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Splitting at the infimum and excursions in half-lines for random walks and Lévy processes

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The central result of this paper is that, for a process X with independent and stationary increments, splitting at the infimum on a compact time interval amounts (in law) to the juxtaposition of the excursions of X in half-lines according to their signs. This identity yields a pathwise construction of X conditioned (in the sense of harmonic transform) to stay positive or negative, from which we recover the extension of Pitman's theorem for downwards-skip-free processes. We also extend for Lévy processes an identity that Karatzas and Shreve obtained for the Brownian motion. In the special case of stable processes, the sample path is studied near a local infimum.

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Random walk * Lévy process * path decomposition * excursions * harmonic transform * downwards skip free * stable processes

1. Introduction

Fluctuation theory of processes with stationary independent increments is concerned with the behaviour of the supremum or/and infimum process(es). It has been developed for real valued random walks in the 50's and 60's, we refer to the classical books of Spitzer [19] and Feller [7]. Splitting at an extremum provides the most natural and efficient approach for numerous results in this field. One of the nice examples is provided by the celebrated Theorem of Sparre-Andersen [18] (which yields the arcsine laws for random walks): if X is a chain with exchangeable increments with a fixed finite number of steps, then the index of the ultimate infimum and the total number of indices $i>0$ such that $X_i \leqslant 0$ have the same law. Sparre-Andersen theorem is enlightened and strengthened by the following pictorial identity due to Feller [7]: the pair of processes obtained by splitting the chain X at the last

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instant when it attains its infimum has the same law as the pair of processes obtained by the juxtaposition of the excursions of X respectively in $(-\infty, 0]$ and in $(0, \infty)$. Here, an excursion in $(0, \infty)$ (respectively in $(-\infty, 0]$) includes the initial positive (negative) jump of X across 0, and excludes the ultimate negative (positive) jump across 0; the meanings of splitting and juxtaposition are made precise in Subsection 2.1. In this framework, Sparre-Andersen theorem follows by comparing the lifetimes of the various processes.

Fluctuation theory can be extended to continuous time, either by approximation based on discrete time skeletons, or directly by excursion theory. See in particular the papers of Bingham [4], Greenwood and Pitman [8] and Silverstein [16]. Here, we deduce by approximation an analogue of the above identity for Lévy processes. The convergence relies crucially on stochastic calculus, and specifically on the Meyer–Tanaka formula for the semimartingale local time. The case of a Brownian motion with drift was previously treated in [3] by excursion theory. Karatzas and Shreve [10] also obtained a related result for Brownian motion, which can be extended to Lévy processes by application of the key identity.

Our interest in this identity is that it provides a simple pathwise construction of the law of the initial Lévy process X conditioned to stay positive (respectively negative). More precisely, Millar [13] established that, when X drifts to $+\infty$ (that is $\lim_{t\uparrow\infty} X_t = +\infty$ a.s.), then the post-infimum process obtained after shifting the path at the instant of its last infimum, has the same law as X conditioned to stay positive, where this a priori heuristic conditioning has to be taken in the sense of a harmonic transform. In the case when X oscillates (that is $\lim\sup_{t\uparrow\infty} X_t = +\infty$ and $\lim\inf_{t\uparrow\infty} X_t = -\infty$ a.s.), one can still consider the law of X conditioned to stay positive/negative in the sense of harmonic transforms; but in general, no pathwise construction of these laws is known. Here, we show that when X oscillates, the processes obtained after the juxtaposition of the excursions of X respectively in $(0, \infty)$ and in $(-\infty, 0]$ are two independent processes whose distributions are respectively those of X conditioned to stay positive and negative.

Williams [21] first stressed the deep connections between the three-dimensional Bessel process (that is Brownian motion conditioned to stay positive) and the Brownian motion. We hope that our construction will be useful to deduce some analogous relations in the Lévy setting. When we specify this construction for the simple random walk and for the Brownian motion, we recover a famous theorem due to Pitman [14]: in these cases, if $\bar{X}$ stands for the past-supremum process of X, then $X - 2\bar{X}$ has the same law as X conditioned to stay negative. Here, the key identity yields an extension of Pitman's Theorem for downwards-skip-free random walks and for spectrally positive Lévy processes, that is with no negative jumps. (A slightly more general extension of Pitman's Theorem for spectrally positive Lévy processes was previously obtained in [1], however, the present proof is much simpler).

This paper is organized as follows: Section 2 is devoted to discrete time. Subsection 2.1 introduces the notation, the main identity is proved in 2.2, 2.3 presents a pathwise construction of randoms walks conditioned to stay positive/negative, and the extension of Pitman's Theorem for downwards-skip-free random walks is presented in 2.4. Subsections 3.1–4 follow the same pattern in continuous time, and the extension of the Karatzas–Shreve decomposition is treated in Subsection 3.5.

2. Discrete time

2.1. Notation

Throughout this section, we will use the following notation. Denote by Ω the set of the sequences $\omega:\mathbb{N}\to\mathbb{R}\cup\{\delta\}$ starting from 0, where δ is a cemetery point, and by $\zeta=\zeta(\omega)=\sup\{i:\,\omega(i)\neq\delta\}$, the lifetime of ω. Let X be the coordinate chain, $X_i(\omega)=\omega(i)$. Introduce for every $i\leq\zeta$, $\underline{X}_i=\inf\{X_j:j\leq i\}$, $\bar{X}_i=\sup\{X_j:j\leq i\}$, and put $\tau=\sup\{i\leq\zeta:\,X_i=\underline{X}_i\}$, the index of the ultimate infimum, and $I=X_\tau$, the value of this infimum. On $\{\tau<\infty\}$, we consider the post-infimum chain $\underset{\rightarrow}{X}$,

$$\underset{\rightarrow}{X}(i)=\begin{cases}X_{\tau+i}-I & \text{for } i\leq\zeta-\tau,\\ \delta & \text{for } i>\zeta-\tau,\end{cases}$$

and the reversed pre-infimum chain $\underset{\leftarrow}{X}$,

$$\underset{\leftarrow}{X}(i)=\begin{cases}X_{\tau-i}-I & \text{for } i\leq\tau,\\ \delta & \text{for } i>\tau.\end{cases}$$

We associate now two new chains, $X^\uparrow$ and $X^\downarrow$, with X: $X^\uparrow$ is obtained by the juxtaposition of the excursions of X in $(0,\infty)$, and $X^\downarrow$ arises similarly from excursions in $(-\infty,0]$. That is to say that the jumps of $X^\uparrow$ correspond to the jumps of X which *end* in $(0,\infty)$, arranged according to their order of appearance, and similarly that the jumps of $X^\downarrow$ correspond to those of X which end in $(-\infty,0]$. Specifically, we introduce the number of indices less than or equal to i at which X is positive:

$$A_i^+=\sum_{j=1}^{i}\mathbf{1}_{\{X_j>0\}},$$

and $\alpha_i^+=\inf\{j:\,A_j^+=i\}$ the index of the ith visit to $(0,\infty)$. Pictorially, the index substitution by α^+ consists of erasing the positive indices at which $X\leq0$, and then closing up the gaps. Denote the jth increment of X by $\Delta X_j=X_j-X_{j-1}$. The chain $X^\uparrow$ is given by

$$X^\uparrow(i)=\sum_{j=1}^{\alpha_i^+}\mathbf{1}_{\{X_j>0\}}\Delta X_j\quad\text{when } i\leq A_\zeta^+,$$

and $X^\uparrow(i)=\delta$ otherwise. Similarly, put

$$X^\downarrow(i)=\sum_{j=1}^{\alpha_i^-}\mathbf{1}_{\{X_j\leq0\}}\Delta X_j\quad\text{when } i\leq A_\zeta^-,$$

where $A_i^- = \sum_{j=1}^{i} \mathbf{1}_{\{X_j \leq 0\}}$, $\alpha_i^- = \inf\{j\colon A_j^- = i\}$, and $X^\downarrow(i) = \delta$ for $i > A_\zeta^-$. We emphasize the *optional* but not predictable nature of the sums which appear in the definition of $X^\uparrow$ and $X^\downarrow$. Also notice that, by construction, $X^\uparrow$ is positive on $(1, \ldots, A_\zeta^+)$ and that $X^\downarrow$ is non-positive on $(1, \ldots, A_\zeta^-)$.

