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Explosion, implosion, and moments of passage times for continuous-time Markov chains: A semimartingale approach☆

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Abstract

We establish general theorems quantifying the notion of recurrence – through an estimation of the moments of passage times – for irreducible continuous-time Markov chains on countably infinite state spaces. Sharp conditions of occurrence of the phenomenon of explosion are also obtained. A new phenomenon of implosion is introduced and sharp conditions for its occurrence are proven. The general results are illustrated by treating models having a difficult behaviour even in discrete time.

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Keywords: Continuous-time Markov chain; Recurrence criteria; Explosion criteria; Moments of passage times; Implosion

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1. Introduction, notation, and main results

1.1. Introduction

In this paper we establish general theorems quantifying the notion of recurrence – by studying which moments of the passage time exist – for irreducible continuous-time Markov chains $\xi = (\xi_t)_{t\in[0,\infty[}$ on a countable space $\mathbb{X}$ in critical regimes.

Models of discrete-time Markov chains with non-trivial behaviour include reflected random walks in wedges of dimension $d = 2$ [1,6,14], Lamperti processes [12,13], etc. These chains exhibit strange polynomial behaviour. In the null recurrent case some (but not all) moments of the random time needed to reach a finite set are obtained by transforming the discrete-time Markov chain into a discrete-time semimartingale via its mapping through a Lyapunov function [6].

There exist in the literature powerful theorems [1], applicable to discrete-time critical Markov chains, allowing to determine which moments of the passage time exist. Beyond their theoretical interest, such results can be used to estimate the decay of the stationary measure [16], and even the speed of convergence towards the stationary measure. The first aim of this paper is to show that theorems concerning moments of passage times can be usefully and instrumentally extended to the continuous time situation.

Continuous-time Markov chains have an additional feature compared to discrete-time ones, namely, on each visited state they spend a random holding time (exponentially distributed) defined as the difference between successive jump times. We consider the space inhomogeneous situation where the parameters $\gamma_x \in \mathbb{R}_+$ of the exponential holding times (the inverse of their expectation) are unbounded, i.e. $\sup_{x\in\mathbb{X}} \gamma_x = +\infty$. In such situations, the phenomenon of *explosion* can occur for transient chains. Chung [3] has established that the condition $\sum_{n=1}^{\infty} 1/\gamma_{\tilde{\xi}_n} < +\infty$, where $\tilde{\xi}_n$ is the position of the chain immediately after the n-th jump has occurred, is equivalent to explosion. However this condition is very difficult to check since is global i.e. requires the knowledge of the entire trajectory of the embedded Markov chain. Later, sufficient conditions for explosion – whose validity can be verified by local estimates – have been introduced. Sufficiently sharp conditions of explosion and non-explosion, applicable only to Markov chains on countably infinite subsets of non-negative reals, are given in [8,10], while Lyapunov functions are used in [2] for the study of Markov chains with state space $\mathbb{Z}$ and time-dependent holding times. In [20], a sufficient condition of explosion is established for Markov chains on general countable sets; similar sufficient conditions of explosion are established in [21] for Markov chains on locally compact separable metric spaces.

In the discrete time case, many results on recurrence/transience, i.e. the estimating of the number of times the Markov chain returns to a given state can be obtained through estimating the moments of the time needed to return to this state. The reason is that (discrete) time flows homogeneously; each step of the chain takes a unit of time to be performed because the internal clock of the chain ticks at constant pace. In the continuous time case, the connection between the number of steps and the time needed to perform them is more subtle because the internal clock of the chain ticks at different pace when the process visits different states. The second aim of this paper is to show that the phenomenon of explosion can also be sharply studied by the use of Lyapunov functions and to establish *locally verifiable* conditions for explosion/non explosion for Markov chains on arbitrary graphs. This method is applied to models that even without explosion are difficult to study. More fundamentally, the development of the semimartingale method has been largely inspired by having these specific critical models in mind (such as the cascade of

k-critically transient Lamperti models or of reflected random walks on quarter planes) that seem refractory to known methods.

Finally, we demonstrate a new phenomenon, we termed *implosion* (see Definition 1.2 below), reminiscent of Döblin's condition for general Markov chains [4], occurring in the case $\sup_{x\in\mathbb{X}} \gamma_x = \infty$. We show that this phenomenon can also be explored with the help of Lyapunov functions.

1.2. Notation

Throughout this paper, $\mathbb{X}$ denotes the state space of our Markov chains; it denotes an abstract denumerably infinite set, equipped with its full σ-algebra $\mathscr{X} = \mathscr{P}(\mathbb{X})$. It is worth stressing here that, generally, this space is not naturally partially ordered. The graph whose edges are the ones induced by the stochastic matrix, when equipped with the natural graph metric on $\mathbb{X}$ need not be isometrically embeddable into $\mathbb{Z}^d$ for some d. Since the definition of a continuous-time Markov chain on a denumerable set is standard (see [3], for instance), we introduce below its usual equivalent description in terms of holding times and embedded Markov chain merely for the purpose of establishing our notation.

Denote by $\Gamma = (\Gamma_{xy})_{x,y\in\mathbb{X}}$ the *generator* of the continuous Markov chain, namely the matrix satisfying: $\Gamma_{xy} \geq 0$ if $y \neq x$ and $\Gamma_{xx} = -\gamma_x$, where $\gamma_x = \sum_{y\in\mathbb{X}\setminus\{x\}} \Gamma_{xy}$. We assume that for all $x \in \mathbb{X}$, we have $\gamma_x < \infty$.

We construct a stochastic Markovian matrix $P = (P_{xy})_{x,y\in\mathbb{X}}$ out of Γ by defining

$$P_{xy} = \begin{cases} \dfrac{\Gamma_{xy}}{\gamma_x} & \text{if } \gamma_x \neq 0 \\ 0 & \text{if } \gamma_x = 0, \end{cases} \quad \text{for } y \neq x, \quad \text{and} \quad P_{xx} = \begin{cases} 0 & \text{if } \gamma_x \neq 0 \\ 1 & \text{if } \gamma_x = 0. \end{cases}$$

The kernel P defines a discrete-time $(\mathbb{X}, P)$-Markov chain $\tilde{\xi} = (\tilde{\xi}_n)_{n\in\mathbb{N}}$ termed the *Markov chain embedded at the moments of jumps*. To avoid irrelevant complications, we always assume that this Markov chain is *irreducible*.

Define a sequence $\sigma = (\sigma_n)_{n\geq 1}$ of *random holding times* distributed, conditionally on $\tilde{\xi}$, according to an exponential law. More precisely, consider

$$\mathbb{P}(\sigma_n \in ds|\tilde{\xi}) = \gamma_{\tilde{\xi}_{n-1}} \exp(-s\gamma_{\tilde{\xi}_{n-1}}) \mathbb{1}_{\mathbb{R}_+}(s) ds, \quad n \geq 1,$$

so that $\mathbb{E}(\sigma_n|\tilde{\xi}) = 1/\gamma_{\tilde{\xi}_{n-1}}$. The sequence $J = (J_n)_{n\in\mathbb{N}}$ of *random jump times* is defined accordingly by $J_0 = 0$ and for $n \geq 1$ by $J_n = \sum_{k=1}^n \sigma_k$. The lifetime is denoted by $\zeta = \lim_{n\to\infty} J_n$ and we say that the (not yet defined continuous-time) Markov chain *explodes* on $\{\zeta < \infty\}$, while it does not explode (or is regular, or conservative) on $\{\zeta = \infty\}$.

Remark 1.1. The parameter γ_x must be interpreted as the proper frequency of the internal clock of the Markov chain multiplicatively modulating the local speed of the chain. We always assume that for all $x \in \mathbb{X}$, $0 < \gamma_x < \infty$. The case $0 < \underline{\gamma} := \inf_{x\in\mathbb{X}} \gamma_x \leq \sup_{x\in\mathbb{X}} =: \overline{\gamma} < \infty$ is elementary because the chain can be stochastically controlled by two Markov chains whose internal clocks tick respectively at constant pace $\underline{\gamma}$ and $\overline{\gamma}$. Therefore the sole interesting cases are

- $\sup_x \gamma_x = \infty$: the internal clock ticks unboundedly fast (leading to an unbounded local speed of the chain),
- $\inf_x \gamma_x = 0$: the internal clock ticks arbitrarily slowly (leading to a local speed that can be arbitrarily close to 0).

To have a unified description of both explosive and non-explosive processes, we can extend the state space into $\hat{\mathbb{X}} = \mathbb{X} \cup \{\partial\}$ by adjoining a special absorbing state ∂. The continuous-time Markov chain is then the càdlàg process $\xi = (\xi_t)_{t\in[0,\infty[}$ defined by

$$\xi_0 = \tilde{\xi}_0 \quad \text{and} \quad \xi_t = \begin{cases} \sum_{n\in\mathbb{N}} \tilde{\xi}_n \mathbb{1}_{[J_n, J_{n+1}[}(t) & \text{for } 0 < t < \zeta \\ \partial & \text{for } t \geq \zeta. \end{cases}$$

Note that although $\mathbb{X}$ is merely a set (i.e. no internal composition rule is defined on it), the above "sum" is well-defined since for every fixed t only one term survives. We refer the reader to standard texts (for instance [3,18]) for the proof of the equivalence between ξ and $(\tilde{\xi}, J)$. The natural right continuous filtration $(\mathscr{F}_t)_{t\in[0,+\infty[}$ is defined as usual through $\mathscr{F}_t = \sigma(\xi_s : s \leq t)$; similarly $\mathscr{F}_{t-} = \sigma(\xi_s : s < t)$, and $\tilde{\mathscr{F}}_n = \sigma(\tilde{\xi}_k, k \leq n)$ for $n \in \mathbb{N}$. For an arbitrary $(\mathscr{F}_t)$-stopping time τ, we denote as usual its past σ-algebra $\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty : A \cap \{\tau \leq t\} \in \mathscr{F}_t\}$ and its strict past σ-algebra $\mathscr{F}_{\tau-} = \sigma\{A \cap \{t < \tau\}; t \geq 0, A \in \mathscr{F}_t\} \vee \mathscr{F}_0$. Since it is immediate to show that τ is $\mathscr{F}_{\tau-}$-measurable, we conclude that the only information contained in $\mathscr{F}_{J_{n+1}}$ but not in $\mathscr{F}_{J_{n+1}-}$ is conveyed by the random variable $\tilde{\xi}_{n+1}$, i.e. the position where the chain jumps at the moment J_{n+1}.

