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Slow diffusion for a Brownian motion with random reflecting barriers

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Abstract

Let β be a positive number: we consider a particle performing a one-dimensional Brownian motion with drift $-\beta$, diffusion coefficient 1, and a reflecting barrier at 0. We prove that the time R, needed by the particle to reach a random level X, has the same distribution tails as $\Gamma(\alpha+1)^{1/\alpha}e^{2\beta X}/2\beta^2$, provided that one of these tails is regularly varying with negative index $-\alpha$. As a consequence, we discuss the asymptotic behaviour of a Brownian motion with random reflecting barriers, extending some results given by Solomon when X is exponential and α belongs to $[\frac{1}{2}, 1]$.

Keywords: Regular variation; Reflected Brownian motion; Random media; Homogenization; Local time

0. Introduction

Let β be a positive number: we consider a particle performing a one-dimensional Brownian motion $(B_t)_{t\geqslant 0}$ on $[0, +\infty[$, with drift $-\beta$, diffusion coefficient 1, and an instantaneously reflecting barrier at 0. In Section 2, we prove that the time R, needed by the particle to reach a random positive level X, independent of $(B_t)_{t\geqslant 0}$, has a regularly varying distribution tail with negative index $-\alpha$ if and only if $e^{2\beta X}$ has a regularly varying distribution tail with the same index, while Section 1 is devoted to preliminary results. Our tools are the theorem of Ray and Knight (Jeulin and Yor, 1978) concerning the local time of a diffusion, and a representation of the local time of the Brownian motion with drift that can be found in Section 6 of Pitman and Yor (1982).

The study of R's tail is motivated by a problem about random walks in random media connected with semi-conductor problems (Molchanov, 1994) and liquid spreading (Collet et al., 1993). This problem was first studied in a paper by Solomon (1975), and will be described in Section 4. Homogenization or slow diffusion for this

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random walk is discussed in Section 5. In his paper, Solomon obtained a slow diffusion behaviour in the case where X is exponential and $e^{2\beta X}$ has a regularly varying distribution tail with index α between $-\frac{1}{2}$ and -1 (i.e. when the mean of X is between $1/2\beta$ and $1/\beta$). We extend his results to $\alpha < 0$, not assuming anymore that X has an exponential law.

1. Preliminary results

In this paper L and L' are slowly varying functions, and α is a nonnegative number. The equivalence of functions f and g is denoted by $f(x) \approx g(x)$ $(x \to a)$. We recall (cf. Bingham et al., 1987, p. 38, Theorem 1.7.1′) that:

Feller's Theorem. *Let X be any positive random variable. The following are equivalent:*

$$P(X \leqslant x) \approx x^{\alpha} L(1/x) \quad (x \to 0_{+}),$$

and

$$E(e^{-\lambda X}) \approx \Gamma(1+\alpha)\lambda^{-\alpha} L(\lambda) \quad (\lambda \to +\infty).$$

Here is an easy consequence that will be useful in Section 3:

Corollary 1. *Let V and U be two positive and independent r.v., and assume that U has density $\lambda e^{-\lambda x} \mathbb{1}_{[0;+\infty[}(x)$. Then relations* (a) *and* (b), *below, are equivalent*

(a) $\lim_{t \to +\infty} t^{\alpha} L(t) \mathbb{P}(V \geqslant t) = \psi$,

(b) $\lim_{t \to +\infty} t^{\alpha} L(t) \mathbb{P}(UV \geqslant t) = \Gamma(\alpha+1)\lambda^{-\alpha}\psi$.

Proof. We have

$$\mathbb{P}(UV \geqslant t) = \int \mathbb{P}[U \geqslant t/x]\,\mathbb{P}_{v}(dx) = \int e^{-\lambda t/x}\,\mathbb{P}_{v}(dx)$$

which is the Laplace transform F^{*} of the random variable $Y = \lambda/V$. From Feller's theorem,

$$F^{*}(t) \approx \Gamma(\alpha+1)\lambda^{-\alpha}\psi t^{-\alpha}/L(t) = Ct^{-\alpha}L'(t) \quad (t \to +\infty)$$

is equivalent to

$$\mathbb{P}(Y \leqslant x) \approx Cx^{\alpha}\frac{1}{\Gamma(1+\alpha)}L'(1/x) \quad (x \to 0^{+}),$$

which can in turn be rephrased:

$$\mathbb{P}(V \geqslant \lambda/x) \approx Cx^{\alpha}\frac{1}{\Gamma(1+\alpha)}L'(1/x) \quad (x \to 0^{+})$$

or

$$\mathbb{P}(V \geqslant y) \approx \psi y^{-\alpha}/L(y) \quad (y \to +\infty). \qquad \square$$

