



On the 3-D stochastic magnetohydrodynamic- α model

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Abstract

We consider the stochastic three dimensional magnetohydrodynamic- α model (MHD- α) which arises in the modeling of turbulent flows of fluids and magnetofluids. We introduce a suitable notion of weak martingale solution and prove its existence. We also discuss the relation of the stochastic 3D MHD- α model to the stochastic 3D magnetohydrodynamic equations by proving a convergence theorem, that is, as the length scale α tends to zero, a subsequence of weak martingale solutions of the stochastic 3D MHD- α model converges to a certain weak martingale solution of the stochastic 3D magnetohydrodynamic equations. Finally, we prove the existence and uniqueness of the probabilistic strong solution of the 3D MHD- α under strong assumptions on the external forces.

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1. Introduction

The three dimensional magnetohydrodynamic (MHD) equations involve coupling Maxwell's equations governing the magnetic field and the Navier–Stokes equations (NSE) governing the fluid motion (see [10]). They play a fundamental role in Astrophysics, Geophysics,

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Plasma Physics, and in many other areas in applied sciences. In many of these, turbulent magnetohydrodynamic flows are typical. Even with the most current sophisticated scientific tools, it is a very challenging task to compute analytically or via direct numerical simulations the turbulent behavior of 3-D incompressible fluids or magnetic fluids due to the large range of scale of motions that need to be resolved. To tackle this issue, models which can capture the physical phenomenon of turbulence in fluid flows at a lower computability cost have been proposed over the past three decades. The Navier–Stokes- α model is considered to be amongst the best as a closure model of turbulence in fluid flows. Several analytical and numerical results seem to confirm this fact (see [11–13]). This excellent performance of the Navier–Stokes- α model has motivated the extensive analysis of alpha subgrid models of turbulence in recent years (see [11–15,22,23,26,27,30–32,35,53]).

Thanks to the success of these alpha models of turbulence, it is natural to consider their extension to the magnetohydrodynamic equations as well. In [33], Linshiz and Titi suggested several models, but in accordance to their analytic comparison it is enough to consider the following magnetohydrodynamic-alpha (MHD- α) model

$$\begin{cases} \frac{\partial v}{\partial t} + (u \cdot \nabla)v + \sum_{j=1}^3 v_j \nabla u_j - \nu \Delta v + \nabla \mathfrak{P} + \frac{1}{2} \nabla |B|^2 = (B \cdot \nabla)B, \\ \frac{\partial B}{\partial t} + (u \cdot \nabla)B - (B \cdot \nabla)u - \eta \Delta B = 0, \\ v = (I - \alpha^2 \Delta)u, \\ \nabla \cdot u = \nabla \cdot v = \nabla \cdot B = 0, \\ u(x, 0) = u_0, \quad B(x, 0) = B_0, \end{cases} \tag{1}$$

where u and B , represent the unknown ‘filtered’ fluid velocity and magnetic field, respectively, \mathfrak{P} is the unknown ‘filtered’ pressure, and $\alpha > 0$ is the length scale parameter that represent the width of the filters.

For a flow region \mathcal{T} , (1) has formally three quadratic invariants in the ideal case, i.e., when $\nu = \eta = 0$:

- the energy $E^\alpha = \frac{1}{2} \int_{\mathcal{T}} (v(x) \cdot u(x) + |B(x)|^2) dx$,
- the cross helicity $H_C^\alpha = \frac{1}{2} \int_{\mathcal{T}} v(x) \cdot B(x) dx$, and
- the magnetic helicity $H_M^\alpha = \frac{1}{2} \int_{\mathcal{T}} A(x) \cdot B(x) dx$, where A is the vector potential so that $B = \nabla \times A$.

These ideal invariants reduce, as $\alpha \rightarrow 0$, to those of 3-D MHD equations under suitable boundary conditions (in periodic boundary conditions or in the whole space \mathbb{R}^3). These facts make (1) more interesting than other MHD- α systems, which are either ill-posed or do not inherit some of the original properties of the 3-D MHD equations, when considering the alpha models as a regularizing numerical scheme. See [33] for justifications of the last statement.

However, in order to consider a more realistic model of turbulent evolutive fluid, it is sensible to consider some kind of stochastic perturbation represented by a *noise* term in the equations. This idea of introducing a noise term has been motivated by Reynolds’ work on stochastic fluids mechanics which stipulated that turbulence is composed by slow (deterministic) and fast (stochastic) components. It is also important to note that a noise term may also reflect, for instance, some environmental effects of the phenomena, some external random forces, etc. Such approach in the mathematical investigation for the understanding of the Newtonian

turbulence phenomenon was pioneered by Bensoussan and Temam in [6] where they studied the Stochastic Navier–Stokes Equation (SNSE). Since then stochastic partial differential equations and stochastic models of Newtonian fluid dynamics have been the object of intense investigations which have [1,5,7,8,17–19,34,38,43,41,42,44,45,21].

In our turn, we shall study the stochastic 3-D MHD- α model in the present work. Let $T > 0$ be a final time and let $\mathcal{T} = [0, 2\pi]^3$, the 3-D stochastic MHD- α is described by the following system

$$\left\{ \begin{aligned} & dv + \left[(u \cdot \nabla)v + \sum_{j=1}^3 v_j \nabla u_j - \nu \Delta v + \nabla \mathfrak{P} + \frac{1}{2} \nabla |B|^2 - (B \cdot \nabla)B \right] dt \\ & \quad = f_1(u, B, t)dt + g_1(u, B, t)dW, \\ & dB + [(u \cdot \nabla)B - (B \cdot \nabla)u - \eta \Delta B] dt = f_2(u, B, t)dt + g_2(u, B, t)dW, \\ & v = (I - \alpha^2 \Delta)u, \\ & \nabla \cdot u = \nabla \cdot v = \nabla \cdot B = 0, \\ & \int_{\mathcal{T}} u(x, t)dx = \int_{\mathcal{T}} B(x, t)dx = 0, \\ & u \text{ and } B \text{ are periodic in space } x, \\ & u(x, 0) = u_0, \quad B(x, 0) = B_0, \end{aligned} \right. \tag{2}$$

where $u = (u_1, u_2, u_3)$, $B = (B_1, B_2, B_3)$ and \mathfrak{P} are unknown random fields defined on $\mathcal{T} \times [0, T]$, representing, respectively, the fluid velocity, the magnetic field and the pressure, at each point of $\mathcal{T} \times [0, T]$. The terms $f_i(u, B, t)$ and $g_i(u, B, t)dW$ ($i = 1, 2$) are external forces depending on u and B , where W is an \mathbb{R}^d -valued standard Wiener process. Finally, u_0 and B_0 are given non random initial velocity and magnetic field, respectively. Although we are working with finite dimensional Wiener process W , all the results of this paper remain valid if W is infinite dimensional with $\mathbb{E}|W(t)|^2 < \infty$. Our argument can be also transferred to the case of two different mutually independent Wiener processes.

At the limit $\alpha = 0$, we formally obtain the stochastic 3-D MHD equations. The study of the stochastic 3-D MHD equations was investigated in [49,2,46]. The authors in [49,2] consider additive noises. Using Galerkin’s approximation and compactness method, the author in [46] proved the existence of probabilistic weak solutions for the stochastic 3-D MHD equations in the presence of nonlinear multiplicative noise which do not satisfy the Lipschitz condition. In [33], the Cauchy problem for the deterministic 3-D MHD- α model (1) with periodic boundary conditions was studied, the global existence, uniqueness and regularity of weak solutions were established. The relation between the solutions of the MHD- α model and the solutions of the MHD equations was proved as α approaches zero.

To the best of our knowledge, there is no known result for the stochastic equations (2). The purpose of the present paper is to prove some results related to problem (2) which are the stochastic version of some of those obtained in [33] for the deterministic case. The following three points are our main goals.

- (1) We prove the existence of weak martingale solution for the stochastic 3-D MHD- α model. We consider a sufficiently general forcing consisting of a regular part and a stochastic part both depending nonlinearly on the velocity of fluid and the magnetic field and we do not require the functions involved in the forcing to satisfy the Lipschitz condition. The method for the proof uses the compactness method in a version which seems to have been introduced initially by Bensoussan in [3–5]. This approach has proved very efficient in works by the present

authors [18,20,43,41,46,45,44]. The compactness method combines Galerkin approximation scheme with sharp compactness results in function spaces of Sobolev type due to Simon [47] and some celebrated compactness results of Prokhorov [40] and Skorokhod [48].

- (2) After obtaining the existence of a weak martingale solution of the stochastic 3-D MHD- α model, we turn our attention to the study of its asymptotic behavior as α tends to zero. For this purpose, we study the weak compactness of weak martingale solutions as α approaches zero. This is not derived directly from a priori estimates obtained in [Theorem 2.3](#) because some explode when α approaches zero. One of the main difficulties lies in obtaining needed a priori estimates in which the constants are independent of α (see the proof of [Lemma 6.1](#)). Once the a priori estimates are secured, the next task is to obtain the tightness of probability measures generated by the sequence $\{u_\alpha\}_{\alpha>0}$ which enables us to make use of Prokhorov and Skorokhod's compactness results. The last main issue is the passage to the limit which turns out to be rather complicated in view of the nature of the nonlinear terms involved in our model (2) (see the proof of (157)).

The question of asymptotic analysis of partial differential equations when some physical parameters converge to some limit has always been of great interest. Notable example is the vanishing viscosity question in Navier–Stokes equations which is still not fully solved when the problem is assigned with Dirichlet boundary conditions for instance. We refer to [16,25,52,51]. In the stochastic case fewer investigation has been carried out; we refer to [9] for relevant investigation.

- (3) We also prove the pathwise uniqueness of the solution of problem (2) when the forcing terms satisfy the Lipschitz condition. The existence of weak martingale solution and the pathwise uniqueness along with Yamada–Watanabe's Theorem imply the unique existence of probabilistic strong solution to (2).

The paper is structured as follows. In [Section 2](#), we gather all the necessary tools, we introduce the definition of the probabilistic weak solution of the problem. In the very same section we formulate the first main result (see [Theorem 2.3](#)). In [Section 3](#), we introduce a Galerkin approximation scheme for the problem (2) and obtain a priori estimates for the approximating solutions. In [Section 4](#), we prove the crucial result of tightness of Galerkin's solutions and apply Prokhorov's and Skorokhod's compactness results. In [Section 5](#), we prove our first main result. In [Section 6](#), we obtain uniform a priori estimates for weak solutions $\{u_\alpha\}_{\alpha>0}$ of the stochastic 3-D MHD- α model. We derive the results of the tightness of the corresponding probability measures and perform the passage to the limit which establishes the convergence of $\{u_\alpha\}_\alpha$ to a probabilistic weak solution of the stochastic 3-D MHD equations. This gives us another proof of the existence of a weak martingale solution to the stochastic 3-D MHD equations. In [Section 7](#), we prove the existence and uniqueness of the probabilistic strong solution for the 3-D stochastic MHD- α model under Lipschitz assumptions on the data.

2. Statement of the problem and the first main result

We introduce some notations and background following the mathematical theory of Navier–Stokes equations (NSE). Let $L^p(\mathcal{T})$ and $H^m(\mathcal{T})$ the L^p -Lebesgue spaces and Sobolev spaces, respectively. We denote by $|\cdot|$ the L^2 -norm, and by (\cdot, \cdot) the L^2 -inner product. Let X be a linear subspace of integrable functions defined on the domain \mathcal{T} , we define

$$\dot{X} = \left\{ \varphi \in X : \int_{\mathcal{T}} \varphi(x) dx = 0 \right\},$$

and

$$\mathcal{V} = \{ \varphi : \varphi \text{ is vector valued trigonometric polynomial defined on } \mathcal{T}, \\ \nabla \cdot \phi = 0 \text{ and } (1, \phi) = 0 \}.$$

The spaces H and V are the closures of \mathcal{V} in $L^2(\mathcal{T})^3$ and $H^1(\mathcal{T})^3$, respectively. We endow H with the inner product of $L^2(\mathcal{T})^3$ and the norm of $L^2(\mathcal{T})^3$. We equip the space V with the inner product

$$((u, v))_V = (u, v) + \alpha^2(\nabla u, \nabla v). \tag{3}$$

Its associated norm is denoted by $\|\cdot\|_V$. The importance of this choice will become clear in the course of the proof of some key estimates in the forthcoming section. On V , the norm generated by the inner product $((\cdot, \cdot))_V$ is equivalent to the gradient norm denoted by $\|\cdot\|$ and the usual H^1 -norm. Indeed, we can deduce from the definition of $\|\cdot\|_V$ and Poincaré’s inequality that

$$(\lambda_1 + \alpha^2)^{-1} \|u\|_V \leq \|u\| \leq \alpha^{-2} \|u\|_V, \quad u \in V. \tag{4}$$

Let $P_\sigma : \dot{L}^2(\mathcal{T})^3 \rightarrow H$ be the Helmholtz–Leray projection, and $A = -P_\sigma \Delta$ be the Stokes operator with the domain $D(A) = H^2(\mathcal{T})^3 \cap V$. In the periodic boundary condition case $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j : j \geq 1\}$ of eigenfunctions of A , $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_j \sim j^{2/3}(2\pi)^{-2}$. Furthermore, it is possible to define the fractional powers of A and it turns out that $D(A^{\frac{1}{2}}) = V$ (see, among others, [50]). Moreover, if Y' is the dual of the topological space Y then it turns out that $D(A^\alpha)' = D(A^{-\alpha})$ for any $\alpha > 0$ and

$$|u|_{D(A^\alpha)} = |A^\alpha u|, \tag{5}$$

for any $\alpha \in \mathbb{R}$. For these facts we refer the reader to [50] for instance. Throughout we will denote by $\langle u, v \rangle$ the values of $u \in Y'$ on $v \in Y$. By the Riesz representation we will identify H with its dual and we will consider the chain

$$D(A) \subset V \subset H \equiv H' \subset V' \subset D(A^{-1}),$$

where each space is densely and compactly embedded into the next one. It follows from this chain that

$$(u, v) = \langle u, v \rangle, \quad \forall u \in H, v \in V.$$

The following relations are very important throughout the article. For all $f \in H$ and $\phi \in V$, we have

$$(((I + \alpha^2 A)^{-1} f, \phi))_V = (f, \phi), \tag{6}$$

$$\|(I + \alpha^2 A)^{-1} f\|_V \leq |f|. \tag{7}$$

We also recall that $(I + \alpha^2 A)^{-1}$ is an isomorphism from H onto $D(A)$.

Following the notation of the NSE, we denote

$$\mathbb{B}(u, v) = P_\sigma [(u \cdot \nabla) v], \\ \tilde{\mathbb{B}}(u, v) = P_\sigma [(\nabla \times v) \times u],$$

for any $u, v, w \in \mathcal{V}$. Note that

$$(\mathbb{B}(u, v), w) = -(\mathbb{B}(u, w), v),$$

and

$$(\tilde{\mathbb{B}}(u, v), w) = (\mathbb{B}(u, v), w) - (\mathbb{B}(w, v), u), \tag{8}$$

for any $u, v, w \in \mathcal{V}$. The Eq. (8) follows from the identity

$$(b \cdot \nabla) a + \sum_{j=1}^3 a_j \nabla b_j = -b \times (\nabla \times a) + \nabla(a \cdot b), \tag{9}$$

where \times is the vector product in \mathbb{R}^3 . We recall that

$$\tilde{\mathbb{B}}(u, u) = \mathbb{B}(u, u).$$

The definitions of $\mathbb{B}(u, v)$ and $\tilde{\mathbb{B}}(u, v)$, and the above algebraic identities may be extended to larger spaces by the density of \mathcal{V} in the appropriate space each time the corresponding trilinear forms are continuous. In the following lemma whose proof can be found in [33], we collect crucial properties of the extensions of the bilinear \mathbb{B} and $\tilde{\mathbb{B}}$ (which we also denote by \mathbb{B} and $\tilde{\mathbb{B}}$.)

