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Global fluctuations in general β Dyson's Brownian motion

Martin Bender*

Department of Mathematics, Royal Institute of Technology (KTH), S-100 44 Stockholm, Sweden

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Abstract

We consider a system of diffusing particles on the real line in a quadratic external potential and with a logarithmic interaction potential. The empirical measure process is known to converge weakly to a deterministic measure-valued process as the number of particles tends to infinity. Provided the initial fluctuations are small, the rescaled linear statistics of the empirical measure process converge in distribution to a Gaussian limit for sufficiently smooth test functions. For a large class of analytic test functions, we derive explicit general formulae for the mean and covariance in this central limit theorem by analyzing a partial differential equation characterizing the limiting fluctuations.

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1. Introduction

We consider the following system of n Itô equations:

$$\mathrm{d}\lambda_t^i = \frac{2\sigma}{\sqrt{n\beta}}\mathrm{d}B_t^i - \lambda_t^i\mathrm{d}t + \frac{2\sigma^2}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{\lambda_t^i - \lambda_t^j}, \quad \text{for } i = 1, \ldots, n. \tag{1}$$

Here $\{B_t^i\}_{i=1}^n$ are independent, standard Brownian motions and σ and $\beta > 0$ are real parameters. These equations model the dynamics of n diffusing particles on the real line with a logarithmic

* Tel.: +46 87906643; fax: +46 87231788.
E-mail address: mbender@math.kth.se.

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interaction potential $u(x) = -\frac{1}{2}\log|x|$, constrained by a quadratic external potential $v_n(x) = \frac{nx^2}{4\sigma^2}$, at inverse temperature β. Cépa and Lépingle [7] prove that the order of the particles is almost surely preserved for all times $t \geq 0$. The distribution of the stationary solution to (1) is given by the associated Gibbs measure

$$\begin{aligned} \mathrm{d}\rho_n^\beta(\lambda) &= \frac{1}{\mathcal{Z}_n^{(\beta)}} \exp\left\{-\beta\left(\sum_{j=1}^n v_n(\lambda_j) + \sum_{i\neq j} u(\lambda_i - \lambda_j)\right)\right\} \prod_{i=1}^n \mathrm{d}\lambda_i \\ &= \frac{1}{\mathcal{Z}_n^{(\beta)}} \prod_{1\leq i<j\leq n} |\lambda_i - \lambda_j|^\beta \exp\left\{-\frac{\beta n}{4\sigma^2}\sum_{j=1}^n \lambda_j^2\right\} \prod_{i=1}^n \mathrm{d}\lambda_i, \end{aligned} \tag{2}$$

where $\mathcal{Z}_n^{(\beta)}$ is a normalizing constant (the partition function) and $\mathrm{d}\lambda$ denotes the Lebesgue measure.

For the specific parameter values $\beta = 1, 2$ and 4 this model can also be interpreted in terms of matrix-valued stochastic processes (Dyson's Brownian motion). Let $\mathcal{M}_n(\beta)$ be the set of all $n \times n$ real ($\beta = 1$), complex ($\beta = 2$) and quaternion ($\beta = 4$) matrices respectively and $\mathcal{S}_n(\beta)$ the set of self-dual (with respect to conjugate transposition) elements in $\mathcal{M}_n(\beta)$. The *Gaussian orthogonal* ($\beta = 1$), *unitary* ($\beta = 2$) and *symplectic* ($\beta = 4$) *ensembles*, denoted $\mathrm{GOE}_n(\sigma_0^2)$, $\mathrm{GUE}_n(\sigma_0^2)$ and $\mathrm{GSE}_n(\sigma_0^2)$ respectively, are the probability distributions

$$\mathrm{d}\mu_n^\beta(M) = \frac{1}{\mathcal{Z}_n^{(\beta)}} \exp\left\{-\frac{\beta n}{4\sigma_0^2}\mathrm{Tr}\, M^2\right\} \mathrm{d}M$$

on $\mathcal{S}_n(\beta)$, where $\mathrm{d}M = \prod_{i=1}^n \mathrm{d}M_{ii} \prod_{1\leq i<j\leq n} \mathrm{d}M_{ij}^{(1)} \cdots \mathrm{d}M_{ij}^{(\beta)}$ is the product Lebesgue measure on the essentially different elements of $M = (M_{ij}^{(1)}, \ldots, M_{ij}^{(\beta)})_{ij}$. Let $M_t = (M_t^{ij})_{ij}$ be an $\mathcal{S}_n(\beta)$-valued Ornstein–Uhlenbeck process, meaning that M_t satisfies the SDE

$$\mathrm{d}M_t = -M_t \mathrm{d}t + \frac{\sigma}{\sqrt{\beta n}} \mathrm{d}(B_t + B_t^*), \tag{3}$$

where B_t is an $n \times n$ matrix, the elements of which are independent standard real ($\beta = 1$), complex ($\beta = 2$) or quaternion ($\beta = 4$) Brownian motions and B_t^* is the conjugate transpose of B_t. Then the eigenvalues $\{\lambda_t^i\}_{i=1}^n$ of M_t satisfy (1) (see [8]). For instance, in the case $\beta = 2$ (and similarly for $\beta = 1, 4$), if $M_0 \in \mathcal{S}_n(2)$ is fixed, Eq. (3) has solution $M_t = \mathrm{e}^{-t} M_0 + N_t$, where $N_t \in \mathrm{GUE}_n(\sigma^2(1-\mathrm{e}^{-2t}))$ and if $M_0 \in \mathrm{GUE}_n(\sigma_0^2)$ we will have $M_t \in \mathrm{GUE}_n(\mathrm{e}^{-2t}(\sigma_0^2 - \sigma^2) + \sigma^2)$ for all $t \geq 0$.

We define the empirical measure process

$$X_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_t^i}.$$

To capture the asymptotic properties of the model on a global scale as $n \to \infty$, one is interested in studying the limiting behaviour of the linear statistics $\langle X_t^n, f\rangle = \frac{1}{n}\sum_{i=1}^n f(\lambda_t^i)$, for test functions f from a suitable class.

Example 1.1. Define a *deformed* GUE to be an ensemble of Hermitian matrices $M_a^{2n} = M^{2n} + D_a^{2n}$ where M^{2n} is distributed according to the $\mathrm{GUE}_{2n}(1)$ and $D_a^{2n} = (d_{ij})_{i,j=1}^{2n}$ is a fixed $2n \times 2n$ diagonal matrix with

$$d_{ii} = \begin{cases} a & \text{for } 1 \le i \le n \\ -a & \text{for } n+1 \le i \le 2n. \end{cases}$$

Then the eigenvalues of the rescaled matrix $M_a^{2n}/\sqrt{a^2+\sigma^{-2}}$ correspond to the particles in our model with initial distribution $X_0 = X_0^{2n} = \frac{1}{2}(\delta_{-1}+\delta_1)$ at time $t = \log\sqrt{1+(a\sigma)^{-2}}$. The local behaviour of the eigenvalues in this model have been studied in [5,2] and it is known that the limiting eigenvalue density of M_a^{2n} as $n \to \infty$ is supported on two disjoint intervals if $0 < a < 1$ and on one single interval if $a \ge 1$. In other words, supp X_t grows from the two starting points $\{-1, 1\}$ at time $t = 0$ into two disjoint intervals that merge at time $t = \log\sqrt{1+\sigma^{-2}}$.

In the stationary case, it is a classical result that for every bounded continuous function f, $\langle X^n, f\rangle$ converges in distribution to $\int f \mathrm{d}\mu$, where

$$\mathrm{d}\mu(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,\chi_{\{|x|<2\sigma\}}\mathrm{d}x$$

is the *Wigner semi-circle law*. More generally, for any initial asymptotic distribution of particles, X_0, the asymptotic particle distribution X_t at each time $t \ge 0$ is uniquely determined by X_0 and converges weakly to μ as $t \to \infty$ (see Theorem 2.1 for a more precise statement).

A natural question is that of whether there is a limiting distribution of the rescaled linear statistics. For ease of notation, we introduce the fluctuation process $Y_t^n = n(X_t^n - X_t)$, which takes signed Borel measures on $\mathbb{R}$ as values. We are interested in the limiting distribution of the random variables

$$\langle Y_t^n, f\rangle = \sum_{i=1}^n f(\lambda_t^i) - n\int_{\mathbb{R}} f(x)\mathrm{d}X_t(x),$$

where f is a test function from an appropriate class, as $n \to \infty$. Note that there is no $\frac{1}{\sqrt{n}}$ normalization of the linear statistics here; this reflects the very regular spacing of the particles and is typical of related models.

Israelsson [11] shows that Y_t^n converges weakly in the space of distributions acting on a class of C^6 test functions to a Gaussian process Y_t, provided the initial distributions X_0^n converge sufficiently fast to X_0 (the full statement is given in Theorem 3.1). Although establishing the existence and uniqueness of Y_t, he does not characterize it very explicitly. In this work we derive explicit general formulae for the mean and covariance of the finite dimensional distributions of Y_t acting on a family of analytic test functions by analyzing the partial differential equation arising in Israelsson's proof. These formulae generalize many similar results obtained for various special cases of our model by completely different methods; some of these are briefly discussed below. In particular it is worth noting that our results hold for all values of the inverse temperature β and in the non-equilibrium case with arbitrary initial particle distribution X_0.