Finally, we have:

Proposition 1. *Let the random variables X and Y, and the function L, satisfy*

$$\lim_{t\to+\infty} t^{\alpha}L(t)\mathbb{P}(|X-Y| \geqslant t) = 0,$$

and let ϕ be a non-negative real number. Then we have

$$\lim_{t\to+\infty} t^{\alpha}L(t)\mathbb{P}(X \geqslant t) = \phi$$

iff

$$\lim_{t\to+\infty} t^{\alpha}L(t)\mathbb{P}(Y \geqslant t) = \phi.$$

The assumption concerning $X - Y$ is satisfied, for instance, when $X - Y$ belongs to some L^p with p greater than α.

2. Regular variation of *R* and *X*'s tails

Set

$$\delta(\alpha) = \frac{\Gamma(\alpha+1)^{1/\alpha}}{2\beta^2}.$$

Theorem 1. *The following relations are equivalent:*

(i) $\exists\,\phi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^{\alpha}L(t)\mathbb{P}(R \geqslant t) = \phi$,
(ii) $\exists\,\psi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^{\alpha}L(t)\mathbb{P}(\delta(\alpha)e^{2\beta X} \geqslant t) = \psi$,

and if (i) *or* (ii) *holds true, then* $\phi = \psi$.

In Section 3, I will give two simple proofs of Theorem 1, both relying on properties of the paths of B_t and I hope, both are of intrinsic interest. The study of the Laplace transform of R was often suggested to me, for the proof of this theorem, and I will explain shortly why I was unable to follow this suggestion. It is well known that

$$\mathbb{E}[\exp(-uR)\,|\,X=x] = \frac{2h(u)\exp[x(-\beta+h(u))]}{h(u)+\beta+(h(u)-\beta)e^{2h(u)x}},$$

where

$$h(u) = \sqrt{\beta^2 + 2u}$$

so that

$$\mathbb{E}[\exp(-uR)] = \int_0^{+\infty} \frac{2h(u)\exp[x(-\beta + h(u))]}{h(u) + \beta + (h(u) - \beta)e^{2hx}} \mathbb{P}_X(dx) \tag{2.1}$$

and is to be compared with

$$\mathbb{E}[\exp(-u\delta(\alpha)e^{2\beta X})] = \int_0^{+\infty} \exp[-u\delta(\alpha)e^{2\beta x}]\mathbb{P}_X(dx). \tag{2.2}$$

Thus, we can hope to deduce Theorem 1 from the fact that the behaviour of the distribution tails of R and $\delta(\alpha)e^{2\beta X}$ are connected to the behaviour of their Laplace transforms at 0^+. That was the approach of Solomon, but his task was greatly facilitated by the explicit knowledge of $\mathbb{P}_X$ (the exponential law), and by the fact that α was less than one. When α is an integer this approach fails (cf. Bingham et al., 1987, Theorem 8.1.6), and if the integer part, say n, of α, is *positive*, one has to study, not the behaviour at 0^+ of the Laplace transforms appearing in (2.1) and (2.2), but the behaviour at 0^+ of their nth derivatives, or, alternatively, the rest of the Taylor formula of this Laplace transform at 0. In order to do so, one has, of course, to compute the n first moments of R in terms of the moments of X. For the results of these computations when $n = 1, 2$, see Propositions 2 and 3 below.

Proposition 2. *R belongs to L^1 iff $e^{2\beta X}$ belongs to L^1, and*

$$\mathbb{E}[R] = (1/2\beta^2)\mathbb{E}[e^{2\beta X} - 2\beta X - 1].$$

Proposition 3. *R belongs to L^2 iff $e^{2\beta X}$ belongs to L^2, and*

$$\mathbb{E}[R^2] = (1/2\beta^4)\mathbb{E}[e^{4\beta X} + e^{2\beta X} - 2 - 6\beta X e^{2\beta X} + 2\beta^2 X^2].$$

For the – tedious – proof, see Section 6. I did not even try to find a general expression of the nth moment. Furthermore, this, together with Theorem 8.1.6 of Bingham et al., would only give, very likely, the (ii) $\Rightarrow$ (i) part of Theorem 1. Instead of the Laplace transform, I would rather suggest to study directly the tail of the distribution of R.

