



# On a weighted embedding for generalized pontograms

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## Abstract

A weighted embedding for the generalized pontogram  $\{K_n(t): 0 \leq t \leq 1\}$  corresponding point-wise to a renewal process  $\{N(s): 0 \leq s < \infty\}$  via  $K_n(t) = n^{-1/2}(N(nt) - tN(n))$  is studied in this paper. After proper normalization, weak convergence results for the processes  $\{K_n(t): 0 \leq t \leq 1\}$  are derived both in sup-norm as well as in  $L_p$ -norm. These results are suggested to serve as asymptotic testing devices for detecting changes in the intensity of the underlying renewal process. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Let  $\{X_k: k \geq 1\}$  be a sequence of independent and identically distributed (i.i.d) random variables with

$$EX_1 = 1/\lambda > 0,$$

and

$$0 < \text{Var}(X_1) = \sigma^2 < \infty.$$

A renewal process based on  $\{X_k: k \geq 1\}$  is defined for  $s > 0$  in the following way:

$$N(s) + 1 = \min\{k: S_k > s\},$$

where  $S_k = X_1 + \dots + X_k$  ( $k = 1, 2, \dots$ ). The *pontogram process* is defined by

$$K_n(t) = \frac{N(nt) - tN(n)}{\sqrt{n}}, \quad 0 \leq t \leq 1 \quad (1)$$

( $n = 1, 2, \dots$ ). The notion of pontograms was introduced and developed by Kendall and Kendall (1980) for re-analyzing the (so-called) “Land’s End data set”. The statistical problem is whether or not an empirical set of  $n$  given data points in the plane could be considered to contain “too many straight line configurations”. Kendall and Kendall introduced a Poisson model for the number of “ $\varepsilon$ -blunt triangles” constructed

from the data points, and they then suggested testing the null hypothesis of “no change in intensity” against the alternative of “an early decrease in intensity” of the homogeneous Poisson process given by  $N(t)$  with  $\{X_k: k \geq 1\}$  being an i.i.d sequence of  $\exp(\lambda)$ -random variables. The Kendall–Kendall pontogram test is based upon the weak limiting behavior

$$\left(\frac{n}{N(n)}\right)^{1/2} \sup_{\varepsilon_1 \leq t \leq 1-\varepsilon_2} \frac{|K_n(t)|}{\sqrt{t(1-t)}} \Rightarrow \sup_{0 \leq s \leq (1/2) \log[(1-\varepsilon_1)/\varepsilon_1][(1-\varepsilon_2)/\varepsilon_2]} |V(s)|, \tag{2}$$

as  $n \rightarrow \infty$ , where  $\Rightarrow$  denotes convergence in distribution and  $\{V(s): 0 \leq s < \infty\}$  is an Ornstein–Uhlenbeck process with covariance function  $\text{Cov}(V(s), V(t)) = \exp(-|t-s|)$ .

Here it is necessary that  $0 < \varepsilon_1 < 1 - \varepsilon_2 < 1$  is fixed. Being interested in detecting early changes of the unknown intensity parameter  $\lambda$  (changepoint estimation), Kendall posed the question whether it is possible to replace  $\varepsilon_i$  in (2) be sequences  $\varepsilon_i(n) \rightarrow 0$  as  $n \rightarrow \infty$  for  $i=1, 2$ . Csörgö and Horváth (1987) gave a sufficient answer concerning this question. They constructed a weighted embedding for Poisson pontograms by showing that, on a rich enough probability space, there exists a sequence  $\{B_n(t): 0 \leq t \leq 1\}$  ( $n = 1, 2, \dots$ ) of Brownian bridges such that

$$\sup_{\delta/n \leq t \leq 1-\delta/n} n^\gamma \frac{|K_n(t) - \lambda^{1/2} B_n(t)|}{(t(1-t))^{1/2-\gamma}} = O_p(1) \tag{3}$$

as  $n \rightarrow \infty$ , for all  $\delta > 0$  and  $0 \leq \gamma < \frac{1}{2}$ . Their proof makes essential use of the Poisson assumption in constructing two *independent* embeddings of  $\{N(s): 0 \leq s \leq n\}$  into two independent Wiener processes, the first embedding over  $[0, \frac{1}{2}]$  starting from 0, the second one over  $[\frac{1}{2}, 1]$  starting from 1.

