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Simulating the ruin probability of risk processes with delay in claim settlement

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Abstract

A risk process with delay in claim settlement is usually described in terms of a Poisson shot-noise process (see Klüppelberg and Mikosch (Bernoulli 1 (1995) 125) and Brémaud (Appl. Probab. 37 (2000) 914)). In particular, Brémaud proves that under suitable conditions the corresponding ruin probability goes to zero not slower than an exponential rate. This yields problems if we want to estimate the ruin probability by a Monte Carlo simulation. In this paper we overcome these difficulties deriving the asymptotically efficient simulation law.

Keywords: Importance sampling; Monte Carlo simulation; Ruin probabilities; Poisson shot-noise process

1. Introduction

A Poisson shot-noise is a stochastic process of the form

$$X(t) = \sum_{n \geqslant 1} h(t - T_n, Z_n)\mathbf{1}_{(0,t]}(T_n),$$

where $\{T_n\}_{n\geqslant 1}$ is the sequence of times of a homogeneous Poisson process with intensity λ, $\{Z_n\}_{n\geqslant 1}$ is a sequence of i.i.d. (independent and identically distributed) non-negative random variables, independent of the Poisson process, and $h : \mathbb{R} \times [0, \infty) \to [0, \infty)$ is a measurable function such that $h(t, z) = 0$ for non-positive times.

We define in a natural way the integrated Poisson shot-noise process by

$$S(t) = \int_0^t X(s)\,\mathrm{d}s = \sum_{n \geqslant 1} H(t - T_n, Z_n)\mathbf{1}_{(0,t]}(T_n),$$

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where

$$H(t,z)=\int_0^t h(s,z)\,\mathrm{d}s.$$

The interpretation of this model in terms of non-life insurance is the following (see Klüppelberg and Mikosch, 1995; Brémaud, 2000). Let us suppose that a claim occurs at time T_n, and the insurance company honours this claim at the rate $h(t-T_n,Z_n)$. The total amount paid in the time interval $(0,t]$ is therefore $S(t)$. Assuming that the insurance company starts with an initial fortune $u>0$, and letting $c>0$ denote the gross premium risk, the corresponding ruin probability is

$$\psi(u)=P(T_u<\infty),$$

where

$$T_u=\inf\{t\geqslant 0: S(t)-ct\geqslant u\}$$

and $\inf\emptyset=+\infty$ by convention.

In Brémaud (2000) it is proved that under the following assumptions:

$$\mathbb{E}[\mathrm{e}^{\theta H(\infty,Z_1)}]<\infty \quad \text{for all } \theta \text{ in a neighbourhood of } 0, \tag{1.1}$$

$$c>\lambda\mathbb{E}[H(\infty,Z_1)] \tag{1.2}$$

and

$$\text{there exists } w>0 \text{ such that } \lambda(\mathbb{E}[\mathrm{e}^{wH(\infty,Z_1)}]-1)-cw=0 \tag{1.3}$$

it holds

$$\psi(u)\leqslant \mathrm{e}^{-wu} \quad \text{for all } u\geqslant 0 \tag{1.4}$$

and

$$\lim_{u\to\infty}\frac{1}{u}\ln\psi(u)=-w. \tag{1.5}$$

Throughout this paper we make the following further assumptions on the model:

$$H(\infty,z)=z \tag{1.6}$$

and

$$P(0<Z_1<\infty)=1. \tag{1.7}$$

As far as assumption (1.6) is concerned we notice that it holds in many cases interesting for applications. For instance, when $h(t,z)=\mathbf{1}_{(0,z]}(t)$ the Poisson shot-noise process can be interpreted as a teletraffic model (see Section 5).

In this work we consider the estimation of the ruin probability $\psi(u)$ by an efficient Monte Carlo simulation. We observe that the direct estimation by the relative frequency

$$\hat{r}=\frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{t_u^{(k)}<\infty},$$

where $t_u^{(1)},\ldots,t_u^{(n)}$ are n independent simulations under P of the random variable T_u, is inefficient. Indeed, for a good relative accuracy, it is required a great number of replications n because, under assumptions (1.1)–(1.3), $\psi(u)$ goes to zero not slower than an exponential rate as $u \to \infty$. Moreover, in our case a direct simulation is even impossible because each realization of the event $\{T_u = +\infty\}$ requires an infinite time in simulation.

To overcome these difficulties we use importance sampling (the reader is referred to the paper by Glynn and Iglehart (1989) and the book by Bucklew (1990) for a thorough treatment on the importance sampling). The idea is to consider independent simulations of the event $\{T_u < \infty\}$ under another suitable law Q which belongs to a class of admissible laws. Allowing a wide class of simulation laws, we derive the unique asymptotically efficient simulation law (as $u \to \infty$) in a sense closely related to large deviations theory.

An early related paper on importance sampling techniques is Siegmund (1976), where it is considered the simulation of probabilities that occur in sequential tests, and optimality results are proved for an exponential family of possible simulation distributions. Similar ideas have been applied by Asmussen (1985) within the framework of insurance risk for the simulation of ruin probabilities. Lehtonen and Nyrhinen (1992) use importance sampling for the simulation of level-crossing probabilities of discrete time random walks. Their techniques involve again exponentially twisted distributions, but while in Siegmund (1976) and Asmussen (1985) the efficiency of a possible simulation distribution is measured directly by the variance of the estimator, in Lehtonen and Nyrhinen (1992) a new criterion is used, based on large deviations theory. A related work is Macci (2001). In this article, we adapt the techniques developed by Lehtonen and Nyrhinen (1992) to the simulation's problem described before. In the particular case of a compound Poisson risk model, the asymptotically efficient simulation law given in this paper coincides with the one given in Asmussen (2000, Chapter X) by importance sampling via Lundberg conjugation.

The paper is organized as follows. In Section 2, we give some preliminaries on the importance sampling technique, introducing the class of admissible laws. An asymptotically efficient law for simulations will be derived in Section 3. In Section 4 we show the uniqueness (or optimality) of such a law. In Section 5, we apply our result to a teletraffic model described in Kostantopoulos and Lin (1998) and Brémaud (2000).