Lemma 2.1. (1) *Let X be either \mathbb{B} or $\tilde{\mathbb{B}}$. The bilinear form X can be extended continuously from $V \times V$ with values in V' (the dual space of V). In particular, for $u, v, w \in V$,*

$$|(X(u, v), w)_{V'}| \leq c|u|^{1/2} \|u\|^{1/2} \|v\| \|w\|. \tag{10}$$

Moreover

$$(\mathbb{B}(u, v), w) = -(\mathbb{B}(u, w), v), \quad u, v, w \in V, \tag{11}$$

which in turn implies that

$$(\mathbb{B}(u, v), v) = 0, \quad u \cdot v \in V. \tag{12}$$

Also,

$$(\tilde{\mathbb{B}}(u, v), w) = (\mathbb{B}(u, v), w) - (\mathbb{B}(w, v), u), \quad u, v, w \in V, \tag{13}$$

and hence

$$(\tilde{\mathbb{B}}(u, v), u) = 0, \quad u, v \in V. \tag{14}$$

(2) *Furthermore, let $u \in D(A)$, $v \in V$, $w \in H$ and let X be either \mathbb{B} or $\tilde{\mathbb{B}}$; then*

$$|(X(u, v), w)| \leq c \|u\|^{1/2} |Au|^{1/2} \|v\| |w|. \tag{15}$$

(3) *Let $u \in V$, $v \in D(A)$, $w \in H$ then*

$$|(\mathbb{B}(u, v), w)| \leq c \|u\| \|v\|^{1/2} |Av|^{1/2} |w|. \tag{16}$$

(4) *Let $u \in D(A)$, $v \in H$, $w \in V$ then*

$$|(\mathbb{B}(u, v), w)_{V'}| \leq c \|u\|^{1/2} |Au|^{1/2} |v| \|w\|. \tag{17}$$

(5) *Let $u, v, w \in V$, then*

$$|(\tilde{\mathbb{B}}(u, v), w)| \leq C \|u\| \|v\| |w|^{1/2} \|w\|^{1/2}. \tag{18}$$

(6) Let $u \in H, v \in V, w \in D(A)$ and let X be either \mathbb{B} or $\tilde{\mathbb{B}}$; then

$$|\langle X(u, v), w \rangle_{D(A)'}| \leq c \|u\| \|v\| \|w\|^{1/2} |Aw|^{1/2}. \tag{19}$$

(7) Let $u \in V, v \in H, w \in D(A)$ then

$$|\langle \tilde{\mathbb{B}}(u, v), w \rangle_{D(A)'}| \leq c (\|u\|^{1/2} \|u\|^{1/2} \|v\| |Aw| + \|v\| \|u\| \|w\|^{1/2} |Aw|^{1/2}), \tag{20}$$

and hence by Poincaré’s inequality,

$$|\langle \tilde{\mathbb{B}}(u, v), w \rangle_{D(A)'}| \leq c (\lambda_1)^{1/4} \|u\| \|v\| |Aw|. \tag{21}$$

(8) Let $u \in D(A), v \in V, w \in V$; then

$$|\langle \tilde{\mathbb{B}}(u, v), w \rangle_{D(A)'}| \leq c (\|u\|^{1/2} |Au|^{1/2} \|v\| \|w\| + |Au\| \|v\| \|w\|^{1/2} \|w\|^{1/2}). \tag{22}$$

In this lemma and throughout the paper, c denotes a positive constant.

Now, we make precise our assumptions on $f_i, g_i, i = 1, 2$.

(A1) We assume that f_i (resp., g_i), $i = 1, 2$, are nonlinear measurable mappings defined on $H \times H \times [0, T]$ taking values on H (resp., $H^{\otimes d}$). We also suppose that they are continuous with respect to their first two arguments and that

$$|f_i(u, B, t)| \leq c(1 + |u| + |B|), \tag{23}$$

and

$$|g_i(u, B, t)|_{H^{\otimes d}} \leq c(1 + |u| + |B|), \tag{24}$$

for $i = 1, 2, t \in [0, T]$ and $u, B \in H$.

Using the above notations and the identity (9), we apply P_σ to the system (2) to obtain, as for the case of the NSE, the equivalent system of equations

$$\begin{cases} dv + [\tilde{\mathbb{B}}(u, v) + vAv - \mathbb{B}(B, B)]dt = f_1(u, B, t)dt + g_1(u, B, t)dW, \\ dB + [\mathbb{B}(u, B) - \mathbb{B}(B, u) + \eta AB]dt = f_2(u, B, t)dt + g_2(u, B, t)dW, \\ v = u + \alpha^2 Au, \\ u(0) = u_0, \\ B(0) = B_0. \end{cases} \tag{25}$$

Before we proceed further we introduce the following notations which will be used frequently in the manuscript. Let X be any Banach space of functions defined on \mathbb{R}^3 . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any complete probability space equipped with a *right-continuous and increasing filtration* $(\mathcal{F}_t)_{t \in [0, T]}$ such that \mathcal{F}_0 contains all null sets of \mathcal{F} ; such kind of filtration will be termed as a “*filtration satisfying the usual condition*”. The mathematical expectation with respect to the probability measure \mathbb{P} is denoted by \mathbb{E} . By the symbol $L^p(\Omega, \mathbb{P}, L^q(0, T, X))$ we denote the space of functions $u = u(\omega, t, x)$ defined on $\Omega \times [0, T]$ with values in X , and such that

(a) $u(\omega, t, x)$ is measurable with respect to (ω, t) and for each t is \mathcal{F}_t -measurable in ω .

(b) $u(\omega, t, x) \in X$, for almost all (ω, t) and

$$\|u\|_{L^p(\Omega, \mathbb{P}, L^q(0, T, X))} = \left(\mathbb{E} \int_0^T (\|u(t)\|_X^q dt)^{p/q} \right)^{1/p} < \infty.$$

If $q = \infty$, we write

$$\|u\|_{L^p(\Omega, \mathbb{P}; L^\infty(0, T, X))} = \left(\mathbb{E} \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X^p \right)^{1/p} < \infty.$$

We now define the concept of weak martingale solution of the problem (25) as follows.

Definition 2.2. A weak martingale solution of (25) is a system $\{(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t \in [0, T]}, W, u, B\}$ where

- (1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space,
- (2) $(\mathcal{F}_t)_{t \in [0, T]}$ is a filtration satisfying the usual condition on $(\Omega, \mathcal{F}, \mathbb{P})$,
- (3) W is a \mathcal{F}_t -adapted \mathbb{R}^d -valued Wiener process,
- (4) $u \in L^p(\Omega, \mathbb{P}; L^\infty(0, T, V)) \cap L^p(\Omega, \mathbb{P}; L^2(0, T, D(A)))$,
 $B \in L^p(\Omega, \mathbb{P}; L^\infty(0, T, H)) \cap L^p(\Omega, \mathbb{P}; L^2(0, T, V))$, for every $p \in [1, \infty)$
- (5) for all $(w, \zeta) \in D(A) \times V$,

$$\begin{cases} (v(t), w) - (v_0, w) + \int_0^t \left(\tilde{\mathbb{B}}(u, v) - \mathbb{B}(B, B), w \right)_{D(A)'} + v(v, Aw) \, ds \\ = \int_0^t (f_1(u, B, s), w) \, ds + \int_0^t (g_1(u, B, s), w) \, dW(s), \\ v(t) = u(t) + \alpha^2 Au(t), \\ v_0 = u_0 + \alpha^2 Au_0, \end{cases} \tag{26}$$

and

$$\begin{cases} (B(t), \zeta) - (B_0, \zeta) + \int_0^t \left(\mathbb{B}(u, B) - \mathbb{B}(B, u), \zeta + \eta((B, \zeta)) \right) \, ds \\ = \int_0^t (f_2(u, B, s), \zeta) + \int_0^t (g_2(u, B, s), \zeta) \, dW(s) \end{cases} \tag{27}$$

hold $dt \otimes d\mathbb{P}$ -almost everywhere.

- (6) The functions $u(t)$ and $B(t)$ takes values in H and are continuous with respect to t \mathbb{P} -almost surely.

The point (6) of our definition can be justified as follows. Owing to Lemma 2.1, point (4) of Definition 2.2 and (26)–(27) $B(t)$ can be written in the following form

$$B(t) = B_0 + \int_0^t G(s) \, ds + \int_0^t S(s) \, W(s), \quad t \in [0, T],$$

where $G \in L^2(\Omega \times [0, T]; D(A)')$ and $S \in L^2(\Omega \times [0, T]; H)$. Now it follows from [29, Chapter I, Theorem 3.2] that there exists $\Omega^* \in \mathcal{F}$ such that $\mathbb{P}(\Omega^*) = 1$ and for $\omega \in \Omega^*$ the function B takes values in H , is continuous in H with respect to t . The same argument applies for u .

The first main result of this paper is given in the next statement.

Theorem 2.3. Assume that $u_0 \in V, B_0 \in H$ and assumptions (A1) are satisfied. Then for any $\alpha > 0$, there exist probabilistic weak solutions $\{(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t \in [0, T]}, W, u, B\}$ of

problem (25) such that the following estimates hold for any $1 \leq p < \infty$:

$$\mathbb{E} \sup_{0 \leq t \leq T} \left(|u(t)|^2 + \alpha^2 \|u(t)\|^2 \right)^{p/2} \leq C_{p,1}, \tag{28}$$

$$\mathbb{E} \left(\int_0^T \left[2\nu \|u_\alpha(s)\|^2 + 2\nu\alpha^2 |Au(s)|^2 \right] ds \right)^{p/2} \leq C_{p,2}, \tag{29}$$

$$\mathbb{E} \sup_{0 \leq t \leq T} |B(t)|^p \leq C_{p,3}, \tag{30}$$

$$\mathbb{E} \left(\int_0^T \|B(s)\|^2 ds \right)^{p/2} \leq C_{p,4}, \tag{31}$$

$$\mathbb{E} \sup_{0 \leq |\theta| \leq \delta \leq 1} \int_0^T |u(t + \theta) - u(t)|^2 dt \leq C_5(\alpha)\delta, \tag{32}$$

$$\mathbb{E} \sup_{0 \leq |\theta| \leq \delta \leq 1} \int_0^T |B(t + \theta) - B(t)|_V^2 dt \leq C_6(\alpha)\delta. \tag{33}$$

Here the constants $(C_{p,i})_{i=1,\dots,4}$ are independent of α , while $C_5(\alpha), C_6(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$. \mathbb{E} is the mathematical expectation with respect to \mathbb{P} .

3. Galerkin approximations and a priori estimates

In this section we introduce the Galerkin approximation of our problem and derive some uniform a priori estimates for the approximating solutions.

3.1. The approximate equations

Let $\{w_j : j = 1, 2, 3 \dots\}$ be an orthonormal basis of H which was already introduced in the previous section. Let us set $H_m = \text{Span}\{w_1, w_2, w_3, \dots, w_m\}$ and let P_m be the L^2 -orthogonal projection from H onto H_m . We look for a sequence of pairs $(u_m; B_m)$ in $H_m^{\otimes 2}$ solutions of the following system of stochastic differential equations

$$\begin{cases} dv_m + P_m[\mathbb{B}(u_m, v_m) + \nu Av_m - \mathbb{B}(B_m, B_m)]dt = P_m f_{m1}dt + P_m g_{m1}d\bar{W} \\ dB_m + P_m[\mathbb{B}(u_m, B_m) - \mathbb{B}(B_m, u_m) + \eta AB_m]dt = P_m f_{m2}dt + P_m g_{m2}d\bar{W}, \\ v_m = u_m + \alpha^2 Au_m \\ u_m(0) = P_m u_0 \\ B_m(0) = P_m B_0, \end{cases} \tag{34}$$

defined on a fixed stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}; (\bar{\mathcal{F}}_t)_{t \in [0, T]}, \bar{\mathbb{P}}, \bar{W})$. The mathematical expectation with respect (wrt) to $\bar{\mathbb{P}}$ is denoted by $\bar{\mathbb{E}}$. Here and throughout we set

$$\begin{aligned} f_{mi} &= f_i(u_m, B_m, t), \\ g_{mi} &= g_i(u_m, B_m, t), \end{aligned}$$

for $i = 1, 2, t \in [0, T]$ and $m = 1, 2, \dots$. By the theory of stochastic differential equations (see, for instance, [24]) there is a local solution $(u_m; B_m)$ defined on $[0, T_m]$. The following a priori estimates will enable us to prove that $T_m = T$.

3.2. A priori estimates

Throughout this section C will denote a positive constant independent of α and m which may change from one term to the next.

Lemma 3.1. *The couple $(u_m; B_m)$ satisfies the estimates*

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} (\|u_m(s)\|_V^2 + |B_m(s)|^2) \\ & + 2\mathbb{E} \int_0^T \left(\nu \left[\|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right] + \eta \|B_m(s)\|^2 \right) ds \leq C. \end{aligned} \tag{35}$$

Proof. It follows from (34) that

$$\begin{aligned} & du_m + (I + \alpha^2 A)^{-1} P_m [\tilde{\mathbb{B}}(u_m, v_m) - \mathbb{B}(B_m, B_m)] dt + Au_m dt \\ & = (I + \alpha^2 A)^{-1} P_m f_{m1} dt + (I + \alpha^2 A)^{-1} P_m g_{m1} d\bar{W}, \end{aligned} \tag{36}$$

and

$$dB_m + P_m [\mathbb{B}(u_m, B_m) - \mathbb{B}(B_m, u_m) + \eta AB_m] dt = P_m f_{m2} dt + P_m g_{m2} d\bar{W}. \tag{37}$$

Thanks to Itô's formula with $\|u_m\|_V^2$ and $|B_m|^2$ we have

$$\begin{aligned} & d\|u_m\|_V^2 + 2 \left\{ \left((\nu Au_m + (I + \alpha^2 A)^{-1} P_m [\tilde{\mathbb{B}}(u_m, v_m) - \mathbb{B}(B_m, B_m)], u_m) \right)_V \right\} dt \\ & = 2 \left((I + \alpha^2 A)^{-1} f_{m1}, u_m \right)_V dt + \|(I + \alpha^2 A)^{-1} P_m g_{m1}\|_V^2 dt \\ & + 2 \left((I + \alpha^2 A)^{-1} g_{m1}, u_m \right)_V d\bar{W} \end{aligned} \tag{38}$$

and

$$\begin{aligned} & d|B_m|^2 + 2 \{ (\eta AB_m + \mathbb{B}(u_m, B_m) - \mathbb{B}(B_m, u_m), u_m) \} dt \\ & = 2(f_{m2}, B_m) dt + |g_{m2}|^2 dt + (g_{m2}, B_m) d\bar{W}. \end{aligned} \tag{39}$$

In view of relation (6) and the definition of $((\cdot, \cdot))_V$, we see that

$$((Au_m, u_m))_V = \|u_m\|^2 + \alpha^2 |Au_m|^2.$$

Thanks to (6) and (14),

$$(((I + \alpha^2 A)^{-1} P_m \tilde{\mathbb{B}}(u_m, v_m), u_m))_V = (\tilde{\mathbb{B}}(u_m, v_m), u_m) = 0.$$