Huse (1988) (cf. also Eastwood, 1990) extended the work of Kendall and Kendall (1980) and Csörgö and Horváth (1987) to pontograms based on general renewal processes. She provided a weighted embedding like that of (3), but with the sup-range restricted to  $[\delta/n, 1 - \varepsilon(n)]$ , where  $\varepsilon(n) \gg \delta/n$  ( $\delta > 0$ ), the order of magnitude depending upon moment assumptions on the underlying distribution. Recently, Steinebach and Zhang (1993) proved that if  $\{X_k: k \geq 1\}$  is a sequence of *nonnegative* i.i.d. r.v.’s, then

$$\sup_{\delta/n \leq t \leq 1-\delta/n} \frac{n^\gamma |K_n(t) - \lambda^{3/2} \sigma B_n(t)|}{(t(1-t))^{1/2-\gamma}} = O_p(1). \tag{4}$$

The aim of this paper is to establish a full extension of the weighted embedding (3) to generalized pontograms as introduced in (1). The latter is given under finite moment generating function of  $X_1$  and under finite  $r$ th moment ( $r > 2$ ). Similarly to Csörgö and Horváth (1987, 1988a), a number of weak asymptotics for  $\{K_n(t): 0 \leq t \leq 1\}$  ( $n = 1, 2, \dots$ ) are immediate from such an embedding, both in sup-norms as well as  $L_p$ -norm.

In setting of changepoint estimations, these asymptotics are suggested to serve as asymptotic testing devices for detecting “changes in the intensity” of a general renewal counting process (for further details see Brodsky and Darkhovsky, 1993; Steinebach, 1993; Szyszkowicz, 1994). Similar asymptotics have been used by Csörgö and Horváth (1988b) to deal with change point problems based on  $U$ -statistics. Furthermore, Ferger (1994, 1995) extended results in Csörgö and Horváth (1988b) to more general  $U$ -statistical processes. For a recent comprehensive survey of changepoint analysis, the readers are referred to Csörgö and Horváth (1998).

The remainder of the paper is organized as follows: Section 2 is devoted to establish the weighted embedding for generalized pontograms given in (1) under finite moment generating function of  $X_1$  and under finite  $r$ th moment, respectively. In Section 3 we briefly discuss the sup-norm and  $L_p$ -norm asymptotics for the pontogram  $\{K_n(t): 0 \leq t \leq 1\}$ .

## 2. The weighted embedding for $\{K_n(t): 0 \leq t \leq 1\}$

In this section we shall establish the weighted embedding for the process  $\{K_n(t): 0 \leq t \leq 1\}$ . By Lemma 4.4.4 in Csörgö and Révész (1981), we can assume, without loss of generality, that our probability space  $(\Omega, \mathcal{F}, P)$  is so rich that all random variables and stochastic processes introduced so far and later on can be defined on it. First, we establish the weighted embedding of the process  $\{K_n(t): 0 \leq t \leq 1\}$  under the condition in which the moment generating function of  $X_1$  is finite.

**Theorem 2.1.** *If  $Ee^{t_0|X_1|} < \infty$  for some  $t_0 > 0$ , then there exists a sequence of Brownian bridges  $\{B_n(t): 0 \leq t \leq 1\}$  such that*

$$\sup_{\delta/n \leq t \leq 1-\delta/n} \frac{n^\gamma |K_n(t) - \lambda^{3/2} \sigma B_n(t)|}{(t(1-t))^{1/2-\gamma}} = O_p(1)$$

as  $n \rightarrow \infty$  for all  $\delta > 0$  and  $0 \leq \gamma < \frac{1}{2}$ .

**Proof.** The general idea of the proof is similar to that one of Steinebach and Zhang (1993) who constructed weighted embeddings for the processes  $\{N(s): \delta \leq s \leq \frac{1}{2}n\}$  and  $\{N(s): \frac{1}{2}n \leq s \leq n-\delta\}$ . They used the renewal process starting at the first renewal point after  $\frac{1}{2}n$  to obtain embedding to  $\{N(s): \frac{1}{2}n \leq s \leq n-\delta\}$ . This provides *independence* of what has happened on  $[\delta, \frac{1}{2}]$ , and consequently, this result makes it possible to obtain small approximation rates near the endpoints  $\delta$  and  $n-\delta$ , respectively.

In the general case (when the renewal process  $\{N(s): 0 \leq s < \infty\}$  is based on a sequence of i.i.d. r.v.'s with positive mean, but possibly with *negative* values), comparing with Steinebach and Zhang's nonnegative case, the difficulty is how to deal with the construction on  $[\frac{1}{2}n, n-\delta]$ . The main idea is to introduce another counting process and to establish the strong approximation for this counting process. Then we use this approximation to obtain small approximation rates near  $n-\delta$ . For sake of clarity, the proof will be given in several steps.

*Step 1:* Similarly to Step 1 in Steinebach and Zhang (1993), from Theorem 3.1 of Csörgö et al. (1987), we get, as  $n \rightarrow \infty$ ,

$$\sup_{\delta \leq t \leq n/2} t^{\gamma-1/2} |N(t) - \lambda t - \lambda^{3/2} \sigma \xi_n^{(1)}(t)| = O_p(1) \quad (5)$$

with a suitably chosen sequence of standard Wiener processes  $\{\xi_n^{(1)}(t): t \geq 0\}$  ( $n = 1, 2, \dots$ ). Later on, (5) will be used to deal with  $\{K_n(t)\}$  on the interval  $[\delta/n, \frac{1}{2}]$ .

