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Extremal behaviour of models with multivariate random recurrence representation[☆]

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Abstract

For the solution Y of a multivariate random recurrence model $Y_n = A_n Y_{n-1} + \zeta_n$ in $\mathbb{R}^q$ we investigate the extremal behaviour of the process $y_n = z_*' Y_n, n \in \mathbb{N}$, for $z_* \in \mathbb{R}^q$ with $|z_*| = 1$. This extends results for positive matrices A_n. Moreover, we obtain explicit representations of the compound Poisson limit of point processes of exceedances over high thresholds in terms of its Poisson intensity and its jump distribution, which represents the cluster behaviour of such models on high levels. As a principal example we investigate a random coefficient autoregressive process.

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1. Introduction

We consider a q-dimensional stochastic recurrence equation

$$Y_n = A_n Y_{n-1} + \zeta_n, \quad n \in \mathbb{N}, \tag{1.1}$$

for some iid sequence $\{(A_n, \zeta_n)\}_{n\in\mathbb{N}}$ of random $q \times q$-matrices A_n and q-dimensional vectors ζ_n. Let $z_* \in \mathbb{R}^q$ be some fixed nonrandom vector with Euclidean norm $|z_*| = 1$. Our goal is to describe the extremal behaviour of the process $y_n = z_*' Y_n$, $n \in \mathbb{N}$, where $'$ denotes transposition, and throughout the paper all vectors are column vectors. The extremal behaviour includes besides the asymptotic behaviour of the running maxima

$$M_n = \max_{1\le j\le n} y_j, \quad n \in \mathbb{N},$$

also a precise description of the limit behaviour of the point processes of exceedances over high thresholds.

Our principal example is the random coefficient autoregressive process

$$y_n = \alpha_{1n} y_{n-1} + \cdots + \alpha_{qn} y_{n-q} + \xi_n, \quad n \in \mathbb{N}, \tag{1.2}$$

with random variables (rvs) $\alpha_{in} = a_i + \sigma_i \eta_{in}$, where $a_i \in \mathbb{R}$ and $\sigma_i \ge 0$ for $i = 1, \ldots, q$. Set

$$\alpha_n = (\alpha_{1n}, \ldots, \alpha_{qn})' \quad \text{and} \quad \eta_n = (\eta_{1n}, \ldots, \eta_{qn})'.$$

We suppose that the sequences of coefficient vectors $\{\eta_n\}_{n\in\mathbb{N}}$ and noise variables $\{\xi_n\}_{n\in\mathbb{N}}$ are independent and that both sequences are iid with

$$\mathbf{E}\xi_1 = \mathbf{E}\eta_{i1} = 0 \quad \text{and} \quad \mathbf{E}\xi_1^2 = \mathbf{E}\eta_{i1}^2 = 1, \quad i = 1, \ldots, q. \tag{1.3}$$

Setting $Y_n = (y_n, \ldots, y_{n-q+1})'$ it follows immediately from (1.2) that the multivariate process $\{Y_n\}_{n\in\mathbb{N}}$ satisfies the random recurrence equation (1.1) with

$$A_n = \begin{pmatrix} \alpha_{1n} & \cdots & \alpha_{qn} \\ I_{q-1} & & 0 \end{pmatrix} \quad \text{and} \quad \zeta_n = (\xi_n, 0, \ldots, 0)', \tag{1.4}$$

where I_{q-1} denotes the identity matrix of order $q-1$. In this case $y_n = z_*' Y_n$ for $z_* = (1, 0, \ldots, 0)'$.

Solutions to random recurrence equations have usually Pareto-like tails, a fact which is based on seminal work by Kesten [12] and was further developed by Goldie [7] for the one-dimensional case, and by Le Page [18] and, more recently, by Klüppelberg and Pergamenchtchikov [13], and De Sapporta, Guivarc'h and Le Page [22] for the multivariate case. Applications of such results appear in various areas; see e.g. Diaconis and Freedman [4] for an overview. Prominent examples in the area of financial time series include the GARCH(1,1) model, which was investigated by Mikosch and Starica [17]. In Klüppelberg and Pergamenchtchikov [14] we investigated model (1.2). We presented conditions such that the process $\{y_n\}_{n\in\mathbb{N}}$ allows for a stationary version, represented by a rv y_∞. We also proved that y_∞ has, under natural conditions, a Pareto-like tail.

The extremal behaviour of solutions to random recurrence equations has been investigated in the one-dimensional case for positive rvs A_n and ζ_n in de Haan et al. [9]. The multivariate case has been studied in Basrak et al. [2] and Mikosch and Starica [17]. A prominent condition in all these papers is that the matrix A_n in (1.1) has a.s. positive entries.

Our paper can be considered as an extension of results of de Haan et al. [9] and Basrak et al. [2] to arbitrary matrices in $\mathbb{R}^q$. De Haan et al. [9] considered the univariate model (1.1) with positive random variables A_n and gave a precise account of the extremal behaviour. In [2] the multivariate model (1.1) is considered with positive entry matrices A_n and its extremal behaviour is studied. In that paper the authors show the existence of a limit process for the point processes of exceedances and existence of an extremal index. In the present paper we give a precise description of this limit process for model (1.1) with general matrices A_n. The limit is a compound Poisson process, and besides the Poisson intensity we also give a complete account of the jump distribution, where jump sizes of the process correspond to the cluster sizes of extremes. We also present an explicit form of the extremal index.

Our paper is organized as follows. In Section 2 we present results on the existence of a stationary solution of the process $\{y_n\}_{n\in\mathbb{N}}$. Stationarity is a usual prerequisite in extreme value theory and we shall work with the stationary model. We also prove strong mixing of the process defined in (1.1), which implies the weaker mixing conditions needed for our results on the extremal behaviour of $\{y_n\}_{n\in\mathbb{N}}$. In Section 3 we state our main results. Starting from the fact that solutions to stochastic recurrence equations have usually Pareto-like tails, we embed our model into the context of multivariate regular variation. We describe the limit distribution of properly normalized running maxima. Furthermore, results on the extremal behaviour of the stationary model include an explicit representation of the limit process of the point processes of exceedances over high thresholds.

In Section 4 we present a new proof of the fact that for such models regular variation of every linear combination of marginals implies multivariate regular variation. This new approach also extends results from Basrak et al. [1] to symmetric distributions, which will prove useful for our principal example (1.2). In Section 5 we prove our main result from Section 3 and present in Section 6 its consequences for the random coefficient autoregressive model. Technical details are summarized in the Appendix.

2. Existence of a stationary solution and the strong mixing property

We consider the model (1.1) and use the following notation to formulate our assumptions. The symbol $\otimes$ denotes the Kronecker product of matrices. Furthermore, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^q$ and $|A|^2 = \mathrm{tr} AA'$ is the corresponding matrix norm.

We make the following assumptions:

($\mathbf{A}_1$) *The sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ are both iid and independent of each other, satisfying*

$$\mathbf{E}|A_1|^2 < \infty, \quad \mathbf{E}\zeta_1 = 0 \quad \text{and} \quad \mathbf{E}|\zeta_1|^2 < \infty.$$

($\mathbf{A}_2$) *The Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ defined in (1.1) is aperiodic and irreducible with respect to some nontrivial measure in $\mathbb{R}^q$.*

Sufficient conditions on $\{(A_n, \zeta_n)\}_{n\in\mathbb{N}}$ to ensure $\mathbf{A}_2$ are well known in Markov chain theory and, for instance, given in Feigin and Tweedie [6]. For example, it suffices in the general model (1.1) that the random vectors ζ_n have a positive Lebesgue density in $\mathbb{R}^q$ on the set $\{x \in \mathbb{R}^q : |x| < R\}$ for some $R \in (0, \infty]$.

In the context of random recurrence equations there exist necessary and sufficient conditions for stationarity, going back to Kesten's seminal work on the subject; see Kesten [12], also Goldie and Maller [8]. Such conditions involve a negative Lyapunov exponent, a condition which is in general difficult to verify. Because of the structure of our model we can give an equivalent condition based on the eigenvalues of a certain matrix.

Assume that the following condition holds:

($\mathbf{A}_3$) *The eigenvalues of the matrix*

$$\mathbf{E}A_1 \otimes A_1 \tag{2.1}$$

have moduli less than one.

As stated in Remark 2.2(ii) of [14], since $\mathbf{E}((A_1 \cdots A_n) \otimes (A_1 \cdots A_n)) = (\mathbf{E}(A_1 \otimes A_1))^n$, condition $\mathbf{A}_3$ guarantees that for some constants $c_*, \gamma > 0$,

$$\mathbf{E}|A_1 \cdots A_n|^2 \le c_* \mathrm{e}^{-\gamma n}. \tag{2.2}$$

Classical Markov chain theory ensures that under $\mathbf{A}_1$ and $\mathbf{A}_3$ the Y_n converge in distribution to its stationary distribution given by the random vector

$$Y_\infty = \zeta_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} \zeta_k \tag{2.3}$$

satisfying $\mathbf{E}|Y_\infty|^2 < \infty$. We denote by π the distribution of Y_∞ in $\mathbb{R}^q$ and by $\mathbf{P}^n(x, \cdot)$ the transition probability

$$\mathbf{P}^n(x, \Gamma) = \mathbf{P}(Y_n \in \Gamma \mid Y_0 = x), \quad x \in \mathbb{R}^q,$$

for every measurable $\Gamma \subseteq \mathbb{R}^q$.

Moreover, for some function $v : \mathbb{R}^q \to [1, \infty)$ we define (see p. 383 in Meyn and Tweedie [16])

$$|||\mathbf{P}^n - \pi|||_v = \sup_{x \in \mathbb{R}^q} \frac{\|\mathbf{P}^n(x, \cdot) - \pi(\cdot)\|_v}{v(x)},$$

where

$$\|\mathbf{P}^n(x, \cdot) - \pi(\cdot)\|_v = \sup_{0 \le f \le v} \left| \int_{\mathbb{R}^q} f(z)\mathbf{P}^n(x, \mathrm{d}z) - \int_{\mathbb{R}^q} f(z)\pi(\mathrm{d}z) \right|.$$

We need the following definitions.

Definition 2.1. (a) A Markov chain $\{Y_n\}_{n \in \mathbb{N}}$ is called *v-uniformly geometric ergodic* if there exist $R > 0$ and $0 < \rho < 1$ such that for every $n \in \mathbb{N}$

$$|||\mathbf{P}^n - \pi|||_v \le R\rho^n.$$

(b) For the stationary process $\{Y_n\}_{n \in \mathbb{N}}$ the *mixing coefficient* is for $k \in \mathbb{N}$ defined as

$$\alpha_k^* = \sup_{f,h} |\mathbf{E} f(\ldots, Y_{-1}, Y_0) h(Y_k, Y_{k+1}, \ldots) - \mathbf{E} f(\ldots, Y_{-1}, Y_0) \mathbf{E} h(Y_k, Y_{k+1}, \ldots)|, \tag{2.4}$$

where the supremum is taken over all measurable functions f and h satisfying $|f|, |h| \le 1$.