2. Preliminaries

In this section, we describe the importance sampling technique in our specific case. For this we start introducing the following class of probability measures. We say that a probability measure Q belongs to the class $\mathscr{D}$ if and only if:

(A1) Q is absolutely continuous with respect to P on the σ-field $\mathscr{F}_t^C$, for each $t \geqslant 0$, where $\{\mathscr{F}_t^C\}_{t\geqslant 0}$ is the filtration generated by the compound Poisson process

$$C(t) = \sum_{n\geqslant 1} Z_n \mathbf{1}_{(0,t]}(T_n).$$

(A2) Under Q the stochastic process $\{T_n\}_{n\geqslant 1}$ is a homogeneous Poisson process with intensity $\lambda^{(Q)}$, independent of the i.i.d. sequence of random variables $\{Z_n\}_{n\geqslant 1}$, whose common law $Q^{(Z)}$ under Q is absolutely continuous with respect to their common law $P^{(Z)}$ under P, with

$$\mathbb{E}_{Q^{(Z)}}\left[\left(\frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(Z_1)\right)^2\right] < \infty. \tag{2.1}$$

To avoid the case of infinite time in simulation, we define the class of admissible laws as

$$\mathscr{C} = \{Q \in \mathscr{D} : Q(T_u < \infty) = 1 \text{ for all } u > 0\}.$$

Remark 2.1. We notice that if $Q \in \mathscr{D}$ then, for each $t > 0$,

$$\gamma_t^{Q,P} = \frac{\mathrm{d}Q^{(Z)}}{\mathrm{d}P^{(Z)}}(Z_1)\cdots\frac{\mathrm{d}Q^{(Z)}}{\mathrm{d}P^{(Z)}}(Z_{N_t})\left(\frac{\lambda^{(Q)}}{\lambda}\right)^{N_t}\exp\{-(\lambda^{(Q)}-\lambda)t\}, \tag{2.2}$$

where $\gamma_t^{Q,P}$ denotes the density of Q with respect to P on the σ-field $\mathscr{F}_t^C$ and

$$N_t = \sum_{n\geqslant 1}\mathbf{1}_{(0,t]}(T_n). \tag{2.3}$$

Moreover, letting $\gamma_t^{P,Q}$ denote the density of P with respect to Q on the σ-field $\mathscr{F}_t^C$ then

$$\gamma_t^{P,Q} = (\gamma_t^{Q,P})^{-1}. \tag{2.4}$$

Remark 2.2. We observe that an unbiased estimator of the ruin probability $\psi(u)$ when we consider n independent simulations $t_u^{(1)},\dots,t_u^{(n)}$ of T_u under $Q \in \mathscr{C}$ is

$$\hat{r}_Q = \frac{1}{n}\sum_{k=1}^{n}\gamma_{t_u^{(k)}}^{P,Q}\mathbf{1}_{t_u^{(k)}<\infty}. \tag{2.5}$$

Indeed, T_u is an $\mathscr{F}_t^C$-stopping time (as can be easily realized noticing that $S(t)$ has continuous trajectories), therefore it is $\mathscr{F}_{T_u}^C$-measurable, and

$$\mathbb{E}_Q[\gamma_{T_u}^{P,Q}\mathbf{1}_{T_u<\infty}] = \psi(u).$$

After straightforward computations it is easily seen that the variance of the estimator $\hat{r}_Q$ defined by (2.5) is

$$\mathrm{var}_Q(\hat{r}_Q) = \frac{\mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2\mathbf{1}_{T_u<\infty}] - [P(T_u<\infty)]^2}{n}.$$

To get an asymptotically efficient simulation law, the idea is to minimize in some sense, for u large, the quantity $\mathrm{var}_Q(\hat{r}_Q)$, varying $Q \in \mathscr{C}$. For this we can concentrate our attention on the only part depending on Q, that is

$$\mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2\mathbf{1}_{T_u<\infty}] = \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2],$$

where the equality follows since

$$Q(T_u < \infty) = 1 \quad \text{for all } Q \in \mathscr{C}.$$

Following the criterion described by Lehtonen and Nyrhinen (1992) (see also Macci, 2001) we say that an admissible law $Q^* \in \mathscr{C}$ is asymptotically efficient for simulations if

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -2w,$$

for all admissible laws $Q \in \mathscr{C}$, and

$$\lim_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_{Q^*}[(\gamma_{T_u}^{P,Q^*})^2] = -2w.$$

In words, this means that for a given number n of replications to obtain the best possible accuracy of the estimate we should perform the simulation of the random variable T_u under the law Q^*. Such a law will be derived in Section 3. The asymptotically efficient simulation law Q^* is unique (or optimal) if

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] > -2w$$

for all $Q \in \mathscr{C}$, $Q \neq Q^*$. In Section 4 we show the uniqueness of the asymptotically efficient simulation law determined in Section 3.

3. Asymptotically efficient simulation law

In a way similar to the paper by Lehtonen and Nyrhinen (1992) (see also Macci, 2001), we introduce a suitable family of conjugate laws

$$\{P_\theta\}_{\theta:\kappa(\theta)<\infty}$$

of the original law P where, for all $\theta \in \mathbb{R}$,

$$\kappa(\theta) = \mathbb{E}[e^{\theta Z_1}].$$

Such a family is defined as follows: the probability measure P_θ is absolutely continuous with respect to P on the σ-field $\mathscr{F}_t^C$, for each $t > 0$, and the corresponding density is

$$\gamma_t^{P_\theta,P} = \frac{e^{\theta C(t)}}{\mathbb{E}[e^{\theta C(t)}]} = \exp\{\theta C(t) - \lambda t(\kappa(\theta) - 1)\}.$$

As it is well-known (see, for instance, Asmussen, 1987, pp. 262–263), under P_θ the process $\{T_n\}_{n\geqslant 1}$ is a Poisson process with intensity $\lambda^{(P_\theta)} = \lambda\kappa(\theta)$, independent of the sequence $\{Z_n\}_{n\geqslant 1}$ of i.i.d. random variables, whose common law $P_\theta^{(Z)}$ is absolutely continuous with respect to their common law $P^{(Z)}$ under P, with density

$$\frac{dP_\theta^{(Z)}}{dP^{(Z)}}(z) = \frac{e^{\theta z}}{\kappa(\theta)}. \tag{3.1}$$

As announced in the introduction, in this section we derive an asymptotically efficient law for simulation. More precisely, we show the following Proposition 3.1.

Proposition 3.1. *Let us assume*

$$\kappa(\theta) < \infty \quad \textit{for all } \theta \textit{ in a neighbourhood of } 0, \textit{ say } (0,\eta) \tag{3.2}$$

and

$$\begin{gathered}\textit{there exists } w \in (0,\eta) \textit{ such that } \lambda(\kappa(w)-1) - cw = 0 \\ \textit{and } \lambda\mathbb{E}[Z_1 e^{wZ_1}] - c > 0.\end{gathered} \tag{3.3}$$

Then P_w is an asymptotically efficient law for simulations.

The proof of Proposition 3.1 is based on the following preliminary lemma.

Lemma 3.2. *Let us assume* (3.2), *then $P_\theta \in \mathscr{C}$ for all $\theta \in (0,\eta)$ such that*

$$\lambda\mathbb{E}[Z_1 e^{\theta Z_1}] - c > 0. \tag{3.4}$$

Proof. Let $\theta \in (0,\eta)$ be such that (3.2) and (3.4) hold. After a straightforward computation it is easily checked that (3.2) implies $P_\theta \in \mathscr{D}$. Therefore, the conclusion follows if we prove

$$P_\theta(T_u < \infty) = 1 \quad \text{for all } u > 0.$$

In particular, this follows if we show

$$\lim_{t\to\infty} \frac{S(t) - ct}{t} = \delta_\theta \quad P_\theta\text{-a.s.}, \tag{3.5}$$

for some positive constant δ_θ.