Most of the previous related results pertain to specific matrix models and are restricted to the cases $\beta = 1$ or $\beta = 2$. The asymptotic global fluctuations for various ensembles of Hermitian and real symmetric matrices have been extensively studied; see e.g. [3,6,10,13,15,17]. In a recent paper [4], Bai and Yao consider $N \times N$ matrices with zero mean, independent, not necessarily identically distributed entries such that the diagonal elements all have the same variance σ^2/N and the off-diagonal elements have variance $1/N$ (real symmetric case) or uncorrelated real and imaginary parts each of variance $1/2N$ (Hermitian case). Such models are known as *Wigner* *ensembles*. Under some fourth-moment conditions, they provide a central limit theorem and give explicit mean and covariance formulae, which agree with those of Corollary 2.6 with $\beta = 2$ and $\Delta t = 0$. Under the assumption of finite moments of all orders of all matrix elements, a more general class of ensembles of symmetric matrices is considered by Anderson and Zeitouni in [1]. Here the variances of all matrix elements, and the means of the diagonal entries, are allowed to depend on position. Spohn [18] derives an expression for the covariance of the Gaussian fluctuations of our model in the hydrodynamic limit, but deals only with the case $\beta = 2$ and (time-dependent) equilibrium fluctuations.

The few previous results available on the general β case are restricted to a stationary situation. For the corresponding model on the circle, Spohn [19] is able to handle the general β case, again in the hydrodynamic limit at equilibrium. By expressing the equilibrium model in terms of ensembles of tridiagonal real matrices, Dumitriu and Edelman [9] manage to find the general β global fluctuations for polynomial test functions; there is work in progress by Dumitriu and Zeitouni extending this to C^1 test functions. Johansson [12] considers a more general model corresponding to the equilibrium measure (2) but with the quadratic external potential v_n replaced by a general polynomial of even degree and with positive leading coefficient. For the case of quadratic v_n his mean and covariance formulae agree with the fixed t equilibrium case of the model we discuss. In Johansson's model the variance is universal in the sense that it does not depend on the details of the potential, provided the support of the equilibrium measure is a single interval. In our model however, the variance at every finite t depends on the initial conditions. For instance, even though the eigenvalue density in Example 1.1 will in finite time be supported on a single interval, the fluctuations remember the initial particle distribution for all $t \geq 0$. Thus the time evolution of the variance is determined by the geometry of the initial distribution; this structure is reminiscent of the role played by the boundary conditions in determining the fluctuations of the height function in discrete plane tiling models such as Kenyon's [14]. There the fluctuations converge to a Gaussian free field for a conformal structure determined by the boundary.

2. Main results

In order to formulate our results we need the following theorem, referred to in the introduction:

Theorem 2.1 (*Rogers and Shi [16], Cépa and Lépingle [7]*). *Suppose that X_0^n converges weakly in $\mathcal{M}$, the space of Borel probability measures on $\mathbb{R}$ with the weak topology, to a point mass X_0 at an arbitrary element of $\mathcal{M}$. Then there is a family $\{X_t\}_{t\geq 0} \subset \mathcal{M}$, depending only on X_0 and converging weakly as $t \to \infty$ to the Wigner semi-circle law, μ, such that for each $t \geq 0$, X_t^n converges weakly to X_t in $\mathcal{M}$ as $n \to \infty$. X_t is uniquely characterized by the property that its Stieltjes transform,*

$$M = M(t, z) = \int \frac{\mathrm{d}X_t(x)}{x - z}, \quad (t, z) \in [0, \infty) \times (\mathbb{C} \setminus \mathbb{R}),$$

solves the initial value problem

$$\begin{cases} M_t = (2\sigma^2 M + z)M_z + M, & t > 0 \\ M(0, z) = \displaystyle\int \frac{\mathrm{d}X_0(x)}{x - z}. \end{cases} \tag{4}$$

We fix some terminology that will be used throughout the rest of this paper. Let X_0 be a given Borel probability measure on $\mathbb{R}$ and define $\Omega = \mathbb{C} \setminus \mathbb{R}$. Put $f(z) = \int \frac{\mathrm{d}X_0(x)}{x-z}$, $z \in \Omega$; f will be a holomorphic function. It follows from Theorem 2.1 that for every $t \geq 0$, $M(t, \cdot) = \int \frac{\mathrm{d}X_t(x)}{x - \cdot}$ is a well defined holomorphic function in Ω, so we can define a family $\{h_t\}_{t \geq 0}$ of holomorphic maps in Ω by the equation

$$h_t(z) = z\mathrm{e}^t + \sigma^2(\mathrm{e}^t - \mathrm{e}^{-t})M(t, z). \tag{5}$$

Proposition 2.2. *For every $t \geq 0$, $h_t(\Omega) \subseteq \Omega$ and the relation $g_t \circ h_t = \mathrm{id}$ holds, where*

$$g_t(w) = \mathrm{e}^{-t}w - \sigma^2(\mathrm{e}^t - \mathrm{e}^{-t})f(w). \tag{6}$$

Define $h_{t_1}^{t_2} = g_{t_2} \circ h_{t_1}$ for $t_1 \geq t_2 \geq 0$. Then $h_{t_1} = h_{t_2} \circ h_{t_1}^{t_2}$.

Proof. This is a step in the proof of Lemma 3.4. □

Recall the definition of the Schwarzian derivative: Let v be a univalent function in some domain of the complex plane. The *Schwarzian derivative* Sv of v is the analytic function

$$(Sv)(z) = \frac{v'''(z)}{v'(z)} - \frac{3}{2}\left(\frac{v''(z)}{v'(z)}\right)^2.$$

We introduce the *generalized Schwarzian derivative*, also denoted as Sv, as the function of two complex variables defined by

$$\frac{1}{6}(Sv)(z_1, z_2) = \begin{cases} \dfrac{\partial^2}{\partial z_1 \partial z_2} \log\left(\dfrac{v(z_1) - v(z_2)}{z_1 - z_2}\right) = \dfrac{v'(z_1)v'(z_2)}{(v(z_1) - v(z_2))^2} - \dfrac{1}{(z_1 - z_2)^2} & (z_1 \neq z_2) \\ \lim\limits_{z \to z_1} \dfrac{v'(z_1)v'(z)}{(v(z_1) - v(z))^2} - \dfrac{1}{(z_1 - z)^2} = \dfrac{1}{6}(Sv)(z_1) & (z_1 = z_2). \end{cases}$$

We can now state the main result, to be proven in Section 3, giving expressions for the mean and covariance of the finite dimensional distributions of the Stieltjes transform of the limiting fluctuation process Y_t.

Theorem 2.3. *Suppose that $Y_0^n = n(X_0^n - X_0)$ satisfies the conditions of Theorem 3.1 so that the sequence $\{Y_t^n\}_{n=1}^{\infty}$ converges weakly to a Gaussian process Y_t. Let $0 \leq t_k \leq t_{k-1} \ldots \leq t_1$ and $z = (z_1, \ldots, z_k) \in (\mathbb{C} \setminus \mathbb{R})^k$. Then the normal random vector*

$$U = (U_1, \ldots, U_k), \quad \text{where } U_j = \left\langle Y_{t_j}, \frac{1}{\cdot - z_j} \right\rangle,$$

has mean

$$\mu_j = \mathbf{E}U_j = \frac{1}{2}\left(\frac{2}{\beta} - 1\right)\frac{h_{t_j}''(z_j)}{h_{t_j}'(z_j)} + \left\langle Y_0, \frac{h_{t_j}'(z_j)}{\cdot - h_{t_j}(z_j)} \right\rangle \tag{7}$$

and covariance matrix

$$\begin{aligned} \Lambda_{lj} = \Lambda_{jl} = \mathbf{Cov}(U_j, U_l) &= \frac{2}{\beta}\frac{\partial^2}{\partial z_j \partial z_l} \log\left(\frac{h_{t_j}(z_j) - h_{t_l}(z_l)}{h_{t_j}^{t_l}(z_j) - z_l}\right) \\ &= \frac{1}{3\beta} h_{t_j}^{t_l\,\prime}(z_j)\left(Sh_{t_l}\right)(h_{t_j}^{t_l}(z_j), z_l), \quad \text{if } l \geq j. \end{aligned} \tag{8}$$

In particular,

$$\mathbf{Var}(U_j) = \frac{1}{3\beta}\left(Sh_{t_j}\right)(z_j). \tag{9}$$

Remark 2.4. An interesting consequence of Eq. (9) is that, for any fixed time $t \geq 0$, the fluctuations contain all the information about the initial conditions, in the sense that the variance of the Stieltjes transform of Y_t as a function of z will uniquely determine the initial particle distribution X_0: By (9) this function is the Schwarzian derivative of some analytic function, which is unique up to composition with an arbitrary Möbius transformation. It is easy to see that this determines h_t (and therefore also $M(t, \cdot)$) uniquely. Note that by finding the inverse g_t of h_t we are immediately provided with the Stieltjes transform f of the initial particle distribution X_0 without having to solve Eq. (4) explicitly for the initial condition.

Remark 2.5. Another model, concerning eigenvalues of non-Hermitian complex matrices, where a similar variance formula involving the Schwarzian derivative occurs is studied in [20].

As an immediate consequence of Theorem 2.3 we can derive expressions for the mean and covariance of the (time-dependent) equilibrium fluctuations. Let

$$f_\mu(z) = \frac{1}{2\pi\sigma^2}\int_{-2\sigma}^{2\sigma} \frac{\sqrt{4\sigma^2 - x^2}}{x - z}\mathrm{d}x = \frac{z}{2\sigma^2}\left(\sqrt{1 - \left(\frac{2\sigma}{z}\right)^2} - 1\right)$$

denote the Stieltjes transform of the Wigner semi-circle law, μ. (Here $\sqrt{\cdot}$ means the branch of the square root for which $\Im(\sqrt{z}) \geq 0$ iff $\Im z \geq 0$, defined for $z \in \mathbb{C} \setminus (-\infty, 0)$.)