Also,

$$(((I + \alpha^2 A)^{-1} P_m \mathbb{B}(B_m, B_m), u_m))_V = (\mathbb{B}(B_m, B_m), u_m).$$

By (6) and (7) the following hold

$$\begin{aligned} & (((I + \alpha^2 A)^{-1} P_m f_{m1}, u_m))_V = (f_{m1}, u_m) \\ & (((I + \alpha^2 A)^{-1} P_m g_{m1}, u_m))_V = (g_{m1}, u_m), \\ & \|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 \leq |g_{m1}|^2. \end{aligned}$$

The above inequalities imply that

$$\begin{aligned}
 & d\|u_m\|_V^2 + 2 \left(v[\|u_m\|^2 + \alpha^2|Au_m|^2] - (\mathbb{B}(B_m, B_m), u_m) \right) dt \\
 & \quad - 2((I + \alpha^2 A)^{-1} g_{m1}, u_m) d\bar{W} \\
 & = 2(f_{m1}, u_m) dt + \|(I + \alpha^2 A)^{-1} P_m g_{m1}\|_V^2 dt.
 \end{aligned} \tag{40}$$

Using (11) and (12) it follows from (39) that

$$\begin{aligned}
 & d|B_m|^2 + 2 \left(\eta\|B_m\|^2 + (\mathbb{B}(B_m, B_m), u_m) \right) dt = 2(f_{m2}, B_m) dt + |g_{m2}|^2 dt \\
 & \quad + (g_{m2}, B_m) d\bar{W}.
 \end{aligned} \tag{41}$$

Now summing up (40) and (41) yields

$$\begin{aligned}
 & d(\|u_m\|_V^2 + |B_m|^2) + 2 \left[v(\|u_m\|^2 + \alpha^2|Au_m|^2) + \eta\|B_m\|^2 \right] dt \\
 & \quad - 2[(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W} \\
 & = 2 \left[(f_{m1}, u_m) + (f_{m2}, B_m) + (1/2)\|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 + (1/2)|g_{m2}|^2 \right] dt.
 \end{aligned} \tag{42}$$

Let

$$\sigma_{mi} = \begin{cases} u_m & \text{if } i = 1 \\ B_m & \text{if } i = 2, \end{cases}$$

and χ_{mi} be either f_{mi} or g_{mi} , $i = 1, 2$. Thanks to the assumptions (A1) and the fact that

$$|\cdot|^2 \leq \|\cdot\|_V^2, \tag{43}$$

we have

$$(\chi_{mi}, \sigma_{mi}) \leq |\chi_{mi}| |\sigma_{mi}| \leq C(1 + \|u_m\|_V^2 + |B_m|^2), \tag{44}$$

and

$$|g_{m2}|^2 \leq C(1 + |B_m|^2 + \|u_m\|_V^2). \tag{45}$$

In view of (7) and (23), there holds

$$\|(I + \alpha^2 A)^{-1} f_{m1}\|_V^2 \leq C(1 + \|u_m\|_V^2 + |B_m|^2). \tag{46}$$

It follows from these and (42) that

$$\begin{aligned}
 & \|u_m(t)\|_V^2 + |B_m(t)|^2 + 2 \int_0^t \left[v(\|u_m\|^2 + \alpha^2|Au_m|^2) + \eta\|B_m\|^2 \right] ds \\
 & \leq \|u_m(0)\|_V^2 + |B_m(0)|^2 + CT + \int_0^t (\|u_m\|_V^2 + |B_m|^2) ds \\
 & \quad + 2 \left| \int_0^t [(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W} \right|.
 \end{aligned} \tag{47}$$

Let us set

$$\gamma_m = 2 \left| \int_0^t [(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W} \right|.$$

By invoking Burkholder–Davis–Gundy’s inequality we get that

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq s \leq t} \gamma_m &\leq 6\bar{\mathbb{E}} \left(\int_0^t [(g_{m1}, u_m) + (g_{m2}, B_m)]^2 ds \right)^{1/2} \\ &\leq 3\bar{\mathbb{E}} \left(\int_0^t [|g_{m1}|^2 |u_m|^2 + |g_{m2}|^2 |B_m|^2] ds \right)^{1/2}. \end{aligned}$$

Using (24) and (43), we obtain

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq s \leq t} \gamma_m &\leq 3C\bar{E} \left(\int_0^t [(1 + \|u_m\|_V^2 + |B_m|^2)(\|u_m\|_V^2 + |B_m|^2)] ds \right)^{1/2}, \\ &\leq \frac{1}{2}\bar{\mathbb{E}} \sup_{0 \leq s \leq t} (\|u_m\|_V^2 + |B_m|^2) + CT + C\bar{\mathbb{E}} \int_0^t (\|u_m\|_V^2 + |B_m|^2) ds. \end{aligned} \tag{48}$$

Taking the supremum, then the mathematical expectation and using (48) we infer from (47) that

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq s \leq t} (\|u_m(s)\|_V^2 + |B_m(s)|^2) + 2\bar{\mathbb{E}} \int_0^t [\nu(\|u_m\|^2 + \alpha^2 |Au_m|^2) + \eta \|B_m\|^2] ds \\ \leq CT + C\bar{\mathbb{E}} \int_0^t [\|u_m\|_V^2 + |B_m|^2] ds. \end{aligned} \tag{49}$$

Dropping off the second term in the left hand side of the last estimate and invoking Gronwall’s lemma yield

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq t} (\|u_m(s)\|_V^2 + |B_m(s)|^2) \leq C.$$

We deduce from (49) that

$$\bar{\mathbb{E}} \int_0^t [\nu(\|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2) + \eta \|B_m(s)\|^2] ds \leq C.$$

These last two equations conclude the proof of the lemma. \square

The following result is related to the higher integrability of u_m and B_m .

Lemma 3.2. *The couple $(u_m; B_m)$ satisfies the following estimates:*

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq T} (\|u_m(s)\|_V^2 + |B_m(s)|^2)^{p/2} \leq C, \tag{50}$$

$$\bar{\mathbb{E}} \left(\int_0^T 2 [\nu(\|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2) + \eta \|B_m(t)\|^2] dt \right)^{p/2} \leq C, \tag{51}$$

for all $1 \leq p < \infty$.

Proof. To simplify the notations let us set $\phi_m = \|u_m\|_V^2 + |B_m|^2$. From (42), we have

$$\begin{aligned} d\phi_m + 2 \left(\nu [\|u_m\|^2 + \alpha^2 |Au_m|^2] + \eta \|B_m\|^2 \right) dt \\ = 2 \left[(f_{m1}, u_m) + (f_{m2}, B_m) + (1/2) \|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 + (1/2) |g_{m2}|^2 \right] dt \\ + 2 [(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W}. \end{aligned} \tag{52}$$

By Itô’s formula,

$$d(\phi_m)^{\frac{p}{2}} = (p/2)d\phi_m + 4(p/2)(p - 2/2)(\phi_m)^{\frac{p-4}{2}} [(g_{m1}, u_m) + (g_{m2}, B_m)]^2 \times dt. \tag{53}$$

By setting $\chi_m = [\nu(\|u_m\|^2 + \alpha^2|Au_m|^2) + \eta\|B_m\|^2]$, we see from the last equation that

$$\begin{aligned} & d(\phi_m)^{\frac{p}{2}} + p(\phi_m)^{\frac{p-2}{2}} \chi_m \\ &= \frac{p}{2}(\phi_m)^{\frac{p-2}{2}} \left\{ 2(f_{m2}, B_m) + \|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 + |g_{m2}|^2 \right\} dt \\ &+ p(\phi_m)^{\frac{p-2}{2}} (f_{m1}, u_m) dt + p(p - 2)(\phi_m)^{\frac{p-4}{2}} [(g_{m1}, u_m) \\ &+ (g_{m2}, B_m)]^2 dt + p(\phi_m)^{\frac{p-2}{2}} [(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W}. \end{aligned} \tag{54}$$

We are going to estimate each term in (54) as follows:

$$\begin{aligned} \|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 + |g_{m2}|^2 &\leq |g_{m1}|^2 + |g_{m2}|^2, \quad (\text{by (7)}) \\ &\leq c(1 + |u_m|^2 + |B_m|^2), \quad (\text{by (24)}) \\ &\leq C(1 + \phi_m), \quad (\text{by (43)}). \end{aligned}$$

Cauchy–Schwarz’s, Cauchy’s inequalities together with (23) and this last estimate imply that

$$\begin{aligned} & \frac{p}{2}(\phi_m)^{\frac{p-2}{2}} \left\{ 2(f_{m1}, u_m) + 2(f_{m2}, B_m) + \|(I + \alpha^2 A)^{-1} g_{m1}\|_V^2 + |g_{m2}|^2 \right\} \\ &\leq C(\phi_m)^{\frac{p}{2}-1} + C(\phi_m)^{\frac{p}{2}}, \\ &\leq C(\phi_m)^{\frac{p}{2}} \quad (\text{by Young’s inequality}). \end{aligned}$$

It follows from Cauchy–Schwarz’s, Cauchy’s inequalities and (24) that

$$\begin{aligned} p(p - 2)(\phi_m)^{\frac{p-4}{2}} [(g_{m1}, u_m) + (g_{m2}, B_m)]^2 &\leq C(\phi_m)^{\frac{p-4}{2}} \\ &\quad \times \left(|u_m| + |u_m|^2 + |B_m| + |B_m|^2 \right)^2, \\ &\leq C(\phi_m)^{\frac{p-4}{2}} (\phi_m + \|u_m\|_V + |B_m|)^2, \\ &\leq 2C(\phi_m)^{\frac{p}{2}} + 4C(\phi_m)^{\frac{p-2}{2}}, \\ &\leq C(\phi_m)^{\frac{p}{2}} \quad (\text{by Young’s inequality}). \end{aligned}$$

For $\gamma_m = p \left| \int_0^s (\phi_m)^{\frac{p-2}{2}} [(g_{m1}, u_m) + (g_{m2}, B_m)] d\bar{W} \right|$, we see from Burkholder–Davis–Gundy’s inequality that

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq s \leq t} &\leq C(p) \bar{\mathbb{E}} \left(\int_0^t (\phi_m)^{p-2} [(g_{m1}, u_m) + (g_{m2}, B_m)]^2 ds \right)^{1/2}, \\ &\leq C(p) \bar{\mathbb{E}} \left(\int_0^t (\phi_m)^{p-2} [(g_{m1}, u_m)^2 + (g_{m2}, B_m)^2] ds \right)^{1/2}, \end{aligned}$$

for all $t \in [0, T]$. Arguing as before, we obtain

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq t} \gamma_m \leq \varepsilon \bar{\mathbb{E}}(\phi_m)^{p/2} + C_\varepsilon \bar{\mathbb{E}} \int_0^t (\phi_m)^{p/2} ds, \quad \text{for any } t \in [0, T].$$

Taking the supremum, the mathematical expectation in (54) and collecting all these inequalities, we have that

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq t} (\phi_m)^{p/2} \leq C + C \bar{\mathbb{E}} \int_0^t (\phi_m)^{p/2} ds,$$

which together with Gronwall’s lemma imply

$$\bar{\mathbb{E}} \sup_{0 \leq s \leq t} \left(\|u_m\|_V^2 + |B_m|^2 \right)^{p/2} \leq C, \quad \text{for any } t \in [0, T]. \tag{55}$$

We derive from (55) and (42) that

$$\bar{\mathbb{E}} \left(\int_0^T \left[\nu (\|u_m\|^2 + \alpha^2 |Au_m|^2) + \eta \|B_m\|^2 \right] ds \right)^{p/2} \leq C. \tag{56}$$

The lemma follows from (55) and (56). \square

The next estimates are very important for the proof of the tightness of the law of the Galerkin solution $(u_m; B_m)$.

Lemma 3.3. *There exists a positive constant $C(\alpha)$ such that $C(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$ and the following inequalities hold*

$$\bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta} \int_0^T |u_m(t + \theta) - u_m(t)|^2 dt \leq C(\alpha)\delta, \tag{57}$$

$$\bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta} \int_0^T |B_m(t + \theta) - B_m(t)|_V^2 dt \leq C(\alpha)\delta, \tag{58}$$

for any $0 < \delta \leq 1$.

Remark 3.4. In the above lemma, u_m and B_m are extended to 0 outside $[0, T]$.

Proof. We can infer from (34) that

$$\begin{aligned} & |v_m(t + \theta) - v_m(t)|_{D(A)'}^2 \\ & \leq 2 \left| \int_t^{t+\theta} \left[P_m (\tilde{\mathbb{B}}(u_m, v_m) + \nu Av_m + \mathbb{B}(B_m, B_m) + f_{m1}) \right] ds \right|_{D(A)'}^2 \\ & \quad + 2 \left| \int_t^{t+\theta} P_m g_{m1} d\bar{W} \right|^2, \end{aligned}$$

which implies

$$\begin{aligned} |v_m(t + \theta) - v_m(t)|_{D(A)'}^2 & \leq 8\theta \int_t^{t+\theta} \left[|P_m \tilde{\mathbb{B}}(u_m, v_m)|_{D(A)'}^2 + \nu^2 |Av_m|_{D(A)'}^2 \right] ds \\ & \quad + \int_t^{t+\theta} |P_m \mathbb{B}(B_m, B_m)|_{D(A)'}^2 ds + \int_t^{t+\theta} |P_m f_{m1}|^2 ds \\ & \quad + 2 \left| \int_t^{t+\theta} P_m g_{m1} d\bar{W} \right|^2. \end{aligned} \tag{59}$$

We have

$$\begin{aligned} |Av_m|_{D(A)'}^2 &= |v_m|^2, \\ |P_m \tilde{\mathbb{B}} u_m v_m|_{D(A)'}^2 &\leq c \|u_m\|^2 |v_m|^2, \\ |P_m \mathbb{B}(B_m, B_m)|_{D(A)'}^2 &\leq c |B_m|^2 \|B_m\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} v^2 |Av_m|^2 ds dt &\leq \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |v_m|^2 ds dt, \\ &\leq \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} (|u_m|^2 + \alpha^4 |Au_m|^2) ds dt, \\ &\leq \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \|u_m\|_V^2 ds dt \\ &\quad + \alpha^2 \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \alpha^2 |Au_m|^2 ds dt, \\ &\leq C\delta + \alpha^2 C. \end{aligned} \tag{60}$$

Also,

$$\bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |P_m \tilde{\mathbb{B}}(u_m, v_m)|_{D(A)'}^2 ds dt \leq c \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \|u_m\|^2 |v_m|^2 ds dt.$$

Hence

$$\begin{aligned} \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |P_m \tilde{\mathbb{B}}(u_m, v_m)|_{D(A)'}^2 ds dt &\leq \delta C \bar{\mathbb{E}} \sup_{0 \leq t \leq T} \|u_m\|^4 \\ &\quad + \delta \bar{\mathbb{E}} \left(\int_0^T (|u_m|^2 + \alpha^4 |Au_m|^2) dt \right)^2, \\ &\leq \frac{C\delta}{\alpha^4} + C\delta \bar{\mathbb{E}} \sup_{-\leq t \leq T} |u_m|^4 \\ &\quad + \alpha^4 \delta \bar{\mathbb{E}} \left(\int_0^T \alpha^2 |Au_m|^2 dt \right)^2. \end{aligned}$$

It follows from the last estimate and (51) that

$$\bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |P_m \tilde{\mathbb{B}}(u_m, v_m)|_{D(A)'}^2 ds dt \leq \frac{C\delta}{\alpha^4} + C\delta + \alpha^4 C\delta. \tag{61}$$

Similarly,

$$\begin{aligned} \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |P_m \mathbb{B}(B_m, B_m)|_{D(A)'}^2 ds dt &\leq c \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \|B_m\|^2 |B_m|^2 ds dt, \\ &\leq C\delta \bar{\mathbb{E}} \sup_{0 \leq t \leq T} |B_m|^2 + c\delta \bar{\mathbb{E}} \left(\int_0^T \|B_m\|^2 dt \right)^2, \\ &\leq C\delta + C\delta \quad (\text{by (50) and (51)}). \end{aligned} \tag{62}$$

In view of (23), we have

$$\begin{aligned} \delta \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |P_m f_{m1}|^2 ds dt &\leq C \delta \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} (1 + |u_m|^2 + |B_m|^2) ds dt, \\ &\leq C \delta^2 T + C \delta^2 \bar{\mathbb{E}} \sup_{0 \leq t \leq T} |u_m|^2 + c \delta^2 \bar{\mathbb{E}} \sup_{0 \leq t \leq T} |B_m|^2, \\ &\leq C \delta^2 \quad (\text{by (50)}). \end{aligned} \tag{63}$$

By Burkholder–Davis–Gundy’s inequality, (24) and by arguing as before we show that

$$\bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta} \left| \int_0^T \int_t^{t+\theta} P_m g_{m1} d\bar{W} \right|^2 dt \leq c \delta. \tag{64}$$

It follows from (60)–(64) and (59) that

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta \leq 1} \int_0^T |v_m(t + \theta) - v_m(t)|_{D(A)}^2 dt &\leq C \delta^2 + C \delta + \alpha^2 C \delta + \alpha^4 C \delta + \frac{C \delta}{\alpha^4}, \\ &\leq C(\alpha) \delta, \end{aligned} \tag{65}$$

where $C(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$. Since

$$\alpha^4 |u_m(t + \theta) - u_m(t)|^2 \leq |A^{-1}(v_m(t + \theta) - v_m(t))|^2 = |v_m(t + \theta) - v_m(t)|_{D(A)'}^2,$$

we deduce from (65) that

$$\bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta \leq 1} \int_0^T |u_m(t + \theta) - u_m(t)|^2 dt \leq C(\alpha) \delta,$$

where $C(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$. To complete the proof of (57), we can use the same argument to prove a similar estimate for the case $\theta \leq 0$.