By the definition of P_θ and formulas in Klüppelberg and Mikosch (1995, p. 127) we have, for each $t > 0$,

$$\mathbb{E}_{P_\theta}[S(t)] = \lambda\kappa(\theta)\int_0^t \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s$$

and

$$\mathrm{var}_{P_\theta}(S(t)) = \lambda\kappa(\theta)\int_0^t \mathbb{E}_{P_\theta}[H^2(s,Z_1)]\,\mathrm{d}s.$$

In order to show (3.5) we start proving

$$\lim_{t\to\infty} \frac{S(t)}{\mathbb{E}_{P_\theta}[S(t)]} = 1 \quad P_\theta\text{-a.s.} \tag{3.6}$$

This follows by Proposition 3.1 in Klüppelberg and Mikosch (1995) if we show

$$\sum_{n\geqslant 1} \frac{\mathrm{var}_{P_\theta}(S(n^2))}{\mathbb{E}^2_{P_\theta}[S(n^2)]} < \infty$$

i.e.

$$\sum_{n\geqslant 1} \frac{\int_0^{n^2} \mathbb{E}_{P_\theta}[H^2(s,Z_1)]\,\mathrm{d}s}{(\int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s)^2} < \infty \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{\int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s}{\int_0^{(n-1)^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s} = 1. \tag{3.8}$$

For (3.7) it suffices to show

$$\lim_{n\to\infty} \frac{\int_0^{n^2} \mathbb{E}_{P_\theta}[H^2(s,Z_1)]\,\mathrm{d}s/(\int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s)^2}{1/n^2} = \xi_\theta \tag{3.9}$$

for some positive constant ξ_θ. We start observing that by assumption (1.6) we have

$$\lim_{s\to\infty} \mathbb{E}_{P_\theta}[H^2(s,Z_1)] = \mathbb{E}_{P_\theta}[Z_1^2]$$

and

$$\lim_{s\to\infty} \mathbb{E}_{P_\theta}[H(s,Z_1)] = \mathbb{E}_{P_\theta}[Z_1].$$

Employing (3.1) we obtain

$$\mathbb{E}_{P_\theta}[Z_1^2] = \frac{\mathbb{E}[Z_1^2 \mathrm{e}^{\theta Z_1}]}{\kappa(\theta)}$$

and

$$\mathbb{E}_{P_\theta}[Z_1] = \frac{\mathbb{E}[Z_1 \mathrm{e}^{\theta Z_1}]}{\kappa(\theta)} \tag{3.10}$$

and these quantities are finite by (3.2). By assumption (1.7) these quantities are also positive. Thus

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^{n^2} \mathbb{E}_{P_\theta}[H^2(s,Z_1)]\,\mathrm{d}s = \mathbb{E}_{P_\theta}[Z_1^2]$$

and

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s = \mathbb{E}_{P_\theta}[Z_1].$$

Therefore

$$\lim_{n\to\infty} \frac{\int_0^{n^2} \mathbb{E}_{P_\theta}[H^2(s,Z_1)]\,\mathrm{d}s/(\int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s)^2}{1/n^2} = \frac{\mathbb{E}_{P_\theta}[Z_1^2]}{\mathbb{E}^2_{P_\theta}[Z_1]} > 0$$

and

$$\lim_{n\to\infty} \frac{\int_0^{n^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s}{\int_0^{(n-1)^2} \mathbb{E}_{P_\theta}[H(s,Z_1)]\,\mathrm{d}s} = \frac{\mathbb{E}_{P_\theta}[Z_1]}{\mathbb{E}_{P_\theta}[Z_1]} = 1$$

which correspond, respectively, to (3.9) and (3.8).

We now observe that

$$\lim_{t\to\infty} \frac{\mathbb{E}_{P_\theta}[S(t)]}{t} = \lambda\kappa(\theta)\mathbb{E}_{P_\theta}[Z_1]$$

and therefore by (3.6) and (3.10) we have

$$\lim_{t\to\infty} \frac{S(t)-ct}{t} = \lim_{t\to\infty} \frac{S(t)}{\mathbb{E}_{P_\theta}[S(t)]} \frac{\mathbb{E}_{P_\theta}[S(t)]}{t} - c = \lambda\mathbb{E}[Z_1 \mathrm{e}^{\theta Z_1}] - c.$$

Relation (3.5) follows by assumption (3.4) setting

$$\delta_\theta = \lambda\mathbb{E}[Z_1 \mathrm{e}^{\theta Z_1}] - c. \qquad \square$$

Finally we show Proposition 3.1.

Proof. In view of Lemma 3.2 we have just to show

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -2w \tag{3.11}$$

for all admissible laws $Q\in\mathscr{C}$, and

$$\lim_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_{P_w}[(\gamma_{T_u}^{P,P_w})^2] = -2w. \tag{3.12}$$

We start proving (3.11). By Jensen's inequality, for all $Q\in\mathscr{C}$,

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant \liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q^2[\gamma_{T_u}^{P,Q}] = 2\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[\gamma_{T_u}^{P,Q}]. \tag{3.13}$$

Since $\mathbb{E}_Q[\gamma_{T_u}^{P,Q}] = \psi(u)$, by (3.13) and (1.5) we have (3.11). We now show (3.12). For this, by virtue of (3.11), it suffices to prove

$$\limsup_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_{P_w}[(\gamma_{T_u}^{P,P_w})^2] \leqslant -2w. \tag{3.14}$$

By the definition of the density $\gamma_t^{P_\theta,P}$ it follows, for all $\theta\in(0,\eta)$,

$$\gamma_t^{P,P_\theta} = \exp\{-\theta(C(t)-ct) + \lambda t(\kappa(\theta)-1) - c\theta t\}. \tag{3.15}$$

Setting $\theta = w$ and $t = T_u$ in (3.15), by assumption (3.3) it follows

$$\gamma_{T_u}^{P,P_w} = \exp\{-w(C(T_u)-cT_u)\}. \tag{3.16}$$

Since, for all $t>0$, $S(t)\leqslant C(t)$ a.s., by the definition of T_u, we have

$$C(T_u) - cT_u \geqslant u \quad \text{a.s.}, \tag{3.17}$$

for all $u>0$. Relation (3.14) it follows by (3.16) and (3.17). $\square$

4. Uniqueness of the asymptotically efficient simulation law

In this section, we prove the uniqueness (or optimality) of the asymptotically efficient simulation law P_w. More precisely, we show the following result.

Proposition 4.1. *Let us assume* (3.2) *and* (3.3). *Then P_w is the unique asymptotically efficient simulation law.*

To prove this proposition we need some preliminaries.