Corollary 2.6. *Let $z_1, z_2 \in \Omega$ and $\Delta t \geq 0$ be given. Put $t_1 = t + \Delta t$ and $t_2 = t$ and let $U_j = \langle Y_{t_j}, \frac{1}{\cdot - z_j}\rangle$ for $j = 1, 2$. Then*

$$\lim_{t\to\infty} \mu_j = \lim_{t\to\infty} \mathbf{E}U_j = \left(\frac{2}{\beta} - 1\right)\frac{\sigma^2 f_\mu(z_j)}{4\sigma^2 - z_j^2}, \tag{10}$$

and

$$\begin{aligned}
\lim_{t\to\infty} \Lambda_{12} &= \lim_{t\to\infty} \mathbf{Cov}(U_1, U_2) \\
&= \mathrm{e}^{-\Delta t}\frac{8\sigma^2\left(\frac{1}{\sqrt{1-\left(\frac{2\sigma}{z_1}\right)^2}} + 1\right)\left(\frac{1}{\sqrt{1-\left(\frac{2\sigma}{z_2}\right)^2}} + 1\right)}{\beta\left(4\sigma^2\mathrm{e}^{-\Delta t} - z_1 z_2\left(\sqrt{1 - \left(\frac{2\sigma}{z_1}\right)^2} + 1\right)\left(\sqrt{1 - \left(\frac{2\sigma}{z_2}\right)^2} + 1\right)\right)^2} \\
&= \mathrm{e}^{-\Delta t}\frac{2\sigma^2 f'_\mu(z_1) f'_\mu(z_2)}{\beta(1 - \sigma^2 f_\mu(z_1) f_\mu(z_2)\mathrm{e}^{-\Delta t})^2}.
\end{aligned} \tag{11}$$

Proof. This is just a calculation using Theorem 2.3 and the fact that X_t converges weakly to the semi-circle law (Theorem 2.1). □

The previous results can be expressed in terms of integral formulae for the fluctuation process acting on more general analytic test functions. By Theorem 3.1, convergence of the fluctuation process Y_t^n is only guaranteed for a class of bounded C^6 test functions, so we will consider analytic functions which are bounded in a domain containing the real line.

Theorem 2.7. *Suppose that F_1 and F_2 are analytic and bounded in a strip $\Omega_\delta = \{z : |\Im(z)| < \delta\}$ for some $\delta > 0$. Let $t_1 \geq t_2 \geq 0$ and define the random variables $Z_1 = \langle Y_{t_1}, F_1\rangle$ and $Z_2 = \langle Y_{t_2}, F_2\rangle$. Then*

$$\mathbf{Cov}(Z_1, Z_2) = \frac{-1}{24\pi^2\beta}\oint_{\Gamma_1}\oint_{\Gamma_2}\left(Sg_{t_2}\right)(w_1, w_2)\big(F_1(g_{t_1}(w_1)) - F_2(g_{t_2}(w_2))\big)^2\,\mathrm{d}w_2\mathrm{d}w_1, \tag{12}$$

where $\Gamma_i = h_{t_i}(\gamma)$, $\gamma = \gamma_- \cup \gamma_+$ and the oriented lines γ_- and γ_+ in Ω_δ are given by the parameterizations $\mathbb{R} \ni s \mapsto s - \mathrm{i}\delta'$ and $\mathbb{R} \ni s \mapsto -s + \mathrm{i}\delta'$ respectively, for any positive $\delta' < \delta$. For $Z_1 = Z_2$ this reduces further to

$$\mathbf{Var}(\langle Y_{t_1}, F_1\rangle) = \frac{1}{4\pi^2\beta}\oint_{\Gamma_1}\oint_{\Gamma_1}\left(\frac{F_1(g_{t_1}(w_1)) - F_1(g_{t_1}(w_2))}{w_1 - w_2}\right)^2\mathrm{d}w_2\mathrm{d}w_1. \tag{13}$$

Proof. Since F_1 and F_2 are bounded in Ω_δ they may be represented by the Cauchy integral formula as contour integrals along γ. We can then use the linearity of Y_t and Fubini's theorem to obtain

$$\begin{aligned}
\mathbf{Cov}(Z_1, Z_2) &= \mathbf{E}(Z_1Z_2) - \mathbf{E}Z_1\mathbf{E}Z_2\\
&= \mathbf{E}\left[\left\langle Y_{t_1}, \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{F_1(z)\mathrm{d}z}{x-z}\right\rangle\left\langle Y_{t_2}, \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{F_2(z)\mathrm{d}z}{x-z}\right\rangle\right]\\
&\quad - \mathbf{E}\left[\left\langle Y_{t_1}, \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{F_1(z_1)\mathrm{d}z}{x-z}\right\rangle\right]\mathbf{E}\left[\left\langle Y_{t_2}, \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{F_2(z)\mathrm{d}z}{x-z}\right\rangle\right]\\
&= \oint_\gamma\oint_\gamma \frac{F_1(z_1)F_2(z_2)}{(2\pi\mathrm{i})^2}\left(\mathbf{E}\left[\left\langle Y_{t_1}, \frac{1}{z_1-\cdot}\right\rangle\left\langle Y_{t_2}, \frac{1}{z_2-\cdot}\right\rangle\right]\right.\\
&\quad \left. -\ \mathbf{E}\left[\left\langle Y_{t_1}, \frac{1}{z_1-\cdot}\right\rangle\right]\mathbf{E}\left[\left\langle Y_{t_2}, \frac{1}{z_2-\cdot}\right\rangle\right]\right)\mathrm{d}z_2\mathrm{d}z_1\\
&= \oint_\gamma\oint_\gamma \frac{F_1(z_1)F_2(z_2)}{(2\pi\mathrm{i})^2}\Lambda_{12}\mathrm{d}z_2\mathrm{d}z_1,
\end{aligned} \tag{14}$$

where

$$\Lambda_{12} = \frac{2}{\beta}\frac{\partial^2}{\partial z_1\partial z_2}\log\left(\frac{h_{t_1}(z_1) - h_{t_2}(z_2)}{h_{t_1}^{t_2}(z_1) - z_2}\right)$$

by Theorem 2.3. Since, for fixed z_2, Λ_{12} is the derivative of an analytic function of z_1 in a domain containing γ, we note that

$$\begin{aligned}
&\oint_\gamma\oint_\gamma (F_2(z_2))^2\Lambda_{12}\mathrm{d}z_2\mathrm{d}z_1\\
&\quad = \oint_\gamma (F_2(z_2))^2\oint_\gamma \frac{\mathrm{d}}{\mathrm{d}z_1}\left(\frac{-h'_{t_2}(z_2)}{h_{t_1}(z_1) - h_{t_2}(z_2)} + \frac{1}{h_{t_1}^{t_2}(z_1) - z_2}\right)\mathrm{d}z_1\mathrm{d}z_2 = 0.
\end{aligned}$$

Similarly,

$$\oint_\gamma \oint_\gamma (F_1(z_1))^2 \Lambda_{12} \mathrm{d}z_1 \mathrm{d}z_2 = 0,$$

so we may substitute $-\frac{1}{2}(F_1(z_1) - F_2(z_2))^2$ for the factor $F_1(z_1)F_2(z_2)$ in Eq. (14), which gives

$$\mathbf{Cov}(Z_1, Z_2) = \frac{1}{4\pi^2\beta} \oint_\gamma \oint_\gamma (F_1(z_1) - F_2(z_2))^2 \times \left(\frac{h'_{t_1}(z_1)h'_{t_2}(z_2)}{(h_{t_1}(z_1) - h_{t_2}(z_2))^2} - \frac{h_{t_1}^{t_2\prime}(z_1)}{(z_2 - h_{t_1}^{t_2}(z_1))^2} \right) \mathrm{d}z_2 \mathrm{d}z_1. \tag{15}$$

In the variance case, $Z_1 = Z_2$, this reduces further since the second term of the integral becomes

$$\oint_\gamma \oint_\gamma \left(\frac{F_1(z_1) - F_1(z_2)}{z_2 - z_1} \right)^2 \mathrm{d}z_2 \mathrm{d}z_1,$$

which vanishes by the analyticity of F_1. Since $g_{t_i} \circ h_{t_i} = \mathrm{id}$ in Ω, the change of variables $w_1 = h_{t_1}(z_1)$, $w_2 = h_{t_2}(z_2)$ transforms formula (15) into (12). □

We note that if in addition F_1 and F_2 are entire functions, the integrand in (12) is analytic in each variable everywhere outside the support of X_0, so the contours of integration Γ_i can be suitably deformed as long as they avoid $\operatorname{supp} X_0$. More specifically, if X_0 has compact support, Γ_i can be replaced by a positively oriented circle $\{z : |z| = R\}$ the interior of which contains $\operatorname{supp} X_0$; outside this circle the integrand is a meromorphic function in each variable with a simple pole at infinity.

Theorem 2.8. *Suppose X_0 is compactly supported and let F be an entire function, bounded in a strip $\Omega_\delta = \{z : |\Im(z)| < \delta\}$ for some $\delta > 0$, with power series expansion $F(z) = \sum_{n=0}^\infty c_n z^n$. Then*

$$\mathbf{Var}(\langle Y_t, F \rangle) = \frac{1}{\beta} \sum_{n=1}^\infty \left(c_n^2 \sum_{s=-n}^{n} |s| A_{-s,n} A_{s,n} + 2 \sum_{m=0}^{n-1} c_n c_m \sum_{s=-m}^{m} |s| A_{-s,n} A_{s,m} \right), \tag{16}$$

where

$$A_{s,n} = \sum_{(k_1,\ldots,k_n)\in I_{sn}} \prod_{i=1}^n a_{k_i},$$

$$I_{sn} = \left\{ (k_1, \ldots, k_n) \in \mathbb{Z}^n : \sum_{i=1}^n k_i = s, k_i \geq -1 \right\},$$

$a_{-1} = 1$, $a_0 = 0$ *and* $a_k = \sigma^2(1 - \mathrm{e}^{-2t})\mathrm{e}^{-(k-1)t} \int x^{k-1} \mathrm{d}X_0(x)$ *for* $k \geq 1$.