Now we consider the construction on  $[\frac{1}{2}, 1-\delta/n]$ .

*Step 2:* Consider

$$N_{n/2}(t) + 1 = \min\{k: X_{N(n/2)+2} + \dots + X_{N(n/2)+1+k} > t\}.$$

Let

$$\begin{aligned}\tilde{Z}_1^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)}I_{\{N_{n/2}(n/2)\geqslant 1\}} + X_{N(n/2)+1+1}I_{\{N_{n/2}(n/2)=0\}}, \\ \tilde{Z}_2^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)-1}I_{\{N_{n/2}(n/2)\geqslant 2\}} + X_{N(n/2)+1+2}I_{\{N_{n/2}(n/2)=1\}}, \\ &\dots\dots\dots \\ \tilde{Z}_k^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)-k+1}I_{\{N_{n/2}(n/2)\geqslant k\}} + X_{N(n/2)+1+k}I_{\{N_{n/2}(n/2)=k-1\}}, \\ &\dots\dots\dots\end{aligned}$$

From the proof of Theorem 2.1 in Steinebach and Zhang (1993), we know that there exists a sequence of suitably chosen standard Wiener processes  $\{\xi_n^{(2)}(t); 0\leqslant t<\infty\}$  such that

$$\max_{1\leqslant t\leqslant N_{n/2}(n/2)+1}t^{\gamma-1/2}\left|\sum_{i=1}^{[t]}\tilde{Z}_i^{(n)}-\lambda^{-1}t-\sigma\xi_n^{(2)}(t)\right|=\text{O}_{\text{P}}(1),\tag{6}$$

where  $[t]$  is the integer part of  $t$ . Furthermore, from the proof of Theorem 2.1 in Steinebach and Zhang (1993) we know that  $\{\xi_n^{(2)}(t); 0\leqslant t<\infty\}$  is independent of  $\{\xi_n^{(1)}(t); t\geqslant 0\}$ . Now let

$$\begin{aligned}Z_1^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)+1}I_{\{N_{n/2}(n/2)\geqslant 0\}}, \\ Z_2^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)}I_{\{N_{n/2}(n/2)\geqslant 1\}}, \\ &\dots\dots\dots \\ Z_k^{(n)} &= X_{N(n/2)+1+N_{n/2}(n/2)+1-k+1}I_{\{N_{n/2}(n/2)\geqslant k-1\}}, \\ &\dots\dots\dots\end{aligned}$$

Thus, we have that for  $t < N_{n/2}(n/2) + 1$

$$\sum_{k=1}^{[t]+1}Z_k^{(n)}=X_{N_{n/2}(n/2)+1}+\sum_{k=1}^{[t]}\tilde{Z}_k^{(n)}.$$

From  $\text{e}^{t_0|X_1|}<\infty$  and (6), we get that

$$\max_{1\leqslant t\leqslant N_{n/2}(n/2)+1}t^{\gamma-1/2}\left|\sum_{i=1}^{[t]}Z_i^{(n)}-\lambda^{-1}t-\sigma\xi_n^{(2)}(t)\right|=\text{O}_{\text{P}}(1).\tag{7}$$

*Step 3:* Define

$$\tilde{N}_{n/2}(s)=\max\{k: Z_1^{(n)}+\cdots+Z_k^{(n)}\leqslant s\},\quad s\geqslant 0.\tag{8}$$

The latter renewal process will be used to establish an approximation for  $\{N(n)-N(s); n/2\leqslant s\leqslant n-\delta\}$ . For this reason, we first derive an approximation of the process  $\{\tilde{N}_{n/2}(s); 0\leqslant s\leqslant n/2\}$ . To do this, we show that for any  $\delta>0$ ,

$$\sup_{\delta\leqslant s\leqslant n/2}\frac{\tilde{N}_{n/2}(s)}{s}=\text{O}_{\text{P}}(1).\tag{9}$$

Let

$$\mathcal{A}_n(K) = \left\{ \omega : \max_{1 \leq s \leq N_{n/2}(n/2)+1} s^{\gamma-1/2} \left| \sum_{k=1}^{[s]} Z_k^{(n)} - \lambda^{-1}s - \sigma_{\xi_n}^{(2)}(s) \right| \leq K \right\},$$

$$\mathcal{B}_n(K) = \left\{ \omega : \sup_{1 \leq s \leq N_{n/2}(n/2)+1} \frac{\sigma|\xi^{(2)}(s)|}{\sqrt{s \log \log s}} \leq K \right\}$$

and

$$\mathcal{C}_n(K) = \left\{ \omega : \sup_{\delta \leq s \leq n/2} \frac{\tilde{N}_{n/2}(s)}{s} \leq K \right\}.$$