Given $Q \in \mathscr{D}$, let us consider a law $L \in \mathscr{D}$ such that under L the process $\{T_n\}_{n\geqslant 1}$ is a Poisson process with intensity

$$\lambda^{(L)} = \frac{\lambda^2}{\lambda^{(Q)}},$$

independent of the i.i.d. sequence of random variables $\{Z_n\}_{n\geqslant 1}$, whose common law $L^{(Z)}$ is absolutely continuous with respect to $Q^{(Z)}$ with density

$$\frac{\mathrm{d}L^{(Z)}}{\mathrm{d}Q^{(Z)}}(z) = K_{A(\beta)}^{-1}\left[\frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(z)\right]^2 \mathbf{1}_{A(\beta)}(z). \tag{4.1}$$

In the above expression

$$A(\beta) = \left\{z \in \mathbb{R}^+ : z \leqslant \beta, \frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(z) \leqslant \beta\right\}$$

for $\beta \in (0,\infty)$ large enough so that

$$K_{A(\beta)} = \int_{A(\beta)} \left[\frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(z)\right]^2 \mathrm{d}Q^{(Z)}(z) > 0.$$

Proposition 4.1 above is consequence of two basic preliminary results: Lemmas 4.2 and 4.6 to be given. In our framework, they correspond, respectively, to Lemmas 3 and 4 in Lehtonen and Nyrhinen (1992). We start stating Lemma 4.2.

Lemma 4.2. *Let $d > 0$ be arbitrarily fixed, then*

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -d\Lambda^{(L)*}\left(\frac{1}{d}\right), \tag{4.2}$$

where

$$\Lambda^{(L)*}(x) = \sup_{\theta\in\mathbb{R}} (\theta x - \Lambda^{(L)}(\theta))$$

is the Fenchel–Legendre transform of the function

$$\Lambda^{(L)}(\theta) = -\theta c + \lambda^{(L)} K_{A(\beta)}\kappa_L(\theta) + \lambda^{(Q)} - 2\lambda, \tag{4.3}$$

being

$$\kappa_L(\theta) = \mathbb{E}_L[\mathrm{e}^{\theta Z_1}].$$

The proof of this result is based on Lemmas 4.3–4.5 below.

Lemma 4.3. *The probability measure* L *is absolutely continuous with respect to* Q *on the* σ*-field* $\mathscr{F}_t^C$, *for each* $t>0$, *with density*

$$\gamma_t^{L,Q} = (\gamma_t^{P,Q})^2 \exp\{-(\lambda^{(Q)} + \lambda^{(L)} - 2\lambda)t - N_t \ln K_{A(\beta)}\} \prod_{i=1}^{N_t} \mathbf{1}\{Z_i \in A(\beta)\},$$

where $\gamma_t^{P,Q}$ *is given by* (2.4), *and* N_t *is defined by* (2.3).

Proof. The conclusion is a straightforward consequence of the definition of L. $\square$

Before stating Lemmas 4.4 and 4.5 we introduce another family $\{L_\theta\}_{\theta\in\mathbb{R}}$ of laws, defined as follows: the probability measure L_θ is absolutely continuous with respect to L on the σ-field $\mathscr{F}_t^C$, for each $t>0$, with density

$$\gamma_t^{L_\theta,L} = \frac{\mathrm{e}^{\theta C(t) + N_t \ln K_{A(\beta)}}}{\mathbb{E}_L[\mathrm{e}^{\theta C(t) + N_t \ln K_{A(\beta)}}]}.$$

By the definition of L it is easily seen that $\gamma_t^{L_\theta,L}$ can be rewritten as

$$\gamma_t^{L_\theta,L} = \exp\{\theta C(t) + N_t \ln K_{A(\beta)} - \lambda^{(L)} t (K_{A(\beta)} \kappa_L(\theta) - 1)\}. \tag{4.4}$$

Lemmas 4.4 and 4.5 give the almost sure convergence (as $t\to\infty$) of the processes $(C(t)-ct)/t$ and $(S(t)-ct)/t$ under the law L_θ. The proof of Lemma 4.4 is similar to the proof of Lemma 3.6 in Macci (2001), and therefore omitted.

Lemma 4.4. *For all* $\theta\in\mathbb{R}$

$$\lim_{t\to\infty} \frac{C(t)-ct}{t} = \Lambda^{(L)'}(\theta) \quad L_\theta\text{-a.s.},$$

where the function $\Lambda^{(L)}(\cdot)$ *is defined by* (4.3) *and* $\Lambda^{(L)'}(\theta)$ *denotes the derivative of* $\Lambda^{(L)}(\theta)$.

The analogous result for the (integrated) Poisson shot-noise process $S(t)$ is the following.

Lemma 4.5. *For all* $\theta\in\mathbb{R}$

$$\lim_{t\to\infty} \frac{S(t)-ct}{t} = \Lambda^{(L)'}(\theta) \quad L_\theta\text{-a.s.}$$

Proof. Let $\theta\in\mathbb{R}$ be arbitrarily fixed. By (4.4) and the result in Asmussen (1987, pp. 262–263) it follows that under $L_\theta S(t)$ is a Poisson shot-noise process where the underlying Poisson process $\{T_n\}_{n\geqslant 1}$ has intensity $\lambda^{(L)} K_{A(\beta)}\kappa_L(\theta)$. In particular (see Klüppelberg and Mikosch, 1995, p. 127), for each $t>0$,

$$\mathbb{E}_{L_\theta}[S(t)] = \lambda^{(L)} K_{A(\beta)} \kappa_L(\theta) \int_0^t \mathbb{E}_{L_\theta}[H(s,Z_1)]\,\mathrm{d}s.$$

Arguing as in the proof of Lemma 3.2, replacing P_θ by L_θ, it can be shown

$$\lim_{t\to\infty} \frac{S(t)}{\mathbb{E}_{L_\theta}[S(t)]} = 1 \quad L_\theta\text{-a.s.} \tag{4.5}$$

We now observe that

$$\lim_{t\to\infty} \frac{\mathbb{E}_{L_\theta}[S(t)]}{t} = \lambda^{(L)} K_{A(\beta)} \kappa_L(\theta) \mathbb{E}_{L_\theta}[Z_1]$$

and moreover

$$\mathbb{E}_{L_\theta}[Z_1] = \frac{\mathbb{E}_L[Z_1 \mathrm{e}^{\theta Z_1}]}{\kappa_L(\theta)} \tag{4.6}$$

as we shall show later. Therefore by (4.5) and (4.6) it follows

$$\lim_{t\to\infty} \frac{S(t)-ct}{t} = \lambda^{(L)} K_{A(\beta)} \mathbb{E}_L[Z_1 \mathrm{e}^{\theta Z_1}] - c = \Lambda^{(L)'}(\theta) \quad L_\theta\text{-a.s.}$$

It remains to show (4.6). By the strong law of the large numbers

$$\lim_{t\to\infty} \frac{C(t)-ct}{t} = \lambda^{(L)} K_{A(\beta)} \kappa_L(\theta) \mathbb{E}_{L_\theta}[Z_1] - c \quad L_\theta\text{-a.s.},$$

and therefore (4.6) follows by Lemma 4.4 and the uniqueness of the limit. □

We now show the announced Lemma 4.2.