Proof. After the change of variables $z_i = \mathrm{e}^t / w_i$, and putting

$$\Psi_n(z_1, z_2) = \frac{(g_t(\mathrm{e}^t/z_1))^n - (g_t(\mathrm{e}^t/z_2))^n}{z_1 - z_2},$$

formula (13) reads

$$\mathbf{Var}(\langle Y_t, F\rangle) = \frac{1}{4\pi^2\beta}\oint_{|z_1|=r}\oint_{|z_2|=r}\left(\sum_{n=0}^{\infty} c_n\,\Psi_n(z_1,z_2)\right)^2 \mathrm{d}z_1\mathrm{d}z_2$$
$$= \frac{1}{4\pi^2\beta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n c_m \oint_{|z_1|=r}\oint_{|z_2|=r} \Psi_n(z_1,z_2)\,\Psi_m(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2, \quad (17)$$

for any $r > 0$ such that supp $X_0 \subset (-\mathrm{e}^t/r, \mathrm{e}^t/r)$. Integrating term by term is justified since the integrand is meromorphic in both variables. By definition of g_t, the Laurent series expansion of $g_t(\mathrm{e}^t/z)$ about $z = 0$ is given by

$$g_t(\mathrm{e}^t/z) = \sum_{k=-1}^{\infty} a_k z^k,$$

which converges for $0 < |z| < r$. Therefore we can write

$$\Psi_n(z_1,z_2) = \frac{\left(\sum\limits_{k=-1}^{\infty} a_k z_1^k\right)^n - \left(\sum\limits_{k=-1}^{\infty} a_k z_2^k\right)^n}{z_1 - z_2}$$
$$= \sum_{k_1,k_2,\ldots,k_n \geq -1} a_{k_1}\cdots a_{k_n}\left(\frac{z_1^{k_1+\cdots+k_n} - z_2^{k_1+\cdots+k_n}}{z_1 - z_2}\right).$$

For given integers K and J a simple combinatorial argument and the residue theorem show that

$$\frac{1}{4\pi^2}\oint_{|z_1|=r}\oint_{|z_2|=r}\left(\frac{z_1^K - z_2^K}{z_1 - z_2}\right)\left(\frac{z_1^J - z_2^J}{z_1 - z_2}\right)\mathrm{d}z_1\mathrm{d}z_2 = \begin{cases}|K| & \text{if } K = -J\\ 0 & \text{otherwise.}\end{cases}$$

This means that we can express the integrals in Eq. (17) in the form

$$\frac{1}{4\pi^2}\oint_{|z_1|=r}\oint_{|z_2|=r} \Psi_n(z_1,z_2)\,\Psi_m(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 = \sum_{|s|\leq n\wedge m} |s| A_{-s,n} A_{s,m},$$

with $A_{s,n}$ as defined in the statement of the theorem. □

Remark 2.9. Theorem 2.8 gives an explicit expression for the variance in terms of moments of the initial distribution X_0 and of the Taylor coefficients of F. To see how the dependence on the initial conditions decays in time and compare with previously known results on the stationary model we consider the limit of formula (16) as t tends to infinity.

First note that each $A_{s,n}$ is a finite sum, all contributing terms of which tend to 0 exponentially in t unless $k_i = \pm 1$ for $i = 1, \ldots, n$. If $n + s$ is odd there are no such terms, and if $n + s$ is even there are $\binom{n}{\frac{n+s}{2}}$ choices of $(k_1, \ldots, k_n)$. Using this we get

$$\rho_{nm} := \lim_{t\to\infty} \frac{1}{\beta} \sum_{|s|\leq n\wedge m} |s| A_{-s,n} A_{s,m}$$

$$
= \begin{cases} \dfrac{2\sigma^{n+m}}{\beta} \displaystyle\sum_{s=0}^{(n\wedge m)/2} 2s \binom{n}{n/2-s}\binom{m}{m/2-s} & m, n \text{ even} \\ \dfrac{2\sigma^{n+m}}{\beta} \displaystyle\sum_{s=0}^{\frac{(n\wedge m)-1}{2}} (2s+1) \binom{n}{n/2-1/2-s}\binom{m}{m/2-1/2-s} & m, n \text{ odd} \\ 0 \quad \text{otherwise.} \end{cases} \tag{18}
$$

We turn to a crude estimate of the remainder term,

$$
R_{nm}(t, X_0) = \frac{1}{\beta} \sum_{|s|\le n\wedge m} |s| A_{-s,n} A_{s,m} - \rho_{nm}.
$$

For $n \ge 1$ and $|s| \le n$ the total number of terms contributing to $A_{s,n}$ is

$$
|I_{sn}| = \binom{(s+n)+(n-1)}{n-1} \le \binom{3n-1}{n-1} \le 2^{3n-1},
$$

and each term, $\prod_{i=1}^{n} a_{k_i}$, is bounded by $(1+\sigma)^{2n}(1+R)^{2n}$, so we have a bound

$$
\begin{aligned}
|R_{nm}(t, X_0)| &= \left| \sum_{|s|\le n\wedge m} |s| A_{-s,n} A_{s,m} - \rho_{nm} \right| \\
&\le e^{-t}(n+m+1)(m+n)2^{3(n+m)-2}(1+\sigma)^{2(n+m)}(1+R)^{2(n+m)} \\
&\le C_1 e^{-t}(C_2(1+R))^{2(m+n)}
\end{aligned}
$$

for constants C_1 and C_2 independent of n, m and X_0. Now

$$
\left| \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n c_m R_{nm}(t, X_0) \right| \le C_1 e^{-t} \left(\sum_{n=0}^{\infty} |c_n| (C_2(1+R))^{2n} \right)^2 = C_R e^{-t}
$$

for some constant C_R since the power series expansion of F is absolutely convergent everywhere, so

$$
\lim_{t\to\infty} \mathbf{Var}(\langle Y_t, F\rangle) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n c_m \rho_{nm}, \tag{19}
$$

with ρ_{nm} given by Eq. (18). Formula (19) agrees with the variance formula of Dumitriu and Edelman [9] (if we formally let F be a polynomial) and that of Johansson [12] which asserts that, in the stationary case,

$$
\mathbf{Var}(\langle Y_t, h\rangle) = \frac{1}{2\beta} \sum_{k=1}^{\infty} k \left(\frac{2}{\pi} \int_0^{\pi} h(2\sigma\cos(\theta))\cos(k\theta) d\theta \right)^2
$$

for an appropriate class of real test functions h. Indeed, rewriting the power series expansion of F in terms of Chebyshev polynomials, the asymptotic variance is recovered from Johansson's result.

3. Proof of Theorem 2.3

The proof of Theorem 2.3 relies on the characterization of Y_t provided in [11] to prove existence and uniqueness of this process. For convenient reference we restate this result.

For each $k \geq 0$, endow $C^k(\mathbb{T})$ with the inner product

$$(f, g)_k = \sum_{i=0}^{k} \int_{-\pi}^{\pi} f^{(i)}(x) g^{(i)}(x) \mathrm{d}x,$$

making it a Hilbert space. The linear operator $T : C^k(\mathbb{T}) \to C^k(\mathbb{R})$ defined by $(Tf)(x) = f(2\arctan(x))$ is one-to-one so

$$H_k = T(C^k(\mathbb{T})) \subset C^k(\mathbb{R})$$

is a Hilbert space with the inner product induced from $C^k(\mathbb{T})$. Let

$$\mathcal{S} = \bigcap_{k=0}^{\infty} H_k$$

be the projective limit of the sequence $\{H_k\}_{k=0}^{\infty}$.

Theorem 3.1 (*Israelsson [11]*). *Suppose that the sequence $\{Y_0^n\}_{n=1}^{\infty}$ converges weakly in $\mathcal{S}'$ to a fixed $Y_0 \in \mathcal{S}'$. Suppose further that there is a constant C such that for every n and $z = a + bi$, $b \neq 0$, the inequality*

$$\mathbf{E}\left| \int \frac{\mathrm{d}Y_0^n(x)}{x - z} \right|^2 \leq \frac{C}{b^2}$$

holds.