Note that Theorem 3 in Feigin and Tweedie [6] and Theorem 16.1.2 in Meyn and Tweedie [16] imply part (a) of the following result; part (b) is proved in the Appendix.

Theorem 2.2. *Let $\{Y_n\}_{n \in \mathbb{N}}$ be as defined in (1.3) such that $\mathbf{A}_1$–$\mathbf{A}_3$ hold.*

(a) *$\{Y_n\}_{n \in \mathbb{N}}$ is v-uniformly geometric ergodic with $v(x) = 1 + x'Tx$, $x \in \mathbb{R}^q$, for some fixed positive definite $q \times q$-matrix T.*

(b) *The stationary process* (1.1) *is strongly mixing with geometric rate, i.e. for some positive constant* C^*,

$$\alpha_k^* \le C^* \rho^k, \quad k \in \mathbb{N}. \tag{2.5}$$

Remark 2.3. Consider two sequences $\{Y_n(Y_\infty)\}_{n\ge 0}$ and $\{Y_n(Y)\}_{n\ge 0}$ given by the same recursion (1.1), but with different initial vectors Y_∞ and Y, where Y_∞ is supposed to have the stationary distribution and $\mathbf{E}|Y|^2 < \infty$. Both vectors Y_∞ and Y are supposed to be independent of the future values $\{(A_n, \zeta_n)\}_{n\in\mathbb{N}}$. For the initial vector Y we have the recursion

$$Y_n(Y) = A_n \cdots A_1 Y + \sum_{k=1}^{n} A_n \cdots A_{k+1}\zeta_k, \quad n \in \mathbb{N},$$

where $A_n A_{n+1} = I_q$, and analogously for initial vector Y_∞. By independence of the matrices A_n for all $n \in \mathbb{N}$ and the vectors Y, Y_∞ we obtain, invoking inequality (2.2),

$$\mathbf{E}|Y_n(Y) - Y_n(Y_\infty)|^2 \le \mathbf{E}\prod_{j=1}^{n}|A_j|^2 \mathbf{E}|Y - Y_\infty|^2 \le c_* \mathbf{E}|Y - Y_\infty|^2 \mathrm{e}^{-\gamma n}. \tag{2.6}$$

Therefore, $\mathbf{E}|Y_n(Y) - Y_n(Y_\infty)|^2$ tends to 0 exponentially fast as $n \to \infty$ for any initial vector Y with $\mathbf{E}|Y|^2 < \infty$. □

3. Extremal behaviour

The main goal of this paper is the investigation of the extremal behaviour of model (1.1). We introduce the unit sphere S in $\mathbb{R}^q$, i.e. $S = \{x \in \mathbb{R}^q : |x| = 1\}$. We assume that the vector (2.3) satisfies the following condition

($\mathbf{H}_0$) *There exists* $\lambda > 0$ *such that*

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(z' Y_\infty > t) = h(z), \quad z \in S,$$

for some strictly positive continuous function h *on* S.

Remark 3.1. We call condition $\mathbf{H}_0$ *regular variation of the vector* Y_∞ *in the Kesten sense*; see for example, [12], Remark 4 on p. 245. Indeed, it means that every linear combination of Y_∞ is one-dimensional regularly varying, where the slowly varying function is a positive constant. Regular variation in the Kesten sense is not necessarily equivalent to multivariate regular variation, which is defined as follows (see Resnick [19,20] for details). The q-dimensional random vector Y is called *regularly varying with index* $\alpha \ge 0$ if there exists a random vector Θ with values on the unit sphere S in $\mathbb{R}^q$ such that for all $t > 0$

$$\frac{\mathbf{P}(|Y| > tx, Y/|Y| \in \cdot)}{\mathbf{P}(|Y| > x)} \xrightarrow{v} t^{-\alpha}\mathbf{P}(\Theta \in \cdot), \quad x \to \infty, \tag{3.1}$$

where $\xrightarrow{v}$ means vague convergence of measures. We shall show in Section 4 that under weak conditions the finite dimensional distributions of $\{y_n\}_{n\in\mathbb{N}}$ are multivariate regularly varying in the sense of Eq. (3.1). □

Condition $\mathbf{H}_0$ implies that for $y_\infty = z_*' Y_\infty$

$$\lim_{n\to\infty} n\mathbf{P}(y_\infty > u_n) = h_* x^{-\lambda}, \quad x > 0, \tag{3.2}$$

where $u_n = n^{1/\lambda} x$ and $h_* = h(z_*)$.

This Poisson condition implies for the so-called associated iid sequence $\{\widehat{y}_k\}_{k\in\mathbb{N}}$ with the same distribution as y_∞ that the partial maxima $\widehat{M}_n = \max_{1\le k\le n} \widehat{y}_k$, $n \in \mathbb{N}$, satisfy

$$\lim_{n\to\infty} \mathbf{P}(n^{-1/\lambda}\widehat{M}_n \le x) = \exp(-h_* x^{-\lambda}), \quad x > 0. \tag{3.3}$$

This is classical extreme value theory and can be found in any textbook on this topic; see e.g. Embrechts, Klüppelberg and Mikosch [5].

For the extremal behaviour of model (1.1) we expect that the running maxima of $\{y_n\}_{n\in\mathbb{N}}$ have a limit of the same type as (3.3), but with different norming constants. Loosely speaking, large values of $\{y_n\}_{n\in\mathbb{N}}$ have a tendency to cluster, which implies that the maximum of M_n behaves as the maximum of θn iid rvs with the same distribution. The constant $\theta \in (0, 1]$ is called *extremal index* of $\{y_k\}_{k\in\mathbb{N}}$. It is a measure of local dependence amongst the exceedances over a high threshold by the process $\{y_k\}_{k\in\mathbb{N}}$ and has a natural interpretation as the reciprocal of the mean cluster size.

To describe the extremal behaviour in more detail we shall also study the point processes of exceedances of $\{y_n\}_{n\in\mathbb{N}}$ over high thresholds. We denote by ϵ the Dirac measure and define for $n \in \mathbb{N}$ and appropriate thresholds u_n the *time normalized point process of exceedances* on the Borel sets of $[0, \infty)$

$$N_n(\cdot) = \sum_{j=1}^{n} \epsilon_{j/n}(\cdot)\mathbf{1}_{\{y_j > u_n\}}. \tag{3.4}$$

We show that the sequences N_n converge for $n \to \infty$ and $u_n \uparrow \infty$ weakly to a compound Poisson process N. Moreover, we derive for the limit process N the intensity and cluster size distribution, which is a discrete distribution, denoted by $\{\nu_j\}_{j\in\mathbb{N}}$. Whereas the intensity describes the frequency of threshold exceedances, the cluster size distribution gives the distribution of the cluster sizes over thresholds. For further background we refer the reader to Leadbetter, Lindgren and Rootzén [15], Section 3.7, and Rootzén [21]; see also Embrechts et al. [5], Chapter 5 and Section 8.1.

Before stating our main results, we prove an analogue of Remark 2.3 for partial maxima.

Remark 3.2. (a) Recall that $M_n = \max\{y_1, \ldots, y_n\} = \max\{z_*' Y_1, \ldots, z_*' Y_n\}$, where the vector $(Y_1, \ldots, Y_n)$ depends on the initial vector Y and we indicate this by writing $M_n(Y)$. Taking into account that $|\max a_k - \max b_k| \le \max |a_k - b_k|$ we obtain for every $\delta > 0$

$$\begin{aligned}\mathbf{P}\left(|M_n(Y) - M_n(Y_\infty)| > \delta n^{1/\lambda}\right) &\le \mathbf{P}\left(\max_{1\le k\le n} |z_*' Y_k(Y) - z_*' Y_k(Y_\infty)| > \delta n^{1/\lambda}\right)\\ &\le \mathbf{P}\left(\max_{1\le k\le n} |Y_k(Y) - Y_k(Y_\infty)| > \delta n^{1/\lambda}\right)\\ &\le \frac{1}{\delta n^{1/\lambda}} \sum_{k=1}^{n} \mathbf{E}|Y_k(Y) - Y_k(Y_\infty)|.\end{aligned}$$

Now inequality (2.6) implies that the right hand side tends to 0, i.e.

$$n^{-1/\lambda}(M_n(Y) - M_n(Y_\infty)) \xrightarrow{P} 0, \quad n \to \infty,$$

for any initial vector Y with $\mathbf{E}|Y|^2 < \infty$. Therefore the weak limit of partial maxima M_n is independent of the initial vector Y.

(b) To show that the point process convergence is independent of the initial vector Y we need to assume condition $\mathbf{H}_0$. For $n \in \mathbb{N}$ we set $u_n = xn^{1/\lambda}$ for some $x > 0$ and denote by

$$N_n^Y(\cdot) = \sum_{j=1}^{n} \epsilon_{j/n}(\cdot) \mathbf{1}_{\{z_*' Y_j(Y) > u_n\}}$$

the point process of exceedances over u_n corresponding to the process $\{Y_j\}_{j\in\mathbb{N}}$ with initial vector Y. For arbitrary $0 < \varepsilon < 1$ we define $\Gamma_{n,\varepsilon} = \{\max_{1\le j\le n} |z_*' Y_j(Y) - z_*' Y_j(Y_\infty)| \le \varepsilon u_n\}$, and then

$$\begin{aligned}
\mathbf{P}(|N_n^Y(\cdot) - N_n^{Y_\infty}(\cdot)| > 0) &\le \mathbf{P}\left(\sum_{j=1}^{n} |\mathbf{1}_{\{z_*' Y_j(Y) > u_n\}} - \mathbf{1}_{\{z_*' Y_j(Y_\infty) > u_n\}}| > 0\right) \\
&\le \mathbf{P}\left(\sum_{j=1}^{n} |\mathbf{1}_{\{z_*' Y_j(Y) > u_n\}} - \mathbf{1}_{\{z_*' Y_j(Y_\infty) > u_n\}}| > 0, \Gamma_{n,\varepsilon}\right) + \mathbf{P}(\Gamma_{n,\varepsilon}^c) \\
&\le \sum_{j=1}^{n} \mathbf{P}(u_n(1-\varepsilon) \le z_*' Y_j(Y_\infty) \le (1+\varepsilon)u_n) + \mathbf{P}(\Gamma_{n,\varepsilon}^c) \\
&= n\mathbf{P}(u_n(1-\varepsilon) \le z_*' Y_\infty \le (1+\varepsilon)u_n) + \mathbf{P}(\Gamma_{n,\varepsilon}^c).
\end{aligned}$$

By definition, $\lim_{n\to\infty} \mathbf{P}(\Gamma_{n,\varepsilon}^c) = 0$ and condition $\mathbf{H}_0$ implies that

$$N_n^Y(\cdot) - N_n^{Y_\infty}(\cdot) \xrightarrow{P} 0, \quad n \to \infty,$$

for any initial vector Y with $\mathbf{E}|Y|^2 < \infty$. Thus the weak limit of $N_n(\cdot)$ is independent of the initial vector Y. $\square$

The extremal index θ and the cluster size distribution $\{\nu_j\}_{j\in\mathbb{N}}$ can be represented by the limit measure Q of the following measures on $\mathbb{R}^q$:

$$Q_t(\Gamma) = \mathbf{P}(t^{-1}Y_\infty \in \Gamma \mid t^{-1}Y_\infty \in W_{z_*}) = \frac{\mathbf{P}(t^{-1}Y_\infty \in \Gamma \cap W_{z_*})}{\mathbf{P}(t^{-1}Y_\infty \in W_{z_*})}, \tag{3.5}$$

for $t \to \infty$, where $W_z = \{y \in \mathbb{R}^q : z'y > 1\}$ for $z \in \mathbb{R}^q$.