2.2. *Splitting at the infimum and juxtaposition of excursions for chains with exchangeable increments*

Consider now a probability measure $\mathbb{P}$ on Ω such that X has exchangeable increments, that is $\mathbb{P}(\zeta = \infty) = 1$ and for every positive integer n and every permutation σ on $\{1, \ldots, n\}$, the n-tuples $(\Delta X_1, \ldots, \Delta X_n)$ and $(\Delta X_{\sigma(1)}, \ldots, \Delta X_{\sigma(n)})$ are equally distributed under $\mathbb{P}$. Denote by $\mathbb{P}^n$ the law under $\mathbb{P}$ of the chain $(X_0, \ldots, X_n, \delta, \ldots)$ with lifetime n, obtained from X by killing at the index $n+1$. The following identity is implicit in Lemma 3 of Section XII.8 in Feller [7]. Since it plays a crucial role in this paper and since the present formulation and notation are different from Feller's, a proof is given.

Theorem 2.1 (Feller). *For every $n \geq 0$, the pairs of chains $(\underset{\rightarrow}{X}, -\underset{\leftarrow}{X})$ and $(X^\uparrow, X^\downarrow)$ have the same law under $\mathbb{P}^n$.*

Proof. Fix $x_1 \leq \cdots \leq x_n$, and let Λ be the subset of the sequences ω with lifetime $\zeta = n$ such that the increasing rearrangement of $(\Delta\omega(1), \ldots, \Delta\omega(n))$ is $(x_1, \ldots, x_n)$, where $\Delta\omega(j) = \omega(j) - \omega(j-1)$, $j = 1$ to n. Since X has exchangeable increments, the law $\mathbb{P}^n$ conditioned on the event $\{(X_0, \ldots, X_n) \in \Lambda\}$ is the equi-probability on Λ, say P_Λ.

Let M be the set of pairs $(\omega', \omega'') \in \Omega \times \Omega$, with $\zeta(\omega') = n'$, $\zeta(\omega'') = n''$, $n' + n'' = n$, $\omega'(i') > 0$ for $1 \leq i' \leq n'$, $\omega''(i'') \leq 0$ for $1 \leq i'' \leq n''$, and such that the increasing rearrangement of $(\Delta\omega'(1), \ldots, \Delta\omega'(n'), \Delta\omega''(1), \ldots, \Delta\omega''(n''))$ is $(x_1, \ldots, x_n)$. The map $\omega \mapsto (\underset{\rightarrow}{\omega}, -\underset{\leftarrow}{\omega})$ being clearly a bijection from Λ to M, the law of $(\underset{\rightarrow}{X}, -\underset{\leftarrow}{X})$ under P_Λ is the equi-probability on M, say P_M.

Note that $X_i = X^\uparrow(A_i^+) + X^\downarrow(A_i^-)$ for every i. Consider a fixed pair $(\omega', \omega'') \in M$, and set $\omega(i) = \omega'(A_i') + \omega''(A_i'')$, where A' and A'' are specified by the reversed induction: $A_n' = n'$, $A_n'' = n''$, and for $0 < i \leq n$,

$$A_{i-1}' = A_i' - 1, \quad A_{i-1}'' = A_i'', \qquad \text{if } \omega(i) > 0,$$

$$A_{i-1}' = A_i', \quad A_{i-1}'' = A_i'' - 1, \qquad \text{if } \omega(i) \leq 0.$$

Then clearly, $\omega \in \Lambda$, $A' = A^+$, $A'' = A^-$, and $(\omega^\uparrow, \omega^\downarrow) = (\omega', \omega'')$. This shows that the mapping $\omega \mapsto (\omega^\uparrow, \omega^\downarrow)$ is a bijection from Λ to M, and the law of $(X^\uparrow, X^\downarrow)$ under P_Λ is P_M. Thus $(\underset{\rightarrow}{X}, -\underset{\leftarrow}{X})$ and $(X^\uparrow, X^\downarrow)$ have the same distribution under $\mathbb{P}^n$ conditioned on $\{(X_0, \ldots, X_n) \in \Lambda\}$. Integrating with respect to the law of the increasing rearrangement of $(\Delta X_1, \ldots, \Delta X_n)$ under $\mathbb{P}^n$, we get that under $\mathbb{P}^n$, $(\underset{\rightarrow}{X}, -\underset{\leftarrow}{X})$ and $(X^\uparrow, X^\downarrow)$ have the same law. $\square$

Remark. It is clear from the proof that the theorem holds for a fixed n whenever under $\mathbb{P}$, the first n increments are exchangeable.

An immediate consequence of Theorem 2.1 is:

Corollary 2.2. *Denote by $\mathscr{F}_n$ the σ-algebra generated by $(X_0, \ldots, X_n)$. For every bounded $\mathscr{F}_n \otimes \mathscr{F}_n$-measurable functional Φ,*

$$\lim_{m\uparrow\infty} \mathbb{E}^m(\Phi(\underrightarrow{X}, \underleftarrow{X})) = \mathbb{E}(\Phi(X^{\uparrow}, -X^{\downarrow})). \qquad \square$$

That the above limit exits is obvious when $(X, \mathbb{P})$ drifts to $+\infty$ (i.e. $\lim_{n\uparrow\infty} X_n = +\infty$ $\mathbb{P}$-a.s.) and, by time reversal, when $(X, \mathbb{P})$ drifts to $-\infty$; so the main interest of Corollary 2.2 is when $(X, \mathbb{P})$ oscillates. Note also that this convergence is much stronger than the usual convergence in law.

2.3. Harmonic transforms in random walks

As stressed by Feller [7, Chapter XI], renewal theory provides a very efficient tool for studying the infimum of a random walk on a finite interval. Below, we briefly recall the arguments for completeness.

Assume that $(X, \mathbb{P})$ is a real-valued random walk, and to avoid triviality, that $\mathbb{P}(X_1 > 0)$ and $\mathbb{P}(X_1 < 0)$ are both positive. Introduce for $c > 0$,

$$\mathbb{P}^{(c)} = \sum_{n=0}^{\infty} \mathrm{e}^{-cn}(1-\mathrm{e}^{-c})\mathbb{P}^n,$$

that is $\mathbb{P}^{(c)}$ is the law of the random walk killed at rate $1-\mathrm{e}^{-c}$. By the lack of memory property, $(X, \mathbb{P}^{(c)})$ satisfies the strong Markov property: if T is a stopping time, then under $\mathbb{P}^{(c)}$, the shifted chain $X \circ \theta_T - X_T$ is independent of $\mathscr{F}_T$, and its law conditionally on $T < \infty$ is again $\mathbb{P}^{(c)}$. Applying the strong Markov property at the descending weak ladder epochs (that are the successive indices at which X attains its infimum), we see that under $\mathbb{P}^{(c)}$, $\underrightarrow{X}$ and $\underleftarrow{X}$ are independent, and that $\underrightarrow{X}$ has the same law as X conditioned on $\{X_i > 0$ for $i = 1$ to $\zeta\}$. It is a Markov chain whose transition function can be expressed for $x \geq 0$ as

$$p_c^{\uparrow}(x, \mathrm{d}y) = \mathbf{1}_{\{y>0\}}(h_c^{\uparrow}(y)/h_c^{\uparrow}(x))\, \mathrm{e}^{-c} p(x, \mathrm{d}y) \tag{1}$$

where $p(x, \mathrm{d}y) = \mathbb{P}(X_1 + x \in \mathrm{d}y)$ is the transition function of the random walk and

$$h_c^{\uparrow}(x) = \mathbb{P}^{(c)}(X_i > -x \quad \text{for } i = 1 \text{ to } \zeta)/\mathbb{P}^{(c)}(X_i > 0 \text{ for } i = 1 \text{ to } \zeta).$$

Now, we want to take the limit as c goes to 0 and apply Corollary 2.2. In this direction, the above formula for $h_c^{\uparrow}$ is not very convenient to work with, so we first re-express it as

$$h_c^{\uparrow}(x) = 1 + \mathbb{E}\left(\sum_{n=1}^{\alpha_1^+ - 1} \mathrm{e}^{-cn}\, \mathbf{1}_{\{X_n > -x\}}\right). \tag{2}$$