Using Theorem 1.2.1 of Csörgő and Révész (1981) and (5), for any  $\varepsilon > 0$ , there exist  $K_1$  and  $n_0$  such that for  $n \geq n_0$

$$P(\mathcal{A}_n(K_1)) \geq 1 - \varepsilon, \quad (10)$$

and

$$P(\mathcal{B}_n(K_1)) \geq 1 - \varepsilon. \quad (11)$$

Therefore, there exist  $s_0$  and  $\delta_1 < \lambda^{-1}$  such that for  $\omega \in \mathcal{A}_n(K_1) \cap \mathcal{B}_n(K_1)$  and  $s_0 \leq s \leq N_{n/2}(n/2) + 1$ ,

$$\left| \sum_{k=1}^{[s]} Z_k^{(n)} - \lambda^{-1}s \right| \leq K_1 s^{1/2-\gamma} + K_1 \sqrt{s \log \log s} < \delta_1 s.$$

This means that for  $\omega \in \mathcal{A}_n(K_1) \cap \mathcal{B}_n(K_1)$  and  $s_0 \leq s \leq t \leq N_{n/2}(n/2) + 1$ ,

$$(\lambda^{-1} - \delta_1)s \leq (\lambda^{-1} - \delta_1)t \leq \sum_{k=1}^{[t]} Z_k^{(n)}. \quad (12)$$

From the definition of the sequence of  $\{Z_k^{(n)}: k \geq 1\}$ ,

$$\sum_{k=1}^{[t]} Z_k^{(n)} = \sum_{k=1}^{N_{n/2}(n/2)+1} Z_k^{(n)} \quad \text{for } t \geq N_{n/2}(n/2) + 1. \quad (13)$$

Consequently, from (12) and (13) we get that for  $\omega \in \mathcal{A}_n(K_1) \cap \mathcal{B}_n(K_1)$  and  $s_0 \leq s \leq N_{n/2}(n/2) + 1$ ,

$$\tilde{N}_{n/2}(s(\lambda^{-1} - \delta_1)) \leq s.$$

Thus, for  $s_0(\lambda^{-1} - \delta_1) \leq s \leq (\lambda^{-1} - \delta_1)(N_{n/2}(n/2) + 1)$  and  $\omega \in \mathcal{A}_n(K_1) \cap \mathcal{B}_n(K_1)$ ,

$$\tilde{N}_{n/2}(s) \leq (\lambda^{-1} - \delta_1)^{-1}s. \quad (14)$$

From the definition of  $N_{n/2}(s)$  and the renewal theorem, we know that there exist  $\delta_2 < \lambda$  and  $n_1$  such that for  $n \geq n_1$

$$P\left(N_{n/2}\left(\frac{n}{2}\right) \geq (\lambda - \delta_2)n\right) \geq 1 - \varepsilon. \quad (15)$$

Using (10)–(11) and (14)–(15), we know that there is  $0 < \alpha < 1$  such that

$$\sup_{\delta \leq s \leq n/2} \frac{\tilde{N}_{n/2}(s)}{s} = O_P(1). \quad (16)$$

Similar to the Step 4 in the proof of Theorem 2.1 in Steinebach and Zhang (1993), we have

$$\sup_{n/2 \leq s \leq n/2} \frac{\tilde{N}_{n/2}(s)}{s} = O_p(1). \quad (17)$$

On combining now (16) and (17) we get (9).

*Step 4:* We are in the position to establish the weighted embedding of  $\{\tilde{N}_{n/2}(s): \delta \leq s \leq n/2\}$ . Here the definition of  $\tilde{N}_{n/2}(s)$  is different from the one given in the proof of Theorem 2.1 in Steinebach and Zhang (1993). Thus we can not use Theorem 3.1 in Csörgö et al. (1987) to get the weighted approximation for  $\tilde{N}_{n/2}(s)$  as what Steinebach and Zhang (1993) gave.

From Step 3, for  $\omega \in \mathcal{A}_n(K)$  and  $1 \leq s \leq N_{n/2}(n/2) + 1$ ,

$$-Ks^{1/2-\gamma} + \lambda^{-1}s + \sigma_{\xi_n}^{(2)}(s) \leq \sum_{k=1}^{[s]} Z_k^{(n)} \leq Ks^{1/2-\gamma} + \lambda^{-1}s + \sigma_{\xi_n}^{(2)}(s) \quad (18)$$

and for  $\omega \in \mathcal{C}_n(K)$  and  $1 \leq s \leq n/2$ ,

$$\tilde{N}_{n/2}(s) \leq Ks. \quad (19)$$

Hence, (18) and (19) ensure that for  $\omega \in \mathcal{A}_n(K) \cap \mathcal{C}_n(K)$  and  $s \leq n/2$ ,

$$\begin{aligned} \tilde{N}_{n/2}(s) &= \max \left\{ k: \sum_{i=1}^k Z_i^{(n)} \leq s \right\} \\ &\leq \sup \left\{ t: \sum_{i=1}^{[t]} Z_i^{(n)} \leq s \text{ and } t \leq Ks \right\} \\ &\leq \sup \{ t: -Kt^{1/2-\gamma} + \lambda^{-1}t + \sigma_{\xi_n}^{(2)}(t) \leq s \text{ and } t \leq Ks \} \\ &\leq \sup \{ t: \lambda^{-1}t + \sigma_{\xi_n}^{(2)}(t) \leq s + K(Ks)^{1/2-\gamma} \}. \end{aligned} \quad (20)$$