Theorem 3.3. *Assume that condition $\mathbf{H}_0$ holds. If the positive exponent λ in this condition is non-integer, then there exists a weak limit Q of the family $\{Q_t\}_{t\ge1}$ of measures (3.5) as $t \to \infty$. It satisfies for measurable $\Gamma \subset \mathbb{R}^q$*

$$Q(\Gamma) = \mu(\Gamma \cap W_{z_*}), \tag{3.6}$$

where μ is some positive σ-finite measure on $\mathbb{R}^q \setminus \{\mathbf{0}\}$ with $\mu(W_{z_*}) = 1$.

In our principal example (1.2) with Gaussian rvs $\{\xi_n\}_{n\in\mathbb{N}}$ the stationary distribution given by the vector (2.3) is symmetric. For such cases we can prove a stronger result.

Theorem 3.4. *Assume that condition $\mathbf{H}_0$ holds and Y_∞ has a symmetric distribution on $\mathbb{R}^q$; i.e. that $Y \stackrel{d}{=} -Y$. If the positive exponent λ in condition $\mathbf{H}_0$ is non-even, then the assertion of Theorem 3.3 holds.*

We shall prove this result in Section 4. The measure Q plays an important role in the description of the joint distribution of the stationary vector $(y_1, \ldots, y_k)$ for every $k \in \mathbb{N}$ on high levels. In this sense it is not surprising that Q describes the partial maxima of $\{y_n\}_{n\in\mathbb{N}}$ and the limit behaviour of point processes of exceedances.

We set

$$\varsigma(y) = \sum_{j=1}^{\infty} \mathbf{1}_{\{z_*' \Pi_j y > 1\}} \quad \text{with} \quad \Pi_j = A_j \cdots A_1. \tag{3.7}$$

and define the following technical conditions:

($\mathbf{H}_1$) $\mathbf{P}(\varsigma(y) = 0) > 0$ *for every* $y \in W_{z_*}$.
($\mathbf{H}_2$) $\mathbf{P}(z_*' \Pi_j y = 1) = 0$ *for every* $y \in W_{z_*}$ *and* $j \in \mathbb{N}$.

$\mathbf{H}_1$ and $\mathbf{H}_2$ are conditions on the distribution of the sequence $(A_n)_{n\in\mathbb{N}}$. Condition $\mathbf{H}_1$ implies that $\max_{j\in\mathbb{N}} z_*' \Pi_j y$ falls with positive probability in the interval $[-1, 1]$. Condition $\mathbf{H}_2$ is for example satisfied if the random variables $z_*' \Pi_j y$ have a density in $\mathbb{R}$ for every $y \in W_{z_*}$ and every $j \in \mathbb{N}$. In Lemma 6.5 with proof in Section 6.2 we shall check these conditions for the random coefficient autoregressive model (1.2).

The following result describes the extremal behaviour of any process with multivariate random recurrence state space representation under natural conditions.

Theorem 3.5. *Assume that the conditions* $\mathbf{A}_1$–$\mathbf{A}_3$ *and* $\mathbf{H}_0$–$\mathbf{H}_2$ *hold. Moreover, let the exponent* λ *in condition* $\mathbf{H}_0$ *be non-even if* Y_∞ *is symmetric, and non-integer otherwise.*
(a) *Then*

$$\lim_{n\to\infty} \mathbf{P}(n^{-1/\lambda} M_n \le x) = \mathrm{e}^{-\theta h_* x^{-\lambda}}, \quad x > 0,$$

where $h_* = h(z_*)$ *and the extremal index* θ *is defined as*

$$\theta = \int_{\mathbb{R}^q} g(y) Q(\mathrm{d}y) > 0. \tag{3.8}$$

The probability measure $Q(\cdot)$ *is the weak limit of the family* (3.5) *as* $t \to \infty$ *and*

$$g(y) = \mathbf{P}(\varsigma(y) = 0), \quad y \in W_{z_*}, \tag{3.9}$$

with the function $\varsigma(\cdot)$ *defined in* (3.7).
(b) *For* $n \in \mathbb{N}$ *let* N_n *be the point process of exceedances over the threshold* $u_n = n^{1/\lambda} x$ *for* $x > 0$ *given by* (3.4). *Then*

$$N_n \xrightarrow{d} N, \quad n \to \infty,$$

where N *is a compound Poisson process with intensity* $\theta\tau$ ($\tau = h_* x^{-\lambda}$) *and the cluster size probabilities*

$$\nu_k = \theta^{-1}(\theta_k - \theta_{k+1}), \quad k \in \mathbb{N},$$

satisfying $\theta_1 = \theta \ge \theta_2 \ge \theta_3 \ge \cdots \ge 0$ *with*

$$\theta_k = \int_{\mathbb{R}^q} g_k(y) Q(\mathrm{d}y) \quad \text{and} \quad g_k(y) = \mathbf{P}(\varsigma(y) = k-1), \quad y \in W_{z_*}, \tag{3.10}$$

for $k \in \mathbb{N}$ *with* $g_1(y) = g(y)$ *as defined in* (3.9).

Remark 3.6. (a) In Appendix B we shall show that $\mathbf{H}_2$ implies that all g_k are continuous.
(b) For $q = 1$ the limit measure Q has a Lebesgue density; more precisely,

$$Q(\mathrm{d}y) = \lambda y^{-\lambda-1}\mathbf{1}_{\{y\geq 1\}}\mathrm{d}y.$$

In this case the extremal index has representation

$$\theta = \lambda \int_1^\infty \mathbf{P}\left(\max_{k\in\mathbb{N}} A_k \cdots A_1 \leq y^{-1}\right) y^{-\lambda-1}\mathrm{d}y.$$

This result has been obtained in Borkovec [3]. □

4. Properties of the measures Q_t — multivariate regular variation

In this section we come back to Remark 3.1. Basrak et al. [1] investigate the various notions of regular variation and their relationships, in particular the relationship between regular variation in the Kesten sense and in the sense of (3.1). They proved in their Theorem 1.1 that for non-integer $\lambda > 0$ regular variation in the Kesten sense implies (3.1). They also proved this result for λ an odd integer and vectors Y in $\mathbb{R}_+^q$. As an important class of models is symmetric – also our principal model (1.2) is in the important Gaussian case symmetric – we reconsider the problem in this context. We present a new proof of Theorem 1.1 of [1], together with an extension of this result for symmetric models.

We follow the point process theory as presented in Kallenberg [11]. Set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and consider in what follows $E = \overline{\mathbb{R}}^q \setminus \{\mathbf{0}\}$ as the state space of the point processes.

We study the properties of the family of measures defined in (3.5). Define

$$m_t(\Gamma) = \frac{\mathbf{P}(Y_\infty \in t\Gamma)}{\mathbf{P}(Y_\infty \in tW_{z_*})}, \quad t \geq 1, \tag{4.1}$$

for any measurable $\Gamma \subseteq \mathbb{R}^q$ and notice that $Q_t(\Gamma) = m_t(\Gamma \cap W_{z_1})$.

Remark 4.1. (a) Regular variation in the Kesten sense as given in $\mathbf{H}_0$ can be rewritten as

$$\lim_{t\to\infty} \frac{P(z'Y_\infty > t)}{P(z_*'Y_\infty > t)} = \tilde{h}(z), \quad z \in S, \tag{4.2}$$

where $\tilde{h}(\cdot) = h(\cdot)/h_*$. Moreover, the function $\tilde{h}$ satisfies for every $t > 0$,

$$\tilde{h}(tz) = t^\lambda \tilde{h}(z), \quad z \in S,$$

where λ is defined in $\mathbf{H}_0$. This means that for all $z \neq \mathbf{0}$ the rv $z'Y_\infty$ is regularly varying with index λ.
(b) The limit relation (4.2) is equivalent to

$$\lim_{t\to\infty} \frac{P(Y_\infty \in tW_z)}{P(Y_\infty \in tW_{z_*})} = \tilde{h}(z), \quad z \in S. \quad \square \tag{4.3}$$

Now define

$$\rho_0(x, y) = \left|\frac{1}{x_+ + 1} - \frac{1}{y_+ + 1}\right| + \left|\frac{1}{x_- + 1} - \frac{1}{y_- + 1}\right|, \quad x, y \in \mathbb{R},$$

were we used the notation $x_+ = \max(x, 0)$ and $x_- = -\min(x, 0)$. With this notation we introduce the following metric on E

$$\rho(x, y) = \sum_{j=1}^{q} \rho_0(x_j, y_j) + \left| \frac{1}{|x|} - \frac{1}{|y|} \right|, \quad x, y \in E.$$

Then (E, ρ) is a separable and complete metric space. Moreover, for every $\delta > 0$ the set $\{x \in E : |x| \geq \delta\}$ is compact in this space, and bounded sets are those that are bounded away from $\mathbf{0} \in \mathbb{R}^q$. The topology on E is discussed in more detail in Resnick [20].

We are interested in vague convergence of measures (4.1) in (E, ρ), i.e. conditions for which there exists a measure m in E such that $m_t \xrightarrow{v} m$ as $t \to \infty$. We recall that $m_t \xrightarrow{v} m$ means vague convergence of m_t to m; i.e.

$$\lim_{t\to\infty} \int_E f(y) m_t(\mathrm{d}y) = \int_E f(y) m(\mathrm{d}y)$$

for all continuous functions f with compact support in (E, ρ). We shall use the following results.

Theorem 4.2 (*Basrak, Davis and Mikosch [1], Theorem 1.1(ii)*). *Let condition $\mathbf{H}_0$ hold for non-integer $\lambda > 0$. Then the family (4.1) of measures has a vague limit in (E, ρ). Moreover, Y_∞ is multivariate regularly varying in the sense of Eq. (3.1).*

The following result is essential for this investigation, in particular, for our principal example (1.2).

Theorem 4.3. *Assume that Y_∞ has a symmetric distribution on $\mathbb{R}^q$. Let condition $\mathbf{H}_0$ hold for non-even $\lambda > 0$. Then the family (4.1) of measures has a vague limit in (E, ρ). Moreover, Y_∞ is multivariate regularly varying in the sense of (3.1).*

Remark 4.4. Note that this theorem does not hold in general for even integers λ. This follows directly from the counterexample given in Hult and Lindskog [10]: take on p. 136 above e.g. $f_\theta = 1/(2\pi)$.

We shall use the following lemma, whose proof is given in Appendix C. To formulate the result we recall the definition of a subsequential vague limit. The measure μ is called a *subsequential vague limit* of m_t if there exists a sequence $t_n \to \infty$ such that $m_{t_n} \xrightarrow{v} \mu$.