Indeed, by time reversal, we have

$$\mathbb{P}^{(c)}(X_i > -x \text{ for } i=1 \text{ to } \zeta) = (1-\mathrm{e}^{-c})\left(1+\sum_{n=1}^{\infty} \mathrm{e}^{-cn}\mathbb{P}(\bar{X}_{n-1}-X_n<x)\right).$$

Denote by $\mathcal{S}=\{0, \mathcal{S}_1, \ldots\}$ the ascending strict ladder epoch set, that is the set of indices when X attains a new supremum. Cutting up $\mathbb{N}$ into the partition generated by $\mathcal{S}$, we get

$$\sum_{n=1}^{\infty} \mathrm{e}^{-cn}\mathbb{P}(\bar{X}_{n-1}-X_n<x)$$
$$=\mathbb{E}\left(\sum_{i=0}^{\infty} \exp(-c\mathcal{S}_i) \sum_{n=\mathcal{S}_i+1}^{\mathcal{S}_{i+1}} \exp -c(n-\mathcal{S}_i)\mathbf{1}_{\{\bar{X}_{n-1}-X_n<x\}}\right).$$

Note that $\bar{X}_{n-1}=\bar{X}(\mathcal{S}_i)$ for every $\mathcal{S}_i<n\leqslant\mathcal{S}_{i+1}$, and apply the strong Markov property. We re-write the above quantity as

$$=\mathbb{E}\left(\sum_{n=1}^{\alpha_1^+} \mathrm{e}^{-cn}\mathbf{1}_{\{X_n>-x\}}\right)(1-\mathbb{E}(\exp -c\alpha_1^+))^{-1}$$

(recall that $\alpha_1^+=\mathcal{S}_1$ is the first positive index when X hits $(0,\infty)$). Putting the pieces together, we find

$$\mathbb{P}^{(c)}(X_i>-x \text{ for } i=1 \text{ to } \zeta)$$
$$=(1-\mathrm{e}^{-c})\left(1+\mathbb{E}\left(\sum_{n=1}^{\alpha_1^+} \mathrm{e}^{-cn}\mathbf{1}_{\{X_n>-x\}}\right)(1-\mathbb{E}(\exp-c\alpha_1^+))^{-1}\right),$$

which yields (2).

Notice that $h_c^\uparrow$ is an increasing function, $h_c^\uparrow(0)=1$, and $h_c^\uparrow(x)$ increases as c goes to 0, to

$$h^\uparrow(x)=1+\mathbb{E}\left(\sum_{i=1}^{\alpha_1^+-1} \mathbf{1}_{\{-x<X_i\}}\right).$$

Moreover, one checks easily that $h^\uparrow(x)$ is finite for every $x\geqslant 0$ (see for instance [20]) and coincides with the renewal function associated with the negative of the first descending ladder height (by the duality lemma). By time-reversal and with obvious notation, we see that $h_c^\downarrow(x)$ increases as c goes to 0, to the finite quantity

$$h^\downarrow(x)=1+\mathbb{E}\left(\sum_{i=1}^{\alpha_1^- -1} \mathbf{1}_{\{X_i\leqslant -x\}}\right).$$

Recall that $\underrightarrow{X}$ and $\underleftarrow{X}$ are independent under $\mathbb{P}^{(c)}$ and apply Corollary 2.2. We finally obtain:

Theorem 2.3. *Under $\mathbb{P}$, $X^\uparrow$ and $X^\downarrow$ are two independent Markov chains with respective transition functions $p^\uparrow(x,\mathrm{d}y)$ and $p^\downarrow(x,\mathrm{d}y)$, where*

$$p^\uparrow(x,\mathrm{d}y)=\mathbf{1}_{\{y>0\}}(h^\uparrow(y)/h^\uparrow(x))p(x,\mathrm{d}y) \quad (x\geqslant 0),$$
$$p^\downarrow(x,\mathrm{d}y)=\mathbf{1}_{\{y\leqslant 0\}}(h^\downarrow(y)/h^\downarrow(x))p(x,\mathrm{d}y) \quad (x\leqslant 0). \qquad \square$$

In the case when $(X, \mathbb{P})$ drifts to $+\infty$, that is $\lim_{n\uparrow\infty} X_n = +\infty$ $\mathbb{P}$-a.s., it can be checked that

$$h^{\uparrow}(x) = \text{const.} \times \mathbb{P}(X_n > -x \text{ for all } n).$$

Thus, $\mathbb{P}^{\uparrow}$ is simply the law of the random walk conditioned to stay positive in the usual sense. Observe also that the transition kernel $p^{\uparrow}$ is Markovian and not just sub-Markovian, since the lifetime is infinite $\mathbb{P}^{\uparrow}$-a.s. That is to say that the function $h^{\uparrow}$ is invariant for the random walk killed as it reaches $(-\infty, 0]$. On the other hand, the lifetime is finite $\mathbb{P}^{\downarrow}$-a.s., so $h^{\downarrow}$ is not invariant, and we shall not refer to $\mathbb{P}^{\downarrow}$ as the law of the random walk conditioned to stay negative.

In the case when $(X, \mathbb{P})$ oscillates, that is drifts neither to $+\infty$ nor to $-\infty$, the lifetime is infinite $\mathbb{P}^{\uparrow}$-a.s. So again the transition kernel $p^{\uparrow}$ is Markovian and the function $h^{\uparrow}$ is invariant. Analogous assertions hold of course for $p^{\downarrow}$ and $h^{\downarrow}$. Moreover, the arguments for Theorem 2.3 show that the law $\mathbb{P}^{\uparrow}$ is the limit as c goes to 0 of the law of the original random walk killed at rate c and then conditioned to stay positive. Thus, we shall refer to $\mathbb{P}^{\uparrow/\downarrow}$ as the law of the random walk conditioned to stay positive/negative. In particular, Theorem 2.3 provides a pathwise construction of these two laws.

Remark. Previously, Tanaka [20] obtained a different construction of the law $\mathbb{P}^{\uparrow}$, based on an infinite number of time reversal at the ladder time set of the dual random walk $(-X, \mathbb{P})$. However, he does not relate his construction with conditioning to stay positive. The fact that the process that he obtains has the same law as the post-infimum process in the case when $(X, \mathbb{P})$ drifts to $+\infty$ is plain from Doney [5, Theorem 1].

2.4. An extension of Pitman's Theorem for downwards-skip-free random walks

In the special case when $(X, \mathbb{P})$ is a simple symmetric random walk, that is $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$, Pitman [14] obtained the following representation of $\mathbb{P}^{\downarrow}$: if we put $R = 2\bar{X} - X$, then $-R$ has law $\mathbb{P}^{\downarrow}$. Noting that the future-infimum chain $\underline{\underline{R}}$ of R coincides with $\bar{X}$, he deduced that conversely, $2\underline{\underline{R}} - R$ is a simple symmetric random walk. In this sub-section, we obtain from Theorem 2.1 an analogous identity when $(X, \mathbb{P})$ is more generally a downwards-skip-free random walk (dsfrw), that is when $X_1 \in \{-1, 0, 1, \ldots\}$ a.s.

First, we introduce the following notation: Let Y be the chain

$$Y_n = X_n - \sum_{i=1}^{n} \mathbf{1}_{\{X_i > \bar{X}_{i-1}\}}(1 + \Delta X_i),$$

that is Y is obtained from X after replacing every jump of X above its previous maximum by a jump of amplitude -1 (in the case when all the jumps of X have amplitude 1 or -1, $Y = X - 2\bar{X}$). Similarly, denote by $\underline{\underline{X}}(n) = \inf\{X_i : i \geq n\}$, the future infimum of X, and consider the chain

$$Z_n = X_n - \sum_{i=1}^{n} \mathbf{1}_{\{\underline{X}(i) < X(i-1)\}}(1 + \Delta X_i).$$

That is Z is obtained from X after replacing every jump of X across its future infimum by a jump of amplitude -1. Recall that $\mathbb{P}^{\uparrow/\downarrow}$ stands for the law of X conditioned to stay positive/negative under $\mathbb{P}$. We claim:

Theorem 2.4. *Assume that* $(X, \mathbb{P})$ *is a dsfrw.*

(i) *If* $\mathbb{E}(X_1) \leq 0$, *then the law of* Y *under* $\mathbb{P}$ *is* $\mathbb{P}^{\downarrow}$.

