{P}[A \vee B > x] \tag{3.10}$$

for some constant $w \ge 0$. Then

$$w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0] \le \liminf_{x\to\infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \vee B > x]} \le \limsup_{x\to\infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \vee B > x]} \le w + n. \tag{3.11}$$

In particular if $R_0 > 0$ a.s. and $B - D > 0$ a.s. then

$$\mathbb{P}[R_n > x] \sim (w + n)\mathbb{P}[A \vee B > x].$$

Furthermore if

- $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$ *then*

$$\mathbb{P}[R_n > x] \sim (w + n)\mathbb{P}[B > x], \tag{3.12}$$

- $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$ *then*

$$\mathbb{P}[R_n > x] \sim \left(w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0]\right) \mathbb{P}[A > x]. \tag{3.13}$$

Remark 3.4. Assume (3.10), (3.1), (3.5), (3.3), $0 \le n < \infty$, and that the function $x \mapsto \mathbb{P}[A \vee B > x]$ is slowly varying. Then

$$w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0] \le \liminf_{x\to\infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \vee B > x]}.$$

3.2. Random difference equation

Suppose, for the rest of this section, that $\Psi(t) = At + B$ and $D = 0$. In the case when $B > 0$ a.s., Theorem 3.1 gives a description of the tail of R in terms of the distribution of $A \vee B$, which allows us to present an example showing that in the case when $\mathbb{P}[A > x] \sim \mathbb{P}[B > x]$, the information about marginal distributions of A and B is not enough to determine the tail asymptotic of R.

Example 3.5. Fix a distribution F on $\mathbb{R}_+$ and consider two types of input: First one $\left(A^{(1)}, B^{(1)}\right)$: with $A^{(1)} = B^{(1)}$ with distribution F. Then, assuming that the assumptions are satisfied, Theorem 3.1 states in (3.8) that the corresponding perpetuity $R^{(1)}$ satisfies

$$\mathbb{P}\left[R^{(1)} > x\right] \sim -\frac{1}{\mathbb{E}\left[\log\left(A^{(1)}\right)\right]} \int_{\log(x)}^{\infty} \mathbb{P}\left[\log\left(A^{(1)}\right) > y\right] \mathrm{d}y.$$

If now we consider the second type of input, namely $\left(A^{(2)}, B^{(2)}\right)$ where $A^{(2)}$, $B^{(2)}$ are independent with the same distribution F, Theorem 3.1 states that the corresponding perpetuity $R^{(2)}$ satisfies

$$\mathbb{P}\left[R^{(2)} > x\right] \sim -\frac{1}{\mathbb{E}\left[\log\left(A^{(2)}\right)\right]} \int_{\log(x)}^{\infty} \mathbb{P}\left[\log\left(A^{(2)} \vee B^{(2)}\right) > y\right] \mathrm{d}y$$

and since $A^{(1)} \stackrel{d}{=} A^{(2)}$ we can write

$$\mathbb{P}\left[A^{(2)} \vee B^{(2)} > x\right] \sim 2\mathbb{P}\left[A^{(2)} > x\right] = 2\mathbb{P}\left[A^{(1)} > x\right]$$

and we see that

$$\mathbb{P}\left[R^{(2)} > x\right] \sim 2\mathbb{P}\left[R^{(1)} > x\right].$$

Even though the marginal distributions of the two types of input are exactly the same, the corresponding perpetuities have different tail asymptotic.

The main result of this paper is closely related to Theorem 4.1 by Maulik and Zwart [23] where so-called exponential functional of Lévy process is treated, i.e. a random variable of the form $\int_0^\infty e^{\xi_s} \mathrm{d}s$ where $\{\xi_s \mid s \geq 0\}$ is a Lévy proses with negative drift. Note that this is a perpetuity corresponding to

$$A = e^{\xi_1} \quad \text{and} \quad B = \int_0^1 e^{\xi_s} \mathrm{d}s.$$

The theorem in question states that

$$\mathbb{P}\left[\int_0^\infty e^{\xi_s} \mathrm{d}s > x\right] \sim -\frac{1}{\mathbb{E}[\xi_1]} \int_{\log(x)}^\infty \mathbb{P}[\xi_1 > y] \mathrm{d}y$$

if $x \mapsto \int_x^\infty \mathbb{P}[\xi_1 > y]\mathrm{d}y$ is subexponential. We see that Theorem 4.1 by Maulik and Zwart [23] is a particular case of the main result of this paper. Next example shows the importance of second condition in (3.5).

Example 3.6. Consider the input (A, B) where $B = \mathbb{1}_{[0,1]}(A) - A$. Assume that $A > 0$ and $\mathbb{E}[\log(A)] < 0$. This ensures the existence of the solution R to

$$R \stackrel{d}{=} AR + \mathbb{1}_{[0,1]}(A) - A \quad R \text{ independent of } A.$$

We see that $\mathbb{P}[B > 0] = \mathbb{P}[A \in [0, 1)] > 0$, but the solution is bounded. Indeed, notice that R also satisfies

$$R - 1 \stackrel{d}{=} A(R - 1) + \mathbb{1}_{[0,1]}(A) - 1 \quad R \text{ independent of } A$$

and so $R - 1$ is a perpetuity obtained from the input $(A, \mathbb{1}_{[0,1]}(A) - 1)$. Since $\mathbb{1}_{[0,1]}(A) - 1 \leq 0$ a.s., we know that $R - 1 \leq 0$ a.s. Whence we can conclude that the perpetuity R obtained from the input (A, B) is bounded above by 1 a.s. This is due to the fact that in this case

$$\mathbb{P}[A > x, B \leq -x] = \mathbb{P}[A > x] = \mathbb{P}[A \vee B > x] \quad \text{for } x > 1.$$

Theorem 3.1 is also related to results from [16,15,27,19] where arising perpetuities exhibit the tail behaviour similar to the tail behaviour of the input. The first one, for example, says that

$$\frac{\mathbb{P}[R > x]}{\mathbb{P}[B > x]} \sim \frac{1}{1 - \mathbb{E}[A^\alpha]}$$

if $\mathbb{E}[A^\alpha] < 1$ and $\mathbb{P}[B > x] \sim x^{-\alpha}L(x)$ for some slowly varying function L and $\alpha > 0$. We see that when $\alpha \to 0$ the constant $(1 - \mathbb{E}[A^\alpha])^{-1}$ tends to infinity. Theorem 3.1 corresponds to the case with $\alpha = 0$ and tells us what is the proper asymptotic. This also gives the reason for the blowup of the constant. By Remark 3.2, $\mathbb{P}[A \vee B > x] = o(\mathbb{P}[R > x])$ and we can write

$$\frac{\mathbb{P}[R > x]}{\mathbb{P}[B > x]} \geq \frac{\mathbb{P}[R > x]}{\mathbb{P}[A \vee B > x]} \to \infty \quad \text{as } x \to \infty.$$