Then $\{Y_t^n\}_{n=1}^{\infty}$ converges weakly to a Gaussian H_6'-valued process Y_t, i.e. for any test functions $f_j \in H_6$ and $t_j \geq 0$, $j = 1, \ldots, k$, the random vector $(\int f_1(x)\mathrm{d}Y_{t_1}^n(x), \ldots, \int f_k(x)\mathrm{d}Y_{t_k}^n(x))$ converges in distribution to the Gaussian vector $(\langle Y_{t_1}, f_1\rangle, \ldots, \langle Y_{t_k}, f_k\rangle)$. Furthermore, Y_t is uniquely characterized by its action on test functions of the form $\frac{1}{\cdot - z}$, $z \in \Omega = \mathbb{C} \setminus \mathbb{R}$, by the following property: Let $0 \leq t_{m+k} \leq t_{m+k-1} \cdots \leq t_{m+1} \leq t_1 \leq T$ be given and for $s = (s_1, \ldots, s_m, \ldots, s_{m+k}) \in \mathbb{C}^{m+k}$, $z = (z_1, \ldots, z_m, \ldots, z_{m+k}) \in \Omega^{m+k}$ and $t_{m+1} \leq t \leq t_1$ define the function

$$\begin{aligned}&\phi(t, s_1, \ldots, s_m, z_1, \ldots, z_m)\\ &\quad = \mathbf{E}\left[\exp\left\{\mathrm{i}\sum_{j=m+1}^{m+k} s_j \left\langle Y_{t_j}, \frac{1}{\cdot - z_j}\right\rangle + \mathrm{i}\sum_{j=1}^{m} s_j \left\langle Y_t, \frac{1}{\cdot - z_j}\right\rangle\right\}\right].\end{aligned}$$

Then ϕ satisfies the PDE

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \sum_{j=1}^{m}\left(s_j\left(1 + 2\sigma^2 \frac{\partial M(t, z_j)}{\partial z_j}\right)\frac{\partial \phi}{\partial s_j} + \left(z_j + 2\sigma^2 M(t, z_j)\right)\frac{\partial \phi}{\partial z_j}\right)\\ &\quad + \left(2\mathrm{i}\sigma^2\left(\frac{2}{\beta} - 1\right)\sum_{j=1}^{m}\int \frac{s_j \mathrm{d}X_t(x)}{(x - z_j)^3} - \frac{2\sigma^2}{\beta}\sum_{j=1}^{m}\sum_{l=1}^{m}\int \frac{s_j s_l \mathrm{d}X_t(x)}{(x - z_j)^2(x - z_l)^2}\right)\phi. \qquad (20)\end{aligned}$$

Remark 3.2. This is a slight reformulation of Israelsson's result: He allows for Y_0 to be random and works with real and imaginary parts of the complex functions $\frac{s}{\cdot - z}$ in order to ensure that the characteristic function ϕ be a priori well defined. However, once it is established that the distributions are Gaussian for such test functions, ϕ will be a well defined entire function of s for test functions $\frac{1}{\cdot - z}$, $z \in \Omega$. The argument leading to Eq. (20) is then identical to that in Israelsson's proof, but this form is convenient for finding explicit solutions.

Remark 3.3. There is a numerical mistake in Israelsson's derivation of Eq. (20) which has been corrected here; all occurrences of the factor $\frac{\alpha}{2}$ in the equations on page 51 and onward in [11] should be replaced by α.

Israelsson's method is similar to, although technically more involved than, that used by Rogers and Shi to prove Theorem 2.1. It is shown that the characteristic function of the Stieltjes transform of any subsequential limit of Y_t^n must satisfy Eq. (20), and a tightness argument then reduces the problem to proving existence and uniqueness of solutions to this equation.

By explicitly solving Eq. (20) under appropriate initial conditions, we will be able to find expressions for the mean and covariance of the finite dimensional distributions of Y_t.

Lemma 3.4. *For any fixed $t_0 \geq 0$, let $\phi_{t_0}(s, z)$ be a given analytic function defined for $s = (s_1, \ldots, s_k) \in \mathbb{C}^k$ and $z = (z_1, \ldots, z_k) \in \Omega^k$ and let $U = \{(t, s, z) : t > t_0, s \in \mathbb{C}^k, z \in \Omega^k\}$ and $\Gamma = \{(t_0, s, z) : s \in \mathbb{C}^k, z \in \Omega^k\} \subseteq \partial U$. The initial value problem*

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \sum_{j=1}^{k} \left(s_j \left(1 + 2\sigma^2 \frac{\partial M(t, z_j)}{\partial z_j} \right) \frac{\partial \phi}{\partial s_j} + \left(z_j + 2\sigma^2 M(t, z_j) \right) \frac{\partial \phi}{\partial z_j} \right) \\
&\quad + \left(2\mathrm{i}\sigma^2 \left(\frac{2}{\beta} - 1 \right) \sum_{j=1}^{k} \int \frac{s_j \mathrm{d}X_t(x)}{(x - z_j)^3} \right. \\
&\quad \left. - \frac{2\sigma^2}{\beta} \sum_{j=1}^{k} \sum_{l=1}^{k} \int \frac{s_j s_l \mathrm{d}X_t(x)}{(x - z_j)^2 (x - z_l)^2} \right) \phi \quad \text{in } U; \\
\phi(t_0, s, z) &= \phi_{t_0}(s, z) \quad \text{on } \Gamma,
\end{aligned} \tag{21}$$

has the following unique solution:

$$\phi(t, s, z) = \phi_{t_0} \left(s \cdot h_t^{t_0\prime}(z), h_t^{t_0}(z) \right) \exp \left\{ \mathrm{i} \sum_{j=1}^{k} s_j \mu_j - \frac{1}{2} \sum_{j=1}^{k} \sum_{l=1}^{k} s_j s_l \Lambda_{jl} \right\}, \tag{22}$$

where

$$\begin{aligned}
\mu_j &= \frac{1}{2} \left(\frac{2}{\beta} - 1 \right) \frac{h_t^{t_0\prime\prime}(z_j)}{h_t^{t_0\prime}(z_j)}, \\
\Lambda_{jl} = \Lambda_{lj} &= \frac{1}{3\beta} \left(S h_t^{t_0} \right) (z_j, z_l),
\end{aligned}$$

and $s \cdot h_t^{t_0\prime}(z)$, $h_t^{t_0}(z)$ is shorthand notation for $(s_1 h_t^{t_0\prime}(z_1), \ldots, s_k h_t^{t_0\prime}(z_k))$ and $(h_t^{t_0}(z_1), \ldots, h_t^{t_0}(z_k))$ respectively.

Proof. The equation is linear and can be solved with the method of characteristics. Fix $(t, s, z) \in U$ and let $\phi(\tau) = \phi(\overline{x}(\tau))$ be the solution along the characteristic $\overline{x}(\tau) = (t(\tau), s(\tau), z(\tau))$ through that point. By (21), the equations for $\overline{x}(\tau)$, if we choose $t(\tau) = \tau$, become

$$\frac{\mathrm{d}z_j(\tau)}{\mathrm{d}\tau} = -z_j(\tau) - 2\sigma^2 M(\tau, z_j(\tau)) \tag{23}$$

and

$$\frac{\mathrm{d}s_j(\tau)}{\mathrm{d}\tau} = -s_j(\tau)\left(1 + 2\sigma^2 \frac{\partial M}{\partial z_j}(\tau, z_j(\tau))\right), \tag{24}$$

while the solution $\phi(\tau) = \phi(\overline{x}(\tau))$ along the characteristic is given by the equation

$$\frac{\mathrm{d}\phi(\tau)}{\mathrm{d}\tau} = \Bigg\{2\mathrm{i}\sigma^2\left(\frac{2}{\beta} - 1\right)\sum_{j=1}^{k}\int \frac{s_j(\tau)\mathrm{d}X_t(x)}{(x - z_j(\tau))^3} - \frac{2\sigma^2}{\beta}\sum_{j=1}^{k}\sum_{l=1}^{k}\int \frac{s_j(\tau)s_l(\tau)\mathrm{d}X_t(x)}{(x - z_j(\tau))^2(x - z_l(\tau))^2}\Bigg\}\phi(\tau). \tag{25}$$

It will be convenient to solve these equations for all $\tau > 0$ and impose the initial condition at the end. It may seem difficult to find solutions in closed form because of the dependence on the evolution of X_t, which is known only through the property of having Stieltjes transform satisfying (4). As we will show however, all dependence on X_t can be expressed in terms of M and, more crucially, the evolution of M along the characteristic is particularly simple. For the first point, it is easy to see by algebraic manipulations that

$$\int \frac{\mathrm{d}X_t(x)}{(x - z_j)^3} = \frac{1}{2}\frac{\partial^2}{\partial z_j^2}\left(\int \frac{\mathrm{d}X_t(x)}{(x - z_j)}\right) = \frac{1}{2}M_{zz}(t, z_j),$$

$$\int \frac{\mathrm{d}X_t(x)}{(x - z_j)^4} = \frac{1}{6}\frac{\partial^3}{\partial z_j^3}\left(\int \frac{\mathrm{d}X_t(x)}{(x - z_j)}\right) = \frac{1}{6}M_{zzz}(t, z_j),$$

and, with a little more effort,

$$\int \frac{\mathrm{d}X_t(x)}{(x - z_j)^2(x - z_l)^2} = \left(\frac{2(M(t, z_j) - M(t, z_l))}{(z_j - z_l)^3} - \frac{M_z(t, z_j) + M_z(t, z_l)}{(z_j - z_l)^2}\right),$$

if $z_j \neq z_l$. (Differentiating under the integral sign is clearly justified here since all integrands are bounded.) Assuming without loss of generality that $z_j \neq z_l$ if $j \neq l$, Eq. (25) can thus be written

$$\begin{aligned}\frac{1}{\phi(\tau)}\frac{\mathrm{d}\phi(\tau)}{\mathrm{d}\tau} &= 2\mathrm{i}\sigma^2\left(\frac{2}{\beta} - 1\right)\sum_{j=1}^{k}\frac{s_j(\tau)}{2}M_{zz}(\tau, z_j(\tau)) - \frac{2\sigma^2}{\beta}\left(\sum_{j=1}^{k}\frac{s_j(\tau)^2}{6}M_{zzz}(\tau, z_j(\tau))\right.\\ &\quad\left. + \sum_{j\neq l}s_j(\tau)s_l(\tau)\left(\frac{2(M(\tau, z_j(\tau)) - M(\tau, z_l(\tau)))}{(z_j(\tau) - z_l(\tau))^3} - \frac{M_z(\tau, z_j(\tau)) + M_z(\tau, z_l(\tau))}{(z_j(\tau) - z_l(\tau))^2}\right)\right).\end{aligned} \tag{26}$$

Eqs. (23) and (24) can now be integrated with the aid of (4) defining the evolution of $M(\tau, z_j(\tau))$. Fix $z_j = z$ and put $M(\tau) \equiv M(\tau, z(\tau))$, $M_z(\tau) \equiv \frac{\partial M(\tau, z(\tau))}{\partial z}$ and so on for all partial derivatives of $M(t, z)$. Differentiating, we have by the chain rule and Eqs. (4) and (23)

$$\frac{\mathrm{d}M(\tau)}{\mathrm{d}\tau} = M_z(\tau)z'(\tau) + M_t(\tau) = M_z(\tau)(z'(\tau) + 2\sigma^2 M(\tau) + z(\tau)) + M(\tau) = M(\tau)$$

or in integrated form simply

$$M(\tau) = M(t, z)\mathrm{e}^{\tau - t}. \tag{27}$$

With (27) substituted into (23), the latter equation can be integrated to yield

$$z(0) = z\mathrm{e}^t + \sigma^2(\mathrm{e}^t - \mathrm{e}^{-t})M(t, z) \equiv h_t(z).$$

Using this initial condition, Eqs. (23) and (27) give the explicit expression

$$z(\tau) = \mathrm{e}^{-\tau} z(0) - \sigma^2(\mathrm{e}^\tau - \mathrm{e}^{-\tau}) f(z(0)) = g_\tau(z(0)) = g_\tau(h_t(z)) \tag{28}$$

for the characteristic. In particular, taking $\tau = t$ gives $z = g_t(h_t(z))$, and since there is a unique characteristic through each point in U it follows that for $t \geq t_1 \geq t_0$, $h_t(z) = h_{t_1}(g_{t_1}(h_t(z)))$, which is the assertion of Proposition 2.2. Note that this provides a method of calculating h_t (and $M(t, z)$) by finding an inverse of the explicitly defined function g_t.