From the definitions of  $N_{n/2}(s)$  and  $\{Z_k^{(n)}, k \geq 1\}$ , we have

$$\sum_{k=1}^{N_{n/2}(n/2)+1} Z_k^{(n)} > \frac{n}{2}.$$

Thus, by (18) and (19), we have that for  $\omega \in \mathcal{A}_n(K) \cap \mathcal{C}_n(K)$  and  $s \leq n/2$ ,

$$\begin{aligned} \tilde{N}_{n/2}(s) &= \max \left\{ k: \sum_{i=1}^k Z_i^{(n)} \leq s \right\} \\ &\geq \min \left\{ k: \sum_{i=1}^k Z_i^{(n)} \geq s \right\} - 1 \\ &\geq \inf \left\{ t - 1: \sum_{i=1}^{[t]} Z_i^{(n)} \geq s \text{ and } t \leq Ks \right\} - 1 \\ &\geq \inf \{ t - 1: \lambda^{-1}t + \sigma_{\xi_n}^{(2)}(t) \geq s - K(Ks)^{1/2-\gamma} \text{ and } t \leq Ks \} - 1 \\ &\geq \inf \{ t - 1: \lambda^{-1}t + \sigma_{\xi_n}^{(2)}(t) \geq s - K(Ks)^{1/2-\gamma} \} - 1. \end{aligned} \quad (21)$$

Now let

$$\tilde{M}_n(s) = \sup\{t: \sigma \xi_n^{(2)}(t) + \lambda^{-1}t = s\}$$

and

$$M_n(s) = \inf\{t: \sigma \xi_n^{(2)}(t) + \lambda^{-1}t = s\}.$$

Then (20) and (21) give

$$M_n(s - K(Ks)^{1/2-\gamma}) \leq \tilde{N}_{n/2}(s) \leq \tilde{M}_n(s + K(Ks)^{1/2-\gamma}). \quad (22)$$

Clearly, we know that  $M_n(s)$  is a Markov time and

$$M_n(s) \leq \tilde{M}_n(s).$$

Thus, for any  $A > 0$ ,

$$\begin{aligned} P(\tilde{M}_n(s) - M_n(s) > A \log s) \\ &= P(\sigma \xi_n^{(2)}(\tilde{M}_n(s)) + \lambda^{-1}\tilde{M}_n(s) = s, \\ &\quad \sigma \xi_n^{(2)}(M_n(s)) + \lambda^{-1}M_n(s) = s \text{ and } \tilde{M}_n(s) - M_n(s) > A \log s) \\ &\leq P\left(\inf_{t > A \log s} \{\sigma \xi_n^{(2)}(t) + \lambda^{-1}t\} \leq 0\right) \\ &\leq P\left(\sup_{t > A \log s} \{-\sigma \xi_n^{(2)}(t) - \lambda^{-1}t\} \geq 0\right) \\ &\leq P\left(\sup_{t > A \log s} \{-\sigma \xi_n^{(2)}(t) - (\lambda^{-1}/2)t\} \geq A(\lambda^{-1}/2) \log s\right) \\ &\leq P\left(\sup_{0 \leq t < \infty} \{-\sigma \xi_n^{(2)}(t) - (\lambda^{-1}/2)t\} \geq A(\lambda^{-1}/2) \log s\right). \end{aligned} \quad (23)$$

By assertion (9.21) on p. 112 in Karatzas and Shreve (1988), (23) implies that

$$P(\tilde{M}_n(s) - M_n(s) > A \log s) \leq s^{-A\lambda^{-1}/4}.$$

Consequently,

$$|\tilde{M}_n([s]) - M_n([s])| =_{\text{a.s.}} O(\log s). \quad (24)$$

From Theorem 2.1 in Csörgö et al. (1987), we have that there is a sequence of suitably chosen standard Wiener process  $\{\xi_n^{(3)}(t); t \geq 0\}$ , which is independent of  $\{\xi_n^{(1)}(t); t \geq 0\}$ , such that

$$\sup_{0 \leq t \leq s} |M_n(t) - \lambda t - \lambda^{3/2} \sigma \xi_n^{(3)}(t)| = O_p(\log s). \quad (25)$$

Note that from Theorem 1.2.1 in Csörgö and Révész (1981),

$$\sup_{0 \leq t \leq s, 0 \leq v \leq 1} |\xi_n^{(3)}(t+v) - \xi_n^{(3)}(t)| =_{\text{a.s.}} O(\log s). \quad (26)$$