If $A \in \mathscr{X}$, we denote by $\tau_A = \inf\{t \geq 0 : \xi_t \in A\}$ the $(\mathscr{F}_t)$-stopping time of reaching A.

A dual notion to explosion is that of *implosion*:

Definition 1.2. Let $(\xi_t)_{t\in[0,\infty[}$ be a continuous-time Markov chain on $\mathbb{X}$ and let $A \subset \mathbb{X}$ be a proper subset of $\mathbb{X}$. We say that the Markov chain *implodes* towards A if $\exists K > 0 : \forall x \in A^c$, $\mathbb{E}_x(\tau_A) \leq K$.

Remark 1.3. It will be shown in Proposition 2.14 that in the case the set A is finite and the chain is irreducible, implosion towards A means implosion towards any state. In this situation, we speak about *implosion of the chain*.

It is worth noticing that some other definitions of implosion can be introduced; all convey the same idea of reaching a finite set from an arbitrary initial point within a random time that can be uniformly (in the initial point) bounded in some appropriate stochastic sense. We stick at the form introduced in the previous definition because it is easier to establish necessary and sufficient conditions for its occurrence and is easier to illustrate on specific problems (see Section 3).

We use the notational conventions of [19] to denote measurable functions, namely $m\mathscr{X} = \{f : \mathbb{X} \to \mathbb{R} | f \text{ is } \mathscr{X}\text{-measurable}\}$ with all possible decorations: $b\mathscr{X}$ to denote bounded measurable functions, $m\mathscr{X}_+$ to denote non-negative measurable functions, etc. For $f \in m\mathscr{X}_+$ and $\alpha > 0$, we denote by $\mathsf{S}_\alpha(f)$ the *sublevel set* of f of height α defined by

$$\mathsf{S}_\alpha(f) := \{x \in \mathbb{X} : f(x) \leq \alpha\}.$$

We recall that a function $f \in m\mathscr{X}_+$ is *unbounded* if $\sup_{x\in\mathbb{X}} f(x) = +\infty$ while *tends to infinity* ($f \to \infty$) when for every $n \in \mathbb{N}$ the sublevel set $\mathsf{S}_n(f)$ is finite. Measurable functions f defined on $\mathbb{X}$ can be extended to functions $\hat{f}$, defined on $\hat{\mathbb{X}}$, by $\hat{f}(x) = f(x)$ for all $x \in \mathbb{X}$ and $\hat{f}(\partial) := 0$ (with obvious extension of the σ-algebra).

We denote by $\mathsf{Dom}(\Gamma) = \{f \in m\mathscr{X} : \sum_{y\in\mathbb{X}\setminus\{x\}} \Gamma_{xy}|f(y)| < +\infty, \forall x \in \mathbb{X}\}$ the domain of the generator and by $\mathsf{Dom}_+(\Gamma)$ the set of non-negative functions in the domain. The action of the generator Γ on $f \in \mathsf{Dom}(\Gamma)$ reads then: $\Gamma f(x) := \sum_{y\in\mathbb{X}} \Gamma_{xy} f(y)$. Similarly we could have defined $\mathsf{Dom}(P)$ and $\mathsf{Dom}_+(P)$. Nevertheless, it is immediate to see that whenever the inequalities $0 < \gamma(x) < \infty$ hold for all $x \in \mathbb{X}$, we have $\mathsf{Dom}(\Gamma) = \mathsf{Dom}(P)$ and $\mathsf{Dom}_+(\Gamma) = \mathsf{Dom}_+(P)$.

1.3. Main results

We recall once more that in the whole paper we make the following

Global assumption 1.4. The chain embedded at the moments of jumps is *irreducible* and $0 < \gamma_x < \infty$ for all $x \in \mathbb{X}$.

We are now in a position to state our main results concerning the use of the Lyapunov function to obtain, through semimartingale theorems, precise and locally verifiable conditions on the parameters of the chain allowing us to establish the existence or non-existence of moments of passage times, explosion or implosion phenomena. The proofs of these results are given in Section 2; Section 3 treats some critical models (especially 3.1 and 3.3) that are difficult to study even in discrete time, illustrating thus both the power of our methods and giving specific examples on how to use them.

1.3.1. Existence or non-existence of moments of passage times

Theorem 1.5. *Let $f \in \mathsf{Dom}_+(\Gamma)$ be such that $f \to \infty$.*

1. *If there exist constants $a > 0$, $c > 0$ and $p > 0$ such that $f^p \in \mathsf{Dom}_+(\Gamma)$ and*
$$\Gamma f^p(x) \leq -c f^{p-2}(x), \quad \forall x \notin \mathsf{S}_a(f),$$
then $\mathbb{E}_x(\tau^q_{\mathsf{S}_a(f)}) < +\infty$ for all $q < p/2$ and all $x \in \mathbb{X}$.
2. *Let $g \in m\mathscr{X}_+$. If there exist*
   (a) *a constant $b > 0$ such that $f \leq bg$,*
   (b) *constants $a > 0$ and $c_1 > 0$ such that $\Gamma g(x) \geq -c_1$ for $x \notin \mathsf{S}_a(g)$,*
   (c) *constants $c_2 > 0$ and $r > 1$ such that $g^r \in \mathsf{Dom}(\Gamma)$ and $\Gamma g^r(x) \leq c_2 g^{r-1}(x)$ for $x \notin \mathsf{S}_a(g)$, and*
   (d) *a constant $p > 0$ such that $f^p \in \mathsf{Dom}(\Gamma)$ and $\Gamma f^p \geq 0$ for $x \notin \mathsf{S}_{ab}(f)$,*
   then $\mathbb{E}_x(\tau^q_{\mathsf{S}_a(f)}) = +\infty$ for all $q > p$ and all $x \notin \mathsf{S}_a(f)$.

Remark 1.6. The condition $f \to \infty$ (together with the majorisation on $\Gamma f^p(x)$) in the first statement of Theorem 1.5 guarantees recurrence of the chain. If we assume recurrence of the chain and we can find a bounded function f we can prove the existence of exponential moments of the time of reaching a finite set. This phenomenon will be discussed later (see Proposition 1.16 below).

In many cases, the function g, whose existence is assumed in statement 2, of the above theorem can be chosen as $g = f$ (with obviously $b = 1$). In such situations we have to check $\Gamma f^r \leq c_2 f^{r-1}$ for some $r > 1$ and find a $p > 0$ such that $\Gamma f^p \geq 0$ on the appropriate sets. However, in the case of the problem studied in Section 3.3, for instance, the full-fledged version of the previous theorem is needed.

Note that the conditions $f \in \mathsf{Dom}_+(\Gamma)$ and $f^p \in \mathsf{Dom}_+(\Gamma)$ for some $p > 0$ holding simultaneously imply that $f^q \in \mathsf{Dom}_+(\Gamma)$ for all q in the interval with end points 1 and p. When τ_A is integrable, the chain is positive recurrent. In the null recurrent situation however, τ_A is almost surely finite but not integrable; nevertheless, some fractional moments $\mathbb{E}(\tau_A^q)$ with $q < 1$ can exist. Similarly, in the positive recurrent case, some higher moments $\mathbb{E}(\tau_A^q)$ with $q > 1$ may fail to exist.

When $p = 2$, the first statement in Theorem 1.5 simplifies to the following: if $\Gamma f(x) \leq -\epsilon$, for some $\epsilon > 0$ and for x outside a finite set F, then the passage time $\mathbb{E}_x(\tau_F^q) < \infty$ for all $x \in \mathbb{X}$ and all $q < 1$. As a matter of fact, in this situation, we have a stronger result, expressed in the form of the following

Theorem 1.7. *Suppose that the chain is recurrent. The following are equivalent:*

1. *The chain is positive recurrent.*
2. *There exist a triple (ϵ, F, f), with $\epsilon > 0$, F a finite non-empty subset of $\mathbb{X}$ and f a function in $\mathsf{Dom}_+(\Gamma)$ verifying $\Gamma f(x) \leq -\epsilon$ for all $x \notin F$.*

Obviously, positive recurrence implies *a fortiori* that $\mathbb{E}_x(\tau_F) < \infty$.

Remark 1.8. It is obvious that the triple (ϵ, F, f) in Theorem 1.7 is not uniquely determined. Mostly, it will be possible to chose a function $f \to \infty$ and F as the sublevel set of f at height a, for some $a > 0$. Sometimes it will be possible to chose the function f uniformly bounded; this case will be further considered in Theorem 1.15 and leads to implosion. It is also immediate that if f verifies the conditions $\Gamma f(x) \leq -\epsilon$ for $x \in F^c$ then the modified function $f + c$, where c is an arbitrary positive constant, also verifies the same condition. Further, if a function f verifies this condition, the function $g = f \mathbb{1}_{F^c}$ verifies *a fortiori* the same condition.