Proof. The proof can be divided in four steps. Let $d>0$ be arbitrarily fixed. As first step we show that there exists $b \in \mathbb{R}$ such that

$$\Lambda^{(L)'}(b) = \frac{1}{d} > 0. \tag{4.7}$$

For this we start noticing that by assumption (1.7) and $L^{(Z)} \ll P^{(Z)}$ it follows $L(Z_1>0) = 1$, and therefore $\Lambda^{(L)''}(\theta) > 0$ for each $\theta \in \mathbb{R}$. Thus $\Lambda^{(L)'}(\theta)$ is an increasing function of θ. Since

$$\Lambda^{(L)'}(\theta) = -c + \lambda^{(L)} K_{A(\beta)} \mathbb{E}_L[Z_1 \mathrm{e}^{\theta Z_1}]$$

we have

$$\lim_{\theta\to-\infty} \Lambda^{(L)'}(\theta) = -c < 0$$

and

$$\lim_{\theta\to+\infty} \Lambda^{(L)'}(\theta) = +\infty.$$

Therefore $\Lambda^{(L)'}(\theta)$ assumes all the values larger than $-c$, and (4.7) is proved.

The second step consists in proving

$$\lim_{u\to\infty} \frac{C(T_u)-cT_u}{C(V_u)-cV_u} = 1 \quad L_b\text{-a.s.}, \tag{4.8}$$

where

$$V_u = \inf\{t \geqslant 0 : C(t) - ct \geqslant u\}.$$

As can be easily realized using Lemma 4.4 and (4.7), relation (4.8) follows if we show

$$\lim_{u\to\infty} \frac{T_u}{u} = d \quad L_b\text{-a.s.} \tag{4.9}$$

and

$$\lim_{u\to\infty} \frac{V_u}{u} = d \quad L_b\text{-a.s.} \tag{4.10}$$

We now prove (4.9). By Lemma 4.5 and (4.7) $L_b(T_u < \infty) = 1$ for all $u > 0$. Since the process $S(t) - ct$ has continuous trajectories, by the definition of T_u we have

$$S(T_u) - cT_u = u \quad L_b\text{-a.s.}$$

We observe that

$$\lim_{u\to\infty} T_u = +\infty \quad L_b\text{-a.s.}$$

and therefore dividing by T_u and letting u tend to $+\infty$, by Lemma 4.5 and (4.7) we have

$$\lim_{u\to\infty} \frac{u}{T_u} = \Lambda^{(L)'}(b) = \frac{1}{d} \quad L_b\text{-a.s.,}$$

which yields (4.9).

We now show (4.10). By Lemma 4.4 and (4.7) $L_b(V_u < \infty) = 1$ for all $u > 0$. Since L_b is absolutely continuous with respect to L, and $L(Z_1 \in (0, \beta]) = 1$ we get $L_b(Z_1 \in (0, \beta]) = 1$, and therefore

$$u \leqslant C(V_u) - cV_u \leqslant u + \beta \quad L_b\text{-a.s.} \tag{4.11}$$

We observe that

$$\lim_{u\to\infty} V_u = +\infty \quad L_b\text{-a.s.}$$

and then dividing all the terms in inequality (4.11) by V_u and letting u tend to $+\infty$, Lemma 4.4 and (4.7) give (4.10).

The third step consists in the following application of Egorov's theorem. By (4.8) and Egorov's theorem we have, for any sequence $\{u_n\}$ such that $u_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{C(T_{u_n}) - cT_{u_n}}{C(V_{u_n}) - cV_{u_n}} = 1$$

almost uniformly with respect to the probability measure L_b, that is for all $\delta \in (0, 1)$ there exists a measurable set E_δ such that $L_b(E_\delta) < \delta$ and

$$\lim_{n\to\infty} \frac{C(T_{u_n}(\omega)) - cT_{u_n}(\omega)}{C(V_{u_n}(\omega)) - cV_{u_n}(\omega)} = 1$$

uniformly on $\Omega - E_\delta$. Thus, employing (4.11) we have that for any $\alpha > 0$ there exists $\bar{n} = \bar{n}(\alpha)$ such that for all $n \geqslant \bar{n}$

$$u_n(1-\alpha) < C(T_{u_n}(\omega)) - cT_{u_n}(\omega) < (u_n + \beta)(1+\alpha), \tag{4.12}$$

for almost all $\omega \in \Omega - E_\delta$, with respect to the probability measure L_b.

Now we perform the final step proving (4.2). By Lemma 4.3 and (4.4) we obtain for any sequence $\{u_n\}$ such that $u_n \to \infty$ as $n \to \infty$

$$\mathbb{E}_Q[(\gamma_{T_{u_n}}^{P,Q})^2] \geqslant \mathbb{E}_Q\left[(\gamma_{T_{u_n}}^{P,Q})^2 \mathbf{1}\{T_{u_n} \leqslant u_n d\} \prod_{i=1}^{N_{T_{u_n}}} \mathbf{1}\{Z_i \in A(\beta)\}\right]$$

$$= \mathbb{E}_L[\exp\{(\lambda^{(Q)} + \lambda^{(L)} - 2\lambda)T_{u_n} + N_{T_{u_n}} \ln K_{A(\beta)}\}\mathbf{1}\{T_{u_n} \leqslant u_n d\}]$$

$$= \mathbb{E}_{L_b}[\exp\{-b(C(T_{u_n}) - cT_{u_n}) + \Lambda^{(L)}(b)T_{u_n}\}\mathbf{1}\{T_{u_n} \leqslant u_n d\}]. \qquad (4.13)$$

Now let $\varepsilon \in (0, d/2)$ be arbitrarily fixed. By (4.13) we obtain

$$\mathbb{E}_Q[(\gamma_{T_{u_n}}^{P,Q})^2]$$
$$\geqslant \mathbb{E}_{L_b}[\exp\{-b(C(T_{u_n}) - cT_{u_n}) + \Lambda^{(L)}(b)T_{u_n}\}$$
$$\times \mathbf{1}\{u_n(d - 2\varepsilon) \leqslant T_{u_n} \leqslant u_n d\}]. \qquad (4.14)$$

By (4.12) it follows, for all $n \geqslant \bar{n}$,

$$\mathbb{E}_{L_b}[\exp\{-b(C(T_{u_n}) - cT_{u_n}) + \Lambda^{(L)}(b)T_{u_n}\}\mathbf{1}\{u_n(d - 2\varepsilon) \leqslant T_{u_n} \leqslant u_n d\}]$$
$$\geqslant \int_{\Omega - E_\delta} \exp\{-b(C(T_{u_n}(\omega)) - cT_{u_n}(\omega)) + \Lambda^{(L)}(b)T_{u_n}(\omega)\}$$
$$\times \mathbf{1}\{u_n(d - 2\varepsilon) \leqslant T_{u_n}(\omega) \leqslant u_n d\}\, \mathrm{d}L_b(\omega)$$
$$\geqslant \exp\{-b(u_n + \beta)(1 + \alpha) + \Lambda^{(L)}(b)u_n d - 2u_n\varepsilon|\Lambda^{(L)}(b)|\}$$
$$\times \int_{\Omega - E_\delta} \mathbf{1}\{u_n(d - 2\varepsilon)_{u_n}(\omega) \leqslant u_n d\}\, \mathrm{d}L_b(\omega), \qquad (4.15)$$

where in the latter inequality we use also that, for any $u > 0$ and ω, if

$$u(d - 2\varepsilon) \leqslant T_u(\omega) \leqslant ud$$

then

$$\Lambda^{(L)}(b)T_u(\omega) \geqslant \Lambda^{(L)}(b)ud - 2u\varepsilon|\Lambda^{(L)}(b)|.$$