Since the function $(t, z) \mapsto g_t(z) = \mathrm{e}^{-t} z - \sigma^2(\mathrm{e}^t - \mathrm{e}^{-t}) f(z)$ is C^∞, it follows from implicit differentiation of the relation $g_t(h_t(z)) = z$ that the order of differentiation can be interchanged in the mixed partial derivatives of $h_t(z)$, in particular $\frac{\partial^{k+1}(h_t(z))}{\partial t \partial z^k} = \frac{\partial^{k+1}(h_t(z))}{\partial z^k \partial t}$ for $k = 1, 2, 3$. Using this and differentiating Eq. (4) gives

$$M_{zt} = M_{tz} = (2\sigma^2 M_z + 1)M_z + (2\sigma^2 M + z)M_{zz} + M_z,$$
$$M_{zzt} = M_{tzz} = (6\sigma^2 M_z + 3)M_{zz} + (2\sigma^2 M + z)M_{zzz},$$

and

$$M_{zzzt} = M_{tzzz} = 6\sigma^2(M_{zz})^2 + (8\sigma^2 M_z + 4)M_{zzz} + (2\sigma^2 M + z)M_{zzzz}.$$

With these equations we can obtain ODEs for M_z, M_{zz} and M_{zzz} in a completely analogous fashion:

$$\frac{\mathrm{d}M_z(\tau)}{\mathrm{d}\tau} = M_{zz}(\tau)z'(\tau) + M_{zt}(\tau) = 2(\sigma^2 M_z(\tau) + 1)M_z(\tau), \tag{29}$$

$$\frac{\mathrm{d}M_{zz}(\tau)}{\mathrm{d}\tau} = M_{zzz}(\tau)z'(\tau) + M_{zzt}(\tau) = (6\sigma^2 M_z(\tau) + 3)M_{zz}(\tau), \tag{30}$$

and

$$\frac{\mathrm{d}M_{zzz}(\tau)}{\mathrm{d}\tau} = M_{zzzz}(\tau)z'(\tau) + M_{zzzt}(\tau) = 6\sigma^2(M_{zz}(\tau))^2 + 4(2\sigma^2 M_z(\tau) + 1)M_{zzz}(\tau). \tag{31}$$

Putting $w \equiv h_t(z)$, Eq. (27) can be expressed as

$$M(\tau) = f(w)\mathrm{e}^\tau, \tag{32}$$

and Eqs. (29) through (31) can be integrated to produce

$$M_z(\tau) = f'(w)\frac{\mathrm{e}^\tau}{g'_\tau(w)}, \tag{33}$$

$$M_{zz}(\tau) = \frac{f''(w)}{(g'_\tau(w))^3}, \tag{34}$$

and

$$M_{zzz}(\tau) = \frac{1}{(g_\tau'(w))^4}\left(f'''(w) + 3\sigma^2(\mathrm{e}^\tau - \mathrm{e}^{-\tau})\frac{(f''(w))^2}{g_\tau'(w)}\right), \tag{35}$$

where $g_\tau(w) = \mathrm{e}^{-\tau}w - \sigma^2(\mathrm{e}^\tau - \mathrm{e}^{-\tau})f(w)$. Inserting into Eq. (24) and integrating we get

$$s(\tau) = s\frac{g_\tau'(w)}{g_t'(w)}. \tag{36}$$

We can now finally express the right hand side of Eq. (26) as an explicit function of τ by plugging in our expressions (28) and (32) through (36) derived for the evolution of s_j, z_j and z-derivatives of M along the characteristic. Integrating we see that

$$\log\left(\frac{\phi(t,s,z)}{\phi(t_0, s(t_0), z(t_0))}\right) = I + II + III, \tag{37}$$

where

$$I = 2\mathrm{i}\sigma^2\left(\frac{2}{\beta} - 1\right)\int_{t_0}^t \sum_{j=1}^k \frac{s_j(\tau)}{2} M_{zz}(\tau, z_j(\tau))\mathrm{d}\tau,$$

$$II = -\frac{2\sigma^2}{\beta}\int_{t_0}^t \left(\sum_{j=1}^k \frac{s_j(\tau)^2}{6} M_{zzz}(\tau, z_j(\tau))\right)\mathrm{d}\tau,$$

and

$$III = -\frac{2\sigma^2}{\beta}\int_{t_0}^t \sum_{j\neq l} s_j(\tau)s_l(\tau)\left(\frac{2(M(\tau, z_j(\tau)) - M(\tau, z_l(\tau)))}{(z_j(\tau) - z_l(\tau))^3}\right.$$
$$\left. - \frac{M_z(\tau, z_j(\tau)) + M_z(\tau, z_l(\tau))}{(z_j(\tau) - z_l(\tau))^2}\right)\mathrm{d}\tau.$$

To calculate these integrals we first note some immediate consequences of the definitions of the function $h_t^{t_0} = g_{t_0} \circ h_t$ and the generalized Schwarzian derivative:

$$\frac{h_t^{t_0\prime\prime}(z)}{h_t^{t_0\prime}(z)} = \frac{1}{g_t'(w)}\left(\frac{g_{t_0}''(w)}{g_{t_0}'(w)} - \frac{g_t''(w)}{g_t'(w)}\right), \tag{38}$$

and

$$(Sh_t^{t_0})(z_1, z_2) = \frac{1}{g_t'(w_1)g_t'(w_2)}\left((Sg_{t_0})(w_1, w_2) - (Sg_t)(w_1, w_2)\right), \tag{39}$$

where $w_i = h_t(z_i)$. Using the change of variables $x = b(\tau) = \sigma^2(\mathrm{e}^{2\tau} - 1)$ we can now calculate the integrals on the right hand side of Eq. (37). First, we note that by (38),

$$\begin{aligned}
\int_{t_0}^t s(\tau)M_{zz}(\tau)\mathrm{d}\tau &= s\int_{t_0}^t \frac{g_\tau'(w)}{g_t'(w)}\frac{f''(w)}{(g_\tau'(w))^3}\mathrm{d}\tau \\
&= \frac{sf''(w)}{2\sigma^2 g_t'(w)}\int_{b(t_0)}^{b(t)} \frac{\mathrm{d}x}{(1 - xf'(w))^2} = \frac{sf''(w)}{2\sigma^2 g_t'(w)f'(w)}\left[\frac{1}{1 - xf'(w)} - 1\right]_{b(t_0)}^{b(t)} \\
&= \frac{s}{2\sigma^2 g_t'(w)}\left(\frac{\mathrm{e}^{-t}b(t)f''(w)}{\mathrm{e}^{-t}(1 - b(t)f'(w))} - \frac{\mathrm{e}^{-t_0}b(t_0)f''(w)}{\mathrm{e}^{-t_0}(1 - b(t_0)f'(w))}\right) = \frac{s}{2\sigma^2}\frac{h_t^{t_0\prime\prime}(z)}{h_t^{t_0\prime}(z)}.
\end{aligned}$$

This means that

$$I = 2\mathrm{i}\sigma^2\left(\frac{2}{\beta}-1\right)\int_{t_0}^{t}\sum_{j=1}^{k}\frac{s_j(\tau)}{2}M_{zz}(\tau,z_j(\tau))\mathrm{d}\tau = \frac{\mathrm{i}}{2}\left(\frac{2}{\beta}-1\right)\sum_{j=1}^{k}s_j\frac{h_t^{t_0\prime\prime}(z_j)}{h_t^{t_0\prime}(z_j)}. \tag{40}$$