If only establishing occurrence of recurrence or transience is sought, the first generalisation of Foster's criteria to the continuous-time case was given in the unpublished technical report [17]. Notice however that the method in that paper is subjected to the same important restriction as in the original paper of Foster [7], namely the semi-martingale condition must be verified everywhere but in one point.

If γ_x is bounded away from 0 and ∞, then since the Markov chain can be stochastically controlled by two Markov chains with constant γ_x reading respectively $\gamma_x = \underline{\gamma}$ and $\gamma_x = \overline{\gamma}$ for all x, the previous result is the straightforward generalisation of Theorems 1 and 2 of [1] established in the case of discrete time; as a matter of fact, in the case of constant γ_x, the complete behaviour of the continuous time process is encoded solely into the jump chain and since results in [1] were optimal, the present theorem introduces no improvement. Only the interesting cases of $\sup_{x \in \mathbb{X}} \gamma_x = \infty$ or $\inf_{x \in \mathbb{X}} \gamma_x = 0$ are studied in the sequel; the models studied in Section 3, illustrate how the theorem can be used in critical cases to obtain locally verifiable conditions of the existence/non-existence of moments of reaching times. The process $X_t = f(\xi_t)$, the image of the Markov chain through the Lyapunov function f, can be shown to be a semimartingale; therefore, the semimartingale approach will prove instrumental as was the case in discrete time chains.

1.3.2. Explosion

The next results concern explosion obtained again using Lyapunov functions. It is worth noting that although explosion can only occur in the transient case, the next result is strongly reminiscent of Foster's criterion [7] for positive recurrence!

Theorem 1.9. *The following are equivalent:*

1. *There exist $f \in \mathsf{Dom}_+(\Gamma)$ strictly positive and $\epsilon > 0$ such that $\Gamma f(x) \leq -\epsilon$ for all $x \in \mathbb{X}$.*
2. *The explosion time ζ satisfies $\mathbb{E}_x \zeta < +\infty$ for all $x \in \mathbb{X}$.*

Remark 1.10. Comparison of statements 1 of Theorem 1.5 and 1 of Theorem 1.9 demands some comments. The conditions of Theorem 1.5 imply that $\mathsf{S}_a(f)$ is a finite set and *necessarily not* *empty*. For $p = 2$ and $F = f^p$, the condition reads $\Gamma F(x) \leq -\epsilon$ outside some finite set and this implies recurrence. In Theorem 1.9 the condition $\Gamma f \leq -\epsilon$ must hold *everywhere* and this implies transience.

The one-side implication $[\Gamma f(x) \leq -\epsilon, \forall x] \Rightarrow [\mathbb{P}_x(\zeta < +\infty) = 1, \forall x]$ is already established, for $f \geq 0$, in the second part of Theorem 4.3.6 of [20]. Here, modulo the (seemingly) slightly more stringent requirement $f > 0$, relying on the powerful ideas developed within the proof of the aforementioned theorem of [20], we strengthen the result from almost sure finiteness to integrability and prove equivalence instead of mere implication.

Proposition 1.11. *Let $f \in \mathsf{Dom}_+(\Gamma)$ be a strictly positive bounded function and denote $b = \sup_{x \in \mathbb{X}} f(x)$; assume there exists an increasing – not necessarily strictly – function $g : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ such that its inverse has an integrable singularity at 0, i.e. $\int_0^b \frac{1}{g(y)} dy < \infty$. If we have $\Gamma f(x) \leq -g(f(x))$ for all $x \in \mathbb{X}$, then $\mathbb{E}_x \zeta < \infty$ for all x.*

The previous proposition, although stating the conditions on Γf quite differently than in Theorem 1.9, will be shown to follow from the former. This proposition is interesting only when $\inf_{x \in \mathbb{X}} g \circ f(x) = 0$ because then the condition required in 1.11 is weaker than the uniform requirement $\Gamma f(x) \leq -\epsilon$ for all x of Theorem 1.9.

If for some $x \in \mathbb{X}$, explosion (i.e. $\mathbb{P}_x(\zeta < +\infty) > 0$) occurs, irreducibility of the chain implies that the process remains explosive for all starting points $x \in \mathbb{X}$. However, since the phenomenon of explosion can only occur in the transient case, examples (see Section 3.4) can be constructed with non-trivial tail boundary so that for some initial $x \in \mathbb{X}$, we have both $0 < \mathbb{P}_x(\zeta < +\infty) < 1$. Additionally, the previous theorems established conditions that guarantee $\mathbb{E}_x \zeta < \infty$ (implying explosion). However examples are constructed where $\mathbb{P}_x(\zeta = \infty) = 0$ (explosion does not occur) while $\mathbb{E}_x \zeta = \infty$. It is therefore important to have results on conditional explosion.

Theorem 1.12. *Suppose that there exists a triple (ϵ, A, f) with $\epsilon > 0$, A a proper (finite or infinite) subset of $\mathbb{X}$ such that $\mathbb{X} \setminus A$ is infinite and $f \in \mathsf{Dom}_+(\Gamma)$ such that*

- *there exists $x_0 \notin A$ with $f(x_0) < \inf_{x \in A} f(x)$,*
- *$\Gamma f(x) \leq -\epsilon$ on A^c.*

Then $\mathbb{E}_{x_0}(\zeta | \tau_A = \infty) < \infty$.

Remark 1.13. It can be shown that the conditioning set appearing in Theorem 1.12 is not negligible, hence the statement about the conditional expectation is not trivial. Additionally, if A is finite, then $\mathbb{E}_x(\zeta) < \infty$. Therefore, we have a much more instrumental criterion for explosion than the one provided by Theorem 1.9; as a matter of fact, except a small number of elementary examples, it is almost impossible to find a function f that maps the Markov chain to a strict supermartingale everywhere, as was required in Theorem 1.9.

The previous results (Theorems 1.9 and 1.12) – through unconditional or conditional integrability of the explosion time ζ – give conditions establishing explosion. For Theorem 1.9 these conditions are even necessary and sufficient. It is nevertheless extremely difficult in general to prove that a function satisfying the conditions of the theorems does not exist. We need therefore a more manageable criterion guaranteeing non-explosion. Such a result is provided by the following

Theorem 1.14. *Let $f \in \mathsf{Dom}_+(\Gamma)$. If*

- *$f \to \infty$,*

– *there exists an increasing (not necessarily strictly) function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *whose inverse is locally integrable but has non integrable tail (i.e.* $G(z) := \int_0^z \frac{dy}{g(y)} < +\infty$ *for all* $z \in \mathbb{R}_+$ *but* $\lim_{z\to\infty} G(z) = \infty$*), and*
– $\Gamma f(x) \leq g(f(x))$ *for all* $x \in \mathbb{X}$,

then $\mathbb{P}_x(\zeta = +\infty) = 1$ *for all* $x \in \mathbb{X}$.

The idea of the proof of Theorem 1.14 relies on the intuitive idea that if $\Gamma f(x) \leq g(f(x))$ for all x, then $\mathbb{E}(f(X_t))$ cannot grow very fast with time and since $f \to \infty$ the process itself cannot grow fast either. The same idea has been used in [5] to prove non-explosion for Markov chains on metric separable spaces. Our result relies on the powerful ideas developed in the proof of Theorems 1 and 2 of [10] and of Theorem 4.1 of [5] but improves the original results in several respects. In first place, our result is valid on arbitrary denumerably infinite state spaces $\mathbb{X}$ (not necessarily subsets of $\mathbb{R}$); in particular, it can cope with models on higher dimensional lattices (like random walks in $\mathbb{Z}^d$ or reflected random walks in quadrants). Additionally, even for processes on denumerably infinite subsets of $\mathbb{R}$, our result covers critical regimes such as those exhibited by the Lamperti model (see Section 3.1), a "crash test" model, recalcitrant to the methods of [10].

1.3.3. Implosion

Finally, we state results about implosion.

Theorem 1.15. *Suppose that the chain is recurrent.*

1. *The following are equivalent:*
   – *There exist a triple* (ϵ, F, f) *with* $\epsilon > 0$, F *a finite set and* $f \in \mathsf{Dom}_+(\Gamma)$ *such that* $\sup_{x\in\mathbb{X}} f(x) = b < \infty$ *and*
   $$x \notin F \Rightarrow \Gamma f(x) \leq -\epsilon.$$
   – *For every finite* $A \in \mathscr{X}$, *there exists a constant* $C := C_A \in]0, \infty[$ *such that the following holds*
   $$x \notin A \Rightarrow \mathbb{E}_x \tau_A \leq C,$$
   (hence there is implosion towards A *and subsequently the chain is implosive).*
2. *Let* $f \in \mathsf{Dom}_+(\Gamma)$ *be such that* $f \to \infty$ *and assume there exist constants* $a > 0$, $c > 0$, $\epsilon > 0$, *and* $r > 1$ *such that* $f^r \in \mathsf{Dom}_+(\Gamma)$. *If further*
   – $\Gamma f(x) \geq -\epsilon$, *for all* $x \notin \mathsf{S}_a(f)$, *and*
   – $\Gamma f^r(x) \leq c f^{r-1}(x)$, *for all* $x \notin \mathsf{S}_a(f)$,
   then the chain does not implode towards $\mathsf{S}_a(f)$.