3.3. Idea of the proof

The key problem is to understand the random difference equation, i.e. the case $\Psi(t) = At + B$. For simplicity, we will focus on that case in the following discussion. The convolution property in Definition 2.3 of the subexponential distributions says that for X_1 and X_2 independent with the same distribution $F \in \mathcal{S}$ it is true that $\mathbb{P}[X_1 + X_2 > x] \sim \mathbb{P}[X_1 \vee X_2 > x]$. It turn's out that the series (1.4) exhibits a similar phenomena, more precisely we are able to approximate

$$\mathbb{P}\left[\sum_{n\geq 0} B_{n+1} \prod_{j=1}^{n} A_j > x\right] \quad \text{by using } \mathbb{P}\left[\sup_{n\geq 0}\left\{B_{n+1}\prod_{j=1}^{n} A_j\right\} > x\right].$$

In order to achieve that we apply technique used in [8,7]. This technique revolves around the idea of grouping the terms of the series of the same order and investigating the sizes of the groups. Then, after obtaining the above relation, we can interpret random variable $\sup_{n\geq 0} B_{n+1} \prod_{j=1}^{n} A_j$ as a supremum of a perturbed random walk and use the known theory, namely Theorem 2.5, to derive upper bound for the desired tail asymptotic. Next, adapting some classical techniques, used for example in [25], we get lower bound for tail asymptotic. Roughly speaking, we find relatively big subsets of $\{R > x\}$ on which we have control over the whole sequence $\left\{B_{n+1} \prod_{j=1}^{n} A_j\right\}_{n\geq 0}$.

4. Proof

In this section we will prove the main result of the paper. Recall that we consider an i.i.d. sequence $\{(\Psi_n, A_n, B_n, D_n)\}_{n\geq 0}$ such that $A_n > 0$, $D_n \geq 0$ and

$$A_n t + B_n - D_n \leq \Psi_n(t) \leq A_n t^+ + B_n^+ + D_n \quad \text{for } n \geq 0 \text{ and } t \in \mathbb{R}.$$

Put $(\Psi, A, B, D) = (\Psi_0, A_0, B_0, D_0)$ and let

$$\mu = -\mathbb{E}[\log(A)].$$

Random walk generated by $\log(A)$ will be very useful, hence define

$$S_n = \sum_{j=1}^{n} \log(A_j) \quad \text{for } n \geq 0 \tag{4.1}$$

and

$$\overline{B}_n = (B_n^+ + D_n) \vee 1, \qquad \underline{B}_n = B_n - D_n \quad \text{for } n \geq 0 \tag{4.2}$$

finally let $\overline{B} = \overline{B}_0$, $\underline{B} = \underline{B}_0$. Notice that (3.5) implies

$$\mathbb{P}[A \vee \underline{B} > x] \sim \mathbb{P}[A \vee B > x] \sim \mathbb{P}[A \vee \overline{B} > x].$$

For $k < n$ define the backward iterations of Ψ by

$$\Psi_{k:n}(t) = \Psi_k \circ \Psi_{k+1} \circ \cdots \circ \Psi_n(t).$$

We will use the convention that for $k > n$ $\Psi_{k:n}(t) = t$. For $n \in \mathbb{N}$ we can put

$$\underline{\Psi}_n(t) = A_n t + \underline{B}_n \quad \text{and} \quad \overline{\Psi}_n(t) = A_n t^+ + \overline{B}_n$$

and define $\underline{\Psi}_{k:n}$ and $\overline{\Psi}_{k:n}$ in the same manner as $\Psi_{k:n}$. Notice that using this notation and the bounds on $\Psi_n(t)$, we get

$$\underline{\Psi}_n(t) \le \Psi_n(t) \le \overline{\Psi}_n(t)$$

and since $\overline{\Psi}$ and $\underline{\Psi}$ are monotone by iteration it gives

$$\underline{\Psi}_{k:n}(t) \le \Psi_{k:n}(t) \le \overline{\Psi}_{k:n}(t).$$

In particular

$$\underline{\Psi}_{1:n}(t) = \sum_{k=0}^{n-1} \underline{B}_{k+1} \prod_{j=1}^{k} A_j + t \prod_{j=1}^{n} A_j \le \Psi_{1:n}(t)$$

and

$$\Psi_{1:n}(t) \le \sum_{k=0}^{n-1} \overline{B}_{k+1} \prod_{j=1}^{k} A_j + t^+ \prod_{j=1}^{n} A_j = \overline{\Psi}_{1:n}(t).$$

We will use the following lemma quite often. The proof follows the idea presented in [25].

Lemma 4.1. *Assume (3.3) and for δ, $K > 0$ consider the sets*

$$E_n = E_n(K,\delta) = \{S_j \in (-j(\mu+\delta) - K, -j(\mu-\delta) + K),\ j \le n\} \tag{4.3}$$

and

$$F_n = F_n(K,\delta) = \left\{ \left|\underline{B}_j\right| \le e^{\delta j + K},\ j \le n \right\}. \tag{4.4}$$

Then the following claim holds

$$\forall \delta, \varepsilon > 0\ \exists K > 0 \quad \mathbb{P}\left[\bigcap_{j \ge 1} (E_j \cap F_j) \right] \ge 1 - \varepsilon. \tag{4.5}$$

Proof. For K large enough it is true that $\mathbb{P}\left[\log\left|\underline{B}\right| > K\right] < 1/2$ and since for $y \in (0, 1/2)$ it holds that $\log(1-y) \ge -2y$, we can write

$$\begin{aligned}\log(\mathbb{P}[F_n]) &= \sum_{j=1}^{n} \log\left(1 - \mathbb{P}\left[\log\left|\underline{B}_j\right| > \delta j + K\right]\right) \ge -2 \sum_{j=1}^{n} \mathbb{P}\left[\log\left|\underline{B}\right| > \delta j + K\right] \\ &\ge -2 \sum_{j=1}^{\infty} \mathbb{P}\left[\delta^{-1}\left(\log\left|\underline{B}\right| - K\right) > j\right] \ge -2\delta^{-1} \mathbb{E}\left[\left(\log\left|\underline{B}\right| - K\right)_+\right]\end{aligned}$$

and so $\mathbb{P}[F_n] \to 1$ as $K \to \infty$ uniformly with respect to n since

$$\mathbb{E}\left[\left(\log\left|\underline{B}\right| - K\right)_+\right] = \mathbb{E}\left[\left(\log^+\left|\underline{B}\right| - K\right)_+\right] < \infty.$$