(ii) *If* $\mathbb{E}(X_1) = 0$, *then the law of* Z *under* $\mathbb{P}^{\uparrow}$ *is* $\mathbb{P}$.

Remarks. 1. Of course, there is an analogous statement for upwards-skip-free random walks.

2. When $\mathbb{P}(X_1 \geq 2) > 0$, one cannot reconstruct X from Y, and (ii) does not merely follow from (i) as in the case of the simple random walk.

Proof of Theorem 2.4. (i) Consider the stochastic integral with *predictable* integrand

$$X'(i) = \sum_{j=0}^{i} \mathbf{1}_{\{X_j \leq 0\}}(X_{j+1} - X_j) \quad \text{for } i \in \mathbb{N}.$$

The proof amounts to showing that $X^{\downarrow}$ coincides with Y evaluated for the chain $(0, X'(\alpha_1^-), \ldots)$ which has the same distribution as X under $\mathbb{P}$.

One checks easily, applying the hypothesis that $(X, \mathbb{P})$ is a dsfrw, that the indices i for which $X_{i-1} \leq 0 < X_i$, are precisely those for which X' jumps over its previous maximum, i.e. $\sup\{X'_j : j < i\} < X'_i$. Moreover, if i is such an index, and if j is the first index after i when X visits $(-\infty, 0]$, then $X_{j-1} = 1$ and $X_j = 0$. Recall that index change α^- is given by $\alpha_n^- = \inf\{i : \sum_{j=1}^{i} \mathbf{1}_{\{X_j \leq 0\}} = n\}$. It is clear from above (or from a picture) that $X^{\downarrow}$ is obtained from the chain $(0, X'(\alpha_1^-), \ldots)$ after replacing every jump across its previous maximum by a jump of amplitude -1. On the other hand, since $\mathbb{E}(X_1) \leq 0$, α^- is a.s. finite (see for instance Feller [7, Section XII-2]), and the chain $(0, X'(\alpha_1^-), \ldots)$ resulting from index change in the stochastic integral, is a random walk with law $\mathbb{P}$. This establishes (i).

(ii) The proof of (ii) is similar. Here, the hypothesis that $(X, \mathbb{P})$ is centered is needed to ensure that the indices i at which $X_{i-1} \leq 0 < X_i$, are precisely those for which $X^{\uparrow} \circ A^{+}(i) = \sum_{j=1}^{i} \mathbf{1}_{\{X_j > 0\}}(X_j - X_{j-1})$ jumps over its future infimum. $\square$

Remark. Assume that $(X, \mathbb{P})$ is an oscillating random walk (without any further assumption on the distribution of X_1). It is easily seen that under $\mathbb{P}$, the two chains $(0, X'(\alpha_1^-), \ldots)$ and $(0, X''(\alpha_1^+), \ldots)$ are independent and have the same law as X. Here, the definition for X' is the one in the proof of Theorem 2.4(i), and the definition for X'' is similar, but with $\mathbf{1}_{\{X_j > 0\}}$ instead of $\mathbf{1}_{\{X_j \leq 0\}}$.

3. Continuous time

3.1. Preliminaries

Throughout this section, we will use the following notation. Denote by Ω the space of càdlàg paths $\omega:[0,\infty)\to\mathbb{R}\cup\{\delta\}$ with lifetime $\zeta=\zeta(\omega)=\inf\{t:\,\omega(t)=\delta\}$. Let X be the coordinate process, $X_t(\omega)=\omega(t)$. Introduce for every $t<\zeta$, $\underline{X}_t=\inf\{X_s:\,s\leqslant t\}$, $\bar{X}_t=\sup\{X_s:\,s\leqslant t\}$, and put $\tau=\sup\{t<\zeta:\,X_t\wedge X_{t-}=\underline{X}_t\}$, the instant of the ultimate infimum and $I=\underline{X}(\zeta-)$, the absolute infimum of the path. On $\{\tau<\infty\}$, we consider the post-infimum process $\underrightarrow{X}$,

$$\underrightarrow{X}(t)=\begin{cases}X_{\tau+t}-I & \text{for } t<\zeta-\tau,\\ \delta & \text{for } t\geqslant\zeta-\tau,\end{cases}$$

and the reversed pre-infimum process $\underleftarrow{X}$,

$$\underleftarrow{X}(t)=\begin{cases}X_{(\tau-t)-}-I & \text{for } t<\tau,\\ \delta & \text{for } t\geqslant\tau.\end{cases}$$

In order to introduce $X^{\uparrow}$ and $X^{\downarrow}$, we briefly recall the notion of local time for semimartingales, and refer to Meyer [12] and Protter [15] for a complete account. Endow Ω with a probability measure P and a filtration $(\mathcal{G}_t)_{t\geqslant0}$ which fulfills the usual conditions, and consider $Y=(Y_t:\,t\geqslant0)$, a $(P,\mathcal{G})$-semimartingale. The (semimartingale) local time L of Y at the level 0 is the continuous increasing process specified by the Meyer–Tanaka formula

$$Y_t^+-Y_0^+=\int_0^t \mathbf{1}_{\{Y_{s-}>0\}}\,\mathrm{d}Y_s+\sum_{0<s\leqslant t}\left(\mathbf{1}_{\{Y_{s-}\leqslant0\}}Y_s^++\mathbf{1}_{\{Y_{s-}>0\}}Y_s^-\right)+\tfrac{1}{2}L_t,$$

where $x^{+/-}$ stands for the positive/negative part of x. Moreover, the following approximation holds:

$$L_t=\lim_{\eta\downarrow0}\,(1/\eta)\int_0^t \mathbf{1}_{\{0<Y_s<\eta\}}\,\mathrm{d}[Y,Y]_s^{\mathrm{c}},\tag{3}$$

where $[Y,Y]^{\mathrm{c}}$ is the continuous part of the bracket of Y (or equivalently the bracket of the continuous local martingale part of Y), and where the limit is uniform over compact intervals, in probability (this is an easy consequence of Itôs formula and of the stochastic theorem of dominated convergence, see for instance [15]).

Consider now a probability measure $\mathbb{P}$ on Ω under which X is a Lévy process; that is X has independent homogeneous increments and $\mathbb{P}(X_0=0)=1$. We denote by $(\mathcal{F}_t)_{t\geqslant0}$ the natural filtration of X after the usual completion. As well-known, X is a $(\mathbb{P},\mathcal{F})$-semimartingale, and we denote by L its (semimartingale) local time at 0. Recall that either the continuous local martingale part of X vanishes (one says that X has no Gaussian component) or it is proportional to a standard Brownian motion. In particular $L\equiv0$ when X has no Gaussian component, and otherwise, L is a Markov local time at 0. Also consider

$$A_t^+=\int_0^t \mathbf{1}_{\{X_s>0\}}\,\mathrm{d}s \quad\text{and}\quad A_t^-=\int_0^t \mathbf{1}_{\{X_s<0\}}\,\mathrm{d}s,$$

the time spent by X respectively in $(0,\infty)$ and in $(-\infty,0]$, and their right-continuous inverses $\alpha^{+/-}(t)=\inf\{s: A_s^{+/-}>t\}$. The time substitution by $\alpha^{+/-}$ consists of erasing the intervals of negative/positive excursion of X and then closing up the gaps.