Turning to the second integral, we have

$$\begin{aligned}
\int_{t_0}^{t}s(\tau)^2M_{zzz}(\tau)\mathrm{d}\tau &= \frac{s^2}{(g_t'(w))^2}\int_{t_0}^{t}\frac{\mathrm{e}^{2\tau}}{(1-b(\tau)f'(w))^2}\left(f'''(w)+\frac{3(f''(w))^2b(\tau)}{1-b(\tau)f'(w)}\right)\mathrm{d}\tau\\
&= \frac{s^2}{2\sigma^2(g_t'(w))^2}\int_{b(t_0)}^{b(t)}\frac{1}{(1-xf'(w))^2}\left(f'''(w)+\frac{3(f''(w))^2x}{(1-xf'(w))}\right)\mathrm{d}x\\
&= \frac{s^2}{2\sigma^2(g_t'(w))^2}\int_{b(t_0)}^{b(t)}\frac{1}{(1-xf'(w))^2}\left(\left(f'''(w)-\frac{3(f''(w))^2}{f'(w)}\right)\right.\\
&\quad\left.+\ \frac{3(f''(w))^2}{f'(w)}\frac{1}{(1-xf'(w))}\right)\mathrm{d}x\\
&= \frac{s^2}{2\sigma^2(g_t'(w))^2}\left[\left(\frac{f'''(w)}{f'(w)}-\frac{3(f''(w))^2}{(f'(w))^2}\right)\left(\frac{1}{(1-xf'(w))}-1\right)\right.\\
&\quad\left.+\ \frac{3}{2}\frac{(f''(w))^2}{(f'(w))^2}\left(\frac{1}{(1-xf'(w))^2}-1\right)\right]_{b(t_0)}^{b(t)}\\
&= \frac{s^2}{2\sigma^2(g_t'(w))^2}\left(\frac{b(t)f'''(w)}{1-b(t)f'(w)}+\frac{3}{2}\left(\frac{b(t)f''(w)}{1-b(t)f'(w)}\right)^2\right.\\
&\quad\left.-\ \frac{b(t_0)f'''(w)}{1-b(t_0)f'(w)}+\frac{3}{2}\left(\frac{b(t_0)f''(w)}{1-b(t_0)f'(w)}\right)^2\right)\\
&= \frac{s^2}{2\sigma^2(g_t'(w))^2}\left(-(Sg_t)(w)+(Sg_{t_0})(w)\right)\\
&= \frac{s^2}{2\sigma^2}(Sh_t^{t_0})(z,z),
\end{aligned}$$

where we used the identity (39) in the last step. Hence

$$II = -\frac{2\sigma^2}{\beta}\int_{t_0}^{t}\left(\sum_{j=1}^{k}\frac{s_j(\tau)^2}{6}M_{zzz}(\tau,z_j(\tau))\right)\mathrm{d}\tau = -\frac{1}{6\beta}\sum_{j=1}^{k}s_j^2(Sh_t^{t_0})(z_j,z_j). \tag{41}$$

To calculate integral III, put $c=\frac{w_j-w_l}{f(w_j)-f(w_l)}$. Then for each $j\neq l$ we get a contribution to the sum in III which takes the form

$$\begin{aligned}
&-\frac{2\sigma^2}{\beta}\int_{t_0}^{t}s_j(\tau)s_l(\tau)\left(\frac{2(M(\tau,z_j(\tau))-M(\tau,z_l(\tau)))}{(z_j(\tau)-z_l(\tau))^3}\right.\\
&\quad\left.-\ \frac{M_z(\tau,z_j(\tau))+M_z(\tau,z_l(\tau))}{(z_j(\tau)-z_l(\tau))^2}\right)\mathrm{d}\tau\\
&= \frac{s_js_l}{\beta g_t'(w_j)g_t'(w_l)\left(f(w_j)-f(w_l)\right)^2}\int_{b(t_0)}^{b(t)}\left(\frac{2cf'(w_j)f'(w_l)-(f'(w_j)+f'(w_l))}{(x-c)^2}\right.
\end{aligned}$$

$$+ \left. \frac{2(cf'(w_j)-1)(cf'(w_l)-1)}{(x-c)^3} \right) \mathrm{d}x. \tag{42}$$

Now for any $s \geq 0$ we can simplify

$$\begin{aligned}
&\int_0^{b(s)} \left(\frac{2cf'(w_j)f'(w_l) - (f'(w_j)+f'(w_l))}{(x-c)^2} + \frac{2(cf'(w_j)-1)(cf'(w_l)-1)}{(x-c)^3} \right) \mathrm{d}x \\
&= \left(2cf'(w_j)f'(w_l) - (f'(w_j)+f'(w_l))\right) \left(\frac{1}{b(s)-c} + \frac{1}{c} \right) \\
&\quad + \left(c^2 f'(w_j)f'(w_l) - c(f'(w_j)+f'(w_l)) + 1 \right) \left(\frac{1}{(b(s)-c)^2} - \frac{1}{c^2} \right) \\
&= \frac{1}{c^2(b(s)-c)^2} \left(2cf'(w_j)f'(w_l) - (f'(w_j)+f'(w_l))\right) (cb(s)^2 - c^2 b(s)) \\
&\quad + \left(c^2 f'(w_j)f'(w_l) - c(f'(w_j)+f'(w_l)) + 1 \right) (2cb(s) - b(s)^2) \\
&= \frac{1}{c^2(b(s)-c)^2} \left(-(x-b(s))^2 + c^2(b(s)f'(w_j)-1)(b(s)f'(w_l)-1) \right) \\
&= (f(w_j)-f(w_l))^2 \left(\frac{g_s'(w_j)g_s'(w_l)}{(g_s(w_j)-g_s(w_l))^2} - \frac{1}{(w_j-w_l)^2} \right) \\
&= \frac{1}{6}(f(w_j)-f(w_l))^2 (Sg_s)(w_j,w_l),
\end{aligned}$$

so by Eqs. (42) and (39),

$$\begin{aligned}
III &= \sum_{j\neq l} \frac{s_j s_l}{6\beta g_t'(w_j)g_t'(w_l)} \left((Sg_t)(w_j,w_l) - (Sg_{t_0})(w_j,w_l) \right) \\
&= -\frac{1}{6\beta} \sum_{j\neq l} s_j s_l (Sh_t^{t_0})(z_j,z_l).
\end{aligned} \tag{43}$$

Inserting the expressions (40), (41) and (43) into Eq. (37) gives (22). $\square$

We are now ready to prove the main result.

Proof (*Proof of Theorem* 2.3). Let $s = (s_1,\ldots,s_k) \in \mathbb{C}^k$, $z = (z_1,\ldots,z_k) \in \Omega^k$ and $t = (t_1,\ldots,t_k)$, where $0 \leq t_k \leq t_{k-1} \ldots \leq t_1$, be given. We will prove that the characteristic function

$$\phi(t,s,z) = \mathbf{E}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^k s_j U_j \right\} \right]$$

of the random vector $U = (\langle Y_{t_1}, \frac{1}{\cdot - z_1}\rangle, \ldots, \langle Y_{t_k}, \frac{1}{\cdot - z_k}\rangle)$ is the characteristic function of a Gaussian vector. Since we have assumed that Y_0 is fixed (non-random), it follows that $\phi_0(s,z) \equiv \phi(0,s,z) = \exp\{\mathrm{i}\sum_{j=1}^k s_j \langle Y_0, \frac{1}{\cdot - z_j}\rangle\}$. With the convention $t_{k+1} = 0$, define the functions $\phi_\tau^{(j)}$, $j = 1,\ldots,k$, depending on the variables $s^{(j)} = (s_1^{(j)},\ldots,s_j^{(j)})$, $z^{(j)} = (z_1^{(j)},\ldots,z_j^{(j)})$ and the single time variable τ, $t_{j+1} \leq \tau \leq t_j$ by the following expression:

$$\phi_\tau^{(j)}(s^{(j)},z^{(j)}) = \mathbf{E}\left[\exp\left\{ \mathrm{i} \sum_{m=j+1}^k s_m \left\langle Y_{t_m}, \frac{1}{\cdot - z_m} \right\rangle + \mathrm{i} \sum_{m=1}^j s_m^{(j)} \left\langle Y_\tau, \frac{1}{\cdot - z_m^{(j)}} \right\rangle \right\} \right].$$

Israelsson's Theorem, 3.1, states precisely that the $\phi_\tau^{(j)}$ satisfy Eq. (21) with initial conditions $\phi_{t_{j+1}}^{(j)}(s^{(j)}, z^{(j)}) = \phi_{t_{j+1}}^{(j+1)}(s^{(j)}, s_{j+1}, z^{(j)}, z_{j+1})$ for $j = 1, \ldots, k-1$, and $\phi_0^{(k)}(s^{(k)}, z^{(k)}) = \phi_0(s^{(k)}, z^{(k)})$. Thus we may successively integrate k times to obtain $\phi(t, s, z) = \phi_{t_1}^{(1)}(s_1, z_1)$ in terms of the initial conditions, using Lemma 3.4 in each step. More explicitly, for $j = 1, \ldots, k-1$ we have by Lemma 3.4

$$\begin{aligned}\phi_{t_j}^{(j)}(s^{(j)}, z^{(j)}) &= \phi_{t_{j+1}}^{(j+1)}(s^{(j)} \cdot h_{t_j}^{t_{j+1}\prime}(z^{(j)}), s_{j+1}, h_{t_j}^{t_{j+1}}(z^{(j)}), z_{j+1}) \\ &\quad \times \exp\left\{ \mathrm{i} \sum_{l=1}^{j} s_l^{(j)} \mu_l^{(j)} - \frac{1}{2} \sum_{l=1}^{j} \sum_{m=1}^{j} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \right\}, \end{aligned} \tag{44}$$

where $\mu_l^{(j)} = \frac{1}{2}(\frac{2}{\beta} - 1) \frac{h_{t_j}^{t_{j+1}\prime\prime}(z_l^{(j)})}{h_{t_j}^{t_{j+1}\prime}(z_l^{(j)})}$ and $\Lambda_{lm}^{(j)} = \Lambda_{ml}^{(j)} = \frac{1}{3\beta}(S h_{t_j}^{t_{j+1}})(z_l^{(j)}, z_m^{(j)})$.