We now observe that the latter term in (4.15) is bigger than or equal to

$$\exp\{-b(u_n + \beta)(1 + \alpha) + \Lambda^{(L)}(b)u_n d - 2u_n\varepsilon|\Lambda^{(L)}(b)|\}$$
$$\times [L_b(u_n(d - 2\varepsilon) \leqslant T_{u_n} \leqslant u_n d) - \delta].$$

Therefore, for all $\alpha \in (0, 1)$ there exists $\bar{n} = \bar{n}(\alpha)$ such that for all $n \geqslant \bar{n}$

$$\mathbb{E}_Q[(\gamma_{T_{u_n}}^{P,Q})^2]$$
$$\geqslant \exp\{-b(u_n + \beta)(1 + \alpha) + \Lambda^{(L)}(b)u_n d - 2u_n\varepsilon|\Lambda^{(L)}(b)|\}$$
$$\times [L_b(u_n(d - 2\varepsilon) \leqslant T_{u_n} \leqslant u_n d) - \delta].$$

Since ε and α are arbitrary, by this latter inequality, (4.9) and (4.7) we have

$$\liminf_{n\to\infty} \frac{1}{u_n} \ln \mathbb{E}_Q[(\gamma^{P,Q}_{T_{u_n}})^2] \geqslant -b + \Lambda^{(L)}(b)d = -d\left(\frac{b}{d} - \Lambda^{(L)}(b)\right)$$
$$= -d\Lambda^{(L)*}\left(\frac{1}{d}\right).$$

The conclusion follows recalling that the sequence $\{u_n\}$ is arbitrary. □

Before stating Lemma 4.6, we introduce another law $R \in \mathscr{D}$ such that under R the process $\{T_n\}_{n\geqslant 1}$ is a Poisson process with intensity $\lambda^{(L)}$, independent of the i.i.d. sequence of random variables $\{Z_n\}_{n\geqslant 1}$, whose common law $R^{(Z)}$ is absolutely continuous with respect to $Q^{(Z)}$ with density

$$\frac{\mathrm{d}R^{(Z)}}{\mathrm{d}Q^{(Z)}}(z) = K^{-1}\left[\frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(z)\right]^2,$$

where

$$K = \mathbb{E}_{Q^{(Z)}}\left[\left(\frac{\mathrm{d}P^{(Z)}}{\mathrm{d}Q^{(Z)}}(Z_1)\right)^2\right].$$

In the following we explicit the dependence of $\Lambda^{(L)}(\cdot)$ from β writing $\Lambda^{(L)}_\beta(\cdot)$ in place of $\Lambda^{(L)}(\cdot)$.

Lemma 4.6. *For all $d > 0$*

$$\lim_{\beta\to\infty} \Lambda^{(L)*}_\beta(d) = \Lambda^{(Q)*}(d), \tag{4.16}$$

where $\Lambda^{(L)*}_\beta(\cdot)$ and $\Lambda^{(Q)*}(\cdot)$ are, respectively, the Fenchel–Legendre transforms of $\Lambda^{(L)}_\beta(\cdot)$ and $\Lambda^{(Q)}(\cdot)$, $\Lambda^{(L)}_\beta(\cdot)$ is defined by (4.3) and, for all $\theta \in \mathbb{R}$,

$$\Lambda^{(Q)}(\theta) = -\theta c + \lambda^{(L)} K \kappa_R(\theta) + \lambda^{(Q)} - 2\lambda \tag{4.17}$$

being

$$\kappa_R(\theta) = \mathbb{E}_R[\mathrm{e}^{\theta Z_1}].$$

Proof. We preliminarly notice that

$$\mathscr{D}_R = \{\theta \in \mathbb{R} : \kappa_R(\theta) < \infty\}$$

is an interval of the following form: $\mathscr{D}_R = (-\infty, a)$ for some $a \in (0,\infty]$ or $\mathscr{D}_R = (-\infty, a]$ for some $a \in [0,\infty)$. Indeed, $R(0 < Z_1 < \infty) = 1$ since $R^{(Z)}$ is absolutely continuous with respect to $P^{(Z)}$ and $P(0 < Z_1 < \infty) = 1$ by assumption (1.7).

We now observe that the proof can be divided in two steps.

Letting $d>0$ denote an arbitrarily fixed positive constant, the first step consists in showing that only the following two cases are possible:

(i) there exists $b\in(-\infty,a)$ so that $\Lambda^{(Q)'}(b)=d$ and then

$$\Lambda^{(Q)*}(d)=bd-\Lambda^{(Q)}(b), \tag{4.18}$$

(ii) $\Lambda^{(Q)*}(d)=ad-\Lambda^{(Q)}(a)$.

Now we are going to prove that (i) holds if $\mathscr{D}_R=(-\infty,a)$ for some $a\in(0,\infty]$. Let us first take $0<a<+\infty$. By (4.17) it is easily realized that, for all $d>0$,

$$\lim_{\theta\to a}(\theta d-\Lambda^{(Q)}(\theta))=-\infty \tag{4.19}$$

and

$$\lim_{\theta\to-\infty}(\theta d-\Lambda^{(Q)}(\theta))=-\infty. \tag{4.20}$$

Therefore, the supremum $\Lambda^{(Q)*}(d)$ is necessarily attained on $(-\infty,a)$, and this gives (4.18). Let us now show (4.18) taking $a=+\infty$. Arguing as above, for this it suffices to prove (4.19) with $a=+\infty$. We first notice that, for all $\theta\in\mathbb{R}$ and for all β,

$$\theta d-\Lambda^{(Q)}(\theta)\leqslant\theta(d+c)-\lambda^{(L)}K_{A(\beta)}\kappa_L(\theta)-\lambda^{(Q)}+2\lambda. \tag{4.21}$$

Since $L(Z_1>0)=1$, there exists $\delta>0$ such that $L(Z_1>\delta)>0$. Therefore by (4.21) it follows, for all $\theta>0$ and for all β,

$$\theta d-\Lambda^{(Q)}(\theta)\leqslant\theta(d+c)-\lambda^{(L)}K_{A(\beta)}\mathrm{e}^{\theta\delta}L(Z_1>\delta)-\lambda^{(Q)}+2\lambda. \tag{4.22}$$

Indeed, $\kappa_L(\theta)\geqslant\mathrm{e}^{\theta\delta}L(Z_1>\delta)$. Passing to the limit as $\theta\to+\infty$ in inequality (4.22) we get (4.19) with $a=+\infty$. Similar arguments show that (i) or (ii) hold if $\mathscr{D}_R=(-\infty,a]$ for some $a\in[0,+\infty)$.