We derive from (34) that

$$\begin{aligned} |B_m(t + \theta) - B_m(t)|_{V'}^2 &\leq C \theta \left(\int_t^{t+\theta} \left[|P_m \mathbb{B}(B_m, u_m)|_{V'}^2 + |P_m \mathbb{B}(u_m, B_m)|_{V'}^2 \right] ds \right) \\ &\quad + C \theta \left(\int_t^{t+\theta} \left[\eta |AB_m|_{V'}^2 + |P_m f_{m2}|^2 \right] ds \right) \\ &\quad + \left| \int_t^{t+\theta} P_m g_{m2} d\bar{W} \right|^2. \end{aligned} \tag{66}$$

It is clear that

$$\bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} |AB_m|_{V'}^2 ds dt \leq \delta \bar{\mathbb{E}} \int_0^T \|B_m\|^2 dt \leq C \delta. \tag{67}$$

Due to the properties of P_m and \mathbb{B} , we have

$$\bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \left[|P_m \mathbb{B}(B_m, u_m)|_{V'}^2 + |P_m \mathbb{B}(u_m, B_m)|_{V'}^2 \right] ds dt \tag{68}$$

$$\leq 2C \bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \|u_m\|^2 \|B_m\|^2 ds dt \tag{69}$$

$$\leq C\delta\bar{\mathbb{E}} \sup_{0 \leq t \leq T} \|u_m\|^4 + C\bar{\mathbb{E}} \left(\int_0^T \|B_m\|^2 ds \right)^2. \tag{70}$$

Thanks to (50) and (51), we obtain

$$\bar{\mathbb{E}} \int_0^T \int_t^{t+\delta} \left[|P_m \mathbb{B}(B_m, u_m)|_{V'}^2 + |P_m \mathbb{B}(u_m, B_m)|_{V'}^2 \right] ds dt \leq \frac{C\delta}{\alpha^4} + C. \tag{71}$$

By similar argument as used before we prove that

$$\bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta \leq 1} \int_0^T \left| \int_t^{t+\theta} P_m g_{m2} d\bar{W} \right|^2 \leq C\delta. \tag{72}$$

Using (67)–(72), we get from (66) that

$$\begin{aligned} \bar{\mathbb{E}} \sup_{0 \leq \theta \leq \delta \leq 1} \int_0^T |B_m(t + \theta) - B_m(t)|_{V'}^2 dt &\leq C\delta^2 + C\delta + \frac{C\delta}{\alpha^4}, \\ &\leq C(\alpha)\delta, \end{aligned}$$

where $C(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$. Arguing as before concludes the proof of (58). \square

4. Tightness property of the probability measures induced by the Galerkin solutions and application of Prokhorov’s and Skorokhod’s theorems

We consider the space $\mathfrak{S} = L^2(0, T; V) \times L^2(0, T; H) \times C(0, T; \mathbb{R}^d)$ equipped with the Borel σ -algebra $\mathcal{B}(\mathfrak{S})$. We denote by Φ the measurable \mathfrak{S} -valued defined on $(\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})$ by

$$\Phi(\bar{\omega}) = (u_m(\bar{\omega}), B_m(\bar{\omega}), \bar{W}(\bar{\omega})).$$

We introduce a probability measure Π_m on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ by

$$\Pi_m(S) = \bar{\mathbb{P}}(\Phi^{-1}(S)), \quad S \in \mathcal{B}(\mathfrak{S}).$$

The next proposition, which can be proved by following the lines of the proof of Theorem 4.6 in [18] (see also [5]), is a result about the tightness of Π_m .

Proposition 4.1. *The family of probability measures $\{\Pi_m : m = 1, 2, 3, \dots\}$ is tight in \mathfrak{S} .*

From the tightness property of $\{\Pi_m : m = 1, 2, 3, \dots\}$ and Prokhorov’s theorem, there exists a subsequence Π_{m_j} and a probability measure Π such that Π_{m_j} weakly converges to Π . By Skorokhod’s embedding theorem, there exists a new probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $(u_{m_j}, B_{m_j}, W_{m_j}), (u, B, W)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in \mathfrak{S} such that

$$\text{the law of } (u_{m_j}, B_{m_j}, W_{m_j}) \text{ is } \Pi_{m_j}, \tag{73}$$

$$\text{the law of } (u, B, W) \text{ is } \Pi, \tag{74}$$

$$(u_{m_j}, B_{m_j}, W_{m_j}) \rightarrow (u, B, W) \text{ strongly in } \mathfrak{S} \text{ } \mathbb{P}\text{-almost surely.} \tag{75}$$

We can see that $\{W_{m_j} : m_j = 1, 2, \dots\}$ is a sequence of d -dimensional standard Brownian Motions. Let

$$\mathcal{F}_t = \sigma \{(u(s), B(s), W(s)) : 0 \leq s \leq t\}.$$

By same argument as in [3], we can prove that W is an d -dimensional \mathcal{F}_t -standard Wiener process. We also show by using the idea in [5] that $(u_{m_j}, B_{m_j}, W_{m_j})$ verifies the following $d\mathbb{P} \otimes dt$ -almost everywhere:

$$\begin{aligned} &(v_{m_j}(t), w) - (v_{m_j}(0), w) + \int_0^t \left(\langle P_{m_j} \tilde{\mathbb{B}}(u_{m_j}, v_{m_j}), w \rangle_{D(A)'} + \nu(v_{m_j}, Aw) \right) ds \\ &= \int_0^t \left[\langle P_{m_j} \mathbb{B}(B_{m_j}, B_{m_j}), w \rangle_{D(A)'} + \langle P_{m_j} f_{m_j1}, w \rangle \right] ds \\ &\quad + \int_0^t \langle P_{m_j} g_{m_j1}, w \rangle dW_{m_j}(s), \end{aligned} \tag{76}$$

and

$$\begin{aligned} &(B_{m_j}(t), \zeta) - (B_{m_j}(0), \zeta) + \int_0^t \left(\langle P_{m_j} \mathbb{B}(u_{m_j}, B_{m_j}), \zeta \rangle + \eta((B_{m_j}, \zeta)) \right) ds \\ &= \int_0^t \left[\langle P_{m_j} \mathbb{B}(B_{m_j}, u_{m_j}), \zeta \rangle + \langle P_{m_j} f_{m_j2}, \zeta \rangle \right] ds + \int_0^t \langle g_{m_j2}, \zeta \rangle dW_{m_j}(s), \end{aligned} \tag{77}$$

for all $(w, \zeta) \in D(A) \times V$.

5. Proof of Theorem 2.3

5.1. Passage to the limit

To complete the proof of Theorem 2.3, we need to pass to the limit in (76) and (77). For that purpose we will pass first to the limit in each term in (76). This procedure is easy for the linear terms so we will just give the details for the nonlinear ones. The couple $(u_{m_j}; B_{m_j})$ satisfies (76)–(77), therefore the following estimates are valid for all $1 \leq p < \infty$

$$\mathbb{E} \sup_{0 \leq s \leq T} (\|u_{m_j}\|_V^2 + |B_{m_j}|^2)^{p/2} \leq C, \tag{78}$$

$$\mathbb{E} \left(\int_0^T 2 \left[\nu(\|u_{m_j}\|^2 + \alpha^2 |Au_{m_j}|^2) + \eta \|B_{m_j}\|^2 \right] dt \right)^{p/2} \leq C, \tag{79}$$

and

$$\mathbb{E} \sup_{0 \leq \theta | \leq \delta \leq 1} \int_0^T |u_{m_j}(t + \theta) - u_{m_j}(t)|^2 dt \leq C(\alpha)\delta, \tag{80}$$

$$\mathbb{E} \sup_{0 \leq \theta | \leq \delta \leq 1} \int_0^T |B_{m_j}(t + \theta) - B_{m_j}(t)|_V^2 dt \leq C(\alpha)\delta, \tag{81}$$

for any $0 < \delta \leq 1$. Here $C(\alpha) \rightarrow \infty$ as α approaches zero. Therefore we can extract from $(u_{m_j}; B_{m_j})$ a subsequence still denoted with the same fashion and a couple $(u; B)$ such that

$$u_{m_j} \rightharpoonup u \quad \text{weakly star in } L^p(\Omega, \mathbb{P}; L^\infty(0, T; V)), \tag{82}$$

$$u_{m_j} \rightharpoonup u \quad \text{weakly in } L^p(\Omega, \mathbb{P}; L^2(0, T; D(A))), \tag{83}$$

$$B_{m_j} \rightharpoonup B \quad \text{weakly star in } L^p(\Omega, \mathbb{P}; L^\infty(0, T; H)), \tag{84}$$

$$B_{m_j} \rightharpoonup B \quad \text{weakly in } L^p(\Omega, \mathbb{P}; L^2(0, T; V)). \tag{85}$$

From (75) we obtain that

$$u_{m_j} \rightarrow u \quad \text{strongly in } L^2(0, T; V), \mathbb{P}\text{-a.s.}, \tag{86}$$

$$B_{m_j} \rightarrow u \quad \text{strongly in } L^2(0, T; H), \mathbb{P}\text{-a.s.} \tag{87}$$

Let us consider the positive nondecreasing function $\phi(x) = x^4$, defined on \mathbb{R}_+ . The function ϕ obviously satisfies

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x} = \infty.$$

Thanks to the estimates (78)–(79) we have

$$\sup_{m_j \geq 1} \mathbb{E}(\phi(\|u_{m_j}\|_{L^2(0,T;V)}^2)) < \infty,$$

$$\sup_{m_j \geq 1} \mathbb{E}(\phi(\|B_{m_j}\|_{L^2(0,T;H)}^2)) < \infty.$$

Thanks to uniform integrability criteria in [28, Chapter 3, Exercise 6] we see that $\|u_{m_j}\|_{L^2(0,T;V)}^2$ and $\|B_{m_j}\|_{L^2(0,T;V)}^2$ is uniform integrable with respect to the probability measure. Hence thanks to (86)–(87) and Vitali’s convergence theorem (see, for instance, [28, Chapter 3, Proposition 3.2]) we have

$$u_{m_j} \rightarrow u \quad \text{strongly in } L^2(\Omega, \mathbb{P}; L^2(0, T; V)), \tag{88}$$

$$B_{m_j} \rightarrow B \quad \text{strongly in } L^2(\Omega, \mathbb{P}; L^2(0, T; H)). \tag{89}$$

It follows from these facts that we can extract again from $(u_{m_j}; B_{m_j})$ a subsequence still denoted by the same symbols such that

$$u_{m_j} \rightarrow u \quad \text{almost everywhere } d\mathbb{P} \otimes dt \text{ in } V \text{ and } H, \tag{90}$$

$$B_{m_j} \rightarrow u \quad \text{almost everywhere } d\mathbb{P} \otimes dt \text{ in } H. \tag{91}$$

For any $w \in D(A)$ we have

$$\left| \mathbb{E} \int_0^t \left[\langle \tilde{\mathbb{B}}(u_{m_j}, v_{m_j}), P_{m_j} w \rangle - \langle \tilde{\mathbb{B}}(u, v), w \rangle_{D(A)'} \right] ds \right| \leq I_{m_j}^{(1)} + I_{m_j}^{(2)} + I_{m_j}^{(3)},$$

where

$$I_{m_j}^{(1)} = \left| \mathbb{E} \int_0^t \langle \tilde{\mathbb{B}}(u_{m_j}, v_{m_j}), P_{m_j} w - w \rangle_{D(A)'} ds \right|$$

$$I_{m_j}^{(2)} = \left| \mathbb{E} \int_0^t \langle \tilde{\mathbb{B}}(u_{m_j} - u, v_{m_j}), w \rangle_{D(A)'} ds \right|$$

$$I_{m_j}^{(3)} = \left| \mathbb{E} \int_0^t \langle \tilde{\mathbb{B}}(u, v_{m_j} - v), w \rangle_{D(A)'} ds \right|.$$

Note first that for any $\zeta \in H$,

$$|P_{m_j} \zeta - \zeta| \rightarrow 0 \quad \text{as } m_j \rightarrow \infty. \tag{92}$$

By (21),

$$\begin{aligned}
 I_{m_j}^{(1)} &\leq c \mathbb{E} \int_0^t \|u_{m_j}\| |v_{m_j}| |P_{m_j}Aw - Aw| ds \\
 &\leq c |P_{m_j}Aw - Aw| \left[\mathbb{E} \sup_{0 \leq t \leq T} \|u_{m_j}\|^2 + \mathbb{E} \int_0^T |v_{m_j}|^2 ds \right].
 \end{aligned}$$

In view of (78), (79) and (92) we see from the last estimate that $I_{m_j}^{(1)} \rightarrow 0$ as $m_j \rightarrow \infty$. Again by (21)

$$\begin{aligned}
 I_{m_j}^{(2)} &\leq c |Aw| \mathbb{E} \int_0^T \|u_{m_j} - u\| |v_{m_j}| ds \\
 &\leq c |Aw| \left(\mathbb{E} \int_0^T \|u_{m_j} - u\|^2 ds \right)^{1/2} \left(\mathbb{E} \int_0^T |v_{m_j}|^2 ds \right)^{1/2}.
 \end{aligned}$$

Due to (4), (79), (88) we have

$$I_{m_j}^{(2)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

For $\chi \in L^2(\Omega, \mathbb{P}; L^2(0, T; H))$, let $\Phi(\chi) = \mathbb{E} \int_0^T \langle \tilde{\mathbb{B}}(u, \chi), w \rangle_{D(A)'} ds$. We see from (21) and Cauchy–Schwarz’s inequality that

$$|\Phi(\chi)| \leq c |Aw| \left(\mathbb{E} \int_0^T \|u\|^2 ds \right)^{1/2} \left(\mathbb{E} \int_0^T |\chi|^2 ds \right)^{1/2}.$$

This shows that Φ is linear continuous. Since $v_{m_j} \rightharpoonup v$ weakly in $L^2(\Omega, \mathbb{P}; L^2(0, T; H))$, then $\Phi(v_{m_j} - v) \rightarrow 0$ as $m_j \rightarrow \infty$. That is, $I_{m_j}^{(3)} \rightarrow 0$ as $m_j \rightarrow \infty$.