In Section 5, Theorem 1 is mainly useful through its immediate consequence:

Corollary 2. *R belongs to the attraction domain of the stable law with index α $(0 < \alpha < 2)$ iff $e^{2\beta X}$ does.*

For the normal law, Proposition 3 yields a similar result.

3. Proofs of Theorem 1

Let $\mathscr{L}(t, x)$ be the local time of a reflected Brownian motion $(B_t)_{t \geq 0}$, with drift $-\beta$, and reflection at 0, starting from 0. Let T_y be the first time at which B_t reaches $y > 0$. Clearly,

$$T_y = \int_0^y \mathscr{L}(T_y, x)\,\mathrm{d}x = \int_0^y \mathscr{L}(T_y, y - x)\,\mathrm{d}x.$$

Let

$$Z_t^{(y)} = \mathscr{L}(T_y, y - t) \quad \text{and} \quad Z^{(y)} = (Z_t^{(y)})_{0 \leq t < y}.$$

According to the Ray and Knight theorem (Jeulin and Yor, 1978), $Z^{(y)}$ is a diffusion, starting from 0, with an infinitesimal generator, $2x(\partial^2/\partial x^2) + 2(1 + \beta x)\partial/\partial x$, that does *not* depend on y. This is the key to our first proof, through the following remark: let $Z = (Z_t)_{t \geq 0}$ be a diffusion with the same infinitesimal generator, starting from 0 too, and independent of X; then

$$R \overset{\mathscr{D}}{=} \int_0^X Z_t\,\mathrm{d}t.$$

As mentioned in Pitman and Yor (1982, Section 6), Z_t can be seen as the sum of squares of 2 independent copies of the standard Ornstein Uhlenbeck process starting from 0, and can thus be written as

$$Z_t = \mathrm{e}^{2\beta t}(W_{1,\rho(t)}^2 + W_{2,\rho(t)}^2), \tag{3.1}$$

where $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions, with $\rho(t)$ defined as

$$\rho(t) = \frac{1 - \mathrm{e}^{-2\beta t}}{2\beta}.$$

Let $U_t = Z_t \mathrm{e}^{-2\beta t}$. We have:

Theorem 2. *U_t is a positive submartingale converging a.s. and in any L^p towards a random variable U with exponential law and mean $1/\beta$. Furthermore,*

$$\mathbb{E}[(U - U_t)^p]^{1/p} \leq C(\beta, p)\mathrm{e}^{-\beta t}. \tag{3.2}$$

Proof. Since

$$\lim_{t \to +\infty} \rho(t) = 1/2\beta,$$

the first part of Theorem 2 follows at once from (3.1). Straightforward computations show that for a standard Brownian motion B_t, and for any $p \geq 0$,

$$\| B_{t+h}^2 - B_t^2 \|_p \approx C_p \sqrt{\text{th}}, \ .$$

leading us to (3.2). Finally, the distribution of Z_t being exponential with mean $(\mathrm{e}^{2\beta t} - 1)/\beta$, the distribution of U is exponential with mean $1/\beta$. $\square$

From these properties of Z_t, we can deduce that:

Lemma 1. *If* $e^{2\beta X}$ *belongs to some* L^p, $p > 0$, *then* $R - (1/2\beta)Ue^{2\beta X}$ *belongs to* L^q *for any* $q < 2p$.

Lemma 2. *If* R *belongs to some* L^p, $p > 0$, *then* $e^{2\beta X}$ *belongs to* L^p, *too.*

Proof of Lemma 1. Indifferently, we prove that $R - (1/2\beta)U(e^{2\beta X} - 1)$ belongs to L^{2p}:

$$\mathbb{E}[|R - (1/2\beta)U(e^{2\beta X} - 1)|^{2p} | X = y] = \mathbb{E}\left[\left|\int_0^y (Z_x - Ue^{2\beta x})\,dx\right|^{2p}\right].$$