For every positive fixed T, we denote by $\mathbb{P}^T$, the law of X killed at time T under $\mathbb{P}$ (so $\mathbb{P}^T(\zeta=T)=1$). Since Lévy processes have no fixed jump time, in order to simplify the notation, we will write $f(\zeta)$ instead of $f(\zeta-)$ whenever f admits a left-limit at ζ. Working under $\mathbb{P}$ or under $\mathbb{P}^T$, we introduce now the process $X^\uparrow$ by

$$X^\uparrow(t)=\Big(X_\cdot+\sum_{0<s\leqslant\cdot}(\mathbf{1}_{\{X_s\leqslant 0\}}X_{s-}^+ +\mathbf{1}_{\{X_s>0\}}X_{s-}^-)+\tfrac12 L_\cdot\Big)(\alpha^+(t))$$

(observe that in the sum, left and right limits are inverted in comparison with the Meyer–Tanaka formula) when $A^+(\zeta)>t$, and $X^\uparrow(t)=\delta$ otherwise. Pictorially, when $(X,\mathbb{P})$ has no Gaussian component, $L\equiv 0$, and $X^\uparrow$ is obtained from X after the juxtaposition of its excursions in $(0,\infty)$ (just as in the discrete case, an excursion in $(0,\infty)$ includes the—possible—initial positive jump across 0 and excludes the—possible—ultimate negative jump across 0). When $(X,\mathbb{P})$ has non-zero Gaussian component, the construction is the same, except that there appears a shift due to the non-vanishing local time. Similarly, we introduce

$$X^\downarrow(t)=\Big(X_\cdot-\sum_{0<s\leqslant\cdot}(\mathbf{1}_{\{X_s\leqslant 0\}}X_{s-}^+ +\mathbf{1}_{\{X_s>0\}}X_{s-}^-)-\tfrac12 L_\cdot\Big)(\alpha^-(t))$$

when $A^-(\zeta)>t$, and $X^\downarrow(t)=\delta$ otherwise. Again, $\mathbb{P}^\uparrow$ and $\mathbb{P}^\downarrow$ will denote the law of $X^\uparrow$ and of $X^\downarrow$ under $\mathbb{P}$.

3.2. Splitting at the infimum and juxtaposition of excursions for Lévy processes

The main result of this section is:

Theorem 3.1. *For every $T\geqslant 0$, the pairs of processes $(\underset{\rightarrow}{X},-\underset{\leftarrow}{X})$ and $(X^\uparrow,X^\downarrow)$ have the same law under $\mathbb{P}^T$.*

Remarks. 1. Taking the limit as T goes to infinity, we see that the same identity also holds under $\mathbb{P}$ whenever $(X,\mathbb{P})$ drifts to $+\infty$ (that is $\lim_{t\to\infty}X_t=+\infty$ $\mathbb{P}$-a.s.).

2. Karatzas and Shreve [10] get a related identity (concerning the pre-infimum process and the reversed post-infimum process) for Brownian motion killed at its last zero before time 1. This will be extended for Lévy processes in Section 3.5.

3. In the special case of a Brownian motion with drift, the author proved this result previously in [3].

4. It would be interesting to decide whether X can be recovered from $(X^\uparrow,X^\downarrow)$. The answer is positive when $(X,\mathbb{P})$ is a Brownian motion with possible drift, see [3].

5. Observing that, with obvious notation, $X(\zeta)=X^\uparrow(\zeta^\uparrow)+X^\downarrow(\zeta^\downarrow)=\underset{\rightarrow}{X}(\zeta)-\underset{\leftarrow}{X}(\zeta)$ $\mathbb{P}^T$-a.s., we see that the identity in law of Theorem 3.1 also holds conditionally on $X(\zeta)$.

Proof of Theorem 3.1. The case when $(X, \mathbb{P})$ is a compound Poisson process (that is finite Lévy measure, no drift and zero Gaussian component) is just a continuous-time version of Theorem 2.1. Henceforth, we assume that $(X, \mathbb{P})$ is not a compound Poisson process, and for simplicity, we take $T = 1$ and we work under $\mathbb{P}^1$. Recall that a.s., the zero-set of X has null Lebesgue measure, that there is no jump of X which starts or ends at 0, that X does not jump at time 1, and that for every $t \neq \tau$, $X_t > I$ (i.e. τ is the unique instant at which X attains its infimum, possibly as left-limit).

Let $\tilde{X}$ be the time-reversed process $\tilde{X}_t = X_{(1-t)-}$ for every $t \in [0, 1]$ and $\tilde{X}_t = \delta$ for $t > 1$, and $(\tilde{\mathscr{F}}_t)_{t \in [0,1]}$ its (completed) natural filtration. Then $\tilde{X} - \tilde{X}_0$ has law $\mathbb{P}^1$, and by a result of Kurtz (see Jacod and Protter [9, Theorem 1.8]), $\tilde{X}$ is a $(\mathbb{P}^1, \tilde{\mathscr{F}})$-semimartingale. Let $\tilde{L}$ stand for its local time at level 0. By the stochastic theorem of dominated convergence, the Riemann sums

$$\sum_{0 \le i < [n(1-t)]} \mathbf{1}_{\{X_{i/n} > 0\}}(X_{(i+1)/n} - X_{i/n}) = \sum_{[nt] \le i < n} \mathbf{1}_{\{\tilde{X}_{i/n} > 0\}}(\tilde{X}_{(i+1)/n} - \tilde{X}_{i/n})$$

(where $[nt]$ stands for the integer part of nt), converge uniformly for $t \in [0, 1]$, in probability as $n \uparrow \infty$ to the $(\tilde{\mathscr{F}})$-stochastic integral

$$\int_t^1 \mathbf{1}_{\{\tilde{X}_{s-} > 0\}} \mathrm{d}\tilde{X}_s := \int_0^1 \mathbf{1}_{\{\tilde{X}_{s-} > 0\}} \mathrm{d}\tilde{X}_s - \int_0^t \mathbf{1}_{\{\tilde{X}_{s-} > 0\}} \mathrm{d}\tilde{X}_s .$$

By the Meyer–Tanaka formula, this quantity is re-expressed as

$$-\tilde{X}_t^+ - \sum_{t < s \le 1} (\mathbf{1}_{\{\tilde{X}_{s-} \le 0\}} \tilde{X}_s^+ + \mathbf{1}_{\{\tilde{X}_{s-} > 0\}} \tilde{X}_s^-) - \tfrac{1}{2}(\tilde{L}_1 - \tilde{L}_t)$$

$$= -X_{(1-t)-}^+ - \sum_{0 < s < 1-t} (\mathbf{1}_{\{X_s \le 0\}} X_{s-}^+ + \mathbf{1}_{\{X_s > 0\}} X_{s-}^-) - \tfrac{1}{2} L_{1-t} ,$$

where the identity $\tilde{L}_1 - \tilde{L}_t = L_{1-t}$ follows from the approximation result (3).

For every positive integer n, consider the chain X^n given by $X^n(i) = X(i/n)$. Let $X^{n\uparrow}$ and $X^{n\downarrow}$ be the two chains obtained from X^n as in Section 2. Applying the property that $\{t: X_t = 0\}$ has zero Lebesgue measure a.s., we have

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{0 \le i < [nt]} \mathbf{1}_{\{X_{i/n} > 0\}} = A_t^+ .$$

We make the convention that $\delta \times a = +\infty$ for $a > 0$ and $\delta \times 0 = 0$. We deduce from above, that for every nonnegative continuous functions f and g,

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{i=0}^{n} f(i/n) X^{n\uparrow}(i) = \int_0^1 X^{\uparrow}(s) f(s) \, \mathrm{d}s,$$

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{i=0}^{n} g(i/n) X^{n\downarrow}(i) = \int_0^1 X^{\downarrow}(s) g(s) \, \mathrm{d}s,$$

where the limits are in probability. On the other hand, let $\underrightarrow{X}^n$ and $\underleftarrow{X}^n$ be the two chains obtained from X^n by splitting at the index of the last infimum. One checks easily (since X attains its infimum only once a.s.) that a.s., for every continuous functions f, g,

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{i=0}^{n}\underset{\rightarrow}{X}^{n}(i)f(i/n)=\int_0^1\underset{\rightarrow}{X}(s)f(s)\,\mathrm{d}s,$$

$$\lim_{n\uparrow\infty}\frac{1}{n}\sum_{i=0}^{n}\underset{\leftarrow}{X}^{n}(i)g(i/n)=\int_0^1\underset{\leftarrow}{X}(s)g(s)\,\mathrm{d}s.$$

But, according to Theorem 2.1, $(X^{n\uparrow}, X^{n\downarrow})$ and $(\underset{\rightarrow}{X}^{n}, -\underset{\leftarrow}{X}^{n})$ have the same law. So

$$\left(\int_0^1 X^{\uparrow}(s)f(s)\,\mathrm{d}s, \int_0^1 X^{\downarrow}(s)g(s)\,\mathrm{d}s\right)$$

and

$$\left(\int_0^1 \underset{\rightarrow}{X}(s)f(s)\,\mathrm{d}s, -\int_0^1 \underset{\leftarrow}{X}(s)g(s)\,\mathrm{d}s\right)$$

have the same laws. Since the processes $\underset{\rightarrow}{X}$, $\underset{\leftarrow}{X}$, $X^{\uparrow}$ and $X^{\downarrow}$ have càdlàg paths, this yields the theorem. □

Remark. It should be clear from the proof that Theorem 3.1 holds more generally if $(X, \mathbb{P}^T)$ is a semimartingale with exchangeable increments, which remains a semimartingale in the filtration expanded by X_T. For instance, one can take $X_t = Y_t - (t/T)Y_T$, where Y is a Lévy process.