Proposition 1.16. *Suppose that the chain is implosive. Then there exists* $\alpha > 0$ *such that for every finite set* F *and every* $x \in F^c$, *we have* $\mathbb{E}_x(\exp(\alpha\tau_F)) < \infty$.

Consequently implosion implies the existence of all moments and even of exponential moments.

We conclude this section by the following

Proposition 1.17. *Suppose that the chain is recurrent. Let* $f \in \mathsf{Dom}_+(\Gamma)$ *be strictly positive and such that* $\sup_{x\in\mathbb{X}} f(x) = b < \infty$; *assume further that for any* a *such that* $0 < a < b$, *the sublevel set* $\mathsf{S}_a(f)$ *is finite. Let* $g : [0, b] \to \mathbb{R}_+$ *be an increasing function such that* $B := \int_0^b \frac{dy}{g(y)} < \infty$. *If* $\Gamma f(x) \leq -g(f(x))$ *for all* $x \notin \mathsf{S}_a(f)$ *then* $\mathbb{E}_x \tau_{\mathsf{S}_a(f)} \leq B$ *for all* $x \notin \mathsf{S}_a(f)$, *i.e. the chain implodes towards* $\mathsf{S}_a(f)$.

In some applications, it is quite difficult to guess immediately the form of the function f satisfying the uniform condition $\Gamma f(x) \leq -\epsilon$ required for the first statement of Theorem 1.15 to apply. It is sometimes more convenient to check merely that $\Gamma f(x) \leq -g(f(x))$ for some function g vanishing at 0 in some controlled way. Proposition 1.17 – although does not improve the already optimal statement 1 of Theorem 1.15 – provides us with a convenient alternative condition to be checked.

2. Proof of the main theorems

We have already introduced the notion of $\mathsf{Dom}(\Gamma)$. A related notion is that of locally p-integrable functions, defined as $\ell^p(\Gamma) = \{f \in m\mathscr{X} : \sum_{y\in\mathbb{X}} \Gamma_{xy}|f(y) - f(x)|^p < +\infty, \forall x \in \mathbb{X}\}$, for some $p > 0$. Obviously $\ell^1(\Gamma) = \mathsf{Dom}(\Gamma)$. In accordance to our notational convention on decorations, $\ell^p_+(\Gamma)$ will denote positive p-integrable functions. For $f \in \ell^1(\Gamma)$, we define the *local f-drift* of the embedded Markov chain as the random variable

$$\Delta^f_{n+1} := \Delta f(\tilde{\xi}_{n+1}) := f(\tilde{\xi}_{n+1}) - f(\tilde{\xi}_{n+1-}) = f(\tilde{\xi}_{n+1}) - f(\tilde{\xi}_n),$$

the *local mean f-drift* by

$$m_f(x) := \mathbb{E}(\Delta^f_{n+1}|\tilde{\xi}_n = x) = \sum_{y\in\mathbb{X}} P_{xy}(f(y) - f(x)) = \mathbb{E}_x \Delta^f_1,$$

and for $\rho \geq 1$ and $f \in \ell^\rho(\Gamma)$ the *ρ-moment of the local f-drift* by

$$v^{(\rho)}_f(x) := \mathbb{E}(|\Delta^f_{n+1}|^\rho\,|\tilde{\xi}_n = x) = \sum_{y\in\mathbb{X}} P_{xy}|f(y) - f(x)|^\rho = \mathbb{E}_x|\Delta^f_1|^\rho.$$

We always write shortly $v_f(x) := v^{(2)}_f(x)$. The action of the generator Γ on f reads

$$\Gamma f(x) := \sum_{y\in\mathbb{X}} \Gamma_{xy} f(y) = \gamma_x m_f(x).$$

Since $(\xi_t)_t$ is a pure jump process, the process $(X_t)_t$ transformed by $f \in \mathsf{Dom}(\Gamma)$, i.e. $X_t = f(\xi_t)$, is also a pure jump process reading, for $t < \zeta$,

$$X_t = f(\xi_t) = \sum_{k=0}^{\infty} f(\tilde{\xi}_k)\mathbb{1}_{[J_k,J_{k+1}[}(t) = X_0 + \sum_{k=0}^{\infty} \Delta^f_{k+1}\mathbb{1}_{]0,t]}(J_k).$$

If there is no explosion, the process (X_t) is a $(\mathscr{F}_t)$-semimartingale admitting the decomposition $X_t = X_0 + M_t + A_t$, where M_t is a martingale vanishing at 0 and the predictable compensator reads [11]

$$A_t = \int_{]0,t]} \Gamma f(\xi_{s-})ds = \int_{[0,t[} \Gamma f(\xi_s)ds.$$

Notice that, although not explicitly marked, (X_t), (M_t), and (A_t) depend on f. We use in the sequel also the infinitesimal form of the above decomposition. For any admissible f we have $dX_t = dM_t + dA_t = dM_t + \Gamma f(\xi_{t-})dt$; in particular, since (M_t) is a $(\mathscr{F}_t)$-martingale, $\mathbb{E}(dX_t|\mathscr{F}_{t-}) = \mathbb{E}(dA_t|\mathscr{F}_{t-}) = \Gamma f(\xi_{t-})dt$ represents the conditional increment of X as an ordinary differential multiplied by a previsible random factor.

Before starting to prove our results on the continuous time chain (ξ_t), recall first the criteria from [6] on recurrence/transience adapted to the embedded discrete time chain $(\tilde{\xi}_n)$ on $\mathbb{X}$.

Theorem 2.1. *The following are equivalent:*

- *The chain* $(\tilde{\xi}_n)$ *is recurrent.*
- *There exists a couple* (F, f)*, where* F *is a finite non-empty subset of* $\mathbb{X}$*, and* $f \in \mathsf{Dom}_+(P)$*, with* $f \to \infty$ *verifying* $m_f(x) \le 0$*, for all* $x \in F^c$*.*

Proof. See Theorem 2.2.1 of [6]. □

Theorem 2.2. *The following are equivalent:*

- *The chain* $(\tilde{\xi}_n)$ *is transient.*
- *There exists a couple* (A, f)*, where* A *is a subset of* $\mathbb{X}$*, and* $f \in \mathsf{Dom}_+(P)$ *is such that there exists* $x_0 \in A^c$ *for which* $f(x_0) < \inf_{x\in A} f(x)$ *and verifies* $m_f(x) \le 0$ *for all* $x \in A^c$*.*

Proof. See Theorem 2.2.2 of [6]. □

2.1. Proof of Theorems 1.5 and 1.7 on moments of passage times

We start by Theorem 1.7.

Lemma 2.3. *Let* $(\Omega, \mathscr{G}, (\mathscr{G}_t), \mathbb{P})$ *be a filtered probability space and* (Y_t) *a* $(\mathscr{G}_t)$*-adapted process taking values in* $[0, \infty[$*. Let* $c \ge 0$ *and denote* $T = \inf\{t \ge 0 : Y_t \le c\}$*. Suppose that*

1. *For some* $y \in]c, \infty[$*, we have* $\mathbb{P}_y(T < \infty) = 1$*.*
2. *There exists* $\epsilon > 0$ *such that* $\mathbb{E}(dY_t|\mathscr{G}_{t-}) \le -\epsilon dt$ *on the event* $\{T \ge t\}$*.*

Then $\mathbb{E}_y(T) \le \frac{y}{\epsilon}$*.*

Proof. Obviously $\mathbb{E}_y(T) = \mathbb{E}_y(T\mathbb{1}_{\{T=\infty\}}) + \mathbb{E}_y(T\mathbb{1}_{\{T<\infty\}})$ and if $\mathbb{P}_y(T = \infty) > 0$, then of course we have $\mathbb{E}_y(T) = \infty$. But this possibility is excluded by the hypothesis of the lemma. It remains thus to study what happens when $T < \infty$ almost surely. Since $\{T \ge t\} \in \mathscr{G}_{t-}$, the hypothesis of the lemma reads $\mathbb{E}(dY_{t\wedge T}|\mathscr{G}_{t-}) \le -\epsilon\mathbb{1}_{\{T\ge t\}}dt$. Taking expectations and integrating over time, yields, for every $t \in \mathbb{R}_+$, $0 \le \mathbb{E}_y(Y_{t\wedge T}) \le y - \epsilon\int_0^t \mathbb{P}_y(T \ge s)ds$ which implies $\epsilon\mathbb{E}_y(T \wedge t) \le y$. Now, since the event $\{T < \infty\}$ has probability 1, by taking the limit $t \to \infty$ in the previous formula yields $\mathbb{E}_y(T) \le \frac{y}{\epsilon}$, by Fatou's lemma. □

Proof of Theorem 1.7. [2 ⇒ 1]: Without loss of generality, using Remark 1.8, may be by modifying f, we can always assume that $\inf_{x\in F^c} f(x) > 0$ and $f(x) = 0$ for $x \in F$. Choose then an arbitrary $c \in]0, \inf_{x\in F^c} f(x)[$. Let $X_t = f(\xi_t)$ and $T = \inf\{t \ge 0 : X_t \le c\}$. Obviously, $T = \tau_F := \inf\{t \ge 0 : \xi_t \in F\}$ and the assumed recurrence of the chain guarantees that $\mathbb{P}_x(T < \infty) = 1$. Then condition 2 reads $\mathbb{E}(dX_{t\wedge T}|\mathscr{G}_{t-}) \le -\epsilon\mathbb{1}_{\{T\ge t\}}dt$. Hence, in accordance with Lemma 2.3, we get $\mathbb{E}_x(\tau_F) = \mathbb{E}_x(T) \le \frac{f(x)}{\epsilon}$ for every $x \notin F$. Now, let $x \in F$. Then

$$\begin{aligned}\mathbb{E}_x(\tau_F) &= \mathbb{E}_x(\tau_F|\tilde{\xi}_1 \in F)\mathbb{P}_x(\tilde{\xi}_1 \in F) + \mathbb{E}_x(\tau_F|\tilde{\xi}_1 \notin F)\mathbb{P}_x(\tilde{\xi}_1 \notin F)\\ &\le \sup_{x\in F}\left(\frac{1}{\gamma_x} + \sum_{y\in\mathbb{X}} P_{xy}\frac{f(y)}{\epsilon}\right) < \infty,\end{aligned}$$

the finiteness of the last expression being guaranteed by the conditions $f \in \mathsf{Dom}_+(\Gamma)$ and the finiteness of the set F.