With the help of (3.2), we obtain that, if $2p \geqslant 1$,

$$\begin{aligned}\mathbb{E}[|R - (1/2\beta)U(e^{2\beta X} - 1)|^{2p} | X = y] &\leqslant y^{2p-1} \int_0^y \mathbb{E}[|Z_x - Ue^{2\beta x}|^{2p}]\,dx \\ &\leqslant C'y^{2p-1}e^{2\beta py}\end{aligned}$$

for some number C' depending only on p and β, and if $2p \leqslant 1$,

$$\begin{aligned}\mathbb{E}[|R - (1/2\beta)U(e^{2\beta X} - 1)|^{2p} | X = y] &\leqslant \mathbb{E}\left[\int_0^y |Z_x - Ue^{2\beta x}|\,dx\right]^{2p} \\ &\leqslant \left(\int_0^y \mathbb{E}[|Z_x - Ue^{2\beta x}|]\,dx\right)^{2p} \\ &\leqslant C''e^{2\beta py}. \qquad \square\end{aligned}$$

Proof of Lemma 2. From the relation

$$E[R^p | X = x] = E\left[\left(\int_0^x Z_t\,dt\right)^p\right],$$

we deduce, if $p \geqslant 1$, that

$$E[R^p | X = x] \geqslant E\left[\int_0^x Z_t\,dt\right]^p = \left(\frac{e^{2\beta x} - 2\beta x - 1}{2\beta^2}\right)^p.$$

And, if $p \leqslant 1$,

$$\begin{aligned}E[R^p | X = x] &\geqslant E\left[x^{p-1}\int_0^x Z_t^p\,dt\right] \\ &\geqslant C_p x^{p-1}\int_0^x (e^{2\beta t} - 1)^p\,dt,\end{aligned}$$

so that

$$e^{2\beta px} = o(E[R^p | X = x]) \quad (x \to +\infty).$$

For the last inequality, we notice that, according to (3.1), Z is stochastically greater than an Ornstein Uhlenbeck process, and use the explicit expressions of moments of a standard Gaussian random variable. □

From Corollary 1 we know that (ii) is equivalent to

(iii) $\lim_{t \to +\infty} t^{\alpha} L(t) \mathbb{P}(1/2\beta) U e^{2\beta X} \geqslant t) = \phi$,

so (iii) entails that $e^{2\beta X}$ belongs to L^p for any $p < \alpha$, but this last fact is a consequence of (i), as well, through Lemma 2. Thus, assuming indifferently (i) or (iii), it follows from Lemma 1 that $W_1 = R - (1/2\beta) U e^{2\beta X}$ belongs to L^q for some $q > \alpha$, and, actually, for any $q < 2\alpha$[. Finally, Proposition 1 then entails that (i) and (iii) are equivalents. This is the end of the first proof of Theorem 1.

The first version of this proof was initially derived from the proof of analogous results in the discrete case (see Alili and Chassaing, 1993), using the approximation of the local time by the number of downcrossings, following the lines of Kawazu and Watanabe (1971), Walsh (1978) and Le Gall (1986). In particular, we have a perfect analogy between Theorem 2 and the results of Section 7 in Alili et al.

The second proof was suggested by Jean Bertoin: Let W_t be the standard Brownian motion associated with B_t, and let L_t be the local time of B_t at 0, i.e. set

$$B_t = W_t - \beta t + \tfrac{1}{2} L_t. \tag{3.3}$$

We then have:

Proposition 4.

$$P(L_R > t) = E\left[\exp\left(\frac{-\beta t}{e^{2\beta X} - 1}\right)\right]. \tag{3.4}$$

Proof. The proposition is a consequence of

$$P(L_{T_x} > t) = \exp\left(\frac{-\beta t}{e^{2\beta x} - 1}\right). \tag{3.5}$$

Let us denote L_{T_x} by L. It is well known that L has an exponential law, due to the Markov property, but it also follows at once from (3.1). Then the computation of L's mean yields (3.5): applying the optional sampling theorem to relation (3.3) with t replaced by T_x, we obtain

$$E(L) = 2(x + \beta E(T_x)) = \frac{e^{2\beta x} - 1}{\beta}. \qquad \square$$

Dividing (3.3) by βt, we obtain directly

$$(1/2\beta) L_t \approx t \quad (t \to +\infty), \text{ a.s.},$$

since B_t and W_t are both $o(t)$ a.s. Thus, we can expect the distribution tails of $(1/2\beta) L_R$ and R to have the same behaviour at $+\infty$.