We deduce immediately the following strong convergence result, which again is mostly interesting when $(X, \mathbb{P})$ oscillates:

Corollary 3.2. *Fix $T>0$, and denote by $\mathscr{F}_T$ the σ-algebra generated by $(X_t : t \leq T)$. For every bounded $\mathscr{F}_T \otimes \mathscr{F}_T$-measurable functional Φ,*

$$\lim_{s\uparrow\infty}\mathbb{E}^s(\Phi(\underset{\rightarrow}{X}, \underset{\leftarrow}{X}))=\mathbb{E}(\Phi(X^{\uparrow}, -X^{\downarrow})) . \qquad \square$$

In the special case when $(X, \mathbb{P})$ is a non-monotone stable process of index $\beta \in (0, 2]$, then $(X, \mathbb{P})$ oscillates, and one can also apply Theorem 3.1 to study the path near its infimum on $[0, 1]$, conditionally on X_1 and on the infimum I. One can also get asymptotic descriptions conditionally on other variables. The choice of X_1 and I here has a special interest because it gives access to laws obtained by killing and superharmonic transforms.

For every $\varepsilon > 0$, put $\underset{\rightarrow}{X}^{\varepsilon}(t) = \varepsilon^{-1/\beta}\underset{\rightarrow}{X}(\varepsilon t)$ and $\underset{\leftarrow}{X}^{\varepsilon}(t) = \varepsilon^{-1/\beta}\underset{\leftarrow}{X}(\varepsilon t)$. Keeping the notation of Corollary 3.2, we have:

Corollary 3.3. *For every bounded $\mathscr{F}_T \otimes \mathscr{F}_T$-measurable functional Φ, and for every bounded Borel function $f : \mathbb{R}^2 \to \mathbb{R}$,*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}^1[\Phi(\underset{\rightarrow}{X}^{\varepsilon}, \underset{\leftarrow}{X}^{\varepsilon})f(I, X_1)] = \mathbb{E}(\Phi(X^{\uparrow}, -X^{\downarrow}))\mathbb{E}(f(I, X_1)).$$

Proof. By Theorem 3.1 and Remark 5, we have with obvious notation

$$\mathbb{E}^1[\Phi(\underline{X}^\varepsilon, \underline{X}^\varepsilon)f(I, X_1)] = \mathbb{E}^1[\Phi((X^\uparrow)^\varepsilon, -(X^\downarrow)^\varepsilon)f(I, X_1)].$$

Let $S_\varepsilon = \inf\{t: A_t^+ > \varepsilon T \text{ and } A_t^- > \varepsilon T\}$. Since $(X, \mathbb{P})$ oscillates, the family of stopping times S_ε decreases to 0 as ε goes to 0 a.s. Observe that $\Phi((X^\uparrow)^\varepsilon, -(X^\downarrow)^\varepsilon)$ is $\mathscr{F}_{S_\varepsilon}$-adapted, and rewrite the right-hand side of the above equality as $\mathbb{E}^1[\Phi((X^\uparrow)^\varepsilon, -(X^\downarrow)^\varepsilon)M_\varepsilon]$, where $M_\varepsilon = \mathbb{E}^1[f(I, X_1)|\mathscr{F}_{S_\varepsilon}]$. By inverse martingale convergence and the Blumenthal 0–1 law, M_ε converges to $\mathbb{E}^1[f(I, X_1)]$ a.s. as ε goes to 0. On the other hand, by the scaling property, we obtain

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^1[\Phi((X^\uparrow)^\varepsilon, -(X^\downarrow)^\varepsilon)] = \mathbb{E}[\Phi(X^\uparrow, -X^\downarrow)],$$

and thus

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^1[\Phi((X^\uparrow)^\varepsilon, -(X^\downarrow)^\varepsilon)M_\varepsilon] = \mathbb{E}[\Phi(X^\uparrow, -X^\downarrow)]\mathbb{E}^1[f(I, X_1)]. \qquad \square$$

3.3. Harmonic transforms in Lévy processes

Just as in the discrete case, we can describe the law of $(X^\uparrow, X^\downarrow)$ under $\mathbb{P}$ in terms of superharmonic transforms. The key is now excursion theory, and specifically Maisonneuve's exit-system formula [11]. The arguments are left to the reader: they are continuous-time analogues of the ones developed in Theorem 2.3, and are merely adapted from Silverstein [16] and Greenwood and Pitman [8]. The special case when $(X, \mathbb{P})$ is a compound Poisson process reduces to a reformulation of the random walk analogue stated in Theorem 2.3, so we shall exclude this case in this subsection (the reason for this is that we do not want to distinguish between hitting of $(-\infty, 0]$ and of $(-\infty, 0)$).

Let $\bar{L}$ (resp. $\underline{L}$) be a Markov local time at 0 for the reflected process $\bar{X} - X$ (resp. $\underline{X} - X$), and $\bar{n}$ (resp. $\underline{n}$), the corresponding excursion measure. The inverse local time is a subordinator, we denote by $\bar{c}$ (resp. $\underline{c}$) its drift coefficient. That is $\bar{c}$ (resp. $\underline{c}$) is the delay coefficient at 0 for the reflected process, and it is known that $\bar{c} > 0$ iff 0 is irregular for $(-\infty, 0)$ (see for instance [16]). Introduce the functions

$$h^\uparrow(x) = \bar{c} + \bar{n}\left(\int_0^\zeta \mathbf{1}_{\{X_s < x\}} \, ds\right) \quad \text{for } x \geq 0,$$

$$h^\downarrow(x) = \underline{c} + \underline{n}\left(\int_0^\zeta \mathbf{1}_{\{X_s > x\}} \, ds\right) \quad \text{for } x \leq 0.$$

Silverstein [16] specifies these functions in terms of ladder processes (the classical time-reversal argument shows that his expressions and ours are the same), and relates them with Fristedt's identity. Denote by $q_t(x, dy)$ and $q_t^*(x, dy)$ the semigroups of the Lévy process killed respectively in $(-\infty, 0]$ and in $[0, \infty)$. Then $h^\uparrow$ and $h^\downarrow$ are superharmonic respectively for q_t and q_t^*, and the kernels

$$p_t^\uparrow(x, dy) = (h^\uparrow(y)/h^\uparrow(x))q_t(x, dy)$$

and

$$p_t^{\downarrow}(x, \mathrm{d}y) = (h^{\downarrow}(y)/h^{\downarrow}(x))q_t^*(x, \mathrm{d}y)$$

define two sub-Markovian semigroups. We state:

Theorem 3.4. *Under $\mathbb{P}$, $X^{\uparrow}$ and $X^{\downarrow}$ are two independent Markov processes with respective semigroups $p_t^{\uparrow}(x, \mathrm{d}y)$ and $p_t^{\downarrow}(x, \mathrm{d}y)$.* □

Again $h^{\uparrow}$ (resp. $h^{\downarrow}$) is invariant as soon as $(X, \mathbb{P})$ does not drift to $-\infty$ (resp. $+\infty$). In this case, we will refer to $\mathbb{P}^{\uparrow}$ (resp. $\mathbb{P}^{\downarrow}$) as the law of the original Lévy process conditioned to stay positive (resp. negative). In particular, when $(X, \mathbb{P})$ oscillates, Theorem 3.4 gives a pathwise construction of two independent processes having the law of the initial Lévy process conditioned respectively to stay positive and negative. In the case when $(X, \mathbb{P})$ drifts to $-\infty$, $h^{\uparrow}$ is not invariant because the lifetime is finite $\mathbb{P}^{\uparrow}$-a.s., and we shall not refer to $\mathbb{P}^{\uparrow}$ as the law of the Lévy process conditioned to stay positive.