[1 ⇒ 2]: Let $F = \{z\}$ for some fixed $z \in \mathbb{X}$; positive recurrence of the chain implies that $\mathbb{E}_x(\tau_F) < \infty$ for all $x \in \mathbb{X}$. Define

$$f(x) = \begin{cases} \mathbb{E}_x(\tau_F) & \text{if } x \notin F, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $m_f(x) = \sum_{y \neq z} P_{xy}\mathbb{E}_y(\tau_F) - \mathbb{E}_x(\tau_F) = \mathbb{E}_x(\tau_F - \sigma_1) - \mathbb{E}_x(\tau_F) = -\mathbb{E}_x(\sigma_1) = -\frac{1}{\gamma_x}$, for all $x \notin F$. It follows that $\Gamma f(x) = \gamma_x m_f(x) = -1$ outside F. By adding constant 1 to the function f determined above (see Remark 1.8), we see that f meets all the requirements. □

The proof of Theorem 1.5 is quite technical and will be split into several steps formulated as independent lemmata and propositions on semimartingales that may have an interest *per se*. As a matter of fact, we use these intermediate results to prove various results of very different nature.

Lemma 2.4. *Let $f \in \mathsf{Dom}_+(\Gamma)$ tending to infinity, $p \geq 2$, and $a > 0$. Use the abbreviation $A := \mathsf{S}_a(f)$. Denote $X_t = f(\xi_t)$ and assume further that $f^p \in \mathsf{Dom}_+(\Gamma)$. If there exists $c > 0$ such that*

$$\Gamma f^p(x) \leq -cf^{p-2}(x), \quad \forall x \notin A,$$

then the process defined by $Z_t = (X^2_{\tau_A \wedge t} + \frac{c}{p/2}\tau_A \wedge t)^{p/2}$ is a non-negative supermartingale.

Proof. Introducing the predictable decomposition $1 = \mathbb{1}_{\{\tau_A < t\}} + \mathbb{1}_{\{\tau_A \geq t\}}$, we get $\mathbb{E}(dZ_t|\mathscr{F}_{t-}) = \mathbb{E}(d(X_t^2 + \frac{c_1}{p/2}t)^{p/2}|\mathscr{F}_{t-})\mathbb{1}_{\{\tau_A \geq t\}}$. Now, (X_t) is a pure jump process, hence by applying the Itô formula, reading for any $g \in C^2$ and (S_t) a semimartingale, $dg(S_t) = g'(S_{t-})dS_t^c + \Delta g(S_t)$, where (S_t^c) denotes the continuous component of (S_t), we get

$$d\left(X_t^2 + \frac{c}{p/2}t\right)^{p/2} = c\left(X_{t-}^2 + \frac{c}{p/2}t\right)^{p/2-1} dt + \left(X_t^2 + \frac{c}{p/2}t\right)^{p/2} - \left(X_{t-}^2 + \frac{c}{p/2}t\right)^{p/2}.$$

Writing the semimartingale decomposition for the process (X_t^p), we remark that the hypothesis of the lemma implies that

$$\mathbb{E}(dX_t^p|\mathscr{F}_{t-}) = \Gamma f^p(\xi_{t-})dt \leq -cX_{t-}^{p-2}dt \quad \text{on the event } \{\tau_A \geq t\}.$$

Applying the conditional Minkowski inequality and the supermartingale hypothesis, we get, on the set $\{\tau_A \geq t\}$,

$$\begin{aligned}
\mathbb{E}\left(\left(X_t^2 + \frac{c}{p/2}t\right)^{p/2} \middle| \mathscr{F}_{t-}\right) &\leq \left[\mathbb{E}(X_t^p|\mathscr{F}_{t-})^{2/p} + \frac{c}{p/2}t\right]^{p/2} \\
&= \left[\left(X_{t-}^p + \mathbb{E}(dX_t^p|\mathscr{F}_{t-})\right)^{2/p} + \frac{c}{p/2}t\right]^{p/2} \\
&\leq \left(X_{t-}^2\left(1 - \frac{c}{X_{t-}^2}dt\right)^{2/p} + \frac{c}{p/2}t\right)^{p/2} \\
&\leq \left(X_{t-}^2 - \frac{c}{p/2}dt + \frac{c}{p/2}t\right)^{p/2}.
\end{aligned}$$

Hence, on the event $\{\tau_A \geq t\}$, we have the estimate

$$\mathbb{E}(dZ_t|\mathscr{F}_{t-}) \leq c\left(X_{t-}^2 + \frac{c}{p/2}t\right)^{p/2-1} dt + \left(X_{t-}^2 + \frac{c}{p/2}t - \frac{c}{p/2}dt\right)^{p/2} - \left(X_{t-}^2 + \frac{c}{p/2}t\right)^{p/2}.$$

Simple expansion of the remaining differential forms (containing now only $\mathscr{F}_{t-}$-measurable random variables) yields $\mathbb{E}(dZ_t|\mathscr{F}_{t-}) \leq 0$. □

Corollary 2.5. *Let $f \in \mathsf{Dom}_+(\Gamma)$ tending to infinity, $p \geq 2$, and $a > 0$. Use the abbreviation $A := \mathsf{S}_a(f)$. Denote $X_t = f(\xi_t)$ and assume further that $f^p \in \mathsf{Dom}_+(\Gamma)$. If there exists $c > 0$ such that*

$$\Gamma f^p(x) \leq -cf^{p-2}(x), \quad \forall x \notin A,$$

then there exists $c' > 0$ such that

$$\mathbb{E}_x(\tau_A^q) \leq c' f(x)^{2q} \quad \textit{for all } q \leq p/2 \textit{ and all } x \in \mathbb{X}.$$

Proof. Without loss of generality, we can assume that $x \in A^c$ since otherwise the corollary holds trivially. Denoting by $Y_t = X_{t\wedge\tau_A}^2 + \frac{c}{p/2}t \wedge \tau_A$, we observe that $Z_t = Y_t^{p/2}$ is a non-negative supermartingale by virtue of Lemma 2.4. Since the function $\mathbb{R}_+ \ni w \mapsto w^{2q/p} \in \mathbb{R}_+$ is increasing and concave for $q \leq p/2$, it follows that Y_t^q is also a supermartingale. Hence,

$$\left(\frac{c}{p/2}\right)^q \mathbb{E}_x[(t \wedge \tau_A)^q] \leq \mathbb{E}_x(Y_t^q) \leq \mathbb{E}_x(Y_0^q) = f(x)^{2q}.$$

We conclude by the monotone convergence theorem on identifying $c' = (\frac{p}{2c})^q$. □

Proposition 2.6. *Let $f \in \mathsf{Dom}_+(\Gamma)$ tending to infinity, $0 < p \leq 2$, and $a > 0$. Use the abbreviation $A := \mathsf{S}_a(f)$. Denote $X_t = f(\xi_t)$ and assume further that $f^p \in \mathsf{Dom}_+(\Gamma)$. If there exists $c > 0$ such that*

$$\Gamma f^p(x) \leq -cf^{p-2}(x), \quad \forall x \notin A,$$

then the process, defined by $Z_t = X_{\tau_A\wedge t}^p + \frac{c}{q}(\tau_A \wedge t)^q$, satisfies

$$\mathbb{E}_x(Z_t) \leq c'' f(x)^p, \quad \textit{for all } q \in]0, p/2].$$

Proof. Since $d(X_t^p + \frac{c}{q}t^q) = dX_t^p + ct^{q-1}dt$, we have

$$\mathbb{E}(dZ_t|\mathscr{F}_{t-}) \leq \mathbb{E}(dZ_t|\mathscr{F}_{t-})\mathbb{1}_{\{\tau_A \geq t\}} \leq cdt\mathbb{1}_{\{\tau_A \geq t\}}(-X_{t-}^{p-2} + t^{q-1}).$$

Now, $\frac{q}{p} \leq \frac{1}{2} \leq \frac{1-q}{2-p}$. Choosing $v \in]\frac{q}{p}, \frac{1-q}{2-p}[$, we write

$$\mathbb{E}(dZ_t|\mathscr{F}_{t-}) \leq cdt\mathbb{1}_{\{\tau_A \geq t\}}(-X_{t-}^{p-2} + t^{q-1})\mathbb{1}_{\{X_{t-} \in]a,t^v]\}} + cdt\mathbb{1}_{\{\tau_A \geq t\}}(-X_{t-}^{p-2} + t^{q-1})\mathbb{1}_{\{X_{t-} \in]t^v,+\infty]\}}.$$

For $X_{t-} \leq t^v$, the first term of the right hand side of the previous inequality is non-positive; as for the second, the condition $X_{t-} > t^v$ implies that $-X_{t-}^{p-2} + t^{q-1} \leq t^{q-1}$. Hence, since