Theorem 3. *The following relations are equivalent:*

(i) $\exists\,\phi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^\alpha L(t)\mathbb{P}(R \geqslant t) = \phi$,
(ii) $\exists\,\psi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^\alpha L(t)\mathbb{P}((1/2\beta)L_R \geqslant t) = \psi$,

and if (i) *or* (ii) *holds true, then* $\phi = \psi$.

On the other hand, (3.4) can be written as

$$P((1/2\beta)L_R > t) = E\left[\exp\left(\frac{-2\beta^2 t}{e^{2\beta X} - 1}\right)\right] = P\left(U\,\frac{e^{2\beta X} - 1}{2\beta^2} > t\right),$$

where U can be any exponential random variable with mean 1, and independent of X, leading us, through Corollary 1, to

Proposition 5. *The following relations are equivalent*

(i) $\exists\,\phi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^\alpha L_{(t)}\mathbb{P}(\delta(\alpha)e^{2\beta X} \geqslant t) = \phi$,
(ii) $\exists\,\psi \geqslant 0$ *such that* $\lim_{t\to+\infty} t^\alpha L_{(t)}\mathbb{P}((1/2\beta)L_R \geqslant t) = \psi$,

and if (i) *or* (ii) *holds true, then* $\phi = \psi$.

This ends the second proof of Theorem 1, provided we prove Theorem 3.

Proof of Theorem 3. We have

$$\begin{aligned} P((1/2\beta)L_R \geqslant t) &\leqslant P(R > t(1-\varepsilon)) + P(R \leqslant t(1-\varepsilon) \text{ and } (1/2\beta)L_R \geqslant t) \\ &\leqslant P(R > t(1-\varepsilon)) + P((1/2\beta)L_{t(1-\varepsilon)} \geqslant t) \end{aligned}$$

and, assuming (i), we deduce that

$$\overline{\lim_{t\to+\infty}}\; t^\alpha L(t) P((1/2\beta)L_R \geqslant t) \leqslant \frac{\phi}{(1-\varepsilon)^\alpha} + \overline{\lim_{t\to+\infty}}\; t^\alpha L(t) P((1/2\beta)L_{t(1-\varepsilon)} \geqslant t). \quad (3.6)$$

But now

$$\begin{aligned} P((1/2\beta)L_{t(1-\varepsilon)} \geqslant t) &= P(L_{t(1-\varepsilon)} - 2\beta t(1-\varepsilon) \geqslant 2\beta t\varepsilon) \\ &\leqslant (2\beta t\varepsilon)^{-p} E(|L_{t(1-\varepsilon)} - 2\beta t(1-\varepsilon)|^p) \end{aligned} \quad (3.7)$$

for any positive p. We have thus to study $L_t - 2\beta t$, or more conveniently, $B_t - W_t$.

On the one hand, we have

$$\| W_t \|_p = C_p\sqrt{t}.$$

On the other hand, the distribution of B_t is stochastically increasing (cf. Asmussen, 1987, p. 83) and converges, as t goes to $+\infty$, towards an exponential law with mean $1/2\beta$, so that

$$\| B_t \|_p \leqslant \beta\delta(p).$$

Finally, for any positive p, and for t greater than 1,

$$E(|L_t - 2\beta t|^p) \leqslant \hat{C}_p t^{p/2}. \tag{3.8}$$

Thus, for $p > 2\alpha$, we deduce from (3.6)–(3.8) that

$$\overline{\lim_{t \to +\infty}} \; t^\alpha L(t) P((1/2\beta) L_R \geqslant t) \leqslant \frac{\phi}{(1-\varepsilon)^\alpha}$$

for any positive ε.

The proof is completed by considering similar inequalities:

$$P((1/2\beta)L_R \geqslant t) \geqslant P(R < t(1+\varepsilon)) - P((1/2\beta)L_{t(1+\varepsilon)} \leqslant t)$$

for (i) $\Rightarrow$ (ii), and

$$P(R \geqslant t) \geqslant P((1/2\beta)L_R \geqslant t(1+\varepsilon)) - P((1/2\beta)L_t \geqslant t(1+\varepsilon)),$$

$$P(R \geqslant t) \leqslant P((1/2\beta)L_R \geqslant t(1-\varepsilon)) + P((1/2\beta)L_t \leqslant t(1-\varepsilon))$$