Combining this fact with the strong law of large numbers for the sequence $\{S_n\}_{n \ge 0}$ we observe that we have shown that for any $\varepsilon, \delta > 0$ we can always take $K > 0$ large enough such that

$$\forall n \quad \mathbb{P}[E_n \cap F_n] \ge 1 - \varepsilon$$

and since the sequence of sets $\{E_n \cap F_n\}_{n\geq 0}$ is decreasing in the sense of inclusion, we can conclude that

$$\mathbb{P}\left[\bigcap_{j\geq 1}(E_j \cap F_j)\right] \geq 1-\varepsilon$$

and hence the proof is complete. $\square$

Note that the statement of Lemma 4.1 remains true if we replace $\underline{B}_j$ by $\overline{B}_j$ in the definition of the set F_n. The bounds on Ψ imply that we can bound the solution of (1.2) by two perpetuities, namely

$$\overline{R} = \sum_{n\geq 0} \overline{B}_{n+1} \prod_{k=1}^{n} A_j \tag{4.6}$$

and

$$\underline{R} = \sum_{n\geq 0} \underline{B}_{n+1} \prod_{k=1}^{n} A_j.$$

The main idea of the proof is to approximate $\mathbb{P}\left[\overline{R} > e^x\right]$ by using $\mathbb{P}[M > x]$, where

$$M = \sup_{n\geq 0}\left\{\log\left(\overline{B}_{n+1}\right) + \sum_{j=1}^{n}\log(A_j)\right\}.$$

Since $\overline{B}_1 \geq 1$ we know that $M > 0$ a.s. Furthermore, we have $e^M \leq \overline{R}$ and the last series is convergent a.s by (3.3). Having introduced this notation, we are ready to prove the main theorem.

Proof of Theorem 3.1. Fix large $x \in \mathbb{R}$. The proof consists of five steps.

Step 1: *Existence, uniqueness and representation of the stationary distribution*. Note that

$$R_n \stackrel{d}{=} \Psi_{1:n}(R_0)$$

so in order to prove that $\{R_n\}_{n\geq 0}$ converges in distribution, it is sufficient to show that the sequence $\{\Psi_{1:n}(R_0)\}_{n\geq 0}$ converges a.s. Recall that (3.1) implies

$$A_n \leq \mathrm{Lip}(\Psi_n)$$

also, by the definition (3.2)

$$\mathrm{Lip}(\Psi_{1:m}) \leq \prod_{j=1}^{m}\mathrm{Lip}(\Psi_j) \quad \text{for } m \in \mathbb{N}.$$

For $n \geq m$ and $t_1, t_2 \in \mathbb{R}$ we have

$$\begin{aligned}
|\Psi_{1:n}(t_1) - \Psi_{1:m}(t_2)| &\leq \mathrm{Lip}(\Psi_{1:m})|\Psi_{m+1:n}(t_1) - t_2| \leq \mathrm{Lip}(\Psi_{1:m})\left(|\Psi_{m+1:n}(t_1)| + |t_2|\right)\\
&\leq \mathrm{Lip}(\Psi_{1:m})\left(\overline{\Psi}_{m+1:n}(t_1) \vee |\underline{\Psi}_{m+1:n}(t_1)| + |t_2|\right)\\
&\leq \mathrm{Lip}(\Psi_{1:m})\left(\sum_{k=m}^{n-1}(\overline{B}_{k+1} \vee |\underline{B}_{k+1}|)\prod_{j=m+1}^{k} A_j + |t_1|\prod_{j=m+1}^{n} A_j + |t_2|\right)
\end{aligned}$$

$$\leq \mathrm{Lip}(\Psi_{1:m}) \left(\sum_{k=m}^{n-1} (\overline{B}_{k+1} \vee |\underline{B}_{k+1}|) \prod_{j=m+1}^{k} \mathrm{Lip}(\Psi_j) + |t_1| \prod_{j=m+1}^{n} \mathrm{Lip}(\Psi_j) + |t_2| \right)$$

$$\leq \sum_{k=m}^{n-1} (\overline{B}_{k+1} \vee |\underline{B}_{k+1}|) \prod_{j=1}^{k} \mathrm{Lip}(\Psi_j) + |t_1| \prod_{j=1}^{n} \mathrm{Lip}(\Psi_j) + |t_2| \prod_{j=1}^{m} \mathrm{Lip}(\Psi_j) \to 0.$$

The first term tends to 0 since the series $\sum_{k\geq 0}(\overline{B}_{k+1} \vee |\underline{B}_{k+1}|) \prod_{j=1}^{k} \mathrm{Lip}(\Psi_j)$ is convergent, for details see [28], and the last two terms tend to 0 by the strong law of large numbers for the sequence $\{\log(\mathrm{Lip}(\Psi_n))\}_{n\geq 0}$. If we take $t_1 = t_2 = R_0$ we see that $\{\Psi_{1:n}(R_0)\}_{n\geq 0}$ is convergent and if we take $t_1 = 0$, $t_2 = R_0$ and $n = m$ we see that the limit does not depend on R_0, hence the stationary distribution is unique and it is the distribution of random variable

$$R = \lim_{n\to\infty} \Psi_{1:n}(R_0). \tag{4.7}$$

For the rest of the proof we will assume that R is given by the limit above.

Step 2: *Upper bound in* (3.7). We claim that

$$\mathbb{P}\left[\overline{R} > e^x, M \leq \log(\varepsilon) + x\right] = \varepsilon^{\gamma/4} O(\mathbb{P}[M > x]) \tag{4.8}$$

for $\varepsilon \in (0, 1)$ sufficiently small and $\gamma > 0$ given in the condition (3.4). To prove (4.8), we will apply the technique from [8,7]. For $k \in \mathbb{Z}$ define random set of integers by

$$\mathcal{Q}(k) := \left\{ s \in \mathbb{N} \middle| \overline{B}_{s+1} \prod_{j=1}^{s} A_j \in \left(e^{-k}e^x, e^{-k+1}e^x\right] \right\}.$$

Notice that if $M \leq \log(\varepsilon) + x$ then $\mathcal{Q}(k) = \emptyset$ for k satisfying $e^{-k} > \varepsilon$. The following inclusion holds

$$\left\{\overline{R} > e^x, M \leq \log(\varepsilon) + x\right\} \subseteq \left\{ \exists k : e^{-k} \leq \varepsilon, \#\mathcal{Q}(k) > \frac{e^k}{5k^2} \right\}. \tag{4.9}$$

Indeed, assume that $\overline{R} > e^x$, $M \leq \log(\varepsilon) + x$ and that for any k such that $e^{-k} \leq \varepsilon$ we have $\#\mathcal{Q}(k) \leq \frac{e^k}{5k^2}$. Since $\mathcal{Q}(k) = \emptyset$ for k satisfying $e^{-k} > \varepsilon$, we can write

$$\overline{R} = \sum_{n\geq 0} \overline{B}_{n+1} \prod_{j=1}^{n} A_j = \sum_{k\in\mathbb{Z}} \sum_{s\in\mathcal{Q}(k)} \overline{B}_{s+1} \prod_{j=1}^{s} A_j$$
$$= \sum_{k\geq -\log(\varepsilon)} \sum_{s\in\mathcal{Q}(k)} \overline{B}_{s+1} \prod_{j=1}^{s} A_j \leq \sum_{k>0} \#\mathcal{Q}(k) e^{-k+1} e^x \leq \sum_{k>0} e^x \frac{e}{5k^2} = \frac{\pi^2 e}{30} e^x < e^x.$$