Lemma 4.5. *Assume that condition $\mathbf{H}_0$ holds. If μ is a subsequential vague limit of $\{m_t\}_{t\geq 1}$, then for every $u > 0$ we have*

$$\mu(y \in \mathbb{R}^q : z'y > u) = u^{-\lambda} \tilde{h}(z), \quad z \in S. \tag{4.4}$$

Moreover, setting $\hat{h}(z) = \tilde{h}(z) + \tilde{h}(-z)$, we obtain for $u > 0$ and $0 < \nu < \lambda$

$$\int_{|z'y|>u} |z'y|^\nu \mu(\mathrm{d}y) = \frac{\lambda}{\lambda - \nu} u^{\nu-\lambda} \hat{h}(z), \quad z \in S, \tag{4.5}$$

and for $\nu > \lambda$

$$\int_{|z'y|<u} |z'y|^\nu \mu(\mathrm{d}y) = \frac{\lambda}{\nu - \lambda} u^{\nu-\lambda} \hat{h}(z), \quad z \in S. \tag{4.6}$$

Proof of Theorem 4.3. We first show that the family (4.1) is relatively compact; i.e. that $\sup_{t\ge 1} m_t(B) < \infty$ for all bounded Borel sets B in E (see Kallenberg [11], Theorem 15.7.5). To see this recall that in the space E bounded sets are those that are bounded away from $\mathbf{0} \in \mathbb{R}^q$, i.e. for every bounded B there exist non-zero vectors $x_1, \dots, x_k$ in $\mathbb{R}^q \setminus \{0\}$ such that

$$B \subseteq \bigcup_{j=1}^{k} W_{x_j} \cup (\overline{\mathbb{R}}^q \setminus \mathbb{R}^q)$$

and, hence, by (4.3)

$$\sup_{t\ge 1} m_t(B) \le \sup_{t\ge 1} \sum_{j=1}^{k} \frac{\mathbf{P}(Y_\infty \in t W_{x_j})}{\mathbf{P}(Y_\infty \in t W_{z_*})} < \infty.$$

This implies that the family $\{m_t\}_{t\ge 1}$ has a subsequential vague limit.

To complete the proof it suffices to show that any two such limits, μ_1 and μ_2, are identical. First we suppose that λ is non-integer, i.e. $\lambda \in (l-1, l)$ for some $l \in \mathbb{N}$. Let μ_1 and μ_2 be two subsequential vague limits.

First note now that (4.4) implies that

$$\mu_1(x \in E : |x| = \infty) = \mu_2(x \in E : |x| = \infty) = 0.$$

Therefore, for the identity of the measures μ_1 and μ_2 it suffices to show that

$$\int_{\mathbb{R}^q} f(x)\mu_1(\mathrm{d}x) = \int_{\mathbb{R}^q} f(x)\mu_2(\mathrm{d}x) \tag{4.7}$$

for every continuous bounded function f satisfying

$$f(x) = 0 \quad \text{if } |x| \le r \tag{4.8}$$

for some positive r. W.l.o.g. we can assume that f is infinitely often differentiable and periodic (with period $2L$) in every component of x. Hence, f has a representation as Fourier series

$$f(x) = \sum_{k\in\mathbb{N}^q} c_k \mathrm{e}^{i(z_k, x)}, \quad x \in \mathbb{R}^q, \tag{4.9}$$

where $z_k = \pi k/L$ and $(z_k, x) = z_k' x$. Now note that condition (4.8) implies for all $d \in \mathbb{N}_0$

$$\sum_{k\in\mathbb{N}^q} c_k (z_k, y)^d = 0, \quad y \in \mathbb{R}^q, \tag{4.10}$$

which implies that f has representation

$$f(x) = \sum_{k\in\mathbb{N}^q} c_k \Delta_{l-1}((z_k, x)), \quad x \in \mathbb{R}^q,$$

where

$$\Delta_l((z_k, x)) = \mathrm{e}^{i(z_k,x)} - \sum_{j=0}^{l} \frac{(i(z_k, x))^j}{j!}, \quad x \in \mathbb{R}^q.$$

Recall from standard analysis that

$$|\Delta_l(x)| \le \min\left(\frac{|x|^{l+1}}{(l+1)!}, \frac{2|x|^l}{l!}\right), \quad x \in \mathbb{R}^q. \tag{4.11}$$

Moreover, we can represent the function f as

$$f(x) = H_{l-1}(x) + \hat{H}_{l-1}(x), \quad x \in \mathbb{R}^q,$$

where, setting $\overline{z} = z/|z|$,

$$H_l(x) = \sum_{k \in \mathbb{N}^q} c_k \Delta_l((z_k, x)) \mathbf{1}_{\{|(\overline{z}_k, x)| \geq 1\}}, \quad \hat{H}_l(x) = \sum_{k \in \mathbb{N}^q} c_k \Delta_l((z_k, x)) \mathbf{1}_{\{|(\overline{z}_k, x)| < 1\}}.$$

Taking (4.5) and (4.11) into account we obtain

$$\begin{aligned}\int_{\mathbb{R}^q} H_{l-1}(x) \mu_1(\mathrm{d}x) &= \sum_{k \in \mathbb{N}^q} c_k \int_{\{|(\overline{z}_k, x)| \geq 1\}} \Delta_{l-1}((z_k, x)) \mu_1(\mathrm{d}x) \\ &= \sum_{k \in \mathbb{N}^q} c_k \int_{\{|(\overline{z}_k, x)| \geq 1\}} \Delta_{l-1}((z_k, x)) \mu_2(\mathrm{d}x) \\ &= \int_{\mathbb{R}^q} H_{l-1}(x) \mu_2(\mathrm{d}x).\end{aligned}$$

We have used that the integrals on the right hand side are finite by Lemma 4.5, which also justifies the interchange of summation and integral. Equality of the two integrals follows from the integrands' dependence on the inner products (z_k, x) only. Analogously, from (4.6) and (4.11) we obtain

$$\int_{\mathbb{R}^q} \hat{H}_{l-1}(x) \mu_1(\mathrm{d}x) = \int_{\mathbb{R}^q} \hat{H}_{l-1}(x) \mu_2(\mathrm{d}x).$$

We show now equality of μ_1 and μ_2 for odd integers $l = \lambda = 2p + 1$ for some $p \in \mathbb{N}_0$. For such l we represent the function (4.9) as

$$f(x) = H_{l-1}(x) + \hat{H}_l(x) - \frac{i^l}{l!} P_l(x), \quad x \in \mathbb{R}^q,$$

where $P_l(x) = \sum_{k \in \mathbb{N}^q} c_k (z_k, x)^l \mathbf{1}_{\{|(\overline{z}_k, x)| < 1\}}$. From the calculations above it follows that (4.7) holds if

$$\int_{\mathbb{R}^q} P_l(x) \mu_1(\mathrm{d}x) = \int_{\mathbb{R}^q} P_l(x) \mu_2(\mathrm{d}x).$$

From the definition of the measures μ_1 and μ_2 there follows the existence of sequences $\{r_{1n}\}_{n \in \mathbb{N}}$ and $\{r_{2n}\}_{n \in \mathbb{N}}$ such that $m_{r_{in}} \to \mu_i$ as $n \to \infty$ for $i = 1, 2$. Let D_{P_l} be the set of discontinuity points of the function P_l, which is given by

$$D_{P_l} = \bigcup_{k \in \mathbb{N}^q} \{x \in \mathbb{R}^q : |(\overline{z}_k, x)| = 1\}.$$

In Appendix C we prove that $\mu_i(D_{P_l}) = 0$ for $i = 1, 2$. Moreover, for every $x \in \mathbb{R}^q$

$$|P_l(x)| \leq \sum_{k \in \mathbb{N}^q} |c_k| < \infty,$$

i.e. P_l is bounded, since the function (4.9) is infinitely often differentiable. Furthermore, if $x \in \mathbb{R}^q$ with $|x| < 1$ then $|(\overline{z}_k, x)| \leq |\overline{z}_k||x| < 1$ for every $k \in \mathbb{N}^q$. Thus (4.10) implies

$$P_l(x) = \sum_{k \in \mathbb{N}^q} c_k (z_k, x)^l \mathbf{1}_{\{|(\overline{z}_k, x)| < 1\}} = \sum_{k \in \mathbb{N}^q} c_k (z_k, x)^l = 0, \quad |x| < 1,$$

i.e. this function has bounded support $\{x \in \mathbb{R}^q : |x| \geq 1\}$. Therefore, by Theorem 15.7.3 of Kallenberg [11] we can write

$$\begin{aligned}\int_{\mathbb{R}^q} P_l(x)\mu_1(\mathrm{d}x) &= \lim_{n\to\infty} \int_{\mathbb{R}^q} P_l(x) m_{r_{1n}}(\mathrm{d}x) \\ &= \lim_{n\to\infty} \frac{1}{\mathbf{P}(Y_\infty \in r_{1n} W_{z_*})} \mathbf{E} P_l(Y_\infty / r_{1n}) \\ &= \lim_{n\to\infty} \frac{1}{r_{1n}^l \mathbf{P}(z_1' Y_\infty > r_{1n})} \sum_{k\in\mathbb{N}^q} c_k |z_k|^l \mathbf{E}(\overline{z}_k, Y_\infty)^l \mathbf{1}_{\{|(\overline{z}_k, Y_\infty)| < r_{1n}\}} \\ &= 0,\end{aligned}$$

by symmetry of Y_∞. Analogously we obtain $\int_{\mathbb{R}^q} P_l(x)\mu_2(\mathrm{d}x) = 0$. Hence, (4.1) converges to a limit μ. □

Proof of Theorems 3.3 and 3.4. We denote by μ the vague limit of the measures (4.1). From (4.4) it follows directly that $\mu(\partial W_{z_1}) = 0$. Therefore, the family (3.5) has also a vague limit Q (see Kallenberg [11], Theorem 15.7.3), which satisfies (3.6). Moreover, by definition, for every $t \geq 1$,

$$Q_t(\mathbb{R}^q) = Q_t(W_{z_1}) = 1 = Q(W_{z_1}) = Q(\mathbb{R}^q).$$

Thus, Theorem 15.7.6 of [11] guarantees weak convergence of the family (3.5) to Q. □

5. The existence of an extremal index and point process convergence

5.1. Definitions and existing results

We consider a stationary process $\{y_k\}_{k\in\mathbb{N}}$ such that for every $\tau > 0$ there exists a sequence $\{u_n(\tau)\}_{n\in\mathbb{N}}$ for which

$$\lim_{n\to\infty} nP(y_1 > u_n(\tau)) = \tau. \tag{5.1}$$

The conditions $\mathbf{D}(u_n(\tau))$ and $\Delta(u_n(\tau))$ are extreme mixing conditions (for the definitions see for example Rootzén [21], p. 379), which are both implied by strong mixing; i.e. they follow immediately from Theorem 2.2 for every appropriate sequence $\{u_n\}_{n\in\mathbb{N}}$.