for (ii) $\Rightarrow$ (i). □

4. Application: A diffusion with random reflecting barriers

Let x_t denote the position at time t of a particle performing a one-dimensional Brownian motion $(B_t)_{t \geqslant 0}$ on $[0, +\infty[$, with drift $-\beta$, diffusion coefficient 1, and with an instantaneously reflecting barrier at 0. Before this particle starts moving, we have placed randomly on $[0, +\infty[$ a set of barriers in such a way that the distances between two consecutive barriers form a sequence $(X_n)_{n \geqslant 1}$ of i.i.d. positive random variables. Let S_n denote the position of the nth barrier: S_n is given by

$$S_n = X_1 + X_2 + \cdots + X_n.$$

We assume that when a particle reaches a barrier S_n, it moves necessarily up and will remain thenceforward in $[S_n, +\infty[$ forever: the barrier has a reflecting upper side and a perfectly porous lower side. The S_n are some kind of random media for the random motion x_t.

According to Molchanov (1994), the knowledge of the asymptotic behavior of x_t is quite relevant for the study of semi-conductor problems. Let us also mention the connection with liquid spreading (Collet et al., 1993). As in the discrete case, the assumptions that come up the most naturally to insure the weak convergence of x_t, properly normalized, are assumptions about the tail of the time R_k that elapses between the first passage at B_{k-1} and the first passage at B_k, R_1 being the first passage time at the first barrier. This is not satisfactory, since the law of R_k is not initial data of our problem. Of course, the $((X_k, R_k))_{k \geqslant 1}$ are i.i.d. and the study of the R_k's tail reduces to the study of R's tail as in Sections 2 and 3.

Section 2 allows us to state fairly complete results of weak convergence for x_t: we prove that there is homogenization for x_t iff $e^{2\beta X_1}$ has a finite second moment, and that there is slow diffusion iff $e^{2\beta X_1}$ belongs to the domain of attraction of a stable law with index $\alpha < 2$. With Alili, we realized that for random walks with random reflecting barriers, a slow diffusion behaviour was impossible: the tail of R_i cannot be regularly varying for the simple reason that the exponential of a lattice r.v., such as X_i, cannot have a regularly varying tail. This unexpected phenomenon led Bougerol to ask us naturally the question of what happens in the continuous case.

One can wonder about the behaviour of x_t when the drift is positive. The case with negative drift seemed more interesting to us, since it is the case where the presence of random barriers induces the more significant change in the behaviour of the diffusion: without the random barriers it would be recurrent, but with these barriers $\lim x_t = +\infty$ a.s. We believe that in the positive drift case, there is homogenization independently of the distribution of the X_i's.

5. Fluctuations of x_t

Let F_α be the distribution function of the stable law with index α and let its Fourier transform be denoted by

$$\gamma_\alpha(x) = \exp - \{|x|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) - \mathrm{i}\,\mathrm{sgn}(x)\sin(\pi\alpha/2))\},$$

and let Φ be the Gaussian distribution function. Furthermore, let:

$$\mu = \mathbb{E}(X), \qquad \mu' = \mathbb{E}(R)$$

and

$$v = \mu/\mu', \qquad d = 1/\mu' \operatorname{Var}(X - vR).$$

Then we have:

Theorem 4. *If* $e^{2\beta X_1}$ *is integrable, then*

$$\lim_{t\to+\infty} (1/t)x_t = v, \quad a.s.$$

Theorem 5. (a) *If* $e^{2\beta X_1}$ *belongs to* L^2, *then*

$$\lim_{t\to+\infty} P((1/\sqrt{dt})(x_t - vt) \leqslant y) = \Phi(y).$$

(b) *If* $e^{2\beta X_1}$ *belongs to the attraction domain of* F_α, $1 < \alpha < 2$, *then*

$$\lim_{t\to+\infty} P((1/t^{1/\alpha}h(t))(x_t - vt) \leqslant y) = 1 - F_\alpha(-y),$$

in which $t^{1/\alpha}h(t) \approx \inf\{x \mid \mathbb{P}(\delta(\alpha)e^{2\beta X_1} \geqslant x) \leqslant 1/t\}$.

(c) *If* $e^{2\beta X_1}$ *belongs to the attraction domain of* F_α, $0 < \alpha < 1$, *then*

$$\lim_{t \to +\infty} P((L(t)/t^\alpha)(x_t - vt) \leqslant y) = 1 - F_\alpha(y^{-1/\alpha}),$$

in which $L(t) \approx t^\alpha \mathbb{P}(\delta(\alpha)e^{2\beta X_1} \geqslant t)$, *and* $v = 0$.