Collecting these convergences we see that

$$\int_0^t \langle \tilde{\mathbb{B}}(u_{m_j}, v_{m_j}), P_{m_j}w \rangle \rightarrow \int_0^t \langle \tilde{\mathbb{B}}(u, v), w \rangle_{D(A)'} ds \quad \text{in } L^1(\Omega \times [0, T]). \tag{93}$$

For

$$\chi_{m_j} = \left| \mathbb{E} \int_0^t \left[\langle P_{m_j} \mathbb{B}(B_{m_j}, B_{m_j}), w \rangle - \langle \mathbb{B}(B, B), w \rangle_{D(A)'} \right] ds \right|, \tag{94}$$

we have

$$\chi_{m_j} \leq J_{m_j}^{(1)} + J_{m_j}^{(2)} + J_{m_j}^{(3)},$$

with

$$\begin{aligned}
 J_{m_j}^{(1)} &= \left| \mathbb{E} \int_0^t \langle \mathbb{B}(B_{m_j}, B_{m_j}), P_{m_j}w - w \rangle_{D(A)'} ds \right|, \\
 J_{m_j}^{(2)} &= \left| \mathbb{E} \int_0^t \langle \mathbb{B}(B_{m_j} - B, B_{m_j}), w \rangle_{D(A)'} ds \right|, \\
 J_{m_j}^{(3)} &= \left| \mathbb{E} \int_0^t \langle \mathbb{B}(B, B_{m_j} - B), w \rangle_{D(A)'} ds \right|.
 \end{aligned}$$

Now, by (10) and Poincaré’s inequality

$$J_{m_j}^{(1)} \leq c \|P_{m_j} w - w\| \mathbb{E} \int_0^T \|B_{m_j}\|^2 ds.$$

Since $\|P_{m_j} w - w\| \rightarrow 0$ as $m_j \rightarrow \infty$ and $\mathbb{E} \int_0^T \|B_{m_j}\|^2 ds \leq C$, we get that

$$J_{m_j}^{(1)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

To deal with $J_{m_j}^{(2)}$ we invoke (19) and Cauchy–Schwarz’s inequality to obtain

$$J_{m_j}^{(2)} \leq |Aw| \left(\mathbb{E} \int_0^T |B_{m_j} - B|^2 ds \right)^{1/2} \left(\mathbb{E} \int_0^T \|B_{m_j}\|^2 ds \right)^{1/2}.$$

This along with (79) and (89) imply that

$$J_{m_j}^{(2)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

Since $B_{m_j} \rightharpoonup B$ weakly in $L^2(\Omega, \mathbb{P}; L^2(0, T; V))$, then we can show as before (see argument for $I_{m_j}^{(3)}$) that

$$J_{m_j}^{(3)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

These show that

$$\int_0^t (P_{m_j} \mathbb{B}(B_{m_j}, B_{m_j}), w) ds \rightarrow \int_0^t \langle \mathbb{B}(B, B), w \rangle_{D(A)'} ds \quad \text{in } L^1(\Omega \times [0, T]). \tag{95}$$

By (90)–(91), the continuity of f_i and g_i , and the applicability of Vitali’s convergence theorem we have

$$P_{m_j} f_i(u_{m_j}, B_{m_j}, t) \rightarrow f_i(u, B, t) \quad \text{strongly in } L^2(\Omega, \mathbb{P}; L^2(0, T; H)), \tag{96}$$

$$P_{m_j} g_i(u_{m_j}, B_{m_j}, t) \rightarrow g_i(u, B, t) \quad \text{strongly in } L^2(\Omega, \mathbb{P}; L^2(0, T; H^{\otimes d})), \tag{97}$$

for $i = 1, 2$. Thanks to (97) we argue as in [5] and show that

$$\int_0^t (P_{m_j} g_i(u_{m_j}, B_{m_j}, s), \chi) dW_{m_j} \rightharpoonup \int_0^t (g_i(u, B, s), \chi) dW$$

weakly in $L^2(\Omega, \mathbb{P}; L^2(0, T))$, (98)

for $i = 1, 2$ and $\chi = w \in D(A)$ if $i = 1$ and $\chi = \zeta \in V$ otherwise.

The convergences (88), (93), (95), (96) and (98) enable us to pass to the limit in (76), from which we see that (26) holds.

It remains to pass to the limit in (77). As above we only work with the nonlinear terms since the linear ones are straightforward. We have that

$$\left| \mathbb{E} \int_0^t \left[\langle \mathbb{B}(u_{m_j}, B_{m_j}), P_{m_j} \zeta \rangle - \langle \mathbb{B}(u, B), \zeta \rangle_{V'} \right] ds \right| \leq S_{m_j}^{(1)} + S_{m_j}^{(2)} + S_{m_j}^{(3)},$$

where

$$S_{m_j}^{(1)} = \left| \mathbb{E} \int_0^t \langle \mathbb{B}(u_{m_j}, B_{m_j}), P_{m_j} \zeta - \zeta \rangle ds \right|,$$

$$S_{m_j}^{(2)} = \left| \mathbb{E} \int_0^t \langle \mathbb{B}(u_{m_j} - u, B_{m_j}), \zeta \rangle_{V'} ds \right|,$$

$$S_{m_j}^{(3)} = \left| \mathbb{E} \int_0^t \langle \mathbb{B}(u, B_{m_j} - B), \zeta \rangle_{V'} ds \right|.$$

By (16), Poincaré’s and Cauchy–Schwarz’s inequalities we have

$$S_{m_j}^{(1)} \leq c |P_{m_j} \zeta - \zeta| \left(\mathbb{E} \int_0^T |Au_{m_j}|^2 ds \right)^{1/2} \left(\mathbb{E} \int_0^T \|B_{m_j}\|^2 ds \right)^{1/2}.$$

Owing to (79) and (92) we see that

$$S_{m_j}^{(1)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

Invoking (10) and Cauchy–Schwarz’s inequality we get

$$S_{m_j}^{(2)} \leq c \|\zeta\| \left(\mathbb{E} \int_0^T \|u_{m_j} - u\|^2 ds \right)^{1/2} \left(\mathbb{E} \int_0^T \|B_{m_j}\|^2 ds \right)^{1/2}.$$

This estimate, (79) and the convergence (88) imply that

$$S_{m_j}^{(2)} \rightarrow 0 \quad \text{as } m_j \rightarrow \infty.$$

The convergence $S_{m_j}^{(3)} \rightarrow 0$ as $m_j \rightarrow \infty$ can be proved by the same idea used for $I_{m_j}^{(3)}$ and $J_{m_j}^{(2)}$. We have just proved that

$$\int_0^t \langle \mathbb{B}(u_{m_j}, B_{m_j}), P_{m_j} \zeta \rangle ds \rightarrow \int_0^t \langle \mathbb{B}(u, B), \zeta \rangle_{V'} ds \quad \text{in } L^1(\Omega \times [0, T]). \tag{99}$$

Similarly, we check that

$$\int_0^t \langle \mathbb{B}(B_{m_j}, u_{m_j}), P_{m_j} \zeta \rangle ds \rightarrow \int_0^t \langle \mathbb{B}(B, u), \zeta \rangle_{V'} ds \quad \text{in } L^1(\Omega \times [0, T]). \tag{100}$$

The convergences (89), (3), (98), (99), (96) imply that (27) holds. The estimates (28)–(33) follow from passing to the limit in (78)–(81), and this concludes the proof of Theorem 2.3.

6. Asymptotic behavior of the 3-D stochastic MHD- α model as $\alpha \rightarrow 0$

In this section, we assume that the hypotheses of Theorem 2.3 hold so that there exists a sequence of weak martingale solutions $\{(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha), (\mathcal{F}_t^\alpha)_{t \in [0, T]}, W_\alpha, u_\alpha, B_\alpha\}$ which satisfies the inequalities in Theorem 2.3. Our goal in this section is to study the behavior of the above sequence when we let α tends to zero. The main ingredient for achieving this target is the use of Prokhorov’s and Skorokhod’s theorems. This procedure mainly relies on some uniform estimates on appropriate norms of the finite differences of u_α and B_α . But the constants $C_5(\alpha), C_6(\alpha)$ explode when $\alpha \rightarrow 0$, then the applicability of Prokhorov’s and Skorokhod’s theorems are not ensured by the estimates (31) and (32). We still need to prove the following inequalities.

Lemma 6.1. *There exists a constant C independent of α such that for all $\alpha \in [0, 1)$ and $0 \leq \delta \leq 1$, we have*

$$\mathbb{E}_\alpha \sup_{0 \leq |\theta| \leq \delta \leq 1} \int_0^T |B_\alpha(t + \theta) - B_\alpha(t)|_{D(A)}^2 dt \leq C\delta, \tag{101}$$

$$\mathbb{E}_\alpha \sup_{0 \leq |\theta| \leq \delta \leq 1} \int_0^T |u_\alpha(t + \theta) - u_\alpha(t)|_{D(A)}^2 dt \leq C\delta. \tag{102}$$

Before we proceed to the proof of these results we should give the following remark.

Remark 6.2. From the Theorem 2.3, the following are valid for any $1 \leq p < \infty$:

$$\mathbb{E}_\alpha \sup_{s \in [0, T]} |u_\alpha(s)|^p + \mathbb{E}_\alpha \sup_{s \in [0, T]} |B_\alpha(s)|^p \leq C_p, \tag{103}$$

$$\mathbb{E}_\alpha \left(\int_0^T \|u_\alpha(s)\|^2 ds \right)^{p/2} + \mathbb{E}_\alpha \left(\int_0^T \|B_\alpha(s)\|^2 ds \right)^{p/2} \leq C_p, \tag{104}$$

where $C_p > 0$ is independent of α .

Proof. We start by proving (102). First recall that $D(A)' = D(A^{-1})$ and $I + \alpha^2 A$ is an isomorphism from $D(A)$ onto H . Moreover

$$\|(I + \alpha^2 A)^{-1}\|_{\mathcal{L}(H, H)} \leq 1. \tag{105}$$

It follows from (25) that

$$\begin{aligned} du_\alpha + \left[vAu_\alpha + (I + \alpha^2 A)^{-1} \tilde{\mathbb{B}}(u_\alpha, u_\alpha + \alpha^2 Au_\alpha) - (I + \alpha^2 A)^{-1} \mathbb{B}(B_\alpha, B_\alpha) \right] dt \\ = (I + \alpha^2 A)^{-1} f_1(u_\alpha, B_\alpha, t) dt + (I + \alpha^2 A)^{-1} g_1(u_\alpha, B_\alpha, t) dW_\alpha, \end{aligned} \tag{106}$$

which implies that

$$\begin{aligned} \sup_{0 \leq \theta \leq \delta} |A^{-1}(u_\alpha(t + \theta) - u_\alpha(t))| \\ \leq \int_t^{t+\delta} \left(|A^{-1} f_1(u_\alpha, B_\alpha, s)| + v|u_\alpha| \right) ds \\ + \int_t^{t+\delta} \left(|A^{-1}(I + \alpha^2 A)^{-1} \tilde{\mathbb{B}}(u_\alpha, v_\alpha)| \right) ds \\ + \int_t^{t+\delta} \left(|A^{-1}(I + \alpha^2 A)^{-1} \mathbb{B}(B_\alpha, B_\alpha)| \right) ds \\ + \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} |A^{-1}(I + \alpha^2 A)^{-1} g_i(u_\alpha, B_\alpha, s)| dW_\alpha \right|. \end{aligned} \tag{107}$$

From (105), (19) and Cauchy’s inequality we have that

$$\begin{aligned} |A^{-1}(I + \alpha^2 A)^{-1} \tilde{\mathbb{B}}(u_\alpha, v_\alpha)| &\leq 2c\lambda_1^{-1/4} |v_\alpha| \|u_\alpha\|, \\ &\leq 2c\lambda_1^{-1/4} \|u_\alpha\| (|u_\alpha| + \alpha^2 |Au_\alpha|), \\ &\leq 2c\lambda_1^{-1/4} \{ |u_\alpha| \|u_\alpha\| + \alpha \|u_\alpha\| \alpha |Au_\alpha| \}, \\ &\leq 2c\lambda_1^{-1/4} \left(|u_\alpha|^2 + \alpha^2 \|u_\alpha\|^2 \right)^{1/2} \\ &\quad \times \left(\|u_\alpha\|^2 + \alpha^2 |Au_\alpha|^2 \right)^{1/2}. \end{aligned}$$

On the other hand

$$|A^{-1}(I + \alpha^2 A)^{-1} X_i(u_\alpha, B_\alpha, t)| \leq |A^{-1} X_i(u_\alpha, B_\alpha, t)| \leq C(1 + |u_\alpha| + |B_\alpha|),$$

where X_i is either f_1 or g_1 . With these inequalities at hand, we derive from (107) that

$$\begin{aligned} & \sup_{0 \leq \theta \leq \delta} |A^{-1}(u_\alpha(t + \theta) - u_\alpha(t))|^2 \\ & \leq C\delta^2 + TC\delta^2 \sup_{s \in [0, T]} |u_\alpha(s)|^2 + C\delta^2 \sup_{s \in [0, T]} |B_\alpha(s)|^2 \\ & \quad + \left(\int_t^{t+\delta} |B_\alpha(s)| \|B_\alpha(s)\| ds \right)^2 + C \sup_{s \in [0, T]} (|u_\alpha(s)| \\ & \quad + \alpha^2 \|u_\alpha(s)\|^2) \left(\int_t^{t+\delta} [\|u_\alpha(s)\|^2 + \alpha^2 |Au_\alpha(s)|^2]^{1/2} ds \right)^2 \\ & \quad + 2 \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} A^{-1}(I + \alpha^2 A)^{-1} g_1(u_\alpha, B_\alpha, s) dW_\alpha \right|^2. \end{aligned}$$