Definition 5.1 (*Extremal Index*). Assume that there exists a constant $0 < \theta \leq 1$ such that for every $\tau > 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\max_{1\leq k\leq n} y_k \leq u_n(\tau)\right) = \mathrm{e}^{-\theta\tau}.$$

Then θ is called the *extremal index* of the sequence $\{y_k\}_{k\in\mathbb{N}}$.

Theorem 5.2 (*Rootzén [21], Theorem 4.1(i)*). *Suppose that condition* $\mathbf{D}(u_n(\tau))$ *holds for each* $\tau > 0$. *Then* $\{y_k\}_{k\in\mathbb{N}}$ *has extremal index* $\theta > 0$ *if and only if*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} |\mathbf{P}(M_{[\epsilon n]} \leq u_n \mid y_0 > u_n) - \theta| = 0 \tag{5.2}$$

for $u_n = u_n(\tau_0)$ *for some* $\tau_0 > 0$.

We consider now the point process of exceedances for the process $\{y_k\}_{k\in\mathbb{N}}$ defined as

$$N_{n,\tau}(\cdot) = \sum_{j=1}^{n} \epsilon_{j/n}(\cdot)\mathbf{1}_{\{y_j > u_n(\tau)\}},$$

where the sequence $\{u_n(\tau)\}$ is given in (5.1).

To study the asymptotic properties of these processes we apply the following criterion.

Theorem 5.3 (*Rootzén [21], Theorem 4.1(ii)*). *Suppose that $\{y_k\}_{k\in\mathbb{N}}$ has extremal index $0 < \theta \le 1$ and the condition $\Delta(u_n(\tau))$ holds for each $\tau > 0$. If for every $k \ge 2$ and some $\tau_0 > 0$*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} |\mathbf{P}(N_{n,\tau_0}((0,\epsilon]) = k-1 \mid y_0 > u_n) - \theta_k| = 0, \tag{5.3}$$

then the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ is decreasing, i.e. $\theta_1 = \theta \ge \theta_2 \ge \theta_3 \ge \dots$, and for every $\tau > 0$ the point processes $N_{n,\tau}$ converge weakly to a compound Poisson process N with intensity $\theta\tau$ and cluster size probabilities $\nu_k = \theta^{-1}(\theta_k - \theta_{k+1})$, $k \in \mathbb{N}$.

5.2. *Proof of Theorem 3.5*

In view of Remarks 2.3 and 3.2 we prove this theorem for the stationary process (1.1), i.e. the process starts with initial vector $Y_0 \overset{d}{=} Y_\infty$ as in (2.3).

5.2.1. *Extremal index*

In this section we prove Theorem 3.5(a). We apply Rootzén's criterion (Theorem 5.2) based on mixing properties of the process (1.1) which immediately follow from Theorem 2.2(b). The most important issue will be representation (3.8) for the extremal index.

First of all, note that condition $\mathbf{H}_1$ implies that θ as defined in (3.8) is strictly positive. We verify property (5.2) for $\{y_k\}_{k\in\mathbb{N}}$ with $u_n = n^{1/\lambda}x$ for arbitrary $x > 0$. Defining again $\Pi_k = A_k \cdots A_1$ we define the auxiliary process

$$\widetilde{Y}_0 = Y_0, \quad \widetilde{Y}_k = \Pi_k Y_0, \quad k \in \mathbb{N},$$

which obviously satisfies

$$\widetilde{Y}_k = A_k \widetilde{Y}_{k-1}.$$

Hence the difference process $V_k = Y_k - \widetilde{Y}_k$, $k \in \mathbb{N}$, satisfies Eq. (1.1) with initial value zero. Define $\widetilde{y}_j = z_*'\widetilde{Y}_j$, $m = [\epsilon n]$ for some $\epsilon > 0$, and $V_m^* = \sup_{1\le k\le m} |V_k|$. To check condition (5.2) notice that for every $0 < \delta < 1$

$$\begin{aligned}\mathbf{P}(M_m \le u_n \mid y_0 > u_n) &= \mathbf{P}\left(\max_{1\le k\le m} (\widetilde{y}_k + z_*' V_k) \le u_n \mid y_0 > u_n\right)\\ &\le \mathbf{P}\left(\max_{1\le k\le m} \widetilde{y}_k \le (1+\delta)u_n \mid y_0 > u_n\right) + \mathbf{P}(V_m^* > \delta u_n)\\ &\le \int_{\mathbb{R}^q} g\left(\frac{y}{1+\delta}\right) Q_{u_n}(\mathrm{d}y) + \Delta_1(n) + \Delta_2(n),\end{aligned}$$

where g is as in (3.9), the measure $Q_t(\cdot)$ is defined in (3.5) and

$$\Delta_1(n) = \mathbf{P}\left(\max_{k>m} \widetilde{y}_k > (1+\delta)u_n \mid y_0 > u_n\right) \quad \text{and} \quad \Delta_2(n) = \mathbf{P}(V_m^* > \delta u_n). \tag{5.4}$$

By Lemma B.1 the function $g(\cdot) = g_1(\cdot)$ is continuous. Moreover, Theorems 3.3–3.4 imply that there exists a weak limit Q for the family $\{Q_t\}_{t\geq 1}$. Therefore,

$$\lim_{t\to\infty} \int_{\mathbb{R}^q} g\left(\frac{y}{1+\delta}\right) Q_t(\mathrm{d}y) = \int_{\mathbb{R}^q} g\left(\frac{y}{1+\delta}\right) Q(\mathrm{d}y).$$

Next we show

$$\lim_{n\to\infty} \Delta_1(n) = 0. \tag{5.5}$$

Indeed, for every $L > 0$ we have

$$\begin{aligned}
\Delta_1(n) &= \frac{\mathbf{P}\left(\max_{k>m} z_*' \Pi_k Y_0 > (1+\delta)u_n,\, z_*' Y_0 > u_n\right)}{\mathbf{P}(z_*' Y_0 > u_n)} \\
&\leq \mathbf{P}\left(\max_{k>m} |\Pi_k| > \frac{1+\delta}{L}\right) + \frac{\mathbf{P}(|Y_0| > L u_n,\, z_*' Y_0 > u_n)}{\mathbf{P}(z_1' Y_0 > u_n)} \\
&\leq \sum_{k>m} \mathbf{P}\left(|\Pi_k| > \frac{1+\delta}{L}\right) + \frac{\mathbf{P}(|Y_0| > L u_n)}{\mathbf{P}(z_*' Y_0 > u_n)}.
\end{aligned}$$

From (2.2) and Chebyshev's inequality we conclude

$$\mathbf{P}\left(|\Pi_k| > \frac{1+\delta}{L}\right) \leq \frac{L^2}{(1+\delta)^2}\mathbf{E}|\Pi_k|^2 \leq \frac{L^2}{(1+\delta)^2} c_* \mathrm{e}^{-\gamma k}.$$

Therefore, condition $\mathbf{H}_0$ yields for every $L > 0$

$$\limsup_{n\to\infty} \Delta_1(n) \leq \text{const}\, L^{-\lambda}.$$

Taking now $L \to \infty$ implies (5.5).
Next we consider the second term in (5.4). We shall show that

$$\Delta_2^* = \limsup_{n\to\infty} \Delta_2(n) \leq \text{const}\, \epsilon. \tag{5.6}$$

Indeed, we have

$$\begin{aligned}
\Delta_2(n) &\leq \sum_{1\leq k\leq m} \mathbf{P}(|V_k| > \delta u_n) \\
&\leq \sum_{1\leq k\leq m} \mathbf{P}(|Y_k| > \delta u_n/2) + \sum_{1\leq k\leq m} \mathbf{P}(|\widetilde{Y}_k| > \delta u_n/2) \\
&\leq m\mathbf{P}(|Y_1| > \delta u_n/2) + \frac{4\mathbf{E}|Y_0|^2}{\delta^2 u_n^2} \sum_{k=1}^{\infty} \mathbf{E}|\Pi_k|^2 \\
&\leq \epsilon n \mathbf{P}(|Y_1| > \delta u_n/2) + 4c_* \frac{\mathbf{E}|Y_0|^2}{\delta^2 u_n^2} \sum_{k=1}^{\infty} \mathrm{e}^{-\gamma k}.
\end{aligned}$$

The last inequality implies (5.6). Taking into account that $g(ry) \to g(y)$ as $r \to 1$ for each $y \in \mathbb{R}^q$, we obtain the following upper bound

$$\limsup_{n\to\infty} \mathbf{P}(M_m \leq u_n \mid y_0 > u_n) \leq \theta + \text{const}\, \epsilon$$

for every $\epsilon > 0$. Analogously, we obtain the lower bound

$$\begin{aligned}
\mathbf{P}(M_m \le u_n | y_0 > u_n) &= \mathbf{P}\left(\max_{1\le k\le m} (\widetilde{y}_k + z_*' V_k) \le u_n \mid y_0 > u_n\right) \\
&\ge \mathbf{P}\left(\max_{1\le k\le m} \widetilde{y}_k \le (1-\delta)u_n,\, V_m^* \le \delta u_n \mid y_0 > u_n\right) \\
&\ge \mathbf{P}\left(\max_{1\le k\le m} z_*' \Pi_k Y_0 \le (1-\delta)u_n \mid y_0 > u_n\right) - \Delta_2(n) \\
&\ge \mathbf{P}\left(\max_{k\in\mathbb{N}} z_*' \Pi_k Y_0 \le (1-\delta)u_n \mid z_*' Y_0 > u_n\right) - \Delta_2(n) \\
&= \int_{\mathbb{R}^q} g\left(\frac{y}{1-\delta}\right) Q_{u_n}(\mathrm{d}y) - \Delta_2(n).
\end{aligned}$$

This implies for every $\epsilon > 0$

$$\liminf_{n\to\infty} \mathbf{P}(M_m \le u_n \mid y_0 > u_n) \ge \theta - \text{const}\,\epsilon.$$

These bounds imply (5.2), i.e. Theorem 3.5(a). □

5.2.2. Point process convergence

In this section we prove Theorem 3.5(b). We invoke Theorem 5.3, which characterizes point process convergence of N_n to a compound Poisson process. As mentioned above the mixing condition $\Delta(u_n)$ immediately follows from Theorem 2.2(b). We verify property (5.3) for $\{y_k\}_{k\in\mathbb{N}}$. As in the proof of Theorem 3.5(a) we set $m = [\epsilon n]$ for some $\epsilon > 0$ and $\widetilde{y}_j = z_*' \widetilde{Y}_j$ for $j \in \mathbb{N}$. For every $0 < \delta < 1$ we get

$$\begin{aligned}
\mathbf{P}(N_n((0,\epsilon]) = k-1 \mid y_0 > u_n) &= \mathbf{P}\left(\sum_{j=1}^m \mathbf{1}_{\{y_j > u_n\}} = k-1 \,\middle|\, y_0 > u_n\right) \\
&\le \mathbf{P}\left(\sum_{j=1}^m \mathbf{1}_{\{y_j > u_n\}} = k-1,\, V_m^* \le \delta u_n \,\middle|\, y_0 > u_n\right) + \Delta_2(n) \\
&\le I_{n,\delta} + D_{n,\delta} + \Delta_2(n), \qquad (5.7)
\end{aligned}$$

where

$$I_{n,\delta} = \mathbf{P}\left(\sum_{j=1}^m \mathbf{1}_{\{\widetilde{y}_j > (1-\delta)u_n\}} = k-1 \,\middle|\, y_0 > u_n\right),$$

$$D_{n,\delta} = \mathbf{P}\left(\sum_{j=1}^m \left(\mathbf{1}_{\{\widetilde{y}_j > (1-\delta)u_n\}} - \mathbf{1}_{\{y_j > u_n\}}\right) \ge 1,\, V_m^* \le \delta u_n \,\middle|\, y_0 > u_n\right)$$

and $\Delta_2(n)$ is defined in (5.4). We estimate $I_{n,\delta}$ first.