From the definition of  $\tilde{M}_n(s)$ ,

$$\begin{aligned} (\tilde{M}_n([t]) - M_n([t])) + M_n([t]) &\leq \tilde{M}_n(t) \\ &\leq (\tilde{M}_n([t] + 1) - M_n([t] + 1)) + M_n([t] + 1). \end{aligned} \quad (27)$$

Combining (24)–(27) yields that

$$\sup_{0 \leq t \leq s} |\tilde{M}_n(t) - \lambda t - \lambda^{3/2} \sigma \xi_n^{(3)}(t)| = O_P(\log s). \tag{28}$$

Similar to argument (3.7) in the proof of Theorem 3.1 in Csörgö et al. (1987), from (22) we have

$$\begin{aligned} &|\tilde{N}_{n/2}(s) - \lambda s - \lambda^{3/2} \sigma \xi_n^{(3)}(s)| \\ &\leq_{\text{a.s.}} |\tilde{M}_n(s + K(Ks)^{1/2-\gamma}) - \lambda(s + K(Ks)^{1/2-\gamma}) - \lambda^{3/2} \sigma \xi_n^{(3)}(s + K(Ks)^{1/2-\gamma})| \\ &\quad + |M_n(s - K(Ks)^{1/2-\gamma}) - \lambda(s - K(Ks)^{1/2-\gamma}) - \lambda^{3/2} \sigma \xi_n^{(3)}(s - K(Ks)^{1/2-\gamma})| \\ &\quad + \lambda^{3/2} \sigma |\xi_n^{(3)}(s + K(Ks)^{1/2-\gamma}) - \xi_n^{(3)}(s)| \\ &\quad + \lambda^{3/2} \sigma |\xi_n^{(3)}(s - K(Ks)^{1/2-\gamma}) - \xi_n^{(3)}(s)| \\ &\quad + 2\lambda K(Ks)^{1/2-\gamma}. \end{aligned}$$

By Theorem 1.2.1 in Csörgö and Révész (1981), (25) and (28) we get that

$$\sup_{\delta \leq s \leq n/2} s^{\gamma-1/2} |\tilde{N}_{n/2}(s) - \lambda s - \lambda^{3/2} \sigma \xi_n^{(3)}(s)| = O_P(1). \tag{29}$$

*Step 5:* In this step we give the strong approximation for  $\{N(n) - N(s): n/2 \leq s \leq n\}$ .  
Let

$$R_{n/2} = X_1 + \cdots + X_{N(n/2)+1} - \frac{n}{2}.$$

$R_{n/2}$  is called the residual waiting time relative to the renewal process generated by the sequence  $\{X_k: k \geq 1\}$ . By Theorem 6.2 on p. 58 in Gut (1988), we know that

$$R_{n/2} = O_P(1). \tag{30}$$

Let

$$N_n(s) + 1 = \min\{k: X_{N(n)+1+1} + \cdots + X_{N(n)+1+k} > s\}.$$

From

$$P(N_n(R_{n/2}) \geq M) \leq P(R_{n/2} \geq s) + P(X_1 + \cdots + X_M \leq s)$$

and the law of large numbers for the sequence  $\{X_k: k \geq 1\}$ , we have

$$N_n(R_{n/2}) = O_P(1). \tag{31}$$

Note that

$$N(n + R_{n/2}) - N(n) \leq N_n(R_{n/2}) + 1. \tag{32}$$

On the other hand, from the definitions of  $N(s)$  and  $\tilde{N}_{n/2}(s)$ , for  $s \in (n/2, n)$

$$N(n + R_{n/2}) - N(s) = \tilde{N}_{n/2}(n - s + R_{n/2} + \tilde{R}_n), \tag{33}$$

where  $\tilde{R}_n = X_1 + \cdots + X_{N(n+R_{n/2})+1} - (n + R_{n/2})$ . Similar to (30), we have

$$\tilde{R}_n = O_P(1). \tag{34}$$



Using (31), (32) and (33), we get that

$$\begin{aligned} & \sup_{n/2 \leq s \leq n-\delta} (n-s)^{\gamma-1/2} |N(n) - N(s) - \tilde{N}_{n/2}(n-s+R_{n/2} + \tilde{R}_n)| \\ & \leq \sup_{n/2 \leq s \leq n-\delta} (n-s)^{\gamma-1/2} |N(n+R_{n/2}) - N(s) - \tilde{N}_{n/2}(n-s+R_{n/2} + \tilde{R}_n)| \\ & \quad + |N(n+R_{n/2}) - N(n)| \\ & = O_P(1). \end{aligned} \quad (35)$$

Furthermore, by (29) and (34),

$$\sup_{\delta \leq s \leq n/2} s^{\gamma-1/2} |\tilde{N}_{n/2}(s+R_{n/2} + \tilde{R}_n) - \lambda s - \lambda^{3/2} \sigma \xi_n^{(4)}(s)| = O_P(1). \quad (36)$$