Integrating with respect to t along $[0, T]$ and taking the mathematical expectation yield

$$\begin{aligned} & \mathbb{E}_\alpha \sup_{0 \leq \theta \leq \delta} \int_0^T |A^{-1}(u_\alpha(t + \theta) - u_\alpha(t))|^2 dt \\ & \leq CT\delta^2 + C\delta^2 + C\mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} \|u_\alpha(s)\|_V^2 \int_0^T \left(\int_t^{t+\delta} [\|u_\alpha\|^2 + \alpha^2 |Au_\alpha|^2]^{1/2} ds \right)^2 dt \right\} \\ & \quad + \mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} |B_\alpha(s)|^2 \int_0^T \left(\int_t^{t+\delta} \|B_\alpha(s)\| ds \right)^2 dt \right\} \\ & \quad + 2\mathbb{E}_\alpha \int_0^T \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} A^{-1}(I + \alpha^2 A)^{-1} g_1(u_\alpha, B_\alpha, s) dW_\alpha \right|^2 dt, \end{aligned} \tag{108}$$

where we have used the uniform estimates (28) and (30). By Hölder’s inequality we have

$$\begin{aligned} & \mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} \|u_\alpha(s)\|_V^2 \int_0^T \left(\int_t^{t+\delta} [\|u_\alpha\|^2 + \alpha^2 |Au_\alpha|^2]^{1/2} ds \right)^2 dt \right\} \\ & \leq \delta^2 T \mathbb{E}_\alpha \left(\sup_{s \in [0, T]} \|u_\alpha(s)\|_V^2 \int_0^T [\|u_\alpha(s)\|^2 + \alpha^2 |Au_\alpha(s)|^2] dt \right), \\ & \leq \delta^2 T \left(\mathbb{E}_\alpha \sup_{s \in [0, T]} \|u_\alpha(s)\|_V^4 \right)^{1/2} \left(\mathbb{E}_\alpha \left[\int_0^T (\|u_\alpha(s)\|^2 + \alpha^2 |Au_\alpha(s)|^2) ds \right]^2 \right)^{1/2}, \end{aligned} \tag{109}$$

and

$$\begin{aligned} & \mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} |B_\alpha(s)|^2 \int_0^T \left(\int_t^{t+\delta} \|B_\alpha(s)\| ds \right)^2 dt \right\} \\ & \leq \delta^2 \mathbb{E}_\alpha \left(\sup_{s \in [0, T]} |B_\alpha(s)|^2 \int_0^T \|B_\alpha(s)\|^2 ds \right), \end{aligned}$$

$$\leq \delta^2 \left(\mathbb{E}_\alpha \sup_{s \in [0, T]} |B_\alpha(s)|^4 \right)^{1/2} \tag{110}$$

$$\times \left(\mathbb{E}_\alpha \left[\int_0^T \|B_\alpha(s)\|^2 ds \right]^2 \right)^{1/2}. \tag{111}$$

Using (28)–(31) implies

$$\mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} \|u_\alpha(s)\|_V^2 \int_0^T \left(\int_t^{t+\delta} [\|u_\alpha\|^2 + \alpha^2 |Au_\alpha|^2]^{1/2} ds \right)^2 dt \right\} \leq C\delta, \tag{112}$$

$$\mathbb{E}_\alpha \left\{ \sup_{s \in [0, T]} |B_\alpha(s)|^2 \int_0^T \left(\int_t^{t+\delta} \|B_\alpha(s)\| ds \right)^2 dt \right\} \leq C\delta, \tag{113}$$

where C is a positive constant independent of α . Next by Burkholder–Davis–Gundy’s inequality we have

$$\begin{aligned} & \mathbb{E}_\alpha \int_0^T \sup_{0 \leq \theta \leq \delta} \left| \int_t^{t+\theta} A^{-1}(I + \alpha^2 A)^{-1} g_1(u_\alpha, B_\alpha, s) dW_\alpha \right|^2 dt \\ & \leq C \mathbb{E}_\alpha \int_0^T \left(\int_t^{t+\delta} |A^{-1}(I + \alpha^2 A)^{-1} g_1(u_\alpha, B_\alpha, s)|^2 ds \right) dt, \end{aligned} \tag{114}$$

$$\leq C \leq \int_0^T \left(\int_t^{t+\delta} (1 + |u_\alpha(s)|^2 + |B_\alpha(s)|^2) ds \right) dt \leq C\delta. \tag{115}$$

By (109)–(115) we derive from (108) that (102) holds, that is,

$$\mathbb{E}_\alpha \sup_{0 \leq |\theta| \leq \delta} \int_0^T |A^{-1}(u_\alpha(t + \theta) - u_\alpha(t))|^2 dt \leq C\delta,$$

where C is a positive constant independent of α .

Now let us deal with (101). We know from (19) that

$$|A^{-1}\mathbb{B}(u_\alpha, B_\alpha)| \leq c|u_\alpha| \|B_\alpha\|.$$

Therefore,

$$\begin{aligned} \mathbb{E}_\alpha \left(\int_t^{t+\delta} |A^{-1}\mathbb{B}(u_\alpha, B_\alpha)| ds \right)^2 & \leq c\delta \mathbb{E}_\alpha \int_t^{t+\delta} |u_\alpha|^2 \|B_\alpha\|^2 ds, \\ & \leq c\delta \mathbb{E}_\alpha \sup_{0 \leq t \leq T} |u_\alpha|^2 \int_t^{t+\delta} \|B_\alpha\|^2 ds, \\ & \leq c\delta \mathbb{E}_\alpha \sup_{0 \leq t \leq T} |u_\alpha|^4 + c\delta \mathbb{E}_\alpha \left(\int_t^{t+\delta} \|B_\alpha\|^2 ds \right)^2. \end{aligned}$$

Thanks to (28) and (31) we have

$$\mathbb{E}_\alpha \left(\int_t^{t+\delta} |A^{-1}\mathbb{B}(u_\alpha, B_\alpha)| ds \right)^2 \leq c\delta. \tag{116}$$

Similarly

$$\mathbb{E}_\alpha \left(\int_t^{t+\delta} |A^{-1} \mathbb{B}(B_\alpha, u_\alpha)| ds \right)^2 \leq c\delta \mathbb{E}_\alpha \sup_{0 \leq t \leq T} |B_\alpha|^4 + c\delta \mathbb{E}_\alpha \left(\int_t^{t+\delta} \|u_\alpha\|^2 ds \right)^2.$$

Because of (29) and (30) we get

$$\mathbb{E}_\alpha \left(\int_t^{t+\delta} |A^{-1} \mathbb{B}(B_\alpha, u_\alpha)| ds \right)^2 \leq c\delta. \tag{117}$$

By using Burkholder–Davis–Gundy’s inequality it is not hard to show that

$$\mathbb{E}_\alpha \sup_{0 \leq \theta \leq \delta \leq 1} \int_0^T \left| \int_t^{t+\theta} g_i(u_\alpha, B_\alpha, s) dW_\alpha \right| dt \leq c\delta, \quad i = 1, 2. \tag{118}$$

In view of (116)–(118), we prove as before that (101) holds. More precisely,

$$\mathbb{E}_\alpha \sup_{0 \leq |\theta| \leq \delta \leq 1} \int_0^T |A^{-1} (B_\alpha(t + \theta) - B_\alpha(t))|^2 dt \leq C\delta. \quad \square \tag{119}$$

The following compactness result plays a crucial role in the proof of the tightness of the probability measures generated by the sequence $(u_\alpha; B_\alpha)_{\alpha \in [0,1]}$.

Lemma 6.3. *Let μ_n, ν_n two sequences of positive real numbers which tend to zero as $n \rightarrow \infty$. Then the injection of*

$$D_{\nu_n, \mu_n} = \left\{ q \in L^\infty(0, T; H) \cap L^2(0, T; V); \right. \\ \left. \sup_n \frac{1}{\nu_n} \sup_{|\theta| \leq \mu_n} \left(\int_0^T |q(t + \theta) - q(t)|_{D(A)}^2 dt \right)^{1/2} < \infty \right\}$$

in $L^2(0, T; H)$ is compact.

The proof is similar to the analogous result in [3,36,37]. The space D_{ν_n, μ_n} is a Banach space with the norm

$$\|q\|_{D_{\nu_n, \mu_n}} = \text{ess sup}_{0 \leq t \leq T} |q(t)| + \left(\int_0^T \|q(t)\|_V^2 dt \right)^{1/2} \\ + \sup_n \frac{1}{\nu_n} \sup_{|\theta| \leq \mu_n} \left(\int_0^T |q(t + \theta) - q(t)|_{D(A)}^2 dt \right)^{1/2}.$$

Alongside D_{ν_n, μ_n} , we also consider the space X_{p, ν_n, μ_n} , $1 \leq p < \infty$, of random variables ζ such that the norm

$$\begin{aligned} \|\zeta\|_{X_{p,v_n,\mu_n}} &= \left(\mathbb{E}_\alpha \operatorname{ess\,sup}_{0 \leq t \leq T} |\zeta(t)|^p \right)^{\frac{1}{p}} + \left(\mathbb{E}_\alpha \left(\int_0^T \|\zeta(t)\|_V^2 dt \right)^{p/2} \right)^{\frac{1}{p}} \\ &\quad + \mathbb{E}_\alpha \sup_n \frac{1}{v_n} \sup_{|\theta| \leq \mu_n} \left(\int_0^T |\zeta(t+\theta) - \zeta(t)|_{D(A)'}^2 dt \right)^{1/2}, \end{aligned}$$

is finite. Endowed with the norm $\|\zeta\|_{X_{p,v_n,\mu_n}}$, X_{p,v_n,μ_n} is a Banach space. Combining the estimates (101)–(102) and those of Remark 6.2, we have

Proposition 6.4. *For any real number $1 \leq p < \infty$ and for any sequences v_n, μ_n converging to 0 such that the series $\sum_n \frac{\sqrt{\mu_n}}{v_n}$ converges, there exists a positive constant C independent of α such that*

$$\begin{aligned} \|u_\alpha\|_{X_{p,v_n,\mu_n}} &< C, \\ \|B_\alpha\|_{X_{p,v_n,\mu_n}} &< C, \end{aligned}$$

for all n .

Now we consider the space $\mathfrak{S} = C(0, T; \mathbb{R}^d) \times L^2(0, T; H) \times L^2(0, T; H)$ equipped with the Borel σ -algebra $\mathcal{B}(\mathfrak{S})$. For $\alpha \in [0, 1)$, let Φ_α be the measurable \mathfrak{S} -valued mapping defined on $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$ by

$$\Phi(\omega) = (W_\alpha(\omega), u_\alpha(\omega), B_\alpha(\omega)).$$

For each α we introduce a probability measure Π_α on $(\mathfrak{S}; \mathcal{B}(\mathfrak{S}))$ by

$$\Pi_\alpha(S) = \mathbb{P}_\alpha(\Phi^{-1}(S)), \quad \text{for any } S \in \mathcal{B}(\mathfrak{S}).$$

Theorem 6.5. *The family of probability measures $\{\Pi_\alpha : \alpha \in [0, 1)\}$ is tight in $(\mathfrak{S}; \mathcal{B}(\mathfrak{S}))$.*

Proof. For $\varepsilon > 0$ we should find compact subsets

$$\Sigma_\varepsilon \subset C(0, T; \mathbb{R}^d); Y_\varepsilon \subset L^2(0, T; H); X_\varepsilon \subset L^2(0, T; H),$$

such that

$$\mathbb{P}_\alpha(\omega : u_\alpha(\omega, \cdot) \notin Y_\varepsilon) + \mathbb{P}_\alpha(\omega : B_\alpha(\omega, \cdot) \notin X_\varepsilon) \leq \frac{\varepsilon}{2}, \tag{120}$$

$$\mathbb{P}_\alpha(\omega : W_\alpha(\omega, \cdot) \notin \Sigma_\varepsilon) \leq \frac{\varepsilon}{2}, \tag{121}$$

for all α .

The quest for Σ_ε is made by taking into account some facts about Wiener process and it is standard so we omit it.

Next we choose Y_ε and X_ε as balls of radii M_ε in D_{v_n,μ_m} centered at 0 and with v_n, μ_n independent of ε , converging to 0 and such that the series $\sum_n \frac{\sqrt{\mu_n}}{v_n}$ converges, from Lemma 6.3, $Y_\varepsilon, X_\varepsilon$ are compact subsets of $L^2(0, T; H)$. Furthermore, we have

$$\begin{aligned} \mathbb{P}_\alpha(\omega : u_\alpha(\omega) \notin Y_\varepsilon) + \mathbb{P}_\alpha(\omega : B_\alpha(\omega) \notin X_\varepsilon) &\leq \mathbb{P}_\alpha(\omega : \|u_\alpha\|_{D_{v_n,\mu_m}} > M_\varepsilon) \\ &\quad + \mathbb{P}_\alpha(\omega : \|B_\alpha\|_{D_{v_n,\mu_m}} > M_\varepsilon) \\ &\leq \frac{1}{M_\varepsilon} (\mathbb{E}_\alpha \|u_\alpha\|_{D_{v_n,\mu_m}} + \mathbb{E}_\alpha \|B_\alpha\|_{D_{v_n,\mu_m}}), \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{M_\varepsilon} (\|u_\alpha\|_{X_{1,v_n,\mu_n}} + \|B_\alpha\|_{X_{1,v_n,\mu_n}}), \\ &\leq \frac{2C}{M_\varepsilon}, \end{aligned}$$

where $C > 0$ is independent of α (see Proposition 6.4 for the justification).

Choosing $M_\varepsilon = 2C\varepsilon^{-1}$, we get (120). From the inequalities (120)–(121) we deduce that

$$\mathbb{P}_\alpha(\omega : W_\alpha(\omega) \in \Sigma_\varepsilon; u_\alpha(\omega) \in Y_\varepsilon; B_\alpha(\omega) \in X_\varepsilon) \geq 1 - \varepsilon,$$

for all $\alpha \in [0, 1)$. This proves that for all $\alpha \in [0, 1)$

$$\Pi_\alpha(\Sigma_\varepsilon \times Y_\varepsilon \times X_\varepsilon) \geq 1 - \varepsilon,$$

from which we deduce the tightness of $\{\Pi_\alpha : \alpha \in [0, 1)\}$ in $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$. \square

6.1. Approximation of the 3-D stochastic MHD equations

In this subsection, we prove that the weak solution of the stochastic 3-D MHD equations are obtained from a sequence of solution of the 3-D stochastic MHD- α model as α approaches 0.

From the tightness of $\{\Pi_\alpha : \alpha \in [0, 1)\}$ in the Polish space \mathfrak{S} and Prokhorov’s theorem, we infer the existence of a subsequence Π_{α_j} of probability measures and a probability measure Π such that $\Pi_{\alpha_j} \rightharpoonup \Pi$ weakly. By Skorokhod’s theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and two triplets of random variables $(W_{\alpha_j}, u_{\alpha_j}, B_{\alpha_j}), (\tilde{W}, \tilde{u}, \tilde{B})$ with values in \mathfrak{S} such that

$$\text{the law of } (W_{\alpha_j}, u_{\alpha_j}, B_{\alpha_j}) \text{ is } \Pi_{\alpha_j}, \tag{122}$$

$$\text{the law of } (\tilde{W}, \tilde{u}, \tilde{B}) \text{ is } \Pi, \tag{123}$$

$$(W_{\alpha_j}, u_{\alpha_j}, B_{\alpha_j}) \rightarrow (\tilde{W}, \tilde{u}, \tilde{B}) \text{ strongly in } \mathfrak{S} \text{ } \tilde{\mathbb{P}}\text{-almost surely.} \tag{124}$$

We can see that W_{α_j} is a sequence of d -dimensional standard Wiener processes.