$$\begin{aligned}
I_{n,\delta} \le\; &\mathbf{P}\left(\sum_{j=1}^m \mathbf{1}_{\{\widetilde{y}_j > (1-\delta)u_n\}} = k-1,\, \max_{j>m} \widetilde{y}_j \le (1-\delta)u_n \,\middle|\, y_0 > u_n\right) \\
&+ \mathbf{P}\left(\max_{j>m} \widetilde{y}_j > (1-\delta)u_n \,\middle|\, y_0 > u_n\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbf{P}\left(\sum_{j=1}^{\infty}\mathbf{1}_{\{z_*'\Pi_j Y_0>(1-\delta)u_n\}}=k-1\,\middle|\, z_*'Y_0>u_n\right)\\
&\quad+\mathbf{P}\left(\max_{j>m}\widetilde{y}_j>(1-\delta)u_n\,\middle|\, y_0>u_n\right)\\
&=\int_{\mathbb{R}^q} g_k\left(\frac{y}{1-\delta}\right)Q_{u_n}(\mathrm{d}y)+\varDelta_1'(n),
\end{aligned}
$$

where

$$
\varDelta_1'(n)=\mathbf{P}\left(\max_{j>m}\widetilde{y}_j>(1-\delta)u_n\,\middle|\, y_0>u_n\right).
$$

Notice that like for (5.5) we obtain $\lim_{n\to\infty}\varDelta_1'(n)=0$. Next we estimate $D_{n,\delta}$. For fixed $l\in\mathbb{N}$, $1\leq l\leq m$, we can write

$$
\begin{aligned}
D_{n,\delta}&=\mathbf{P}\left(\sum_{j=1}^{m}\mathbf{1}_{\{y_j\leq u_n,\widetilde{y}_j>(1-\delta)u_n\}}\geq 1,\, V_m^*\leq\delta u_n\,\middle|\, y_0>u_n\right)\\
&=\mathbf{P}\left(\bigcup_{j=1}^{m}\{y_j\leq u_n,\widetilde{y}_j>(1-\delta)u_n\},\, V_m^*\leq\delta u_n\,\middle|\, y_0>u_n\right)\\
&\leq\sum_{j=1}^{l}\mathbf{P}\left(y_j\leq u_n,\widetilde{y}_j>(1-\delta)u_n,\, V_m^*\leq\delta u_n\,\middle|\, y_0>u_n\right)\\
&\quad+\mathbf{P}\left(\bigcup_{j=l+1}^{m}\{y_j\leq u_n,\widetilde{y}_j>(1-\delta)u_n\},\, V_m^*\leq\delta u_n\,\middle|\, y_0>u_n\right).
\end{aligned}
$$

Moreover, setting $\varGamma_n=\bigcup_{j=l+1}^m\{\widetilde{y}_j>(1-\delta)u_n\}$ we obtain

$$
\begin{aligned}
D_{n,\delta}&\leq\sum_{j=1}^{l}\frac{\mathbf{P}((1-2\delta)u_n<y_j\leq u_n)}{\mathbf{P}(y_0>u_n)}+\mathbf{P}\left(\varGamma_n|y_0>u_n\right)\\
&=l\frac{\mathbf{P}((1-2\delta)u_n<y_0\leq u_n)}{\mathbf{P}(y_0>u_n)}+\mathbf{P}\left(\varGamma_n|y_0>u_n\right).
\end{aligned}
$$

Taking into account that for every fixed $L>0$

$$
\varGamma_n\cap\{|Y_0|\leq Lu_n\}\subseteq\bigcup_{j=l+1}^{m}\{|\Pi_j|>L^{-1}(1-\delta)\}
$$

and that $\{A_j\}_{j\in\mathbb{N}}$ is independent of Y_0, we deduce

$$
\begin{aligned}
\mathbf{P}\left(\varGamma_n|\, y_0>u_n\right)&\leq\mathbf{P}\left(\varGamma_n,|Y_0|\leq Lu_n|\, y_0>u_n\right)+\mathbf{P}\left(|Y_0|>Lu_n|\, y_0>u_n\right)\\
&\leq\sum_{j=l+1}^{m}\mathbf{P}\left(|\Pi_j|>\frac{1-\delta}{L}\right)+\frac{\mathbf{P}(|Y_0|>Lu_n)}{\mathbf{P}(y_0>u_n)}\\
&\leq\frac{L}{1-\delta}\sum_{j=l+1}^{\infty}\mathbf{E}|\Pi_j|+\frac{\mathbf{P}(|Y_0|>Lu_n)}{\mathbf{P}(y_0>u_n)}.
\end{aligned}
$$

Hence inequality (2.2) yields

$$\mathbf{P}\,(\Gamma_n|\, y_0 > u_n) \le \frac{Lc_*}{1-\delta} \sum_{j=l+1}^{\infty} \mathrm{e}^{-\gamma j} + \frac{\mathbf{P}(|Y_0| > Lu_n)}{\mathbf{P}(y_0 > u_n)}.$$

Thus we obtain the following upper bound:

$$D_{n,\delta} \le l \frac{\mathbf{P}((1-2\delta)u_n < y_0 \le u_n)}{\mathbf{P}(y_0 > u_n)} + \frac{Lc_*}{1-\delta} \sum_{j=l+1}^{\infty} \mathrm{e}^{-\gamma j} + \frac{\mathbf{P}(|Y_0| > Lu_n)}{\mathbf{P}(y_0 > u_n)}.$$

By condition $\mathbf{H}_0$ there exists some universal constant $c > 0$ such that for every $\delta > 0$, $l \ge 1$ and $L > 0$

$$\limsup_{n\to\infty} D_{n,\delta} \le l \left(\frac{1}{(1-2\delta)^{\lambda}} - 1 \right) + \frac{Lc_*}{1-\delta} \sum_{j=l+1}^{\infty} \mathrm{e}^{-\gamma j} + cL^{-\lambda}.$$

Taking in this inequality the limits $\lim_{L\to\infty} \lim_{l\to\infty} \lim_{\delta\to 0}$ implies

$$\lim_{\delta\to 0} \limsup_{n\to\infty} D_{n,\delta} = 0.$$

Therefore, by (5.7) we get the following upper bound

$$\mathbf{P}(N_n((0,\epsilon]) = k-1 \mid y_0 > u_n) \le \int_{\mathbb{R}^q} g_k \left(\frac{y}{1-\delta} \right) Q_{u_n}(\mathrm{d}y) + \Delta_1'(n) + \Delta_2(n) + D_{n,\delta}.$$

Analogously, we obtain a lower bound

$$\mathbf{P}(N_n((0,\epsilon]) = k-1 \mid y_0 > u_n) \ge \int_{\mathbb{R}^q} g_k \left(\frac{y}{1-\delta} \right) Q_{u_n}(\mathrm{d}y) - \Delta_1'(n) - \Delta_2(n) - D_{n,\delta}.$$

This concludes the proof of Theorem 3.5(b). □

6. The random coefficient autoregressive model

6.1. Extreme behaviour

In this section we consider model (1.2) satisfying (1.3). We can represent this process in the form (1.1) with the sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ defined in (1.4). We suppose that $\{A_n\}_{n\in\mathbb{N}}$ satisfies condition $\mathbf{A}_3$.

Example 6.1. We start with an example satisfying $\mathbf{A}_3$. Consider model (1.2) for $q = 2$ with $a_1 = 0$ and $\sigma_2 = 0$. In this case the corresponding matrix (1.4) has the following form:

$$A_n = \begin{pmatrix} \sigma_1 \eta_{1n} & a_2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E} A_1 \otimes A_1 = \begin{pmatrix} \sigma_1^2 & 0 & 0 & a_2^2 \\ 0 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix can be calculated as

$$z_1 = a_2,\ z_2 = -a_2,\ z_3 = \frac{\sigma_1^2}{2} + \sqrt{\frac{\sigma_1^4}{4} + a_2^2},\ z_4 = \frac{\sigma_1^2}{2} - \sqrt{\frac{\sigma_1^4}{4} + a_2^2}.$$

Hence, condition $\mathbf{A}_3$ holds if $\sigma_1^2 + a_2^2 < 1$. □

Theorem 3 in Feigin and Tweedie [6] in combination with Theorem 2.2(a) and (b) implies immediately the following result.

Theorem 6.2. *Consider model (1.2)–(1.3). We assume that ξ_1 has a positive Lebesgue density on $(-R, R)$ for some $R \in (0, \infty]$. If $\mathbf{A}_3$ holds, then $Y_n = (y_n, \dots, y_{n-q+1})$ converges in distribution to the random vector Y_∞ in (2.3) for which $\mathbf{E}|Y_\infty|^2 < \infty$. The process $\{Y_n\}_{n\in\mathbb{N}}$ is v-uniformly geometric ergodic, where $v(x) = 1 + x'Tx$, $x \in \mathbb{R}^q$, for some positive definite matrix T. Moreover, $\{Y_n\}_{n\in\mathbb{N}}$ is strongly mixing with geometric rate.*

To derive the tail behaviour of the stationary rv $y_\infty = z_*'Y_\infty$ for $z_* = (1, 0, \dots, 0)'$ we require the following additional conditions for the distributions of the coefficient vectors $\{\eta_{in}\}_{n\in\mathbb{N}}$ and the noise variables $\{\xi_n\}_{n\in\mathbb{N}}$ in model (1.2).

($\mathbf{D}_1$) *The rvs $\{\eta_{in}, 1 \le i \le q, n \in \mathbb{N}\}$ are iid with symmetric continuous positive density $\phi(\cdot)$ which is non-increasing on $\mathbb{R}_+$ and moments of all order exist.*
($\mathbf{D}_2$) *For some $m \in \mathbb{N}$ we assume that $\mathbf{E}(\alpha_{11} - a_1)^{2m} = \sigma_1^{2m}\mathbf{E}\eta_{11}^{2m} \in (1, \infty)$. In particular, $\sigma_1 > 0$.*
($\mathbf{D}_3$) $\mathbf{E}|\xi_1|^m < \infty$ *for all $m \ge 2$.*
($\mathbf{D}_4$) *For every real sequence $\{c_k\}_{k\in\mathbb{N}}$ with $0 < \sum_{k=1}^\infty |c_k| < \infty$, the rv $\tau = \sum_{k=1}^\infty c_k\xi_k$ has a symmetric density, which is non-increasing on $\mathbb{R}_+$.*
As stated in Proposition 2.3 of [14] condition $\mathbf{D}_4$ is satisfied if the following simpler condition holds:
($\mathbf{D}_4'$) *The rv ξ_1 has bounded symmetric density f, which is continuously differentiable with bounded derivative $f' \le 0$ on $[0, \infty)$.*

An important example is the following.