This is a contradiction. Using the inclusion (4.9) one gets instantly that

$$\left\{\overline{R} > e^x, M \leq \log(\varepsilon) + x\right\} \subseteq \left\{ M \leq \log(\varepsilon) + x, \exists k \geq -\log(\varepsilon), \#\mathcal{Q}(k) > \frac{e^k}{5k^2} \right\}. \tag{4.10}$$

Let us focus our interest on the set RHS (4.10). Define the sequence $\tau(k) = \inf \mathcal{Q}(k)$ (we use the convention that $\inf \emptyset = +\infty$). On the set RHS (4.10) there exists $k \geq -\log(\varepsilon)$ such that

$\tau(k) < \infty$ and from the fact that $\tau(k) \in \mathcal{Q}(k)$ and $\#\mathcal{Q}(k) > \frac{e^k}{5k^2}$, we conclude that

$$\overline{B}_{\tau(k)+1} \prod_{j=1}^{\tau(k)} A_j, \qquad \overline{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j \in (e^{-k}e^x, e^{-k+1}e^x] \quad \text{for some } p > \frac{e^k}{5k^2} - 1.$$

By taking $\varepsilon > 0$ sufficiently small we can ensure that $\frac{e^k}{5k^2} - 1 > \frac{e^k}{10k^2}$ for $k \geq -\log(\varepsilon)$. By dividing the two quantities above we obtain that

$$\frac{A_{\tau(k)+1}}{\overline{B}_{\tau(k)+1}} \overline{B}_{\tau(k)+p+1} \prod_{j=\tau(k)+2}^{\tau(k)+p} A_j \in (e^{-1}, e^1) \quad \text{for some } p > \frac{e^k}{10k^2}. \tag{4.11}$$

The quotient $A_{\tau(k)+1}/\overline{B}_{\tau(k)+1}$ is bounded on the set RHS (4.10), because

$$\frac{A_{\tau(k)+1}}{\overline{B}_{\tau(k)+1}} = \frac{\prod_{j=1}^{\tau(k)+1} A_j}{\overline{B}_{\tau(k)+1} \prod_{j=1}^{\tau(k)} A_j} \leq \frac{\overline{B}_{\tau(k)+2} \cdot \prod_{j=1}^{\tau(k)+1} A_j}{e^{-k}e^x} \leq \frac{e^M}{e^{-k}e^x} \leq \frac{\varepsilon e^x}{e^{-k}e^x} \leq e^k. \tag{4.12}$$

Combining bounds in (4.11) and (4.12) we can conclude: on the set RHS (4.10) there exists an integer $k \geq -\log(\varepsilon)$ for which $\tau(k) < \infty$ and

$$\overline{B}_{\tau(k)+p+1} \prod_{j=\tau(k)+2}^{\tau(k)+p} A_j > e^{-k-1} \quad \text{for some } p > \frac{e^k}{10k^2}.$$

Whence

$$\log\left(\overline{B}_{\tau(k)+p+1}\right) + \sum_{l=\tau(k)+2}^{\tau(k)+p} \frac{\mu}{2} + \log(A_l) > \frac{\mu}{2}\left(\frac{e^k}{10k^2} - 1\right) - k - 1$$

from which me may infer that

$$M_k^* = \sup_{j \geq \tau(k)+2} \left\{ \log\left(\overline{B}_j\right) + \sum_{l=\tau(k)+2}^{j-1} \frac{\mu}{2} + \log(A_l) \right\} > \frac{\mu}{2}\left(\frac{e^k}{10k^2} - 1\right) - k - 1 > e^{k/2}$$

if ε is small enough (recall that $k > -\log(\varepsilon)$). So the following inclusion is also correct

$$\left\{\overline{R} > e^x, M \leq \log(\varepsilon) + x\right\} \subseteq \bigcup_{k \geq -\log(\varepsilon)} \left\{\tau(k) < \infty, M_k^* > e^{k/2}\right\}. \tag{4.13}$$

Notice that by the strong Markov property, conditioned in $\tau(k)$, the distribution of M_k^* is the same as the distribution of

$$M^* = \sup_{j \geq 2} \left\{ \log\left(\overline{B}_j\right) + \sum_{l=2}^{j-1} \frac{\mu}{2} + \log(A_l) \right\}.$$

Theorem 2.5 says that

$$\mathbb{P}\left[M^* > x\right] \sim \frac{2}{\mu} \int_x^\infty \mathbb{P}[\log(A \vee B) > y] dy \sim \frac{2}{\mu} \overline{F}_I(x) \leq c_1 x^{-\gamma}, \tag{4.14}$$

for some $c_1 > 0$. In terms of probability (4.13) yields

$$\begin{aligned}\mathbb{P}\left[\overline{R} > e^x, M \le \log(\varepsilon) + x\right] &\le \sum_{k \ge -\log(\varepsilon)} \mathbb{P}\left[\tau(k) < \infty, M_k^* > e^{k/2}\right] \\ &= \sum_{k \ge -\log(\varepsilon)} \mathbb{P}\left[M_k^* > e^{k/2} | \tau(k) < \infty\right] \mathbb{P}[\tau(k) < \infty] \\ &= \sum_{k \ge -\log(\varepsilon)} \mathbb{P}\left[M^* > e^{k/2}\right] \mathbb{P}[\tau(k) < \infty]\end{aligned}$$

by the strong Markov property of the sequence $\{(A_n, \overline{B}_n)\}_{n \ge 0}$. Using (4.14) we obtain for $\eta > 0$

$$\begin{aligned}&\mathbb{P}\left[\overline{R} > e^x, M \le \log(\varepsilon) + x\right] \\ &\le c_1 \sum_{k \ge -\log(\varepsilon)} \mathbb{P}[\tau(k) < \infty] e^{-k\gamma/2} \le c_1 \sum_{k \ge -\log(\varepsilon)} \mathbb{P}[M > x - k] e^{-k\gamma/2} \\ &\le c_1 \sum_{x - \eta \ge k \ge -\log(\varepsilon)} \mathbb{P}[M > x - k] e^{-k\gamma/2} + c_1 \sum_{k > x - \eta} \mathbb{P}[M > x - k] e^{-k\gamma/2} \\ &=: c_1 I_1(x) + c_1 I_2(x).\end{aligned}$$

Now we will investigate I_1 and I_2 separately. From Theorem 2.5 we can conclude that the distribution of the random variable M belongs to the class $\mathcal{S} \subseteq \mathcal{L}$ and so we can use Potter bounds (Corollary 2.2) for $\mathbb{P}[M > t]$ to find $\eta > 0$, such that for $t, s > \eta$ we have

$$\frac{\mathbb{P}[M > t]}{\mathbb{P}[M > s]} \le 2 \exp\left\{\gamma \frac{|t - s|}{4}\right\}.$$