Examples. 1. When $\mathbb{P}$ is the Wiener measure, then for $x \geqslant 0$, $h^{\uparrow}(x) = x$, $h^{\downarrow}(-x) = -x$, $\mathbb{P}^{\uparrow}$ is the law of the 3-dimensional Bessel process starting from 0, and $\mathbb{P}^{\downarrow}$ is the image of $\mathbb{P}^{\uparrow}$ by the mapping $x \mapsto -x$.

2. When $(X, \mathbb{P})$ is a spectrally positive Lévy process, that is when it has no negative jumps, the superharmonic functions $h^{\uparrow}$ and $h^{\downarrow}$ can be expressed in terms of the characteristic exponent and the so-called scale function of the Lévy process, see [1] and [4]. When moreover $(X, \mathbb{P})$ does not drift to $+\infty$, then law $\mathbb{P}^{\downarrow}$ of the Lévy process conditioned to stay negative coincides with the law denoted by P in [1], which is the limit as $K\uparrow\infty$ of the measure corresponding to conditioning X to leave the interval $[-K, 0]$ at the lower boundary (this identity is seen from the explicit expression of $h^{\downarrow}$). On the other hand, the limit law P exists even when $(X, \mathbb{P})$ drifts to $+\infty$. Nevertheless in this case, $P \neq \mathbb{P}^{\downarrow}$ (because the lifetime is finite $\mathbb{P}^{\downarrow}$-a.s. and infinite P-a.s.). More precisely, there exists a new Lévy law $\mathbb{P}^*$ obtained by conditioning X to drift to $-\infty$ (see [2] and the references therein), and it holds that $P = (\mathbb{P}^*)^{\downarrow}$.

3.4. An extension of Pitman's Theorem for spectrally negative Lévy processes

Pitman [14] deduced from his representation of the law of the simple random walk conditioned to stay positive, the following famous construction of the 3-dimensional Bessel process: if B is a standard one-dimensional Brownian motion and $\bar{B}$ its supremum process, then $R = 2\bar{B} - B$ is a 3-dimensional Bessel process. Since the future-infimum process $\underline{R}$ of R coincides with $\bar{B}$, conversely $2\underline{R} - R = B$ is a standard Brownian motion. The first identity of Pitman has been recently extended to spectrally positive (that is with no negative jumps) Lévy processes in [1]. We will see here that Theorem 3.1 yields a simple proof of this extension, and also provides an extension of the second identity of Pitman. The key for this is an analogue of Paul Lévy Theorem (Lemma 3.5 below), which permits, just as in the discrete case, to re-express $X^{\uparrow}$ and $X^{\downarrow}$ in terms of time-changes of a stochastic integral with a predictable integrand. However, it is simpler in our framework to work with processes with no positive jumps, so the present statements concern spectrally negative Lévy processes.

Lemma 3.5. *Assume that the Lévy process $(X, \mathbb{P})$ has no positive jumps and that $\mathbb{E}(X_1) \geqslant 0$. Denote by $\underline{X}^{\mathrm{c}}$, the continuous part of the decreasing process $\underline{X}$. Then the trivariate processes*

$$\left(X_{\cdot}, -\tfrac{1}{2}L_{\cdot}, -\sum_{0<s\leqslant \cdot} \mathbf{1}_{\{X_{s-}>0\}} X_s^-\right)\circ\alpha^+$$

and

$$(X-\underline{X}, \underline{X}^{\mathrm{c}}, \underline{X}-\underline{X}^{\mathrm{c}})$$

have the same law under $\mathbb{P}$.

Proof. We work under $\mathbb{P}$. Since $\mathbb{E}(X_1)\geqslant 0$, X does not drift to $-\infty$ and $\alpha^+ < \infty$ a.s. Since X has no positive jump, we can re-write the Meyer–Tanaka formula as

$$X_t^+ = \int_0^t \mathbf{1}_{\{X_{s-}>0\}}\,\mathrm{d}X_s + \sum_{0<s\leqslant t} \mathbf{1}_{\{X_{s-}>0\}} X_s^- + \tfrac{1}{2}L_t.$$

Keep in mind that the time-substitution based on α^+ consists of erasing the intervals on which X is negative and then closing up the gaps. In particular, $X\circ\alpha^+ = X^+\circ\alpha^+$ has no positive jumps. Put

$$\mathscr{X}_t = -\int_0^{\alpha_t^+} \mathbf{1}_{\{X_{s-}>0\}}\,\mathrm{d}X_s \quad \text{and} \quad \bar{\mathscr{X}}_t = \sup\{\mathscr{X}_s : s\leqslant t\}.$$

Then one verifies easily that $\mathscr{X}$ has the same law as $-X$ (see e.g. Doney [6]). Moreover, we deduce from the classical arguments of Skorohod [17] that $\mathbb{P}$-a.s., for every $t\geqslant 0$,

$$\bar{\mathscr{X}}_t = \sum_{0<s\leqslant\alpha_t^+} \mathbf{1}_{\{X_{s-}>0\}} X_s^- + \tfrac{1}{2}L(\alpha_t^+). \tag{4}$$

Indeed, the inequality $\leqslant$ is obvious from the Meyer–Tanaka formula. On the other hand, consider $g(t) = \sup\{s<t : X\circ\alpha_s^+ = 0\}$, the last zero of $X\circ\alpha^+$ before t. Since $X\circ\alpha^+$ has no positive jumps, it does not jump at $g(t)$. Moreover, one deduces easily from the absence of positive jumps of X that $(\sum_{0<s\leqslant\cdot} \mathbf{1}_{\{X_{s-}>0\}} X_s^- + \frac{1}{2}L\cdot)\circ\alpha^+$ only increases when $X\circ\alpha^+$ visits 0, so

$$\left(\sum_{0<s\leqslant\cdot} \mathbf{1}_{\{X_{s-}<0\}} X_s^- + \tfrac{1}{2}L_{\cdot}\right)\circ\alpha_t^+ = \left(\sum_{0<s\leqslant\cdot} \mathbf{1}_{\{X_{s-}>0\}} X_s^- + \tfrac{1}{2}L_{\cdot}\right)\circ\alpha_{g(t)}^+ = \mathscr{H}_{g(t)},$$

which proves (4).

Observe now that when $\mathbb{P}$ has non-vanishing Gaussian component, $L\circ\alpha^+$ is a continuous positive process. Indeed, L is then a Markov local time at 0 for X, 0 is regular for $(0,\infty)$. Thus the measure of the excursions of X from 0 which spend a positive time in $(0,\infty)$ is infinite and $L\circ\alpha^+$ is continuous. When $\mathbb{P}$ has no Gaussian component, $L\circ\alpha^+$ vanishes. Hence, in both cases, (4) is the canonical decomposition of $\bar{\mathscr{X}}$ as the sum of its discontinuous part and of its continuous part $\bar{\mathscr{X}}^c$, and the Meyer–Tanaka formula is a representation of the reflected process $\bar{\mathscr{X}}-\mathscr{X}$ as $X\circ\alpha^+$. This is an analogue of P. Lévy's theorem for the reflected Brownian motion which is stated in the Lemma 3.5. $\square$

We will use the following notation: let $\bar{\bar{X}}(t)=\sup\{X_s: s\geq t\}$ be the future-supremum process of X, $\underline{X}^c$ and $\bar{\bar{X}}^c$ respectively the continuous part of the decreasing processes $\bar{\bar{X}}$ and $\underline{X}$. Finally, put

$$\underline{J}(t)=\sum_{0\leq s\leq t}(X_s-X_{s-})\mathbf{1}_{\{X(s)<\underline{X}(s-)\}},$$

$$\bar{\bar{J}}(t)=\sum_{0\leq s\leq t}(X_s-X_{s-})\mathbf{1}_{\{X(s-)<\bar{\bar{X}}(s-)\}}$$

(recall the convention $X_{0-}=0$), that is $\underline{J}$ is the sum of the jumps of X across its previous infimum, and $\bar{\bar{J}}$ is the sum of the jumps of X across its future supremum. Observe that $\underline{X}^c$, $\bar{\bar{X}}^c$, $\underline{J}$ and $\bar{\bar{J}}^c$ are all decreasing processes. We have:

Theorem 3.6. *Assume that the Lévy process $(X,\mathbb{P})$ has no positive jumps.*
(i) *If $\mathbb{E}(X_1)\geq 0$, then under $\mathbb{P}$, $X-2\underline{X}^c-\underline{J}$ has law $\mathbb{P}^\uparrow$.*
(ii) *If $\mathbb{E}(X_1)=0$, then under $\mathbb{P}^\downarrow$, $X-2\bar{\bar{X}}^c-\bar{\bar{J}}$ has law $\mathbb{P}$.*

Remarks. 1. The analogue of statements (i) is obtained in [1] for Lévy processes with no negative jumps without any assumption on the first moment. A result related to (ii) is proved in [2] for Lévy processes with no negative jumps and $\mathbb{E}(X_1)>0$.