Example 6.3 (*Gaussian Model*). Recall from Proposition 2.6 of [14] that for η_{i1}, $i = 1, \dots, q$, and ξ_1 Gaussian rvs and $\sigma_1 > 0$ the conditions $\mathbf{D}_1 - \mathbf{D}_4$ hold. We call this model the *Gaussian linear random coefficient model* or simply *Gaussian model*. As was shown in Lemma 2.7 of [14], this model is equivalent in distribution to an autoregressive model (with deterministic coefficients a_i) and ARCH(q) error term. □

A first step in studying the extremal behaviour of any stationary time series model is the tail behaviour of the stationary distribution.

Theorem 6.4 (*Klüppelberg and Pergamenchtchikov [14], Theorem 2.4*). *Consider model (1.1) and (1.4). We assume that the sequences $\{\eta_{in}, 1 \le i \le q\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are independent, that conditions $\mathbf{A}_3$ and $\mathbf{D}_1$–$\mathbf{D}_4$ hold, and that $a_q^2 + \sigma_q^2 > 0$. Then the distribution of the vector (2.3) satisfies*

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(z'Y_\infty > t) = h(z), \quad z \in S.$$

The function $h(\cdot)$ is strictly positive and continuous on S and the parameter λ is given as the unique positive solution of

$$\kappa(\lambda) = 1, \tag{6.1}$$

where for some probability measure ν on S

$$\kappa(\lambda) := \lim_{n\to\infty} \left(\mathbf{E}|A_1 \cdots A_n|^\lambda\right)^{1/n} = \int_S \mathbf{E}|x'A_1|^\lambda \nu(\mathrm{d}x),$$

and the solution of (6.1) satisfies $\lambda > 2$.

For the stationary process (1.1) this means that for every marginal rv $y_k = z_*' Y_k$ with $z_* = (1, 0, \dots, 0)'$

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(y_k > t) = h(z_*) =: h_*.$$

Thus Theorem 6.4 implies condition $\mathbf{H}_0$ for the model (1.2). The following Lemma guarantees $\mathbf{H}_1$ and $\mathbf{H}_2$; its proof can be found in Section 6.2.

Lemma 6.5. *Assume that condition $\mathbf{D}_1$ holds and that $\sigma_1^2 > 0$. Then the sequence of matrices $\{A_n\}_{n\in\mathbb{N}}$ given in (1.4) satisfies $\mathbf{H}_1$ and $\mathbf{H}_2$.*

Thus Theorem 3.4 implies immediately the following result.

Theorem 6.6. *Consider model (1.2)–(1.3). Assume that conditions $\mathbf{A}_3$ and $\mathbf{D}_1 - \mathbf{D}_4$ hold and that $\sigma_1^2 > 0$ and $a_q^2 + \sigma_q^2 > 0$. Assume, furthermore, that the positive solution λ of (6.1) is non-even. Then Theorem 3.4 holds for the process (1.2).*

Example 6.7 (*Continuation of Example* 6.3). We derive sufficient conditions for the coefficients in the Gaussian model (1.1) such that the solution λ of Eq. (6.1) is non-even. To this end we calculate $\kappa(4)$. For every matrix we denote its elements by $\langle\cdot\rangle$. This yields $\Pi_n = A_n \cdots A_1 = (\langle \Pi_n\rangle_{ij})_{1\le i,j\le q}$. We represent the (1, 1)-element of this matrix by its recurrence form

$$\langle \Pi_n\rangle_{11} = \sigma_1 \eta_{1n} \langle \Pi_{n-1}\rangle_{11} + m_{n-1},$$

where $m_{n-1} = a_1\langle \Pi_{n-1}\rangle_{11} + \sum_{j=2}^q \alpha_{jn}\langle \Pi_{n-1}\rangle_{j1}$ is independent of η_{1n}. Therefore, from the Newton formula we get for $n \in \mathbb{N}$

$$\begin{aligned}\mathbf{E}(\langle \Pi_n\rangle_{11})^4 &= \mathbf{E}(\sigma_1 \eta_{1n} \langle \Pi_{n-1}\rangle_{11} + m_{n-1})^4\\ &\ge \sigma_1^4 \mathbf{E}(\eta_{1n})^4 \mathbf{E}(\langle \Pi_{n-1}\rangle_{11})^4\\ &= 3\sigma_1^4 \mathbf{E}(\langle \Pi_{n-1}\rangle_{11})^4.\end{aligned}$$

This implies that $\mathbf{E}(\langle \Pi_n\rangle_{11})^4 \ge (3\sigma_1^4)^n$ for all $n \in \mathbb{N}$. Then it is easy to show that $\kappa(4) \ge 3\sigma_1^4 > 1$ for all $\sigma_1 > \sigma_* = 3^{-1/4} \approx 0.76$. From Theorem 6.4 we know that $\lambda > 2$; therefore, for $\sigma_1 > 3^{-1/4}$ the value λ is non-even. □

6.2. Distributional properties of the random coefficient autoregressive model

In this section we prove Lemma 6.5. Condition $\mathbf{D}_1$ ensures that the rv η_{11} has symmetric positive density ϕ with certain additional properties. In the following lemma we show that this implies that $\rho_q(y) = \Pi_q y$ has also a density, which can be given explicitly in terms of ϕ. As before we denote by $\langle\cdot\rangle$ the components of the corresponding vector.

Lemma 6.8. *Assume that condition $\mathbf{D}_1$ holds and $\sigma_1 > 0$. Then for $y = (y_1, \dots, y_q) \in \Gamma_0 = \{y \in \mathbb{R}^q : y_1 \ne 0\}$ the vector $\rho_q(y) = (\langle \Pi_q y\rangle_1, \dots, \langle \Pi_q y\rangle_q)'$ has a density given by*

$$\begin{aligned}p(x, y) &= p(x_1, \dots, x_q, y)\\ &= \varphi(x_q, y) \prod_{j=1}^{q-1} \mathbf{1}_{\{x_{j+1}\ne 0\}} \varphi(x_j, m_{j+1}(x, y)), \quad x \in \mathbb{R}^q, \end{aligned} \tag{6.2}$$

where $m_j(x,y) = (x_j,\ldots,x_q,y_1,\ldots,y_{j-1})'$ *and*

$$\varphi(v,y) = \frac{1}{\sigma_1|y_1|}\mathbf{E}\phi\left(\frac{v - a_1 y_1 - \sum_{i=2}^{q}\alpha_{j1}y_j}{\sigma_1|y_1|}\right), \quad v\in\mathbb{R},\ y=(y_1,\ldots,y_q)\in\mathbb{R}^q.$$

Proof. The special form (1.4) of the matrices A_j implies that for $j = 1,\ldots,q$ the vector $\rho_j(y)\in\mathbb{R}^q$ has the following components:

$$\rho_j(y) = (\langle A_j\cdots A_1 y\rangle_1,\ldots,\langle A_1 y\rangle_1, y_1,\ldots,y_{q-j})'.$$

In particular,

$$\rho_q(y) = (\langle A_q\cdots A_1 y\rangle_1,\ldots,\langle A_1 y\rangle_1)'. \tag{6.3}$$

Notice now that for $k\in\mathbb{N}$ every linear combination $\sum_{j=1}^{q}\alpha_{jk}y_j$ with $y_1\neq 0$ has the density $\varphi(v,y)$. Moreover, for $j\geq 2$ the rv $\langle\Pi_j y\rangle_1$ has a conditional (conditioned on $A_1,\ldots,A_{j-1}$) density

$$\mathbf{1}_{\{\langle\Pi_{j-1}y\rangle_1\neq 0\}}\varphi(v,\rho_{j-1}(y)).$$

Here we took into account that $\mathbf{P}(\langle\Pi_{j-1}y\rangle_1 = 0) = 0$ for $j\geq 2$. Now it is easy to show by induction on j that the random vector $(\langle A_1 y\rangle_1,\ldots,\langle\Pi_j y\rangle_1)'$ has for every $y\in\Gamma_0$ the following density on $\mathbb{R}^j$

$$f_j(z_1,\ldots,z_j,y) = \varphi(z_1,y)\prod_{i=2}^{j}\mathbf{1}_{\{z_{j-1}\neq 0\}}\varphi(z_j,\vartheta_{j-1}(z,y)),$$

where

$$\vartheta_j(z,y) = (z_j,\ldots,z_1,y_1,\ldots,y_{q-j})'.$$

Therefore,

$$p(x_1,\ldots,x_q,y) = f_q(x_q,\ldots,x_1,y)$$

and we obtain (6.2). □

Next we prove $\mathbf{H}_1$: Notice that for $z_* = (1,0,\ldots,0)'$ we have $W_{z_*}\subset\Gamma_0$, where Γ_0 is defined in Lemma 6.8. Define $\Pi^*(y) = \sup_{j\in\mathbb{N}} z_*'\Pi_j y$. We shall show by contradiction that for every $y\in\Gamma_0$

$$\mathbf{P}(\varsigma(y)=0) = \mathbf{P}(\Pi^*(y)\leq 1) > 0. \tag{6.4}$$

So assume that $\mathbf{P}(\varsigma(y)=0) = 0$ for some $y\in\Gamma_0$. Then, immediately, $\mathbf{P}(\Pi^*(y)>1) = 1$. Now note that for the matrices $\{A_n\}_{n\in\mathbb{N}}$ of type (1.4) the vector $\rho_q(y) = \Pi_q y$ has the form (6.3), i.e. $\rho_q(y) = (z_*'\Pi_q y,\ldots,z_*'\Pi_1 y)'$. Therefore,

$$\mathbf{P}(\Pi^*(y)>1) = \mathbf{E}\mathbf{P}(\Pi^*(y)>1 \mid \rho_q(y)) = \mathbf{E}F(\rho_q(y)),$$

where the function F is defined as

$$F(x) = \mathbf{P}\left(\left(\max_{1\leq j\leq q} x_j\right)\vee\Pi^*(x)>1\right), \quad x=(x_1,\ldots,x_q)'\in\mathbb{R}^q,$$

with $a \vee b = \max(a, b)$. Moreover, for $|x| \le 1$ by (2.2)

$$F(x) = \mathbf{P}(\Pi^*(x) > 1) \le \sum_{j=1}^{\infty} P(z_1' \Pi_j x > 1)$$
$$\le \sum_{j=1}^{\infty} \mathbf{E}|\Pi_j x|^2 \le c_* \sum_{j=1}^{\infty} \mathrm{e}^{-\gamma j} |x|^2.$$