Then for $x > \eta - \log(\varepsilon)$ we have

$$\frac{I_1(x)}{\mathbb{P}[M > x]} = \sum_{x - \eta \ge k \ge -\log(\varepsilon)} \frac{\mathbb{P}[M > x - k]}{\mathbb{P}[M > x]} e^{-\gamma k/2} \le 2 \sum_{x - \eta \ge k \ge -\log(\varepsilon)} e^{-\gamma k/4} \le C\varepsilon^{\gamma/4}$$

and for the second term

$$I_2(x) \le \sum_{k > x - \eta} e^{-k\gamma/2} \le c_2 e^{-x\gamma/2} = o\left(\mathbb{P}[M > x]\right)$$

for some $c_2 > 0$, since the distribution of M is long-tailed. Thus claim (4.8) follows. Now we need to notice that since $R \le \overline{R}$ we have

$$\left\{R > e^x\right\} \subseteq \left\{\overline{R} > e^x, M \le \log(\varepsilon) + x\right\} \cup \{M > \log(\varepsilon) + x\}$$

and thus using (4.8), we get

$$\frac{\mathbb{P}[R > e^x]}{\mathbb{P}[M > x]} \le \varepsilon^{\gamma/4} \frac{O(\mathbb{P}[M > x])}{\mathbb{P}[M > x]} + \frac{\mathbb{P}[M > x + \log(\varepsilon)]}{\mathbb{P}[M > x]}.$$

First let $x \to \infty$ and notice that from Theorem 2.5

$$\mathbb{P}[M > x] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_x^\infty \mathbb{P}[\log(A \vee B) > y] \mathrm{d}y \sim \frac{1}{\mu} \overline{F}_I(x). \tag{4.15}$$

From this we can conclude that

$$\limsup_{x \to \infty} \frac{\mathbb{P}[R > e^x]}{\overline{F}_I(x)} \le C\varepsilon^{\gamma/4} + \frac{1}{\mu}$$

for some finite constant $C > 0$ independent of $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary small we get the upper bound.

Step 3: *Lower bound in* (3.7). Fix $0 < \varepsilon$ and $0 < \delta < \frac{\mu}{2} \wedge 1$. For $K > 0$ consider the sets E_n and F_n given by (4.3) and (4.4) respectively. Choose $K > 0$ large enough for (4.5) to be satisfied. Consider also the random variables

$$R_n^* = \lim_{N\to\infty} \Psi_n \circ \cdots \circ \Psi_N(R_0). \tag{4.16}$$

Note that $R_n^* \stackrel{d}{=} R$ and

$$R = \Psi_{1:n+1}\left(R_{n+2}^*\right).$$

Finally put

$$G_n = E_n \cap F_n \cap \left\{A_{n+1} \vee \underline{B}_{n+1} > e^{n(\mu+\delta)+L+K+x}, \underline{B}_{n+1} \geq -e^{n(\mu-\delta)-K+x}\right\} \\ \cap \left\{R_{n+2}^* > \delta\right\}$$

where $L > 0$ is a constant independent of x and n. We see that the sets $\{G_n\}_{n\geq 0}$ are disjoint if we take $L = L(K, \delta, \mu)$ sufficiently large. Moreover on the set G_n we have

$$\begin{aligned} R &= \Psi_{1:n+1}(R_{n+2}^*) \geq \underline{\Psi}_{1:n+1}(R_{n+2}^*) = \sum_{k=0}^{n-1} \underline{B}_{k+1} \prod_{j=1}^{k} A_j + \left(\underline{B}_{n+1} + R_{n+2}^* A_{n+1}\right) \prod_{j=1}^{n} A_j \\ &\geq -\sum_{k=0}^{n-1} \left|\underline{B}_{k+1}\right| \prod_{j=1}^{k} A_j \\ &\quad + \left(\underline{B}_{n+1} + e^{n(\mu-\delta)-K+x} + A_{n+1} R_{n+2}^*\right) \prod_{k=1}^{n} A_k - e^{n(\mu-\delta)-K+x} \prod_{j=1}^{n} A_j \\ &\geq -\frac{e^{2K}}{1-e^{-\mu+2\delta}} + \delta\left(A_{n+1} \vee \left(\underline{B}_{n+1} + e^{n(\mu-\delta)-K+x}\right)\right) e^{-n(\mu+\delta)-K} - e^x \\ &\geq -\frac{e^{2K}}{1-e^{-\mu+2\delta}} + \delta(A_{n+1} \vee \underline{B}_{n+1}) e^{-n(\mu+\delta)-K} - e^x \geq -\frac{e^{2K}}{1-e^{-\mu+2\delta}} \\ &\qquad + \delta e^{x+L} - e^x > e^x \end{aligned}$$

and the last inequality is valid for all $x > 0$ and all $n \in \mathbb{N}$ if $L = L(K, \delta, \mu)$ is sufficiently large. We see that $G_n \subseteq \{R > e^x\}$ and this allows us to write

$$\begin{aligned} \mathbb{P}[R > e^x] &\geq \sum_{n\geq 0} \mathbb{P}[G_n] \\ &\geq (1-\varepsilon) \sum_{n\geq 0} \mathbb{P}\Big[A_{n+1} \vee \underline{B}_{n+1} > e^{n(\mu+\delta)+L+x+K}, \\ &\qquad \underline{B}_{n+1} \geq -e^{x+n(\mu-\delta)-K}\Big] \mathbb{P}\left[R_{n+2}^* > \delta\right] \\ &\geq (1-\varepsilon)\mathbb{P}[R > \delta] \sum_{n\geq 0} \left\{\mathbb{P}\left[A \vee \underline{B} > e^{n(\mu+\delta)+L+K+x}\right] \right. \\ &\qquad \left. - \mathbb{P}\left[A > e^{n(\mu+\delta)+L+K+x}, \underline{B} < -e^{x+n(\mu-\delta)+K}\right]\right\} \\ &\sim \frac{(1-\varepsilon)\mathbb{P}[R > \delta]}{\mu+\delta} \int_x^\infty \mathbb{P}[\log(A \vee B) > y] \mathrm{d}y. \end{aligned}$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R > e^x\right]}{\int_x^\infty \mathbb{P}[\log(A \vee B) > y]\mathrm{d}y} \geq \frac{(1-\varepsilon)\mathbb{P}[R > \delta]}{\mu+\delta}.$$

If we allow $\varepsilon, \delta \to 0$ we see that we have proven the lower estimate for the desired limit.