Applying formula (44) $k-1$ times, starting with $\phi(t, s, z) = \phi_{t_1}^{(1)}(s_1, z_1)$, we obtain

$$\phi(t, s, z) = \phi_{t_k}^{(k)}(s^{(k)}, z^{(k)}) \prod_{j=1}^{k-1} \exp\left\{ \mathrm{i} \sum_{l=1}^{j} s_l^{(j)} \mu_l^{(j)} - \frac{1}{2} \sum_{l=1}^{j} \sum_{m=1}^{j} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \right\}, \tag{45}$$

where $z_j^{(j)} = z_j$, $s_j^{(j)} = s_j$ and

$$\begin{cases} z_m^{(j)} = h_{t_{j-1}}^{t_j} \circ \cdots \circ h_{t_m}^{t_{m+1}}(z_m) = h_{t_m}^{t_j}(z_m) \\ s_m^{(j)} = s_m \prod_{l=m}^{j-1} h_{t_l}^{t_{l+1}\prime}(z_m^{(l)}) = s_m h_{t_m}^{t_j\prime}(z_m) \quad \text{for } m = 1, \ldots, j-1. \end{cases}$$

After a final application of Lemma 3.4, Eq. (45) becomes

$$\begin{aligned}\phi(t, s, z) &= \phi_0(s \cdot h_t'(z), h_t(z)) \exp\left\{ \sum_{j=1}^{k} \left(\mathrm{i} \sum_{l=1}^{j} s_l^{(j)} \mu_l^{(j)} - \frac{1}{2} \sum_{l=1}^{j} \sum_{m=1}^{j} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \right) \right\} \\ &= \phi_0(s \cdot h_t'(z), h_t(z)) \exp\left\{ \mathrm{i} \sum_{l=1}^{k} \sum_{j=l}^{k} s_l^{(j)} \mu_l^{(j)} - \frac{1}{2} \sum_{l=1}^{k} \sum_{m=1}^{k} \sum_{j=l \vee m}^{k} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \right\}, \end{aligned} \tag{46}$$

where $s \cdot h_t'(z) = (s_1 h_{t_1}'(z_1), \ldots, s_k h_{t_k}'(z_k))$ and $h_t(z) = (h_{t_1}(z_1), \ldots, h_{t_k}(z_k))$. Now

$$\begin{aligned} s_l^{(j)} \mu_l^{(j)} &= \frac{1}{2}\left(\frac{2}{\beta} - 1\right) s_l h_{t_l}^{t_j\prime}(z_l) \frac{h_{t_j}^{t_{j+1}\prime\prime}(h_{t_l}^{t_j}(z_l))}{h_{t_j}^{t_{j+1}\prime}(h_{t_l}^{t_j}(z_l))} \\ &= \frac{1}{2}\left(\frac{2}{\beta} - 1\right) s_l \frac{\frac{\mathrm{d}}{\mathrm{d}z_l}\left(h_{t_j}^{t_{j+1}\prime}(h_{t_l}^{t_j}(z_l))\right)}{h_{t_j}^{t_{j+1}\prime}(h_{t_l}^{t_j}(z_l))} \\ &= \frac{1}{2}\left(\frac{2}{\beta} - 1\right) s_l \frac{\mathrm{d}}{\mathrm{d}z_l}\left(\log h_{t_j}^{t_{j+1}\prime}(h_{t_l}^{t_j}(z_l))\right), \end{aligned}$$

so since

$$\frac{\mathrm{d}}{\mathrm{d}z_l}\left(\log \prod_{j=l}^{k} h_{t_j}^{t_{j+1}\prime}(h_{t_l}^{t_j}(z_l))\right) = \frac{\mathrm{d}}{\mathrm{d}z_l} \log((h_{t_k}^{t_{k+1}} \circ \cdots \circ h_{t_l}^{t_{l+1}})'(z_l)) = \frac{\mathrm{d}}{\mathrm{d}z_l} \log(h_{t_l}'(z_l)),$$

we have found that

$$\sum_{j=l}^{k} s_l^{(j)} \mu_l^{(j)} = \frac{1}{2}\left(\frac{2}{\beta}-1\right) s_l \frac{\mathrm{d}}{\mathrm{d}z_l} \log(h_{t_l}'(z_l)). \tag{47}$$

To evaluate $s_l s_m \Lambda_{lm} \equiv \sum_{j=l\vee m}^{k} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)}$ we distinguish between two cases. First, suppose that $h_{t_l}(z_l) \neq h_{t_m}(z_m)$. In this case, if we assume $l > m$, we can write

$$\begin{aligned}
s_l s_m \Lambda_{lm} &= \sum_{j=l\vee m}^{k} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \\
&= s_l \sum_{j=l}^{k} h_{t_l}^{t_j\,\prime}(z_l) s_m h_{t_m}^{t_j\,\prime}(z_m) \frac{1}{3\beta}\left(S h_{t_j}^{t_{j+1}}\right)(h_{t_l}^{t_j}(z_l), h_{t_m}^{t_j}(z_m)) \\
&= s_l s_m \frac{2}{\beta} \sum_{j=l}^{k} h_{t_l}^{t_j\,\prime}(z_l) h_{t_m}^{t_j\,\prime}(z_m) \\
&\quad \times \frac{\partial^2}{\partial(h_{t_l}^{t_j}(z_l))\partial(h_{t_m}^{t_j}(z_m))} \log\left(\frac{h_{t_j}^{t_{j+1}}(h_{t_l}^{t_j}(z_l)) - h_{t_j}^{t_{j+1}}(h_{t_m}^{t_j}(z_m))}{h_{t_l}^{t_j}(z_l) - h_{t_m}^{t_j}(z_m)}\right) \\
&= s_l s_m \frac{2}{\beta} \sum_{j=l}^{k} \frac{\partial^2}{\partial z_l \partial z_m} \log\left(\frac{h_{t_j}^{t_{j+1}}(h_{t_l}^{t_j}(z_l)) - h_{t_j}^{t_{j+1}}(h_{t_m}^{t_j}(z_m))}{h_{t_l}^{t_j}(z_l) - h_{t_m}^{t_j}(z_m)}\right) \\
&= s_l s_m \frac{2}{\beta} \frac{\partial^2}{\partial z_l \partial z_m} \log \prod_{j=l}^{k} \left(\frac{h_{t_l}^{t_{j+1}}(z_l) - h_{t_m}^{t_{j+1}}(z_m)}{h_{t_l}^{t_j}(z_l) - h_{t_m}^{t_j}(z_m)}\right) \\
&= s_l s_m \frac{2}{\beta} \frac{\partial^2}{\partial z_l \partial z_m} \log\left(\frac{h_{t_l}^{t_{k+1}}(z_l) - h_{t_m}^{t_{k+1}}(z_m)}{h_{t_l}^{t_l}(z_l) - h_{t_m}^{t_l}(z_m)}\right) \\
&= s_l s_m \frac{1}{3\beta} h_{t_m}^{t_l\,\prime}(z_m) \left(S h_{t_l}\right)(z_l, h_{t_m}^{t_l}(z_m)),
\end{aligned}$$

as claimed. Secondly, consider the case $h_{t_l}(z_l) = h_{t_m}(z_m)$. Using the identity $(S(f \circ g))(z) = (g'(z))^2 (Sf)(g(z)) + (Sg)(z)$ for the Schwarzian derivative of a composition, we have

$$\begin{aligned}
s_l s_m \Lambda_{lm} &= \sum_{j=l\vee m}^{k} s_l^{(j)} s_m^{(j)} \Lambda_{lm}^{(j)} \\
&= \sum_{j=l}^{k} s_l h_{t_l}^{t_j\,\prime}(z_l) s_m h_{t_m}^{t_j\,\prime}(z_m) \frac{1}{3\beta}\left(S h_{t_j}^{t_{j+1}}\right)(h_{t_l}^{t_j}(z_l), h_{t_m}^{t_j}(z_m)) \\
&= s_l s_m \frac{1}{3\beta} \sum_{j=l}^{k} h_{t_l}^{t_j\,\prime}(z_l) h_{t_m}^{t_j\,\prime}(z_m) \left(S h_{t_j}^{t_{j+1}}\right)(h_{t_l}^{t_j}(z_l)) \\
&= s_l s_m \frac{1}{3\beta} \sum_{j=l}^{k} h_{t_l}^{t_j\,\prime}(z_l) h_{t_m}^{t_j\,\prime}(z_m) \left(\frac{\left(S(h_{t_j}^{t_{j+1}} \circ h_{t_l}^{t_j})\right)(z_l) - \left(S h_{t_l}^{t_j}\right)(z_l)}{(h_{t_l}^{t_j\,\prime}(z_l))^2}\right)
\end{aligned}$$

$$= s_l s_m \frac{1}{3\beta} \sum_{j=l}^{k} h_{t_m}^{t_l\,\prime}(z_m) \left(\left(S h_{t_l}^{t_{j+1}} \right)(z_l) - \left(S h_{t_l}^{t_j} \right)(z_l) \right)$$
$$= s_l s_m \frac{1}{3\beta} h_{t_m}^{t_l\,\prime}(z_m) \left(S h_{t_l} \right)(z_l, z_l).$$

Thus Eq. (46) can be expressed as

$$\phi(t, s, z) = \exp\left\{ \mathrm{i} \sum_{j=1}^{k} s_j \left(h'_{t_j}(z_j) \left\langle Y_0, \frac{1}{\cdot - h_{t_j}(z_j)} \right\rangle + \frac{1}{2}\left(\frac{2}{\beta} - 1\right) \frac{\mathrm{d}}{\mathrm{d}z_j} \log(h'_{t_j}(z_j)) \right) - \frac{1}{2} \sum_{l=1}^{k} \sum_{j=1}^{k} s_l s_j \frac{1}{3\beta} h_{t_j}^{t_l\,\prime}(z_j) \left(S h_{t_l} \right)(z_l, h_{t_j}^{t_l}(z_j)) \right\}, \tag{48}$$

which shows that U is Gaussian with mean and covariance as claimed. □

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