$(X_t^p)_{t\in\mathbb{R}_+}$ is a supermartingale,

$$\mathbb{E}_x(dZ_t) \le ct^{q-1}\mathbb{P}_x(X_{t-}^p \ge t^{vp}; \tau_A \ge t)dt \le ct^{q-1}\frac{\mathbb{E}_x(X_{t-}^p)}{t^{vp}}dt \le ct^{q-1-vp}f(x)^p dt.$$

Integrating this differential inequality yields $\mathbb{E}_x(Z_t) \le cf^p(x)\int_{a^{1/v}}^\infty \frac{dt}{t^{vp+1-q}}$; the condition $v > q/p$ ensures the finiteness of the last integral proving thus the lemma with $c'' = c\int_{a^{1/v}}^\infty \frac{dt}{t^{vp+1-q}}$. □

Corollary 2.7. *Under the same conditions as in Proposition 2.6, there exists $c''' > 0$ such that $\mathbb{E}_x(\tau_A^q) \le c''' f(x)^p, \forall q \in]0, p/2]$.*

Proof. Since X_t is non-negative, $\frac{q}{c}Z_t \ge (t\wedge\tau_A)^q$. By the previous proposition, $\mathbb{E}_x[(t\wedge\tau_A)^q] \le \frac{q}{c}\mathbb{E}_x(Z_t) \le c''\frac{q}{c}f(x)^p$. We conclude by the monotone convergence theorem on identifying $c''' = \frac{c''q}{c}$. □

Remark 2.8. All the propositions, lemmata, and corollaries shown so far allow to prove statement 1 of Theorem 1.5. The subsequent propositions are needed for statement 2 of this theorem. Notice also that the following Proposition 2.9 is very important and tricky. It provides us with a generalisation of Theorem 3.1 of Lamperti [13] and serves twice in this paper: one first time to establish conditions for some moments of passage time to be infinite (statement 2 of Theorem 1.5) and once more in a very different context, namely for finding conditions for the chain not to implode (statement 2 of Theorem 1.15).

Proposition 2.9. *Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_t, \mathbb{P})$ be a filtered probability space and (Y_t) be a $(\mathcal{G}_t)$-adapted process taking values in an unbounded subset of $\mathbb{R}_+$. Let $a > 0$ and $T_a = \inf\{t \ge 0 : Y_t \le a\}$. Suppose that there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

1. *$\mathbb{E}(dY_t|\mathcal{G}_{t-}) \ge -c_1 dt$ on the event $\{T_a > t\}$, and*
2. *there exists $r > 1$ such that $\mathbb{E}(dY_t^r|\mathcal{G}_{t-}) \le c_2 Y_{t-}^{r-1}dt$ on the event $\{T_a > t\}$.*

Then, for all $\alpha \in]0, 1[$, there exist $\epsilon > 0$ and $\delta > 0$ such that

$$\forall t > 0 : \mathbb{P}(T_a > t + \epsilon Y_{t\wedge T_a}|\mathcal{G}_t) \ge 1-\alpha, \quad \text{on the event } \{T_a > t; Y_t > a(1+\delta)\}.$$

Remark 2.10. The meaning of Proposition 2.9 is the following. If the process (Y_t) has average increments bounded from below by a constant $-c_1$, it is intuitively appealing to suppose that the average time of reaching 0 is of the same order of magnitude as Y_0. However this intuition proves false if the increments are wildly unbounded since then 0 can be reached in one step. Condition 2, by imposing some control on r-moments of the increments with $r > 1$, prevents this from occurring. It is in fact established that if $T_a > t$, the remaining time $T_a - t$ to reach $A_a := [0, a]$ exceeds ϵY_t with probability $1-\alpha$; more precisely, for every α we can chose ϵ such that $\mathbb{P}(T_a - t > \epsilon Y_t|\mathcal{G}_t) \ge 1-\alpha$.

Proof of Proposition 2.9. Let $\sigma = (T_a - t)\mathbb{1}_{\{T_a\ge t\}}$; then for all $s > 0$ we have $\{\sigma < s\} = \{T_a \ge t\}\cap\{T_a < t+s\} \in \mathcal{G}_{t+s-}$. To prove the proposition, it is enough to establish $\mathbb{P}(\sigma > \epsilon Y_t|\mathcal{G}_t) \ge 1-\alpha$ on the set $\{\sigma > 0; Y_t > a(1+\delta)\}$. On this latter set: $\mathbb{P}(\sigma > \epsilon Y_t|\mathcal{G}_t) = \mathbb{P}(Y_{t+(\epsilon Y_t)\wedge\sigma} > a|\mathcal{G}_t)$, because once the process $Y_{t+(\epsilon Y_t)\wedge\sigma}$ enters in A_a, it remains there forever, due to the stopping by σ. On defining $U := Y_{(\epsilon Y_t)\wedge\sigma+t}$ one has

$$\begin{aligned}\mathbb{E}(U|\mathcal{G}_t) &= \mathbb{E}(U\mathbb{1}_{\{U\le a\}}|\mathcal{G}_t) + \mathbb{E}(U\mathbb{1}_{\{U>a\}}|\mathcal{G}_t)\\ &\le a + (\mathbb{E}(U^r|\mathcal{G}_t))^{1/r}(\mathbb{P}(U > a|\mathcal{G}_t))^{1-1/r};\end{aligned}$$

therefore

$$\mathbb{P}(U > a|\mathcal{G}_t) \geq \left[\frac{(\mathbb{E}(U|\mathcal{G}_t) - a)_+}{(\mathbb{E}(U^r|\mathcal{G}))^{1/r}} \right]^{r/(r-1)}.$$

To minorise the numerator, we observe that on the set $\{\sigma > 0, Y_t > (1+\delta)a\}$,

$$\mathbb{E}(U|\mathcal{G}_t) = \mathbb{E}(Y_{t+\epsilon Y_{t\wedge\sigma}} - Y_t|\mathcal{G}_t) + Y_t = \int_t^{t+\epsilon Y_t} \mathbb{E}(dY_{s\wedge\sigma}|\mathcal{G}_t) + Y_t \geq -c_1\epsilon Y_t + Y_t.$$

To majorise the denominator $\mathbb{E}(U^r|\mathcal{G}) = \mathbb{E}(Y^r_{t+(\epsilon Y_t)\wedge\sigma}|\mathcal{G}_t)$, we must be able to majorise $\mathbb{E}(Y^r_{t+s\wedge\sigma}|\mathcal{G}_t)$ for arbitrary $s > 0$. Let $t > 0$ be arbitrary and S be a $\mathcal{G}_t$-optional random variable, $S > 0$. For $c_3 = c_2/r$ and any $s \in]0, S]$, define

$$F_S(s) = \mathbb{E}[(Y_{t+s\wedge\sigma} + c_3 S - c_3 s \wedge \sigma)^r|\mathcal{G}_t].$$

We shall show that $F_S(s) \leq F_S(s-)$ for all $s \in]0, S]$. It is enough to show this inequality on $\{\sigma > s\}$ since otherwise $F_S(s) = F_S(s-)$ and there is nothing to prove. To show that F_S is decreasing in $]0, S]$, it is enough to show that $\mathbb{E}(d\Xi_s|\mathcal{G}_{t+s-}) \leq 0$ for all $s \in]0, S]$, where $\Xi_s = (Y_{t+s\wedge\sigma} + c_3 S - c_3 s \wedge \sigma)^r$. Now, on $\{\sigma > s\}$, by use of Itô's formula, we get

$$d\Xi_s = -rc_3(Y_{t+s-} + c_3 S - c_3 s)^{r-1} ds + (Y_{t+s} + c_3 S - c_3 s)^r - (Y_{t+s-} + c_3 S - c_3 s)^r.$$

Moreover, using the Minkowski inequality, we get

$$\mathbb{E}[(Y_{t+s} + c_3 S - c_3 s)^r|\mathcal{G}_{t+s-}] \leq \left[\mathbb{E}(Y^r_{t+s}|\mathcal{G}_{t+s-})^{1/r} + c_3 S - c_3 s \right]^r,$$

and by use of the hypothesis

$$\begin{aligned}\mathbb{E}(Y^r_t|\mathcal{G}_{t-}) &\leq Y^r_{t-} + c_2 Y^{r-1}_{t-} \mathbb{1}_{\{T_a > t\}} dt = Y^r_{t-} \left(1 + \frac{c_2}{Y_{t-}} \mathbb{1}_{\{T_a > t\}} dt \right) \\ &\leq \left(Y_{t-} + c_3 \mathbb{1}_{\{T_a > t\}} dt \right)^r.\end{aligned}$$

Therefore, $\mathbb{E}(d\Xi_s|\mathcal{G}_{t+s-}) \leq 0$ for all $s \in]0, S]$. Subsequently, for all $S > 0$,

$$Y^r_{t+S} = F_S(S) \leq \lim_{s\to 0+} F_S(s) = (Y_t + c_3 S)^r.$$

Since S is an arbitrary $\mathcal{G}_t$-optional random variable, on choosing $S = \epsilon Y_t$, we get finally

$$(\mathbb{E}(Y^r_{t+\epsilon Y_t}|\mathcal{G}_t))^{1/r} \leq Y_t + c_3\epsilon Y_t.$$

Substituting yields, that for any $\alpha \in]0, 1[$, parameters $\epsilon > 0$ and $\delta > 0$ can be chosen so that the following inequality holds:

$$\begin{aligned}\mathbb{P}(U > a|\mathcal{G}_t) &\geq \left[\frac{\left(1 - c_1\epsilon - \frac{a}{Y_t}\right)_+}{1 + c_3\epsilon} \right]^{\frac{r}{r-1}} \geq \left(1 - c_1\epsilon - \frac{1}{1+\delta} \right)_+^{\frac{r}{r-1}} (1 + c_3\epsilon)^{-\frac{r}{r-1}} \\ &\geq 1 - \alpha. \quad \square\end{aligned}$$

Lemma 2.11. *Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$ be a filtered probability space, (Y_t) and (Z_t) two $\mathbb{R}_+$-valued, $(\mathcal{G}_t)$-adapted processes on it. For $a > 0$, we denote $S_a = \inf\{t \geq 0 : Y_t \leq a\}$ and $T_a = \inf\{t \geq 0 : Z_t \leq a\}$. Suppose that there exist positive constants a, b, p, K_1 such that*

1. *$Y_t \leq bZ_t$ almost surely, for all t, and*

2. $\mathbb{E}(Z^p_{t\wedge T_a}) \le K_1$.