Step 4: *The case* $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$. Firstly, notice that we need only to prove the lower bound and that in this case Theorem 2.5 yields

$$\mathbb{P}\left[M > x\right] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_x^\infty \mathbb{P}[\log^+(B) > y]\mathrm{d}y. \tag{4.17}$$

For $0 < \varepsilon$, $0 < \delta < \mu/2$ and $K > 0$ consider the sets E_n and F_n given by (4.3) and (4.4) respectively with $K > 0$ large enough for (4.5) to be satisfied. Finally, put

$$J_n = E_n \cap F_n \cap \left\{\underline{B}_{n+1} > e^{x+n(\mu+\delta)+K+L},\, A_{n+1} \leq e^{n(\mu-\delta)-K+x}\right\} \cap \left\{|R^*_{n+2}| \leq \delta^{-1}\right\},$$

for some large $L > 0$ independent of x. We see that the sets $\{J_n\}_{n\geq 0}$ are disjoint. Moreover on the set J_n we have

$$\begin{aligned}
R &= \Psi_{1:n+1}(R^*_{n+2}) \geq \underline{\Psi}_{1:n+1}(R^*_{n+2}) = \sum_{k=0}^{n-1} \underline{B}_{k+1} \prod_{j=1}^{k} A_j + \underline{B}_{n+1} \prod_{j=1}^{n} A_j \\
&\quad + R^*_{n+2} A_{n+1} \prod_{j=1}^{n} A_j \\
&\geq -\sum_{k=0}^{n-1} |\underline{B}_{k+1}| \prod_{j=1}^{k} A_j + \underline{B}_{n+1} \prod_{j=1}^{n} A_j - A_{n+1}|R^*_{n+2}| \prod_{k=1}^{n} A_k \\
&\geq -\frac{e^{2K}}{1-e^{-\mu+2\delta}} + e^{x+L} - \delta^{-1}e^x > e^x
\end{aligned}$$

and the last inequality is valid for all $x > 0$ if $L = L(K, \delta, \mu)$ is sufficiently large. Therefore $J_n \subseteq \{R > e^x\}$ and this allows us to write

$$\begin{aligned}
\mathbb{P}[R > e^x] &\geq \sum_{n\geq 0} \mathbb{P}[J_n] \\
&\geq (1-\varepsilon) \sum_{n\geq 0} \mathbb{P}\Big[\underline{B}_{n+1} > e^{x+n(\mu+\delta)+K+L}, \\
&\qquad A_{n+1} \leq e^{n(\mu-\delta)-K+x}\Big] \mathbb{P}\left[|R^*_{n+2}| \leq \delta^{-1}\right] \\
&\geq (1-\varepsilon)\mathbb{P}\left[|R| \leq \delta^{-1}\right] \sum_{n\geq 0} \Big\{ \mathbb{P}\left[\underline{B} > e^{x+n(\mu+\delta)+K+L}\right] \\
&\qquad - \mathbb{P}\left[A > e^{n(\mu-\delta)-K+x}\right] \Big\} \\
&\sim \frac{(1-\varepsilon)\mathbb{P}\left[|R| \leq \delta^{-1}\right]}{\mu+\delta} \int_x^\infty \mathbb{P}[\log^+(B) > y]\mathrm{d}y.
\end{aligned}$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R > e^x\right]}{\int_x^\infty \mathbb{P}[\log^+(B) > y]\mathrm{d}y} \geq \frac{(1-\varepsilon)\mathbb{P}\left[|R| \leq \delta^{-1}\right]}{\mu+\delta}.$$

If we allow $\varepsilon, \delta \to 0$ we get the lower estimate.

Step 5: *The case* $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$. Notice that we only need to prove the upper estimate and that in this case

$$\mathbb{P}[M > x] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_x^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y. \tag{4.18}$$

Fix $\varepsilon \in (0.1)$ and notice that since (4.8) holds we only need to focus on the set LHS (4.19):

$$\left\{R > e^x, M > \log(\varepsilon) + x\right\} \subseteq \{M \in (\log(\varepsilon) + x, x]\} \cup \left\{R > e^x, M > x\right\} \tag{4.19}$$

and since the distribution of M is long-tailed and (4.18) is valid we have

$$\mathbb{P}\left[M \in (\log(\varepsilon) + x, x]\right] = o\left(\int_x^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y\right). \tag{4.20}$$

For the other set we have

$$\left\{R > e^x, M > x\right\} = \{M > x\} \setminus \left\{R \leq e^x, M > x\right\}$$

so by (4.18) now we only need to prove that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R \leq e^x, M > x\right]}{\int_x^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y} \geq -\frac{\mathbb{P}[R \leq 0]}{\mathbb{E}[\log(A)]}.$$

We achieve that using the same technique, but this time we consider the sets

$$H_n = E_n \cap F_n' \cap \left\{\overline{B}_{n+1} \leq \frac{1}{2}e^{x+n(\mu-\delta)-K}, A_{n+1} > e^{n(\mu+\delta)+K+x}\right\} \cap \{R^*_{n+2} \leq 0\},$$

where

$$F_n' = F_n'(\delta, K) = \left\{\left|\overline{B}_j\right| < e^{\delta j + K}, j \leq n\right\}.$$

Note that (4.5) also holds true for big $K > 0$ if we replace F_n by F_n'. Fix such K. We see that the sets $\{H_n\}_{n\geq 0}$ are disjoint if x is sufficiently large. Moreover on the set H_n we have

$$\begin{aligned} R &= \Psi_{1:n+1}(R^*_{n+2}) \leq \overline{\Psi}_{1:n+1}(R^*_{n+2}) \\ &= \sum_{j=0}^{n-1} \overline{B}_{j+1} \prod_{k=1}^{j} A_k + \overline{B}_{n+1} \prod_{k=1}^{n} A_k + (R^*_{n+2})^+ \prod_{k=1}^{n+1} A_k \\ &\leq \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + \frac{1}{2}e^x + 0 \leq e^x \end{aligned}$$

and the last inequality is valid for all $x > x_0 = x_0(K, \delta, \mu)$. Therefore $H_n \subseteq \{R \leq e^x\}$. Moreover on the set H_n

$$M \geq \sum_{k=1}^{n+1} \log(A_k) > -n(\mu+\delta) - K + n(\mu+\delta) + K + x = x$$

and this proves that $H_n \subseteq \{R \le e^x, M > x\}$, which allows us to write

$$\begin{aligned}
\mathbb{P}\left[R \le e^x, M > x\right] &\ge \sum_{n\ge 0} \mathbb{P}[H_n] \\
&\ge (1-\varepsilon) \sum_{n\ge 0} \mathbb{P}\left[\overline{B}_{n+1} \le \frac{1}{2} e^{x+n(\mu-\delta)-K}, A_{n+1} > e^{n(\mu+\delta)+K+x}\right] \mathbb{P}[\overline{R}_{n+2} \le 0] \\
&\ge (1-\varepsilon)\mathbb{P}[R \le 0] \sum_{n\ge 0} \left\{ \mathbb{P}\left[A > e^{x+n(\mu+\delta)+K}\right] - \mathbb{P}\left[\overline{B} > \frac{1}{2} e^{n(\mu-\delta)+x-K}\right] \right\} \\
&\sim \frac{(1-\varepsilon)\mathbb{P}[R \le 0]}{\mu+\delta} \int_x^\infty \mathbb{P}[\log(A) > y] \mathrm{d}y.
\end{aligned}$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R \le e^x, M > x\right]}{\int_x^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y} \ge \frac{(1-\varepsilon)\mathbb{P}[R \le 0]}{\mu+\delta}. \tag{4.21}$$