The final step consists in proving (4.16). We give the details just in the case when (i) holds (the proof of (4.16) under (ii) is similar). For each β, let $\theta_L^*(\beta)$ be such that

$$\Lambda_\beta^{(L)'}(\theta_L^*(\beta))=d$$

(the existence of $\theta_L^*(\beta)$, for each β, can be proved as at the beginning of the proof of Lemma 4.2). We now show that

$$\theta_L^*(\beta)\downarrow b \tag{4.23}$$

as $\beta\uparrow+\infty$. Reasoning by contradiction let us suppose that $\theta_L^*(\beta_1)<\theta_L^*(\beta_2)$ if $\beta_1<\beta_2$. Then since the functions

$$\beta\to\Lambda_\beta^{(L)'}(\theta)=-c+\lambda^{(L)}K_{A(\beta)}\mathbb{E}_L[Z_1\mathrm{e}^{\theta Z_1}] \tag{4.24}$$

and

$$\theta\to\Lambda_\beta^{(L)'}(\theta) \tag{4.25}$$

are increasing we have

$$d=\Lambda_{\beta_1}^{(L)'}(\theta_L^*(\beta_1))<\Lambda_{\beta_2}^{(L)'}(\theta_L^*(\beta_2))=d$$

which is impossible. Therefore

$$\theta_L^*(\beta) \downarrow \theta^* = \inf_\beta \theta_L^*(\beta)$$

as $\beta \uparrow +\infty$.

We now observe that, for all $\theta \in (-\infty, a)$, the function (4.25) converges monotonically increasing, as $\beta \uparrow \infty$, to the function

$$\Lambda^{(Q)'}(\theta) = -c + \lambda^{(L)} K \mathbb{E}_R[Z_1 e^{\theta Z_1}].$$

Reasoning by contradiction, let us assume that there exists $\bar\beta$ such that $\theta_L^*(\bar\beta) < b$, then

$$d = \Lambda_{\bar\beta}^{(L)'}(\theta_L^*(\bar\beta)) < \Lambda^{(Q)'}(b) = d$$

which is impossible. Therefore $\theta^* \geqslant b$. It remains to show $\theta^* = b$. Reasoning again by contradiction, let us assume $\theta^* > b$. We notice that

$$\lim_{\beta\to\infty} [\theta^* d - \Lambda_\beta^{(L)}(\theta^*)] = \theta^* d - \Lambda^{(Q)}(\theta^*) < bd - \Lambda^{(Q)}(b).$$

Moreover, since $\theta_L^*(\beta) \geqslant \theta^* > b$ for each β, and the function $\theta \to \theta d - \Lambda_\beta^{(L)}(\theta)$ is increasing on $(-\infty, \theta_L^*(\beta)]$ we get

$$\theta^* d - \Lambda_\beta^{(L)}(\theta^*) > bd - \Lambda_\beta^{(L)}(b).$$

Taking the limit as $\beta \uparrow \infty$, it follows

$$\theta^* d - \Lambda^{(Q)}(\theta^*) \geqslant bd - \Lambda^{(Q)}(b)$$

which is impossible. Thus, we have (4.23).

We now notice that, as $\theta_L^*(\beta) \downarrow b$,

$$\Lambda_\beta^{(L)}(\theta_L^*(\beta)) - \Lambda_\beta^{(L)}(b) = \Lambda_\beta^{(L)'}(b)(\theta_L^*(\beta) - b) + o(\theta_L^*(\beta) - b).$$

Therefore

$$\Lambda_\beta^{(L)*}(d) = \theta_L^*(\beta) d - \Lambda_\beta^{(L)}(\theta_L^*(\beta)) = \theta_L^*(\beta) d - \Lambda_\beta^{(L)}(b) - \Lambda_\beta^{(L)'}(b)(\theta_L^*(\beta) - b)$$
$$+ o(\theta_L^*(\beta) - b),$$

and this latter quantity converges to $bd - \Lambda^{(Q)}(b) = \Lambda^{(Q)*}(d)$ as $\beta \uparrow \infty$. □

Next Lemma 4.7, whose proof can be found in Macci (2001), is also needed to show Proposition 4.1.

Lemma 4.7. *For all* $\theta \in \mathbb{R}$ *and* $Q \in \mathscr{D}$

$$\Lambda^{(Q)}(2\theta) \geqslant 2\Lambda^{(P)}(\theta), \tag{4.26}$$

where the function $\Lambda^{(Q)}(\cdot)$ *is defined by* (4.17) *and*

$$\Lambda^{(P)}(\theta) = -\theta c + \lambda(\kappa(\theta) - 1). \tag{4.27}$$

Moreover, if there exists $\hat\theta \in \mathbb{R}$ *such that*

$$\Lambda^{(Q)}(2\hat\theta) = 2\Lambda^{(P)}(\hat\theta) < \infty,$$

then $Q = P_{\hat\theta}$.

Now we prove Proposition 4.1.

Proof. Arguing as in the proof of Proposition 3.1 it follows that for all $Q \in \mathscr{C}$

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -2w$$

and therefore the result follows if we show that for any $Q \in \mathscr{C}$ such that

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] = -2w \tag{4.28}$$

then $Q = P_w$. We first show that the following inequalities hold:

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -g\Lambda^{(Q)*}\left(\frac{1}{g}\right) \geqslant -2g\Lambda^{(P)*}\left(\frac{1}{g}\right) = -2w, \tag{4.29}$$

where

$$g = (-c + \lambda\mathbb{E}[Z_1 e^{wZ_1}])^{-1}.$$

By Lemma 4.7

$$\Lambda^{(Q)}(2\theta) \geqslant 2\Lambda^{(P)}(\theta),$$

for all $Q \in \mathscr{C}$ and $\theta \in \mathbb{R}$. Therefore, for all $x \in \mathbb{R}$,

$$\Lambda^{(Q)*}(x) \leqslant 2\Lambda^{(P)*}(x).$$

In particular,

$$\Lambda^{(Q)*}\left(\frac{1}{d}\right) \leqslant 2\Lambda^{(P)*}\left(\frac{1}{d}\right),$$

for all $d > 0$. Thus, by Lemmas 4.2 and 4.6 we have

$$\liminf_{u\to\infty} \frac{1}{u} \ln \mathbb{E}_Q[(\gamma_{T_u}^{P,Q})^2] \geqslant -d\Lambda^{(Q)*}\left(\frac{1}{d}\right) \geqslant -2d\Lambda^{(P)*}\left(\frac{1}{d}\right). \tag{4.30}$$

Since for all $\theta \in (-\infty, \eta)$

$$\Lambda^{(P)'}(\theta) = -c + \lambda\mathbb{E}[Z_1 e^{\theta Z_1}],$$

we have $\Lambda^{(P)'}(w) = g^{-1}$, thus

$$\Lambda^{(P)*}\left(\frac{1}{g}\right) = \frac{w}{g} - \Lambda^{(P)}(w) = \frac{w}{g},$$

where the latter equality follows by (3.3). Therefore, setting $d = g$ in (4.30) we obtain (4.29).