Let $\tilde{\mathcal{F}}_t = \sigma\{\tilde{W}(s), \tilde{u}, \tilde{B}(s) : 0 \leq s \leq t\}$. Arguing as in [3,5] we prove that \tilde{W} is a d -dimensional $\tilde{\mathcal{F}}_t$ -standard Wiener process and the triplet $(W_{\alpha_j}, u_{\alpha_j}, B_{\alpha_j})$ satisfies

$$\begin{aligned} &(v_{\alpha_j}(t), w) - (v_{\alpha_j}(0), w) \\ &\quad + \int_0^t \left(\langle \tilde{\mathbb{B}}(u_{\alpha_j}, v_{\alpha_j}) - \mathbb{B}(B_{\alpha_j}, B_{\alpha_j}), w \rangle_{D(A)'} + \nu(v_{\alpha_j}, Aw) \right) ds \\ &= \int_0^t (f_1(u_{\alpha_j}, B_{\alpha_j}, s), w) ds + \int_0^t (g_1(u_{\alpha_j}, B_{\alpha_j}, s), w) dW_{\alpha_j}(s), \end{aligned} \tag{125}$$

and

$$\begin{aligned} &(B_{\alpha_j}(t), \zeta) - (B_{\alpha_j}(0), \zeta) + \int_0^t \left[\langle \mathbb{B}(u_{\alpha_j}, B_{\alpha_j}) - \mathbb{B}(B_{\alpha_j}, u_{\alpha_j}), \zeta \rangle + \eta((B_{\alpha_j}, \zeta)) \right] ds \\ &= \int_0^t (f_2(u_{\alpha_j}, B_{\alpha_j}, s), \zeta) + \int_0^t (g_2(u_{\alpha_j}, B_{\alpha_j}, s), \zeta) dW_\alpha(s), \end{aligned} \tag{126}$$

for $\tilde{\mathbb{P}} \otimes dt$ -almost everywhere, for all $(w; \zeta) \in D(A) \times V$ and for any α_j . In (125), $v_{\alpha_j} = u_{\alpha_j} + \alpha_j^2 Au_{\alpha_j}$. Before we state the second main result of our work we repeat here the definition of weak martingale solution of the 3-D stochastic MHD equations. The following is taken from [46].

Definition 6.6. A weak martingale solution of the stochastic MHD equations is a system $\{(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{W}, \tilde{u}, \tilde{B}\}$ where

- (1) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a complete probability space,
- (2) $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ is a filtration satisfying the usual condition on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$,
- (3) \tilde{W} is a $\tilde{\mathcal{F}}_t$ -adapted \mathbb{R}^d -valued Wiener process,
- (4) $\tilde{u} \in L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T, H)) \cap L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T, V))$,

$$\tilde{B} \in L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T, H)) \cap L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T, V)), \quad \text{for every } p \in [1, \infty)$$

- (5) for all $(w, \zeta) \in D(A) \times V$,

$$\begin{cases} (\tilde{u}(t), w) - (\tilde{u}_0, w) + \int_0^t (\langle \mathbb{B}(\tilde{u}, \tilde{u}) - \mathbb{B}(\tilde{B}, \tilde{B}), w \rangle_{D(A)'} + \nu(\tilde{u}, Aw)) ds \\ = \int_0^t (f_1(\tilde{u}, \tilde{B}, s), w) ds + \int_0^t (g_1(\tilde{u}, \tilde{B}, s), w) d\tilde{W}(s), \end{cases} \quad (127)$$

and

$$\begin{cases} (\tilde{B}(t), \zeta) - (\tilde{B}_0, \zeta) + \int_0^t (\langle \mathbb{B}(\tilde{u}, \tilde{B}) - \mathbb{B}(\tilde{B}, \tilde{u}), \zeta \rangle + \eta((\tilde{B}, \zeta))) ds \\ = \int_0^t (f_2(\tilde{u}, \tilde{B}, s), \zeta) + \int_0^t (g_2(\tilde{u}, \tilde{B}, s), \zeta) d\tilde{W}(s) \end{cases} \quad (128)$$

hold $dt \otimes d\tilde{\mathbb{P}}$ -almost everywhere.

- (6) The functions $\tilde{u}(t)$ and $\tilde{B}(t)$ takes values in H and are continuous with respect to t $\tilde{\mathbb{P}}$ -almost surely.

The Justification of point (6) of Definition 6.6 can be done exactly in the same way as for the item (6) of Definition 2.2.

Theorem 6.7. Assume that the set of hypotheses (A1) holds and $u_0 \in V$ and $B_0 \in H$. Then as α_j tends to 0, the sequence $u_{\alpha_j}, v_{\alpha_j}, B_{\alpha_j}$ (obtained) above satisfy

$$u_{\alpha_j} \rightarrow \tilde{u} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; H)), \quad (129)$$

$$u_{\alpha_j} \rightharpoonup \tilde{u} \quad \text{weakly star in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T; H)), \quad (130)$$

$$u_{\alpha_j} \rightharpoonup \tilde{u} \quad \text{weakly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; V)), \quad (131)$$

$$v_{\alpha_j} \rightarrow \tilde{v} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; V')), \quad (132)$$

$$v_{\alpha_j} \rightharpoonup \tilde{v} \quad \text{weakly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; H)), \quad (133)$$

$$B_{\alpha_j} \rightarrow \tilde{B} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; H)), \quad (134)$$

$$B_{\alpha_j} \rightharpoonup \tilde{B} \quad \text{weakly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; V)), \quad (135)$$

where $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_{t \in [0, T]}; \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}, \tilde{B})$ is a weak martingale solution of the 3-D stochastic MHD equations with initial values $\tilde{u}(0) = u_0$ and $\tilde{B}(0) = B_0$.

Proof. From (125) and (126), it follows that $(u_{\alpha_j}; B_{\alpha_j})$ satisfies the estimates

$$\tilde{\mathbb{E}} \sup_{s \in [0, T]} \|u_{\alpha_j}(s)\|_V^p + \tilde{\mathbb{E}} \sup_{s \in [0, T]} |B_{\alpha_j}(s)|^p \leq C_p, \quad (136)$$

$$\mathbb{E} \left(\int_0^T \|u_{\alpha_j}(s)\|^2 ds \right)^{p/2} + \mathbb{E} \left(\int_0^T \|B_{\alpha_j}(s)\|^2 ds \right)^{p/2} \leq C_p, \tag{137}$$

$$\mathbb{E} \left(\int_0^T \left[2\nu \|u_{\alpha_j}\|^2 + 2\nu\alpha^2 |Au_{\alpha_j}|^2 \right] ds \right)^{p/2} \leq C_p, \tag{138}$$

where \mathbb{E} denotes the mathematical expectation with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Hence, modulo the extraction of a subsequence denoted again $u_{\alpha_j}, v_{\alpha_j}, B_{\alpha_j}$ we have

$$u_{\alpha_j} \rightharpoonup \tilde{u} \quad \text{weakly star in } L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T; H)), \tag{139}$$

$$u_{\alpha_j} \rightharpoonup \tilde{u} \quad \text{weakly in } L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; V)), \tag{140}$$

$$v_{\alpha_j} \rightharpoonup \tilde{v} \quad \text{weakly in } L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T; H)), \tag{141}$$

$$B_{\alpha_j} \rightharpoonup \tilde{B} \quad \text{weakly star in } L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T; H)), \tag{142}$$

$$B_{\alpha_j} \rightharpoonup \tilde{B} \quad \text{weakly star in } L^p(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\infty(0, T; V)). \tag{143}$$

By Vitali’s convergence theorem and (124), we have

$$u_{\alpha_j} \rightarrow \tilde{u} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; H)), \tag{144}$$

$$B_{\alpha_j} \rightarrow \tilde{B} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; H)). \tag{145}$$

Therefore, modulo the extraction of a subsequence denoted again with the same symbols,

$$u_{\alpha_j} \rightarrow \tilde{u} \quad dt \otimes d\tilde{\mathbb{P}}\text{-almost everywhere in } H, \tag{146}$$

$$B_{\alpha_j} \rightarrow \tilde{B} \quad dt \otimes d\tilde{\mathbb{P}}\text{-almost everywhere in } H. \tag{147}$$

Let $i = 1, 2, \chi_1 = w \in D(A)$ and $\chi_2 = \zeta \in V$. The convergences (146)–(147) together with the assumptions on f_i, g_i and Vitali’s convergence theorem imply that

$$\int_0^t (f_i(u_{\alpha_j}, B_{\alpha_j}, s), \chi_i) ds \rightarrow \int_0^t (f_i(\tilde{u}, \tilde{B}, s), \chi_i) ds \quad \text{in } L^2(\tilde{\Omega} \times [0, T]), \tag{148}$$

$$(g_i(u_{\alpha_j}, B_{\alpha_j}, t), \chi_i) \rightarrow (g_i(\tilde{u}, \tilde{B}, t), \chi_i) \quad \text{in } L^2(\tilde{\Omega} \times [0, T]). \tag{149}$$

Thanks to (149) we prove as in [5] that

$$\int_0^t (g_i(u_{\alpha_j}, B_{\alpha_j}, s), \chi_i) dW_{\alpha_j} \rightarrow \int_0^t (g_i(\tilde{u}, \tilde{B}, s), \chi_i) d\tilde{W} \quad \text{in } L^2(\tilde{\Omega} \times [0, T]). \tag{150}$$

We also have

$$\mathbb{E} \int_0^t \|v_{\alpha_j} - u_{\alpha_j}\|_V^2 ds = \alpha_j^2 \mathbb{E} \int_0^t \alpha_j^2 |Au_{\alpha_j}(s)|^2 ds.$$

From this and (134) we deduce that

$$v_{\alpha_j} \rightarrow \tilde{u} \quad \text{strongly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; V')), \tag{151}$$

and $\tilde{v}(t) = \tilde{u}(t)$ almost everywhere in $dt \otimes d\tilde{\mathbb{P}}$, since $\mathbb{E} \int_0^T \alpha_j^2 |Au_{\alpha_j}(s)|^2 ds$ is uniformly bounded in α_j .

Using (144)–(145), we have

$$\int_0^t \langle \mathbb{B}(B_{\alpha_j}, B_{\alpha_j}), w \rangle_{D(A)'} ds \rightarrow \int_0^t \langle \mathbb{B}(\tilde{B}, \tilde{B}), w \rangle_{D(A)'} ds \quad \text{in } L^2(\tilde{\Omega} \times [0, T]), \tag{152}$$

$$\int_0^t \langle \mathbb{B}(u_{\alpha_j}, B_{\alpha_j}), \zeta \rangle_{D(A)'} ds \rightarrow \int_0^t \langle \mathbb{B}(\tilde{u}, \tilde{B}), \zeta \rangle_{D(A)'} ds \quad \text{in } L^2(\tilde{\Omega} \times [0, T]), \tag{153}$$

$$\int_0^t \langle \mathbb{B}(B_{\alpha_j}, u_{\alpha_j}), \zeta \rangle_{D(A)'} ds \rightarrow \int_0^t \langle \mathbb{B}(\tilde{B}, \tilde{u}), \zeta \rangle_{D(A)'} ds \quad \text{in } L^2(\tilde{\Omega} \times [0, T]). \tag{154}$$

We have

$$\int_0^t \tilde{\mathbb{B}}(u_{\alpha_j}, v_{\alpha_j}) ds \rightarrow \int_0^t \tilde{\mathbb{B}}(\tilde{u}, \tilde{u}) ds \quad \text{weakly in } L^\beta(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\beta(0, T; D(A)')), \tag{155}$$

for some $1 < \beta < 2$. To prove this, it is sufficient to show that

$$\tilde{\mathbb{B}}(u_{\alpha_j}, v_{\alpha_j}) \rightharpoonup \tilde{\mathbb{B}}(\tilde{u}, \tilde{u}) = \mathbb{B}(\tilde{u}, \tilde{u}) \quad \text{weakly in } L^\beta(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\beta(0, T; D(A)')), \tag{156}$$

for some $1 < \beta \leq 2$. For that purpose let us recall first that $v_{\alpha_j} = u_{\alpha_j} + \alpha^2 Au_{\alpha_j}$ and

$$\tilde{\mathbb{B}}(u_{\alpha_j}, v_{\alpha_j}) = \mathbb{B}(u_{\alpha_j}, u_{\alpha_j}) + \alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j}).$$

Hence we need to show that for $1 < \beta < 2$

$$\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j}) \rightarrow 0 \quad \text{strongly in } L^\beta(\tilde{\Omega}, \tilde{\mathbb{P}}; L^\beta(0, T; D(A)')), \tag{157}$$

and

$$\mathbb{B}(u_{\alpha_j}, u_{\alpha_j}) \rightharpoonup \mathbb{B}(\tilde{u}, \tilde{u}) \quad \text{weakly in } L^2(\tilde{\Omega}, \tilde{\mathbb{P}}; L^2(0, T; D(A)')), \tag{158}$$

as $\alpha_j \rightarrow 0$. Owing to (140) and (144) the convergence (158) can be easily proved as in the case of 3-D stochastic Navier–Stokes equations, so we will deal only with (157). Thanks to (20) there holds

$$\|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'} \leq c \alpha_j^2 \|u_{\alpha_j}\| |Au_{\alpha_j}|.$$

Fixing an arbitrary β , $1 < \beta < 2$, we obtain the following chain of inequalities

$$\begin{aligned} & \int_0^T \|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'}^\beta dt \\ & \leq c^\beta \alpha_j^{2\beta} \int_0^T \|u_{\alpha_j}(t)\|^\beta |Au_{\alpha_j}|^\beta dt, \\ & \leq c^\beta \alpha_j^{2\beta} \left(\sup_{t \in [0, T]} \|u_{\alpha_j}(t)\|^\gamma \right) \int_0^T \|u_{\alpha_j}\|^{\beta-\gamma} |Au_{\alpha_j}|^\beta dt, \\ & \leq c^\beta \alpha_j^{2\beta} \left(\sup_{t \in [0, T]} \|u_{\alpha_j}(t)\|^\gamma \right) \left[\int_0^T \|u_{\alpha_j}(t)\|^{q(\beta-\gamma)} dt \right]^{1/q} \end{aligned} \tag{159}$$

$$\times \left[\int_0^T |Au_{\alpha_j}(t)|^{p\beta} dt \right]^{1/p}, \tag{160}$$

where γ is an arbitrary number such that $0 < \gamma < \beta$, and in (160) we applied Hölder’s inequality with $\frac{1}{p} + \frac{1}{q} = 1$ (these numbers will be determined later on).

Continuing the chain of inequalities, we have

$$\begin{aligned} & \int_0^T \|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'}^\beta dt \\ & \leq c^\beta \alpha_j^{2\beta} \left(\sup_{t \in [0, T]} \|u_{\alpha_j}(t)\|^2 \right)^{\gamma/2} \\ & \quad \times \left[\int_0^T \|u_{\alpha_j}\|^{q(\beta-\gamma)} dt \right]^{1/q} \left[\int_0^T |Au_{\alpha_j}|^{p\beta} dt \right]^{1/p}. \end{aligned} \tag{161}$$

Now we set $p = \frac{2}{\beta}$, $q = \frac{2}{2-\beta}$. Let the number γ satisfies the equation $q(\beta - \gamma) = 2$, that is, $\gamma = 2(\beta - 1)$. We see from this that $0 < \gamma < \beta$ holds.