Then, there exists $K_2 > 0$ *such that* $\mathbb{E}(Y^p_{t\wedge S_{ab}}) \le K_2$.

Proof. For arbitrary $s > 0$, the condition $Z_s < a$ implies $X_s \le bZ_s < ab$ almost surely. Hence, $\{S_{ab} \ge t\} \subseteq \{T_a \ge t\}$. Then,

$$\begin{aligned}\mathbb{E}(Y^p_{t\wedge S_{ab}}) &\le \mathbb{E}\left[Y^p_{t\wedge S_{ab}}(\mathbb{1}_{\{S_{ab}\ge t\}} + \mathbb{1}_{\{S_{ab}<t\}})\right]\\ &\le \mathbb{E}(Y^p_t \mathbb{1}_{\{S_{ab}\ge t\}}) + (ab)^p \le b^p K_1 + (ab)^p := K_2. \quad \square\end{aligned}$$

Proposition 2.12. *Let* $(\Omega, \mathscr{G}, (\mathscr{G}_t), \mathbb{P})$ *be a filtered probability space,* (Y_t) *and* (Z_t) *two* $\mathbb{R}_+$*-valued,* $(\mathscr{G}_t)$*-adapted processes on it. For* $a > 0$*, we denote* $S_a = \inf\{t \ge 0 : Y_t \le a\}$ *and* $T_a = \inf\{t \ge 0 : Z_t \le a\}$*. Suppose that*

1. *there exist positive constants* a, c_1, c_2, r *such that*
   - $\mathbb{E}(dZ_t|\mathscr{G}_{t-}) \ge -c_1 dt$ *on the event* $\{T_a \ge t\}$,
   - $\mathbb{E}(dZ^r_t|\mathscr{G}_{t-}) \le c_2 Z^{r-1}_{t-} dt$ *on the event* $\{T_a \ge t\}$,
   - $Z_0 = z > a$.
2. $Y_0 = y$ *and there exists a constant* $b > 0$ *such that*
   - $ab < y < bz$ *and*
   - $Y_t \le bZ_t$ *almost surely for all* t.

If for some $p > 0$*, the process* $(Y^p_{t\wedge S_{ab}})$ *is a submartingale, then* $\mathbb{E}(T^q_a) = \infty$ *for all* $q > p$.

Proof. We can without loss of generality examine solely the case $\mathbb{P}(S_{ab} < \infty) = 1$, since otherwise $\mathbb{E}(T^q_a) = \infty$ holds trivially for all $q > 0$. Assume further that for some $q > p$, it happens $\mathbb{E}(T^q_a) < \infty$. Hypothesis 1 allows applying Proposition 2.9; for $\alpha = 1/2$, we can thus determine positive constants ϵ and δ such that $\mathbb{P}(T_a > t + \epsilon Z_{t\wedge T_a}|\mathscr{G}_t) \ge 1/2$ on the event $\{Z_{t\wedge T_a} > a(1+\delta)\}$. Hence

$$\begin{aligned}\mathbb{E}(T^q_a) &\ge \mathbb{E}(T^q_a \mathbb{1}_{\{Z_{t\wedge T_a} > a(1+\delta)\}})\\ &\ge \frac{1}{2}\mathbb{E}\left[(t + Z_{t\wedge T_a})^q \mathbb{1}_{\{Z_{t\wedge T_a} > a(1+\delta)\}}\right]\\ &\ge \frac{\epsilon^q}{2}\mathbb{E}(Z^q_{t\wedge T_a}) - \frac{\epsilon^q}{2} a^q (1+\delta)^q.\end{aligned}$$

Now, finiteness of the q moment of T_a implies the existence of a constant $K_1 > 0$ such that $\mathbb{E}(Z^q_{t\wedge T_a}) \le K_1$. From Lemma 2.11 it follows that there exists some $K_2 > 0$ such that $\mathbb{E}(Y^q_{t\wedge S_{ab}}) \le K_2$. The previous majorisation – holding for $q > p$ – implies that the family $(Y^p_{t\wedge S_{ab}})_{t\in\mathbb{R}^+}$ is uniformly integrable and since the time S_{ab} is assumed almost surely finite, we get $\lim_{t\to\infty} \mathbb{E}(Y^p_{t\wedge S_{ab}}) = \mathbb{E}(Y^p_{S_{ab}}) \le (ab)^p$. On the other hand, $(Y^p_{t\wedge S_{ab}})$ is a submartingale, so is thus a fortiori $(Y^q_{t\wedge S_{ab}})$. Hence, $\mathbb{E}(Y^q_{t\wedge S_{ab}}) \ge \mathbb{E}(Y^q_0) = y^q$, leading to a contradiction if we chose $y > (ab)^{p/q}$. $\square$

Corollary 2.13. *Let* $(\Omega, \mathscr{G}, (\mathscr{G}_t), \mathbb{P})$ *be a filtered probability space,* (X_t) *a* $\mathbb{R}_+$*-valued,* $(\mathscr{G}_t)$*-adapted process on it. For* $a > 0$*, we denote* $T_a = \inf\{t \ge 0 : X_t \le a\}$*. Suppose that there exist positive constants* a, c_1, c_2, p, r *such that*

- $X_0 = x > a$,
- $\mathbb{E}(dX_t|\mathscr{G}_{t-}) \ge -c_1 dt$ *on the event* $\{T_a \ge t\}$,

– *$\mathbb{E}(dX_t^r|\mathscr{G}_{t-}) \leq c_2 X_{t-}^{r-1} dt$ on the event $\{T_a \geq t\}$,*
– *$(X_{t\wedge T_a}^p)$ is a submartingale.*

Then $\mathbb{E}(T_a^q) = \infty$ for all $q > p$.

After all this preparatory work, the proof of Theorem 1.5 is now immediate.

Proof of Theorem 1.5. Write $X_t = f(\xi_t)$ and use the abbreviation $A := \mathsf{S}_a(f)$. Notice moreover that $\tau_A = T_a$.

1. Since $f \to \infty$ the set A is finite. The integrability of the passage time follows from Corollaries 2.5 and 2.7.
2. On identifying Z_t and Y_t in Proposition 2.12 with $g(\xi_t)$ and $f(\xi_t)$ respectively, we see that the conditions of the theorem imply the hypotheses of the proposition. The non-existence of moments immediately follows. □

2.2. *Proof of theorems on explosion and implosion*

As stated in the introduction, the result concerning integrability of the explosion time is reminiscent to Foster's criterion for positive recurrence! The reason lies in Lemma 2.3.

Proof of Theorem 1.9. $1 \Rightarrow 2$: Condition $\Gamma f(x) \leq -\epsilon$ implies that $m_f(x) < 0$. Since this occurs for all $x \in \mathbb{X}$, the function f cannot be constant. Therefore $i := \inf_{x\in\mathbb{X}} f(x) < \sup_{x\in\mathbb{X}} f(x) =: s$. On choosing a $a \in]i, s[$, and defining $A^c = \{x \in \mathbb{X} : f(x) \leq a\}$, we are in the situation described by Theorem 2.2, hence transience follows from that theorem. Let $(F_n)_{n\in\mathbb{N}}$ be an arbitrary nested increasing sequence of finite sets exhausting $\mathbb{X}$. Since the chain is transient, it follows that it leaves almost surely any finite set in finite time, hence $\mathbb{P}_x(\tau_{F_n^c} < \infty) = 1$, for every $x \in F_n$. Since the condition $\Gamma f(x) \leq -\epsilon$ holds everywhere for a strictly positive f, using Remark 1.8, we can change the function f into $f_n = f\mathbb{1}_{F_n}$ that vanishes outside F_n and still verifies the condition $\Gamma f_n(x) \leq -\epsilon$, for $x \in F_n$. Additionally, since the set F_n is finite, we have $\min_{x\in F_n} f(x) > 0$. Consider now the process $Y_t^{(n)} = f_n(\xi_t)$; the condition $\Gamma f_n(x) \leq -\epsilon$ implies that the process is a strong supermartingale on F_n. Choose an arbitrary $c \in]0, \min_{x\in F_n} f(x)[$ and observe that $T_n = \inf\{t \geq 0 : Y_t^{(n)} \leq c\} = \tau_{F_n^c}$. Lemma 2.3 guarantees that $\epsilon\mathbb{E}_x(\tau_{F_n^c}) \leq f_n(x) = f(x)$ on F_n. But $F_n \nearrow \mathbb{X}$ and $\tau_{F_n^c} \nearrow \zeta$; therefore, by Fatou's lemma, $\mathbb{E}_x(\zeta) \leq \frac{f(x)}{\epsilon}$ on $\mathbb{X}$.