So if we put everything together, we notice that since $R \le \overline{R}$ we have

$$\begin{aligned}
\{R > e^x\} \subseteq &\left\{\overline{R} > e^x, M \le \log(\varepsilon) + x\right\} \cup \{M \in (\log(\varepsilon) + x, x]\} \\
&\cup \left(\{M > x\} \setminus \left\{R \le e^x, M > x\right\}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbb{P}[R > e^x] \le\ &\mathbb{P}\left[\overline{R} > e^x, M \le \log(\varepsilon) + x\right] + \mathbb{P}\left[M \in (\log(\varepsilon) + x, x]\right] \\
&+ \mathbb{P}[M > x] - \mathbb{P}\left[R \le e^x, M > x\right]
\end{aligned}$$

and so using (4.8), (4.20), (4.18) and (4.21) we get

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left[R > e^x\right]}{\int_x^\infty \mathbb{P}[\log(A) > y]\mathrm{d}y} \le C\varepsilon^{\gamma/4} + 0 + \frac{1}{\mu} - \frac{(1-\varepsilon)\mathbb{P}[R \le 0]}{\mu+\delta}.$$

If we allow $\varepsilon, \delta \to 0$ we see that we achieved the desired upper bound and hence the proof is complete. □

Now we can turn our attention to the finite time horizon. Notice that Theorem 3.3 follows by induction form the following lemma.

Lemma 4.2. *Assume (3.1), (3.3), (3.5) and that F defined by (3.9) is subexponential. Assume additionally that*

$$w_1 = \liminf_{x\to\infty} \frac{\mathbb{P}[R_0 > x]}{\mathbb{P}[A \vee B > x]} \le \limsup_{x\to\infty} \frac{\mathbb{P}[R_0 > x]}{\mathbb{P}[A \vee B > x]} = w_2$$

for some finite constants $w_1, w_2 \ge 0$. Then, for $R_1 = \Psi_1(R_0)$

$$w_1 + \mathbb{P}[R_0 > 0] \le \liminf_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[A \vee B > x]} \le \limsup_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[A \vee B > x]} \le 1 + w_2. \tag{4.22}$$

Furthermore if

- $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$ *then*

$$w_1 + 1 \le \liminf_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[B > x]} \le \limsup_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[B > x]} \le 1 + w_2. \tag{4.23}$$

• $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$ *then*

$$w_1 + \mathbb{P}[R_0 > 0] \le \liminf_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[A > x]} \le \limsup_{x\to\infty} \frac{\mathbb{P}[R_1 > x]}{\mathbb{P}[A > x]} \le \mathbb{P}[R_0 > 0] + w_2. \tag{4.24}$$

Proof. The proof mimics the one of the main result. Fix $x \in \mathbb{R}$.

Step 1: *Upper bound in* (4.22). Notice that

$$\begin{aligned} R_1 &= \Psi_1(R_0) \le \overline{\Psi}_1(R_0) = \overline{B}_1 + R_0^+ A_1 \\ &\le A_1 \vee \overline{B}_1 + R_0^+(A_1 \vee \overline{B}_1) \le (1 + R_0^+)(A_1 \vee \overline{B}_1) \end{aligned}$$

and so

$$\mathbb{P}[R_1 > x] \le \mathbb{P}\left[(1 + R_0^+)(A_1 \vee \overline{B}_1) > x\right] \le (1 + w_2 + o(1))\mathbb{P}[A \vee B > x]$$

as $x \to \infty$, since F is subexponential.

Step 2: *Lower bound in* (4.22). Fix $0 < \varepsilon$ and $0 < \delta < \frac{\mu}{2} \wedge 1$. For $K, L > 0$ consider the sets

$$G_0 = \left\{A_1 \vee \underline{B}_1 > e^{L+K+x}, \underline{B}_1 \ge -e^{-K+x}\right\} \cap \{R_0 > \delta\}$$

and

$$G_1 = \left\{e^{-K} \le A_1 \le e^K, \left|\underline{B}_1\right| \le e^K\right\} \cap \left\{R_0 > e^{L+K+x}\right\}.$$

Take $K > 0$ sufficiently large such that

$$\mathbb{P}\left[e^{-K} \le A_1 \le e^K, \left|\underline{B}_1\right| \le e^K\right] \ge 1 - \varepsilon. \tag{4.25}$$

We see that the sets G_0 and G_1 are disjoint. Moreover on the set G_0 we have

$$\begin{aligned} R_1 &= \Psi_1(R_0) \ge \underline{\Psi}_1(R_0) = \underline{B}_1 + A_1 R_0 \\ &= \left(\underline{B}_1 + e^{-K+x} + A_1 R_0\right) - e^{-K+x} \ge \delta(A_1 \vee \underline{B}_1) - e^x \ge \delta e^{x+L} - e^x > e^x \end{aligned}$$

and the last inequality is valid for all $x > 0$ if $L = L(K, \delta)$ is sufficiently large. On the set G_1 we have

$$\begin{aligned} R_1 &= \Psi_1(R_0) \ge \underline{\Psi}_1(R_0) = \underline{B}_1 + R_0 A_1 \ge -|\underline{B}_1| + R_0 A_1 \\ &\ge -e^K + R_0 e^{-K} \ge -e^K + e^{x+L} > e^x \end{aligned}$$

and again, the last inequality holds if we take $L = L(K, \delta)$ sufficiently large. Therefore $G_0 \cup G_1 \subseteq \{R_1 > e^x\}$ and since

$$\begin{aligned} \mathbb{P}[G_0] &\ge \mathbb{P}\left[A_1 \vee \underline{B}_1 > e^{L+x+K}, \underline{B}_1 \ge -e^{-K+x}\right]\mathbb{P}[R_0 > \delta] \\ &\ge \mathbb{P}[R_0 > \delta]\left(\mathbb{P}\left[A_1 \vee \underline{B}_1 > e^{L+x+K}\right] - \mathbb{P}\left[A_1 > e^{L+x+K}, \underline{B}_1 > -e^{-K+x}\right]\right) \\ &\sim \mathbb{P}[R_0 > \delta]\mathbb{P}\left[A \vee B > e^x\right] \end{aligned}$$

and

$$\mathbb{P}[G_1] \ge (1 - \varepsilon)\mathbb{P}\left[R_0 > e^{L+K+x}\right] \ge ((1 - \varepsilon)w_1 + o(1))\mathbb{P}[A \vee B > e^x].$$

We can write

$$\mathbb{P}[R_1 > e^x] \geq \mathbb{P}[G_0] + \mathbb{P}[G_1] \geq ((1-\varepsilon)w_1 + \mathbb{P}[R_0 > \delta] + o(1))\,\mathbb{P}[A \vee B > e^x].$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R_n > e^x\right]}{\mathbb{P}[A \vee B > e^x]} \geq ((1-\varepsilon)w_1 + \mathbb{P}[R_0 > \delta]) .$$

If we allow $\varepsilon, \delta \to 0$ we see that we have proven the lower estimate for the desired limit.