As can be easily realized after an elementary study of the function $\theta \to \theta g^{-1} - \Lambda^{(Q)}(\theta)$, there exists $\tilde{\theta} \in \mathbb{R}$ such that

$$\Lambda^{(Q)*}\left(\frac{1}{g}\right) = \frac{\tilde{\theta}}{g} - \Lambda^{(Q)}(\tilde{\theta}). \tag{4.31}$$

By Lemma 4.7 again it follows

$$\Lambda^{(Q)*}\left(\frac{1}{g}\right) = \frac{\tilde{\theta}}{g} - \Lambda^{(Q)}(\tilde{\theta}) \leqslant \frac{\tilde{\theta}}{g} - 2\Lambda^{(P)}\left(\frac{\tilde{\theta}}{2}\right) \leqslant 2\Lambda^{(P)*}\left(\frac{1}{g}\right). \tag{4.32}$$

We now observe that if $Q \in \mathscr{C}$ is such that (4.28) holds, then all the inequalities in (4.29) turn into equalities, and therefore

$$\Lambda^{(Q)*}\left(\frac{1}{g}\right) = 2\Lambda^{(P)*}\left(\frac{1}{g}\right).$$

In particular, by (4.32) it follows

$$\Lambda^{(Q)*}\left(\frac{1}{g}\right) = \frac{\tilde{\theta}}{g} - \Lambda^{(Q)}(\tilde{\theta}) = 2\left[\frac{\tilde{\theta}}{2g} - \Lambda^{(P)}\left(\frac{\tilde{\theta}}{2}\right)\right] = 2\Lambda^{(P)*}\left(\frac{1}{g}\right). \tag{4.33}$$

Since the supremum $\Lambda^{(P)*}\left(g^{-1}\right)$ is attained for $\theta = w$, (4.33) yields

$$\tilde{\theta} = 2w. \tag{4.34}$$

Therefore by (4.33) and (4.34) we have

$$\frac{2w}{g} - \Lambda^{(Q)}(2w) = 2\left[\frac{w}{g} - \Lambda^{(P)}(w)\right],$$

that is

$$\Lambda^{(Q)}(2w) = 2\Lambda^{(P)}(w)(=0).$$

The conclusion follows by Lemma 4.7. □

5. Application to a teletraffic model

We now apply Proposition 4.1 to the teletraffic model described in Kostantopoulos and Lin (1998) and Brémaud (2000). In this model the points of a Poisson process $\{T_n\}_{n\geqslant 0}$ with intensity $\lambda > 0$ are the times when a new ON-period of an individual source in a computer network starts. The i.i.d. lengths $\{Z_n\}_{n\geqslant 1}$ of the ON-periods are independent of the Poisson process. During an ON-period the source sends a signal at unit rate. At time t the number of active computers in the network is given by the Poisson shot-noise process

$$X(t) = \sum_{n\geqslant 1} \mathbf{1}_{(0,Z_n]}(t - T_n)\mathbf{1}_{(0,t]}(T_n) = \sum_{n\geqslant 1} \mathbf{1}_{(T_n,T_n+Z_n]}(t).$$

Now let us consider a single server queue with service rate $c > 0$ and Poisson shot-noise traffic intensity $X(t)$. The corresponding workload process in the time interval $(0,t]$ is then given by the process

$$S(t) - ct,$$

where $S(t)$ is the integrated Poisson shot-noise process

$$S(t) = \int_0^t X(s)\,\mathrm{d}s = \sum_{n\geqslant 1} \min\{t - T_n, Z_n\}\mathbf{1}_{(0,t]}(T_n).$$

Table 1
Simulation results

θ	$10^5 \hat{r}_{P_\theta}$	$10^{10} s(\hat{r}_{P_\theta})$	p_θ
0.8	1.18	3.69	1.62
0.9	1.17	1.37	1
1	1.29	0.95	0.75
1.1	1.22	2.83	1.37
1.2	0.89	2.50	1.77

We observe that if $T_n + Z_n \leqslant t$, then the full period Z_n contributes to the workload. Otherwise, only the length of the unfinished ON-period $t - T_n$ is taken into account. When the buffer is not finite the queue length is

$$\sup_{t \geqslant 0} \{S(t) - ct\},$$

and therefore for a finite buffer with capacity $u > 0$ the overflow probability is over-estimated by

$$\psi(u) = P\left(\sup_{t \geqslant 0} \{S(t) - ct\} > u\right) = P(T_u < \infty).$$

Assuming that the random variables Z_n are exponentially distributed with mean β^{-1} such that $\beta > \lambda c^{-1}$, by Proposition 4.1 it follows that $P_{\beta - \lambda c^{-1}}$ is the unique asymptotically efficient simulation law. Indeed, for the model described above assumption (1.6) is satisfied since $h(t,z) = \mathbf{1}_{(0,z]}(t)$. Moreover, since Z_1 has an exponential distribution with mean β^{-1}, (1.7) is trivially satisfied and (3.2) holds with $\eta = \beta$. Finally, as it is easily realized after straightforward calculations, setting $w = \beta - \lambda c^{-1}$ we have

$$\lambda(\kappa(w) - 1) - cw = 0$$

and

$$\lambda \mathbb{E}[Z_1 e^{wZ_1}] - c > 0,$$

and therefore (3.3) is satisfied.

Below we give a numerical illustration. We consider the estimation of the probability $\psi(u)$ by using different importance sampling estimators $\hat{r}_{P_\theta}$, where P_θ are admissible laws for simulation, and we compute their sample variances $s(\hat{r}_{P_\theta})$ and their estimated precisions $p_\theta = [s(\hat{r}_{P_\theta})]^{1/2} / \hat{r}_{P_\theta}$. In Table 1 we summarize our simulation results in the case when $n = 200$ is the number of replications in each simulation, $u = 10$, $\beta = 3$, $\lambda = 2$, $c = 1$, and the parameter θ is varied in the set $\{0.8, 0.9, 1, 1.1, 1.2\}$. Simulation results are in accordance with the theoretical ones, indeed the asymptotically efficient importance sampling estimator $\hat{r}_{P_1}$ has the lowest sample variance and estimated precision. We also notice that the importance sampling estimators considered in Table 1 lead to interesting simulated values, in that they are all less than 4.54×10^{-5}, which corresponds to the upper bound on $\psi(10)$ given in Brémaud (2000) (see inequality (1.4)). It is worthwhile to point out that this is not a general property of the importance sampling estimators, as the following numerical argument shows. If Z_1 has a gamma distribution with shape parameter 2 and inverse scale parameter 5, $\lambda = 2$, and $c = 1$, a straightforward computation gives $w = 0.68$. Therefore, by inequality (1.4), for $u = 10$ we have $\psi(10) \leqslant 1.11 \times 10^{-3}$. However, setting $n = 2$ and $\theta = 0.5$, we found that for different computer simulations $10^3 \hat{r}_{P_{0.5}}$ assumes the values $\{1.4, 1.6, 1.8, 1.9, 2.4\}$.

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