This implies that there exists $0 < r < 1$ such that $F(x) \le 1/2$ on the set $B_r = \{x \in \mathbb{R}^q : |x| \le r\}$. But by our assumption above we get

$$1 = \mathbf{P}(\Pi^*(y) > 1) = \mathbf{E}F(\rho_q(y)) = \int_{\mathbb{R}^q} F(x) p(x, y) \mathrm{d}x, \tag{6.5}$$

where the density $p(x, y)$ is defined in (6.2) and, therefore,

$$\int_{B_r} p(x, y) \mathrm{d}x > 0.$$

Hence, the right hand side of equality (6.5) is strictly less than 1. This contradiction proves for all $y \in \Gamma_0$ inequality (6.4), which implies condition $\mathbf{H}_1$.
Finally, we check condition $\mathbf{H}_2$. By condition $\mathbf{D}_1$ and $\sigma_1^2 > 0$ the rv $z_*' A_1 y$ has a density for all $y \in \Gamma_0$, and thus $\mathbf{P}(z_*' A_1 y = a) = 0$ for $a \in \mathbb{R}$. Suppose for some $j \in \mathbb{N}$ that $\mathbf{P}(z_*' \Pi_j y = a) = 0$ for every $a \in \mathbb{R}$. Then for $a \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{P}(z_*' \Pi_{j+1} y = a) &\le \mathbf{P}(z_*' A_{j+1} \Pi_j y = a, z_*' \Pi_j y = 0) + \mathbf{P}(z_*' A_{j+1} \Pi_j y = a, z_*' \Pi_j y \ne 0) \\
&\le \mathbf{P}(z_*' \Pi_j y = 0) + \int_{\Gamma_0} \mathbf{P}(z_*' A_{j+1} x = a) \mathbf{P}(\Pi_j y \in \mathrm{d}x) \\
&= \mathbf{P}(z_*' \Pi_j y = 0) + \int_{\Gamma_0} \mathbf{P}(z_*' A_1 x = a) \mathbf{P}(\Pi_j y \in \mathrm{d}x).
\end{aligned}$$

The first probability is equal to zero by assumption. The second term is equal to zero as the rv $z_*' A_1 x$ has a density for every $x \in \Gamma_0$. This means that for all $j \in \mathbb{N}$ and all $a \in \mathbb{R}$ we have $\mathbf{P}(z_*' \Pi_j y = a) = 0$. This implies $\mathbf{H}_2$.
This concludes the proof of Lemma 6.5. □

Appendix A. Proof of Theorem 2.2(b)

For a stationary Markov process we have for functions f and h as in Definition 2.1,

$$\begin{aligned}
&\mathbf{E}[f(\ldots, Y_{-1}, Y_0) h(Y_k, Y_{k+1}, \ldots)] \\
&\quad = \mathbf{E}\left[f(\ldots, Y_{-1}, Y_0) \mathbf{E}[h(Y_k, Y_{k+1}, \ldots) \mid Y_j, j \le k]\right] \\
&\quad = \mathbf{E}[f(\ldots, Y_{-1}, Y_0) H(Y_k)] \\
&\quad = \mathbf{E}\left[f(\ldots, Y_{-1}, Y_0) \mathbf{E}_{Y_0} H(Y_k)\right],
\end{aligned}$$

where $\mathbf{E}_x$ denotes the expectation, given the process starts in x, and

$$\mathbf{E}h(Y_k, Y_{k+1}, \ldots) = \mathbf{E}H(Y_k) = \mathbf{E}H(Y_\infty),$$

where $H(y) = \mathbf{E}_y h(y, Y_1, \ldots)$. Thus

$$|\mathbf{E}f(\ldots, Y_{-1}, Y_0) h(Y_k, Y_{k+1}, \ldots) - \mathbf{E}f(\ldots, Y_{-1}, Y_0) \mathbf{E}h(Y_k, Y_{k+1}, \ldots)|$$

$$
\begin{aligned}
&= |\mathbf{E}[f(\ldots, Y_{-1}, Y_0)(\mathbf{E}_{Y_0} H(Y_k) - \mathbf{E} H(Y_\infty))]| \\
&\le \mathbf{E}|\mathbf{E}_{Y_0} H(Y_k) - \mathbf{E} H(Y_\infty)| \\
&\le R\rho^k \mathbf{E} v(Y_0) = R\rho^k \mathbf{E} v(Y_\infty) \le R\rho^k (1 + |T|\mathbf{E}|Y_\infty|^2),
\end{aligned}
$$

and this implies inequality (2.5) with $C^* = R(1 + |T|\mathbf{E}|Y_\infty|^2) < \infty$. $\square$

Appendix B. Properties of the functions g_k

Recall the functions g_k from (3.10).

Lemma B.1. *Under condition* $\mathbf{H}_2$ *the functions* $g_k(\cdot)$ *are continuous for all* $k \in \mathbb{N}$.

Proof. By the definition of ς in (3.7) we have for every $k \in \mathbb{N}$

$$
|g_k(y) - g_k(y_0)| \le 2\mathbf{P}(\nu(y, y_0) \ge 1), \quad y, y_0 \in W_{z*},
$$

where $\nu(y, y_0) = \sum_{j=1}^{\infty} |\mathbf{1}_{\{\xi_j(y)>1\}} - \mathbf{1}_{\{\xi_j(y_0)\ge 1\}}|$ and $\xi_j(y) = z_*' \Pi_j y$. Therefore, it suffices to show that

$$
\lim_{y \to y_0} \mathbf{P}(\nu(y, y_0) \ge 1) = 0. \tag{B.1}
$$

For every fixed $0 < \varepsilon < 1/2$ we set $\Gamma_\varepsilon = \bigcap_{j=1}^{\infty} \{|\xi_j(y) - \xi_j(y_0)| \le \varepsilon\}$. Taking into account that

$$
\{\nu(y, y_0) \ge 1\} \cap \Gamma_\varepsilon \subseteq \bigcup_{j=1}^{\infty} \{1 - \varepsilon \le \xi_j(y_0) \le 1 + \varepsilon\}
$$

we get

$$
\begin{aligned}
\mathbf{P}(\nu(y, y_0) \ge 1) &\le \mathbf{P}(\nu(y, y_0) \ge 1, \Gamma_\varepsilon) + \mathbf{P}(\Gamma_\varepsilon^c) \\
&\le \mathbf{P}\left(\bigcup_{j=1}^{\infty} \{1 - \varepsilon \le \xi_j(y_0) \le 1 + \varepsilon\} \right) + \mathbf{P}(\Gamma_\varepsilon^c) \\
&\le \sum_{j=1}^{l} \mathbf{P}(1 - \varepsilon \le \xi_j(y_0) \le 1 + \varepsilon) + \sum_{j=l+1}^{\infty} \mathbf{P}(\xi_j(y_0) \ge 1/2) + \mathbf{P}(\Gamma_\varepsilon^c) \\
&\le \sum_{j=1}^{l} \mathbf{P}(1 - \varepsilon \le \xi_j(y_0) \le 1 + \varepsilon) + 4 \sum_{j=l+1}^{\infty} \mathbf{E}|\xi_j(y_0)|^2 + \mathbf{P}(\Gamma_\varepsilon^c).
\end{aligned}
$$

Moreover, by Chebyshev's inequality and (2.2) we can estimate the last probability as

$$
\begin{aligned}
\mathbf{P}(\Gamma_\varepsilon^c) &\le \sum_{j=1}^{\infty} \mathbf{P}(|\xi_j(y) - \xi_j(y_0)| > \varepsilon) \le \frac{1}{\varepsilon^2} \sum_{j=1}^{\infty} \mathbf{E}|\xi_j(y) - \xi_j(y_0)|^2 \\
&\le \frac{1}{\varepsilon^2} |y - y_0|^2 \sum_{j=1}^{\infty} \mathbf{E}|\Pi_j|^2 \le \frac{c_*}{\varepsilon^2} |y - y_0|^2 \sum_{j=1}^{\infty} \mathrm{e}^{-\gamma j}.
\end{aligned}
$$

Therefore,

$$
\mathbf{P}(\nu(y, y_0) \ge 1) \le \sum_{j=1}^{l} \mathbf{P}(1 - \varepsilon \le \xi_j(y_0) \le 1 + \varepsilon)
$$

$$+ 4c_*|y_0|^2 \sum_{j=l+1}^{\infty} \mathrm{e}^{-\gamma j} + \frac{c_*}{\varepsilon^2}|y - y_0|^2 \sum_{j=1}^{\infty} \mathrm{e}^{-\gamma j}.$$

By condition $\mathbf{H}_2$ and taking here limits $y \to y_0$, $\varepsilon \to 0$ and $l \to \infty$ we obtain (B.1). □

Appendix C. Proof of Lemma 4.5

Let μ be a subsequential vague limit, i.e. if there exists a sequence $t_n \to \infty$ such that $m_{t_n} \xrightarrow{v} \mu$, then $\mu(\partial B_{u,x}) = 0$ for the set $B_{u,x} = uW_x$ for every $u > 0$ and $x \in \mathbb{R}^q \setminus \{0\}$. Indeed, in this case

$$\partial B_{u,x} = \{y \in E : (x, y) = u\} \subset \{y \in E : (1-\delta)u < (x, y) < (1+\delta)u\} =: G_\delta$$

for all $0 < \delta < 1$. Therefore, by the property of vague convergence (see Kallenberg [11], Theorem 15.7.2(iii)) and the limiting relationship (4.3) we have for every $0 < \delta < u$ and G_δ as above,

$$\begin{aligned}
\mu(\partial B_{u,x}) &\le \mu(G_\delta) \le \liminf_{n\to\infty} m_{t_n}(G_\delta) \\
&= \lim_{t_n\to\infty} \frac{\mathbf{P}(x'Y_\infty > t_n(1-\delta)u) - \mathbf{P}(x'Y_\infty \ge t_n(1+\delta)u)}{\mathbf{P}(z_*'Y_\infty > t_n)} \\
&= \lim_{t_n\to\infty} \frac{\mathbf{P}(x'Y_\infty > (1-\delta)ut_n) - \mathbf{P}(x'Y_\infty \ge (1+\delta)ut_n)}{\mathbf{P}(z_*'Y_\infty > t_n)} \\
&= \tilde{h}(x)u^{-\lambda}\left((1-\delta)^{-\lambda} - (1+\delta)^{-\lambda}\right).
\end{aligned}$$

Taking the limit for $\delta \to 0$ implies that $\mu(\partial B_{u,x}) = 0$. By Theorem 15.7.2(ii) of Kallenberg [11] and condition $\mathbf{H}_0$ we get (4.4). Next we show (4.5). An application of (4.4) yields

$$\begin{aligned}
\int_{|z'y|>u} |z'y|^\nu \mu(\mathrm{d}y) &= \nu \int_0^\infty t^{\nu-1}\mu(y \in \mathbb{R}^q : |z'y| > \max(u,t))\mathrm{d}t \\
&= u^\nu \mu(y \in \mathbb{R}^q : |z'y| > u) + \nu \int_u^\infty t^{\nu-1}\mu(y \in \mathbb{R}^q : |z'y| > t)\mathrm{d}t \\
&= \hat{h}(z)u^{\nu-\lambda} + \hat{h}(z)\nu \int_u^\infty t^{\nu-\lambda-1}\mathrm{d}t.
\end{aligned}$$

This implies (4.5). Analogous reasoning yields (4.6). □

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