Theorem 6 claims that, as in the discrete case, the assumptions in (b) and (c) are not only sufficient, but also, in a way, necessary.

Theorem 6. (a) *If the* X_i *are integrable, and if there exists* $f(t)$ *such that* $x_t/f(t)$ *weakly converges towards a nondegenerate limit law, then* $e^{2\beta X_k}$ *belongs to the attraction domain of* F_α, *for an index* $\alpha \leqslant 1$.

(b) *If the* $e^{2\beta X_k}$ *are integrable, and if, being given an index* α *in* $]1, 2[$ *and a slowly varying function* L, $(x_t - vt)/t^{1/\alpha}L(t)$ *weakly converges towards a nondegenerate limit law, then* $e^{2\beta X_k}$ *belongs to the attraction domain of* F_α.

(c) *If the* $e^{2\beta X_k}$ *are integrable, and if* $(x_t - vt)/t^{1/2}$ *weakly converges towards a non-degenerate limit law, then* $e^{2\beta X_k}$ *has a finite second moment.*

The proofs of these theorems are quite similar to the proofs for the discrete case. As in the discrete case, if the X_i's are lattice and if the tail of $e^{2\beta X_i}$ is weakly regular, x_t will show a partial attraction phenomenon. Note that the assumption in (a) is rather weak, since it holds true as soon as R_1 belongs to some L^p, $p > 0$ (cf. Lemma 2). Finally, Propositions 2 and 3 allow one to give expressions for v and d in term of X alone.

Proposition 6. *When they are defined, the speed,* v, *and the diffusion coefficient,* d, *are given by*

$$v = \mathbb{E}[X]/\mathbb{E}[R],$$

and

$$d = \mathrm{Var}(X - vR)/\mathbb{E}[R],$$

where

$$\mathrm{Var}(X - vR) = \mathbb{E}[X^2] - v/\beta^2 \mathbb{E}[Xe^{2\beta X} - 2\beta X^2 - X] + v^2\mathbb{E}[R^2].$$

6. Computations of speed and diffusion coefficients

Proposition 2 follows from

$$\mathbb{E}[R \mid X = x] = \int_0^x \mathbb{E}(Z_t)\,\mathrm{d}t = (1/2\beta^2)(e^{2\beta x} - 2\beta x - 1),$$

using (3.1), and gives v:

$$v = \frac{2\beta^2 \mathbb{E}[X]}{\mathbb{E}[e^{2\beta X} - 2\beta X - 1]}.$$

Computation of the diffusion coefficient d

From Section 5 we have that

$$d = (1/\mu')\operatorname{Var}(X - vR),$$

where

$$\mu' = \mathbb{E}[R] = (1/2\beta^2)\mathbb{E}[e^{2\beta X} - 2\beta X - 1]$$

and

$$\operatorname{Var}(X - vR) = \mathbb{E}[X^2] - v/\beta^2 \mathbb{E}[X e^{2\beta X} - 2\beta X^2 - X] + v^2 \mathbb{E}[R^2].$$

Let us compute $\mathbb{E}[R^2]$:

$$u(x) = \mathbb{E}[R^2 \mid X = x] = \mathbb{E}\left[\left(\int_0^x Z_t \, dt\right)^2\right],$$

so we see at once that

$$u'(x) = 2\mathbb{E}\left[Z_x \int_0^x Z_t \, dt\right]$$

and

$$1/2u''(x) = \mathbb{E}[Z_x^2] + \mathbb{E}\left[\int_0^x Z_t \, dt \times \lim_{h \to 0+} (1/h)\mathbb{E}(Z_{x+h} - Z_x \mid \mathscr{F}_x)\right].$$

Using the infinitesimal generator of $(Z_t)_{t \geq 0}$, we get that

$$u''(x) = 2\mathbb{E}[Z_x^2] + 4\mathbb{E}\left[\int_0^x Z_t \, dt\right] + 2\beta u'(x).$$

Finally,

$$\mathbb{E}[R^2 \mid X] = (1/2\beta^4)(e^{4\beta X} + e^{2\beta x} - 2 - 6\beta X e^{2\beta X} + 2\beta^2 X^2),$$

and Proposition 3 follows. □

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