Replacing such p, q and γ in (161), we obtain the following results

$$\begin{aligned} & \int_0^T \|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'}^\beta dt \\ & \leq c^\beta \alpha_j^{2-\beta} \left(\sup_{t \in [0, T]} \alpha_j^2 \|u_{\alpha_j}(t)\|^2 \right)^{\beta-1} \cdot \left[\int_0^T \|u_{\alpha_j}(t)\|^2 dt \right]^{\frac{2-\beta}{2}} \\ & \quad \times \left[\int_0^T \alpha_j^2 |Au_{\alpha_j}(t)|^2 dt \right]^{\beta/2}. \end{aligned} \tag{162}$$

Taking the mathematical expectation in (162) and using Hölder’s inequality, we have

$$\begin{aligned} \tilde{\mathbb{E}} \int_0^T \|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'}^\beta dt & \leq c^\beta \alpha_j^{2-\beta} \left[\tilde{\mathbb{E}} \left(\sup_{t \in [0, T]} \alpha_j^2 \|u_{\alpha_j}(t)\|^2 \right) \right]^{\beta-1} \\ & \quad \times \left[\tilde{\mathbb{E}} \left(\int_0^T \|u_{\alpha_j}(t)\|^2 dt \right) \right]^{\frac{2-\beta}{2}} \\ & \quad \times \left[\tilde{\mathbb{E}} \left(\int_0^T \alpha_j^2 |Au_{\alpha_j}(t)|^2 dt \right) \right]^{\frac{\beta}{2-\beta}}. \end{aligned}$$

Using the estimates (136)–(138), we have

$$\tilde{\mathbb{E}} \int_0^T \|\alpha_j^2 \tilde{\mathbb{B}}(u_{\alpha_j}, Au_{\alpha_j})\|_{D(A)'}^\beta dt \leq c^\beta \alpha_j^{2-\beta}, \quad 1 < \beta < 2. \tag{163}$$

Therefore, we have proved (157).

Collecting all the above convergences, especially (139)–(143), (144), (145), (148)–(155), we pass to the limit in (125)–(126). We see from this passage to the limit that the processes \tilde{u}, \tilde{B} satisfy the 3-D stochastic MHD equations. That is,

$$\begin{aligned} & (\tilde{u}(t), w) - (\tilde{u}(0), w) + \int_0^t \left(\langle \mathbb{B}(\tilde{u}, \tilde{u}) - \mathbb{B}(\tilde{B}, \tilde{B}), w \rangle_{D(A)'} + \nu(\tilde{u}, Aw) \right) ds \\ & = \int_0^t (f_1(\tilde{u}, \tilde{B}, s), w) ds + \int_0^t (g_1(\tilde{u}, \tilde{B}, s), w) d\tilde{W}(s), \end{aligned} \tag{164}$$

and

$$\begin{aligned}
 & (\tilde{B}(t), \zeta) - (\tilde{B}(0), \zeta) + \int_0^t \left[(\mathbb{B}(\tilde{u}, \tilde{B}) - \mathbb{B}(\tilde{B}, \tilde{u}), \zeta) + \eta((\tilde{B}, \zeta)) \right] ds \\
 &= \int_0^t (f_2(\tilde{u}, \tilde{B}, s), \zeta) + \int_0^t (g_2(\tilde{u}, \tilde{B}, s), \zeta) d\tilde{W}(s),
 \end{aligned} \tag{165}$$

for $dt \otimes \tilde{\mathbb{P}}$ -everywhere and for all $(w; \zeta) \in D(A) \times V$.

Then, we have proved that the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \tilde{\mathcal{F}}_t, \tilde{W}, \tilde{u}, \tilde{B})$ is a weak martingale solution of the 3-D stochastic MHD equations (see Definition 6.6 for the definition of weak martingale solution of stochastic MHD equations). \square

7. Pathwise uniqueness of the solutions

In this section we study the pathwise uniqueness of the solutions of the 3-D stochastic MHD- α models. In addition to the assumptions (A1) we introduce new set of assumptions on the nonlinear mappings f_i and g_i .

(A2) In addition to (A1) we suppose that f_i and $g_i, i = 1, 2$, are Lipschitz continuous, that is, there exist a constant C such that

$$\begin{aligned}
 & |f_i(u_1, B_1, t) - f_i(u_2, B_2, t)| \leq C(|u_1 - u_2| + |B_1 - B_2|), \\
 & |g_i(u_1, B_1, t) - g_i(u_2, B_2, t)|_{H^{\otimes d}} \leq C(|u_1 - u_2| + |B_1 - B_2|).
 \end{aligned}$$

In contrast to the 3-D stochastic MHD equations a pathwise uniqueness holds for the 3-D stochastic MHD- α models. Mainly we have the following result:

Theorem 7.1. *In addition to assumptions (A1) we suppose that (A2) are valid. Then two solutions $(u_1; B_1), (u_2; B_2)$ of the 3-D stochastic MHD- α defined on the same prescribed stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \in [0, T]}, W)$ starting with the same initial condition $(u_0; B_0)$ coincide \mathbb{P} -almost surely.*

From Yamada–Watanabe’s Theorem for infinite dimensional stochastic equations (see [39, Theorem E.1.8]) which states that the existence of weak martingale solution and the pathwise uniqueness imply the existence of unique probabilistic strong solution, we have the following consequence of the above theorem.

Corollary 7.2. *Assume that (A1) and (A2) hold. There exists a unique probabilistic strong solution of the 3-D stochastic MHD- α models. That is, for a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \in [0, T]}, W)$ and for any $\alpha > 0$, there exists a unique couple $(u; B)$ satisfying the points (4)–(6) of Definition 2.2.*

Proof of Theorem 7.1. Let $v_i = u_i + \alpha^2 Au_i$ ($i = 1, 2$), $\delta v = v_1 - v_2, \delta u = u_1 - u_2$ and $\delta B = B_1 - B_2$. We see that

$$\begin{cases}
 d\delta v + \left[vA\delta v + \tilde{\mathbb{B}}(\delta u, v_1) + \tilde{\mathbb{B}}(u_2, \delta v) \right] dt \\
 \quad = [\mathbb{B}(\delta B, B_1) + \mathbb{B}(B_2, \delta B) + \delta f_1] dt + \delta g_1 dW, \\
 d\delta B + [\eta A\delta B + \mathbb{B}(\delta u, B_1) + \mathbb{B}(u_2, \delta B)] dt \\
 \quad = [\mathbb{B}(\delta B, u_1) + \mathbb{B}(B_2, \delta u) + \delta f_2] dt + \delta g_2 dW, \\
 \delta v(0) = \delta u(0) = \delta B(0) = 0,
 \end{cases} \tag{166}$$

where

$$\delta f_i = f_i(u_1, B_1, t) - f_i(u_2, B_2, t),$$

and

$$\delta g_i = g_i(u_1, B_1, t) - g_i(u_2, B_2, t).$$

Applying $(I + \alpha^2 A)^{-1}$ to the first equation in (166) gives

$$\begin{aligned} d\delta u + \left[vA\delta u + \tilde{\mathbb{B}}^\#(\delta u, v_1) + \tilde{\mathbb{B}}^\#(u_2, \delta v) \right] dt \\ = \left[\mathbb{B}^\#(\delta B, B_1) + \mathbb{B}^\#(B_2, \delta B) + \delta f_1^\# \right] dt + \delta g_1^\# dW, \end{aligned} \tag{167}$$

where $(I + \alpha^2 A)^{-1} \chi = \chi^\#$. By Itô’s formula, we infer from this equation that

$$\begin{aligned} d\|\delta u\|_V^2 + 2 \left[v((A\delta u, \delta u))_V + ((\tilde{\mathbb{B}}^\#(\delta u, v_1), \delta u))_V + ((\tilde{\mathbb{B}}^\#(u_2, \delta v), \delta u))_V \right] dt \\ = 2 \left[((\mathbb{B}^\#(\delta B, B_1), \delta u))_V + ((\mathbb{B}^\#(B_2, \delta B), \delta u))_V \right] dt \\ + 2 \left[((\delta f_1^\#, \delta u))_V + (1/2)\|\delta g_1^\#\|_V^2 \right] dt + 2((\delta g_1^\#, \delta u))_V dW, \end{aligned} \tag{168}$$

or equivalently

$$\begin{aligned} d\|\delta u\|_V^2 + 2 \left[v((A\delta u, \delta u))_V + \langle \tilde{\mathbb{B}}(\delta u, v_1), \delta u \rangle_{D(A)'} + \langle \tilde{\mathbb{B}}(u_2, \delta v), \delta u \rangle_{D(A)'} \right] dt \\ = 2 \left[\langle \mathbb{B}(\delta B, B_1), \delta u \rangle_{D(A)'} + \langle \mathbb{B}(B_2, \delta B), \delta u \rangle_{D(A)'} \right] dt \\ + 2 \left[\langle \delta f_1, \delta u \rangle + (1/2)\|\delta g_1^\#\|_V^2 \right] dt + 2(\delta g_1, \delta u) dW. \end{aligned} \tag{169}$$

There also holds

$$\begin{aligned} d|\delta B|^2 + 2\eta\|\delta B\|^2 dt = 2 \left[\langle \mathbb{B}(\delta u, B_1), \delta B \rangle_{V'} + \langle \mathbb{B}(\delta B, u_1), \delta B \rangle_{V'} \right] dt \\ + 2 \left[\langle \mathbb{B}(B_2, \delta u), \delta B \rangle_{V'} + (\delta f_2, \delta B) + (1/2)|\delta g_2|^2 \right] dt \\ + 2(\delta g_2, \delta B) dW. \end{aligned} \tag{170}$$

Summing up (169) and (170) yields

$$\begin{aligned} d\phi + \left(2v((A, \delta u))_V + 2\eta\|\delta B\|^2 \right) dt \\ - 2 \left[\langle \delta f_1, \delta u \rangle + (\delta f_2, \delta B) + \|\delta g_1^\#\|_V^2 + |\delta g_2|^2 \right] dt \\ = 2 \left(-\langle \tilde{\mathbb{B}}(u_2, \delta v), \delta u \rangle_{D(A)'} + \langle \tilde{\mathbb{B}}(\delta B, B_1), \delta u \rangle_{D(A)'} + \langle \mathbb{B}(\delta B, u_1), \delta B \rangle_{V'} \right) dt \\ + 2 \left[(\delta g_1, \delta u) + (\delta g_2, \delta B) \right] dW. \end{aligned} \tag{171}$$

Here and for the rest of the proof, $\phi = \|\delta u\|_V^2 + |\delta B|^2$. Now let us set

$$\sigma(s) = \exp \left(\int_0^s -z(s)\phi(s) ds \right),$$

where $z(s)$ is a real valued function that will be fixed later on. We apply Itô’s formula to $\sigma(t)\phi(t)$ and find that

$$\begin{aligned}
 & \sigma(t)\phi(t) + 2 \int_0^t \sigma(s) \left[v((A\delta u, \delta u))_V + \eta \|\delta B\|^2 \right] ds - \int_0^t z(s)\sigma(s)\phi(s) ds \\
 &= 2 \int_0^t \sigma(s) \left(-\langle \tilde{\mathbb{B}}(u_2, \delta v), \delta u \rangle_{D(A)'} + \langle \tilde{\mathbb{B}}(\delta B, B_1), \delta u \rangle_{D(A)'} \right) ds \\
 & \quad + 2 \int_0^t \langle \mathbb{B}(\delta B, u_1), \delta B \rangle_{V'} ds + 2 \int_0^t \sigma(s) [(\delta g_1, \delta u) + (\delta g_2, \delta B)] dW \\
 & \quad + 2 \int_0^t \sigma(s) \left[(\delta f_1, \delta u) + (\delta f_2, \delta B) + \|\delta g_1^\# \|_V^2 + |\delta g_2|^2 \right] ds. \tag{172}
 \end{aligned}$$

By (22) and by Young’s inequality we have

$$2|\langle \tilde{\mathbb{B}}(u_2, \delta v), \delta u \rangle_{D(A)'}| \leq \frac{2c}{v\lambda_1^{1/2}} |Au_2|^2 (|\delta u|^2 + \alpha^2 \|\delta u\|^2) + v\|\delta u\|^2 + \frac{v}{2}\alpha^2 |\delta u|^2. \tag{173}$$

By (19) and Young’s inequality, we obtain

$$2|\langle \tilde{\mathbb{B}}(\delta B, B_1), \delta u \rangle_{D(A)'}| \leq 2c|\delta B|^2 \|B_1\|^2 + \frac{2}{v\alpha^2} \|\delta u\|^2 + \frac{v}{2}\alpha^2 |A\delta u|^2. \tag{174}$$

Also, by (16) and Young’s inequality, we see that

$$2|\langle \mathbb{B}(\delta B, u_1), \delta B \rangle_{V'}| \leq \frac{2c}{\eta\lambda_1^{1/2}} |Au_1|^2 |\delta B|^2 + \eta \|\delta B\|^2. \tag{175}$$

By summing up (173)–(175) and by using the resulting estimates in (172), we derive that

$$\begin{aligned}
 & \sigma(t)\phi(t) + 2 \int_0^t \sigma(s) \left(v[\|\delta u\|^2 + \alpha^2 |A\delta u|^2] + \eta \|\delta B\|^2 \right) ds \\
 & \leq \int_0^t \left(\frac{2c}{v\lambda_1^{1/2}} |Au_2|^2 + \frac{2}{v\alpha^4} + 2c\|B_1\|^2 + \frac{c}{\eta\lambda_1^{1/2}} |Au_1|^2 \right) \sigma s\phi(s) ds \\
 & \quad - \int_0^t z(s)\sigma(s)\phi(s) ds \\
 & \quad + 2 \int_0^t \sigma(s) \left((\delta f_1, \delta u) + (\delta f_2, \delta B) + \frac{1}{2} \|\delta g_1\|_V^2 + \frac{1}{2} |\delta g_2|^2 \right) ds \\
 & \quad + 2 \int_0^t \sigma(s) ((\delta g_1, \delta u) + (\delta g_2, \delta B)) dW. \tag{176}
 \end{aligned}$$

We choose

$$z(s) = \frac{2c}{v\lambda_1^{1/2}} |Au_2(s)|^2 + \frac{2}{v\alpha^4} + 2c\|B_1(s)\|^2 + \frac{c}{\eta\lambda_1^{1/2}} |Au_1(s)|^2,$$

and infer from (176) that

$$\begin{aligned}
 & \mathbb{E}\sigma(t)\phi(t) + 2\mathbb{E} \int_0^t \sigma(s) \left(v[\|\delta u\|^2 + \alpha^2 |A\delta u|^2] + \eta \|\delta B\|^2 \right) ds \\
 & \leq 2\mathbb{E} \int_0^t \sigma(s) \left((\delta f_1, \delta u) + (\delta f_2, \delta B) + \frac{1}{2} \|\delta g_1^\# \|_V^2 + \frac{1}{2} |\delta g_2|^2 \right) ds, \tag{177}
 \end{aligned}$$

where the property of the stochastic integral was used. By using the Lipschitz properties of f_i and g_i , $i = 1, 2$, we deduce from (177) that

$$\begin{aligned} \mathbb{E}\sigma(t)\phi(t) + 2\mathbb{E}\int_0^t \sigma(s) \left(\nu[\|\delta u\|^2 + \alpha^2|A\delta u|^2] + \eta\|\delta B\|^2 \right) ds \\ \leq 2\mathbb{E}\int_0^t \sigma(s) \left(C|\delta u|[\|\delta u\| + |\delta B|] + |\delta B|[\|\delta u\| + |\delta B|] + C[\|\delta u\|^2 + |\delta B|^2] \right) ds. \end{aligned}$$

Since $|\delta u| \leq \|\delta u\|_V$, we obtain that

$$\begin{aligned} \mathbb{E}\sigma(t)\phi(t) + 2\mathbb{E}\int_0^t \sigma(s) \left(\nu[\|\delta u\|^2 + \alpha^2|A\delta u|^2] + \eta\|\delta B\|^2 \right) ds \\ \leq C\mathbb{E}\int_0^t \sigma(s) \left(\|\delta u\|_V^2 + |\delta B|^2 \right) ds, \end{aligned}$$

from which we infer that

$$E\sigma(t)\phi(t) \leq C\mathbb{E}\int_0^t \sigma(s)\phi(s)ds.$$

We conclude the proof of the pathwise uniqueness theorem by the application of Gronwall's lemma. \square

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