Consequently, (35) and (36) ensure that

$$\sup_{n/2 \leq s \leq n-\delta} (n-s)^{\gamma-1/2} |N(n) - N(s) - \lambda(n-s) - \lambda^{3/2} \sigma \xi_n^{(4)}(n-s)| = O_P(1). \quad (37)$$

**Step 6:** In this step we combine the two independent embeddings of (5) and (37). Define

$$\xi_n(s) = \begin{cases} \xi_n^{(1)}(s) & \text{for } 0 \leq s \leq \frac{1}{2}n, \\ \xi_n^{(1)}(\frac{1}{2}n) + \xi_n^{(4)}(\frac{1}{2}n) - \xi_n^{(4)}(n-s) & \text{for } \frac{1}{2}n \leq s \leq n. \end{cases}$$

By the independence of  $\{\xi_n^{(1)}(s); 0 \leq s < \infty\}$  and  $\{\xi_n^{(4)}(s); 0 \leq s < \infty\}$ , it is easy to check for the covariance function  $\xi_n(s)$  and conclude that  $\{\xi_n(s); 0 \leq s < \infty\}$  is Wiener process.

Now by (5) and (37),

$$\begin{aligned} & \sup_{\delta/n \leq s \leq 1-\delta/n} n^{\gamma-1/2} (t(1-t))^{\gamma-1/2} |N(nt) - tN(n) - \lambda^{3/2} \sigma (\xi_n(nt) - t\xi_n(n))| \\ & \leq \sup_{\delta/n \leq s \leq 1/2} n^{\gamma-1/2} (t(1-t))^{\gamma-1/2} |N(nt) - \lambda nt - \lambda^{3/2} \sigma \xi_n^{(1)}(nt)| \\ & \quad + \sup_{\delta/n \leq s \leq 1/2} n^{\gamma-1/2} t(t(1-t))^{\gamma-1/2} |N(\frac{1}{2}n) - \frac{1}{2}\lambda n - \lambda^{3/2} \sigma \xi_n^{(1)}(\frac{1}{2}n)| \\ & \quad + \sup_{\delta/n \leq s \leq 1/2} n^{\gamma-1/2} t(t(1-t))^{\gamma-1/2} |N(n) - N(\frac{1}{2}n) - \frac{1}{2}\lambda n - \lambda^{3/2} \sigma \xi_n^{(4)}(\frac{1}{2}n)| \\ & \quad + \sup_{\delta/n \leq s \leq 1-\delta/n} n^{\gamma-1/2} (t(1-t))^{\gamma-1/2} |N(n) - N(nt) - \lambda n(1-t) \\ & \quad - \lambda^{3/2} \sigma \xi_n^{(4)}(n-nt)| \\ & \quad + \sup_{\delta/n \leq s \leq 1/2} n^{\gamma-1/2} (1-t)(t(1-t))^{\gamma-1/2} |N(\frac{1}{2}n) - \frac{1}{2}\lambda n - \lambda^{3/2} \sigma \xi_n^{(1)}(\frac{1}{2}n)| \\ & \quad + \sup_{\delta/n \leq s \leq 1/2} n^{\gamma-1/2} (1-t)(t(1-t))^{\gamma-1/2} |N(n) - N(\frac{1}{2}n) \\ & \quad - \frac{1}{2}\lambda n - \lambda^{3/2} \sigma \xi_n^{(4)}(\frac{1}{2}n)| \\ & = O_P(1). \end{aligned}$$

Noticing that  $B_n(t) = n^{-1/2}(\xi_n(nt) - t\xi_n(n))$  is a Brownian bridge for each  $n \geq 1$ , Theorem 2.1 is proved.  $\square$

Finally, we give the weighted embedding of the process  $\{K_n(t): 0 \leq t \leq 1\}$  when only  $r$ th moment of  $X_1$  exists.

**Theorem 2.2.** *Assuming that  $E|X_1|^r < \infty$  (for some  $r > 2$ ), we can define a sequence of Brownian bridges  $\{B_n(s): 0 \leq s \leq 1\}$  such that*

$$\sup_{\delta/n \leq t \leq 1-\delta/n} \frac{n^\gamma |K_n(t) - \lambda^{3/2} \sigma B_n(t)|}{(t(1-t))^{1/2-\gamma}} = O_P(1)$$

for all  $\delta > 0$  and  $0 \leq \gamma < \frac{1}{2} - 1/r$ .

The proof of this theorem is similar to the proof of Theorem 2.1, so the details are omitted.