Step 3: *The case* $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$. For $0 < \varepsilon$, $0 < \delta < \mu/2$ and $K, L > 0$ consider the sets

$$J_0 = \left\{\underline{B}_1 > e^{x+K+L}, A_1 \leq e^{-K+x}\right\} \cap \left\{|R_0| \leq \delta^{-1}\right\}$$

and

$$J_1 = \left\{e^{-K} \leq A_1 \leq e^K, \left|\underline{B}_1\right| \leq e^K\right\} \cap \left\{R_0 > e^{x+K+L}\right\}$$

with K such that (4.25) is satisfied. We see that the sets J_0 and J_1 are disjoint. Moreover for $L = L(K, \delta)$ large enough, on the set J_0

$$R_1 = \Psi_1(R_0) \geq \underline{\Psi}_1(R_0) = \underline{B}_1 + R_0 A_1 \geq \underline{B}_1 - |R_0| A_1 \geq e^{x+L} - \frac{1}{\delta} e^x > e^x$$

and on the set J_1

$$R_n = \Psi_1(R_0) \geq \underline{\Psi}_1(R_0) = \underline{B}_1 + R_0 A_1 \geq -|\underline{B}_1| + R_0 A_1 \geq -e^K + e^{x+L} > e^x$$

and the last inequalities are valid for all $x > 0$ if $L = L(K, \delta, \mu)$ is sufficiently large. Therefore $J_0 \cup J_1 \subseteq \{R > e^x\}$.

$$\begin{aligned}
\mathbb{P}[J_0] &\geq \mathbb{P}\left[\underline{B}_1 > e^{x+K+L}, A_1 \leq e^{-3K+x}\right] \mathbb{P}\left[|R_0| \leq \delta^{-1}\right] \\
&= \mathbb{P}\left[|R_0| \leq \delta^{-1}\right]\left(\mathbb{P}\left[\underline{B} > e^{x+K+L}\right] - \mathbb{P}\left[A > e^{-K+x}\right]\right) \\
&\geq \left(\mathbb{P}\left[|R_0| \leq \delta^{-1}\right] + o(1)\right)\mathbb{P}[B > e^x]
\end{aligned}$$

and

$$\mathbb{P}[J_1] \geq (1-\varepsilon)\mathbb{P}[R_0 > e^{x+K+L}] \geq ((1-\varepsilon)w_1 + o(1))\mathbb{P}[B > e^x].$$

This allows us to write

$$\mathbb{P}[R_1 > e^x] \geq \mathbb{P}[J_0] + \mathbb{P}[J_1] \geq \left((1-\varepsilon)w_1 + \mathbb{P}\left[|R_0| \leq \delta^{-1}\right] + o(1)\right)\mathbb{P}[B > e^x].$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R > e^x\right]}{\mathbb{P}[\log^+(B) > x]} \geq (1-\varepsilon)\left(w_1 + \mathbb{P}\left[|R_0| \leq \delta^{-1}\right]\right).$$

If we allow $\varepsilon, \delta \to 0$ we get the lower estimate.

Step 4: *The case* $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$. Notice that we only need to prove the upper estimate. Let

$$M_1 = \left\{\log(\overline{B}_1)\right\} \vee \left\{\log^+(R_0) + \log(A_1)\right\}.$$

Next, notice that

$$\left\{\overline{\Psi}_1(R_0) > e^x, M_1 \le x - \log(2)\right\} = \emptyset$$

and so

$$\begin{aligned}\left\{\overline{\Psi}_1(R_0) > e^x\right\} &= \left\{\overline{\Psi}_1(R_0) > e^x, M_1 > x - \log(2)\right\} \\ &= \{M_1 > x - \log(2)\} \setminus \left\{\overline{\Psi}_1(R_0) \le e^x, M_1 > x - \log(2)\right\}. \end{aligned} \tag{4.26}$$

Since $M_1 \le \log^+(R_0) + \log(A_1 \vee \overline{B}_1)$ we can write

$$\begin{aligned}\mathbb{P}\left[M_1 > x - \log(2)\right] &\le \mathbb{P}\left[\log^+(R_0) + \log(A_1 \vee \overline{B}_1) > x - \log(2)\right] \\ &\ge (w_1 + 1 + o(1))\mathbb{P}[\log(A \vee B) > x] \end{aligned} \tag{4.27}$$

so we only need to prove that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[\overline{\Psi}_1(R_0) \le e^x, M_1 > x - \log(2)\right]}{\mathbb{P}[\log(A) > x]} \ge \mathbb{P}[R_0 \le 0].$$

We achieve that using the same technique, but this time we consider the set

$$H_0 = \left\{\overline{B}_1 \le e^x, A_1 > e^x\right\} \cap \{R_0 \le 0\}$$

on which we have

$$\Psi_1(R_0) \le \overline{\Psi}_1(R_0) = \overline{B}_1 + R_0^+ A_1 \le e^x + 0 \le e^x$$

and

$$M_1 \ge \log(A_1) > x.$$

We see that $H_1 \subseteq \{\Psi_1(R_0) \le e^x, M > x - \log(2)\}$ and this allows us to write

$$\begin{aligned}\mathbb{P}\left[\Psi_1(R_0) \le e^x, M_1 > x - \log(2)\right] &\ge \mathbb{P}[H_0] \ge \mathbb{P}\left[\overline{B}_1 \le e^x, A_1 > e^x\right]\mathbb{P}[R_0 \le 0] \\ &\ge \left\{\mathbb{P}\left[A_1 > e^x\right] - \mathbb{P}\left[\overline{B}_1 > e^x\right]\right\}\mathbb{P}[R_0 \le 0] \\ &\ge \mathbb{P}[A > e^x]\left(\mathbb{P}[R_0 \le 0] + o(1)\right). \end{aligned}$$

This yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left[R_1 \le e^x, M_1 > x - \log(2)\right]}{\mathbb{P}[\log(A) > x]} \ge \mathbb{P}[R_0 \le 0].$$

Putting everything together, that is (4.26), (4.27) and the last inequality we get

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left[R_1 > e^x\right]}{\mathbb{P}[\log(A) > x]} \le 1 + w_2 - \mathbb{P}[R_0 \le 0]$$

which is the desired upper bound and hence the proof is complete in this case. □

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