2. When X possesses negative jumps, the transformation $X\mapsto X-2\bar{\bar{X}}^c-\underline{J}$ cannot be inverted, so (ii) does not follow from (i) as in the Brownian case.

Proof of Theorem 3.6. (i) Keeping the notation and the arguments of the proof of Lemma 3.5, we have furthermore that the pure-jump process

$$\left(\sum_{0<s\leq\cdot}\mathbf{1}_{\{X_s>0\}}(X_s-X_{s-})\right)\circ\alpha^+=\bar{J}$$

is the sum of the jumps of $\mathscr{X}$ across its previous supremum. So, finally, we can write $X^\uparrow$ as $-\mathscr{X}+2\bar{\mathscr{X}}^c+\bar{\mathscr{J}}$, and (i) follows from Theorem 3.1.

(ii) The proof of (ii) follows the same lines. $\square$

3.5. A path decomposition of the type of Karatzas and Shreve

We conclude this work with an extension for Lévy processes of a result that Karatzas and Shreve [10] obtained for Brownian motion. Introduce $g=\inf\{s<1: X_s=0\}$, the last zero of X before time 1, and denote by $\mathbb{P}^g$, the $\mathbb{P}$-law of X killed at time g. To avoid triviality, we shall assume in this sub-section that 0 is not polar and that $\mathbb{P}$ is not the law of a compound Poisson process. First, we observe that Theorem 3.1 still holds when we replace $\mathbb{P}^T$ by $\mathbb{P}^g$:

Lemma 3.7. *$(\underset{\leftarrow}{X}, \underset{\rightarrow}{X})$ and $(X^{\uparrow}, -X^{\downarrow})$ have the same law under $\mathbb{P}^g$.*

Proof. Under $\mathbb{P}^g$, X is an inhomogeneous Markov process with transition probability given for $0 \leqslant s \leqslant t < 1$ by

$$\mathbb{P}^g(X_t \in \mathrm{d}y \,|\, X_s = x) = \mathbb{P}(x + X_{t-s} \in \mathrm{d}y) \times \mathbb{P}(\sigma(-x-y) < 1-t),$$

where $\sigma(a) = \inf\{s\colon X_s = a\}$. In particular, $\mathbb{P}^g$ is an inhomogeneous superharmonic transform of $\mathbb{P}^1$. Applying Theorem 3.1 and remark 5 below it, we deduce that $(\underset{\leftarrow}{X}, \underset{\rightarrow}{X})$ and $(X^{\uparrow}, -X^{\downarrow})$ have the same law under $\mathbb{P}^t$, conditionally on $t \leqslant g$. Thus, for any positive ε, the same identity holds conditionally on $t \leqslant g < t + \varepsilon$. Denoting by k_n the operator killing at time $2^{-n}[2^n \cdot g]$, where $[\,\cdot\,]$ stands for the integer part, we get that $(\underset{\leftarrow}{X}, \underset{\rightarrow}{X})$ and $(X^{\uparrow}, -X^{\downarrow})$ have the same law under $\mathbb{P}(\,\cdot \circ k_n)$, which proves the Lemma by taking the limit as n goes to infinity. $\square$

Now we construct from the path of X on the time interval $[0, 1]$, a new path denoted by Y as follows:

$$Y_t = \Big(X_{\cdot} - \sum_{0 < s \leqslant \cdot} (\mathbf{1}_{\{X_{s-} < 0\}} X_s^+ + \mathbf{1}_{\{X_{s-} > 0\}} X_s^-) - \tfrac{1}{2} L_{\cdot} \Big) \circ \alpha^+(t)$$

for $t \leqslant A^+(g)$,

$$Y_{(g-t)} = -\Big(X_{\cdot} + \sum_{0 < s \leqslant \cdot} (\mathbf{1}_{\{X_{s-} < 0\}} X_s^+ + \mathbf{1}_{\{X_{s-} > 0\}} X_s^-) + \tfrac{1}{2} L_{\cdot} \Big) \circ \alpha^-(t)$$

for $t < A^-(g)$, and $Y_t = X_t$ for $t \in [g, 1]$. Note that (by the Meyer–Tanaka formula) the above quantities can also be expressed as stochastic integrals. In particular, g is the last zero of Y before time 1, and $A^+(g)$ is the instant of the ultimate infimum of Y before time g. Karatzas and Shreve proved the following result in the Brownian case:

Corollary 3.8. *The law of Y under $\mathbb{P}$ is $\mathbb{P}^1$.*

Remark. There is of course an analogue of this identity for random walks. The proof can easily be performed by arguments close to the ones of Subsection 2.2.

Proof of Corollary 3.8. We know from excursion theory that the pre-g process $(X_t : t \leqslant g)$ and the post-g process $(X_{g+t} : 0 \leqslant t \leqslant 1 - g)$ are independent conditionally on g under $\mathbb{P}^1$, so all what we need is to check that the processes $(X_t : t \leqslant g)$ and $(Y_t : t \leqslant g)$ have the same distribution under $\mathbb{P}$.

Recall that the processes $(X_s\colon s \leqslant t)$ and $(X_t - X_{(t-s)-} : s \leqslant t)$ are equally distributed under $\mathbb{P}$ and have the same final value. Since $\mathbb{P}^g$ is an inhomogeneous super-harmonic transform of $\mathbb{P}$, it follows from the arguments of Lemma 3.7 above that $(X_t\colon t<g)$ and $(\check{X}_t := -X_{(g-t)-}\colon t<g)$ have the same law under $\mathbb{P}$ (because $X_g=0$ and X is continuous at time g $\mathbb{P}$-a.s.). In particular, with obvious notation, it holds that

$$(X^{\uparrow}, X^{\downarrow}) \text{ and } (\check{X}^{\uparrow}, \check{X}^{\downarrow}) \text{ have the same law under } \mathbb{P}^g. \tag{5}$$

Moreover, it can be checked (after some lengthy but simple calculation) that

$$\begin{aligned}
&\check{X}^{\downarrow}(A^+(g)) - \check{X}^{\downarrow}(A^+(g)-t)\\
&= \Big(X_\cdot - \sum_{0<s\leqslant\cdot} (\mathbf{1}_{\{X_{s-}\leqslant 0\}}X_s^+ + \mathbf{1}_{\{X_{s-}>0\}}X_s^-) - \tfrac{1}{2}L_\cdot\Big)\circ\alpha^+(t)\\
&= Y_t \quad (\text{for } t\leqslant A^+(g))
\end{aligned}$$

and

$$\begin{aligned}
&\check{X}^{\uparrow}(A^-(g)) - \check{X}^{\uparrow}(A^-(g)-t)\\
&= \Big(X_\cdot + \sum_{0<s\leqslant\cdot} (\mathbf{1}_{\{X_{s-}\leqslant 0\}}X_s^+ + \mathbf{1}_{\{X_{s-}>0\}}X_s^-) + \tfrac{1}{2}L_\cdot\Big)\circ\alpha^-(t)\\
&= -Y_{(g-t)-} \quad (\text{for } t\leqslant A^-(g)).
\end{aligned}$$

That is to say that $\check{X}^{\uparrow}$ and $-\check{X}^{\downarrow}$ are respectively the post-infimum process and the reversed pre-infimum process of Y. Applying (5) and then Lemma 3.7, this shows that Y has the same law as $(X_t\colon t<g)$ under $\mathbb{P}$. $\square$

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