### 3. Supremum of the pontograms

The sup-norm and  $L_p$ -norm asymptotics for pontograms given in Csörgö and Horváth (1987) for the Poisson pontograms and Steinebach and Zhang (1993) for the pontograms with the nonnegative assumption  $X_k \geq 0$  can be extended to the case of general pontograms and general  $p, 1 \leq p < \infty$ , using the same technique. Let

$$\begin{aligned} a(x) &= (2 \log x)^{1/2}, \\ b(x) &= 2 \log x + \frac{1}{2} \log \log x - \frac{1}{2} \log \pi, \\ c(x) &= \log \frac{1-x}{x}, \end{aligned}$$

and let  $Y$  be a random variable with distribution function  $\exp\{-2 \exp(-y)\}, -\infty < y < \infty$ . Then, we have

**Theorem 3.1.** *Assume that the distribution of  $X_1$  is non-arithmetic and  $E|X_1|^r < \infty$  (for some  $r > 2$ ). If  $k_n \rightarrow \infty, k_n/n \rightarrow 0$  ( $n \rightarrow \infty$ ), then,*

$$\begin{aligned} a(\log n)(\lambda^{3/2} \sigma)^{-1} \sup_{0 < t < 1} |K_n(t)|/(t(1-t))^{1/2} - b(\log n) &\Rightarrow Y, \\ a(\tfrac{1}{2} \log k_n)(\lambda^{3/2} \sigma)^{-1} \sup_{0 < t \leq k_n/n} |K_n(t)|/(t(1-t))^{1/2} - b(\tfrac{1}{2} \log k_n) &\Rightarrow Y, \\ a(\tfrac{1}{2} \log k_n)(\lambda^{3/2} \sigma)^{-1} \sup_{1-k_n/n \leq t < n} |K_n(t)|/(t(1-t))^{1/2} - b(\tfrac{1}{2} \log k_n) &\Rightarrow Y \end{aligned}$$

and

$$a(c(\log k_n/n))(\lambda^{3/2} \sigma)^{-1} \sup_{k_n/n \leq t \leq 1-k_n/n} |K_n(t)|/(t(1-t))^{1/2} - b(c(\log k_n)) \Rightarrow Y$$

as  $n \rightarrow \infty$ .

**Proof.** See the proof of Theorem 3.1 in Steinebach and Zhang (1993).  $\square$

Now we consider  $L_p$ -norm asymptotics for pontograms. Note that from Theorem 2.2,

$$\int_0^{|X_1|/n} |K_n(t)|^p / (t(1-t))^{p/2+1} dt = O_p((N(n)/n)^p |X_1|^{p/2}) = O_p(1), \quad (38)$$

$$\int_{|S_{N(n)}|/n}^1 |K_n(t)|^p / (t(1-t))^{p/2+1} dt = O_p((N(n)/n)^p (n - S_N(n))^{p/2}) = O_p(1). \quad (39)$$

Similarly to the  $L_2$ -case in Csörgö and Horváth (1987), Theorem 3.1. Instead of Lemma 3.1 there, using Theorem 3.4 and assertion (4.27) of Csörgö and Horváth (1988a), here, from (38) to (39), we get

**Theorem 3.2.** Assume that the distribution of  $X_1$  is non-arithmetic and  $E|X_1|^r < \infty$  (for some  $r > 2$ ). If  $k_n \rightarrow \infty$ ,  $k_n/n \rightarrow 0$  ( $n \rightarrow \infty$ ), then

$$(4D \log n)^{-1/2} \left\{ (\lambda^{3/2} \sigma)^{-p} \int_0^1 |K_n(t)|^p / (t(1-t))^{p/2+1} dt - 2m \log n \right\} \Rightarrow N(0, 1),$$

$$(2D \log k_n)^{-1/2} \left\{ (\lambda^{3/2} \sigma)^{-p} \int_0^{k_n/n} |K_n(t)|^p / (t(1-t))^{p/2+1} dt - m \log k_n \right\} \\ \Rightarrow N(0, 1),$$

$$(2D \log k_n)^{-1/2} \left\{ (\lambda^{3/2} \sigma)^{-p} \int_{1-k_n/n}^1 |K_n(t)|^p / (t(1-t))^{p/2+1} dt - m \log k_n \right\} \\ \Rightarrow N(0, 1)$$

and

$$(4D \log n/k_n)^{-1/2} \left\{ (\lambda^{3/2} \sigma)^{-p} \int_{k_n/n}^{1-k_n/n} |K_n(t)|^p / (t(1-t))^{p/2+1} dt - m \log(n/k_n) \right\} \\ \Rightarrow N(0, 1),$$

as  $n \rightarrow \infty$ , where  $D = D(p)$  is a positive constant,

$$m = m(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |x|^p \exp(-\tfrac{1}{2}x^2) dx,$$

and  $N(0, 1)$  stands for the standard normal variable.

**Remark.** Here the assumption that the distribution of  $X_1$  is non-arithmetic is necessary in Theorems 3.1 and 3.2, otherwise the sup-norm and  $L_p$ -norm asymptotics of Csörgö and Horváth (1988a) cannot be obtained, see Landau (1993).

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