$2 \Rightarrow 1$: Let $f(x) := \mathbb{E}_x(\zeta) \in]0, \infty[$. Compute then $m_f(x) = \sum_{y\in\mathbb{X}} P_{xy} f(y) - f(x) = \sum_{y\in\mathbb{X}} P_{xy}\mathbb{E}_y(\zeta) - \mathbb{E}_x(\zeta) = \mathbb{E}_x(\zeta - \sigma_1) - \mathbb{E}_x(\zeta) = -\frac{1}{\gamma_x}$ (the penultimate equality holding because the kernel P is stochastic on $\mathbb{X}$, hence it is impossible to reach the boundary ∂ in one step). It follows that $\Gamma f(x) = -1$ for all $x \in \mathbb{X}$. □

Proof of Proposition 1.11. Let $G(z) = \int_0^z \frac{dy}{g(y)}$. Then G is differentiable, with $G'(z) = \frac{1}{g(z)} > 0$ hence an increasing function of $z \in [0, b]$. Since g is increasing, G' is decreasing and hence G is concave satisfying $\lim_{z\to 0} G(z) = 0$ and $\lim_{z\to\infty} G(z) < \infty$. Additionally, boundedness of G implies that $G \circ f \in \ell_+^1(\Gamma)$. Due to differentiability and concavity of G, we have:

$$\begin{aligned}\Gamma G \circ f(x) &= \gamma_x \mathbb{E}\left[G(f(\tilde{\xi}_n) + \Delta_{n+1}^f) - G(f(\tilde{\xi}_n))|\tilde{\xi}_n = x\right]\\ &\leq \gamma_x \frac{1}{g(f(x))} m_f(x) = \frac{\Gamma f(x)}{g(f(x))} \leq -c;\end{aligned}$$

we conclude by Theorem 1.9 because $G \circ f$ is strictly positive and bounded. □

Proof of Theorem 1.12. The conditions, thanks to Theorem 2.2, imply transience of the chain. Additionally, it can be shown that $\mathbb{P}_{x_0}(\tau_A = \infty) > 0$ (see proof of Theorem 2.2.2 of [6]).

Let $(F_n)_{n\in\mathbb{N}}$ be a nested increasing sequence of finite sets exhausting $\mathbb{X} \setminus A$. Since every F_n is finite and the chain is transient, $\mathbb{P}_{x_0}(\tau_{F_n^c} < \infty) = 1$; the very same arguments used in the proof of Theorem 1.9 and Lemma 2.3 imply here that $\mathbb{E}_{x_0}(\tau_{F_n^c}) \le \frac{f(x_0)}{\epsilon}$. Now $\tau_{F_n^c} \nearrow \zeta \wedge \tau_A$; hence by Fatou's lemma, $\mathbb{E}_{x_0}(\zeta \wedge \tau_A) \le \frac{f(x_0)}{\epsilon}$. But $\mathbb{E}_{x_0}(\zeta \wedge \tau_A) \ge \mathbb{E}_{x_0}(\zeta \wedge \tau_A | \tau_A = \infty)\mathbb{P}_{x_0}(\tau_A = \infty)$. We conclude that

$$\mathbb{E}_{x_0}(\zeta \wedge \tau_A | \tau_A = \infty) \le \frac{f(x_0)}{\epsilon \mathbb{P}_{x_0}(\tau_A = \infty)}. \quad \square$$

Proof of Theorem 1.14. Let $G(z) = \int_0^z \frac{1}{g(y)} dy$; this function is differentiable with $G'(z) = \frac{1}{g(z)} > 0$, hence increasing. Since g is increasing, G' is decreasing, hence G is concave. Non integrability of the infinite tail means that G is unboundedly increasing towards ∞ as $z \to \infty$, hence $G \circ f \to \infty$. Concavity and differentiability of G imply that $G(f(x) + \Delta) - G(f(x)) \le \Delta G'(f(x))$; integrability of f guarantees then $0 \le \mathbb{E}(G(f(\tilde{\xi}_n) + \Delta^f_{n+1}) | \tilde{\xi}_n = x) \le G(f(x)) + \frac{m_f(x)}{g(f(x))} < \infty$ so that $F := G \circ f \in \ell^1_+(\Gamma)$ as well. Now

$$\begin{aligned}
\Gamma G \circ f(x) &= \gamma_x \mathbb{E}\left[G(f(\tilde{\xi}_n) + \Delta^f_{n+1}) - G(f(\tilde{\xi}_n)) | \tilde{\xi}_n = x\right] \\
&= \gamma_x \mathbb{E}\left[\int_{f(x)}^{f(x)+\Delta^f_{n+1}} \frac{1}{g(y)} dy | \tilde{\xi}_n = x\right] \\
&\le \gamma_x \frac{1}{g(f(x))} m_f(x) \le c.
\end{aligned}$$

Let $X_t = F(\xi_t)$ be the process obtained form the Markov chain after transformation by F. Using the semimartingale decomposition of $X_t = X_0 + M_t + \int_{]0,t]} \Gamma F(\xi_{s-}) ds$, it becomes obvious then that $\mathbb{E}_x X_t \le F(x) + ct$, showing that for every finite t, $\mathbb{P}_x(X_t = \infty) = 0$. But since $F \to \infty$ the process itself ξ_t cannot explode. $\square$

Proposition 2.14. *Suppose the chain is recurrent and there exist a finite set $A \in \mathscr{X}$ and a constant $C = C_A > 0$ such that, for all $x \in \mathbb{X}$, the uniform bound $\mathbb{E}_x \tau_A \le C$ holds. Then the chain implodes towards any state $z \in \mathbb{X}$.*

Proof. We only sketch the proof since it relies on the same ideas serving to prove that an irreducible Markov chain that positively recurs to a finite set, positively recurs everywhere. First remark that obviously $\sigma_0 \le \tau_A$, where σ_0 is the holding time at the initial state, i.e. $\frac{1}{\gamma_x} = \mathbb{E}_x(\sigma_0) \le \mathbb{E}_x \tau_A \le C$. Hence $\underline{\gamma} := \inf_{x\in\mathbb{X}} \gamma_x > 0$. Let a be an arbitrary state in A and z an arbitrary fixed state in $\mathbb{X}$. Irreducibility means that there exists a path of finite length, say $k = k_a$, satisfying $a \equiv x_0, \ldots, x_k \equiv z$ and $\delta = \delta_a = \prod_{i=0}^{k-1} P_{x_i, x_{i+1}} > 0$. Now it is straightforward to show that the probability that starting on A, we recur n times to A before reaching $z \in \mathbb{X}$ decays exponentially with n. Therefore, once we have reached A, with probability 1 we will reach z. Standard arguments then allow to show that $\sup_x \mathbb{E}_x(\tau_z) \le K_{A,z}$ for some constant $K_{A,z} < \infty$ and this guarantees implosion towards any $z \in \mathbb{X}$. $\square$

Proof of Theorem 1.15. 1. We first show implosion. Since f is bounded, the condition $\Gamma f(x) = \gamma_x m_f(x) \le -\epsilon$ guarantees that $\underline{\gamma} = \inf_x \gamma_x > 0$, hence the chain has always strictly positive speed.

[$\Rightarrow$:] Since F is finite and the chain is recurrent, it follows that $\mathbb{P}_x(\tau_F < \infty) = 1$ for all $x \notin F$; associated with the condition $\Gamma f(x) \leq -\epsilon$ for $x \notin F$ these properties guarantee – by virtue of Lemma 2.3 – that $\mathbb{E}_x(\tau_F) \leq \frac{f(x)}{\epsilon} \leq \sup_{z\in\mathbb{X}} \frac{f(z)}{\epsilon} \leq \frac{b}{\epsilon}$. Now, due to recurrence and irreducibility, if there exists a constant $C' = C'_F$ such that $x \notin F \Rightarrow \mathbb{E}_x(\tau_F) < C'$, then, for every $z \in \mathbb{X}$, there exists a constant C such that $x \neq z \Rightarrow \mathbb{E}_x(\tau_z) < C$, by virtue of Proposition 2.14.

[$\Leftarrow$:] Suppose now that for a finite $A \in \mathscr{X}$, there exists a constant C such that $\mathbb{E}_x(\tau_A) \leq C$. Consequently $\frac{1}{\gamma_x} \leq \mathbb{E}_x\tau_A \leq C$, leading necessarily to the lower bound $\underline{\gamma} > 0$. Define

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ \mathbb{E}_x\tau_A & \text{if } x \notin A. \end{cases}$$

Then it is immediate to show that for $x \notin A$, we have $\Gamma f(x) \leq -1$.

2. To show non-implosion, use Proposition 2.9 to guarantee that if at some time t the process ξ_t is at some point x_0, then the time needed for the process $X_t = f(\xi_t)$ to reach $\mathsf{S}_a(f)$ exceeds $\epsilon f(x_0)$ with some substantially large probability. More precisely, $\mathbb{P}_{x_0}(\tau_{\mathsf{S}_a(f)} - t > \epsilon f(x_0)) \geq 1-\alpha$, for some $\alpha \in]0, 1[$. Therefore $\mathbb{E}_{x_0}(\tau_{\mathsf{S}_a(f)}) \geq (1-\alpha)\epsilon f(x_0)$ and since $f \to \infty$ then this expectation cannot be bounded uniformly in x_0. □

Proof of Proposition 1.16. Since the chain is implosive, for every finite set F, there exists a constant $C = C_F$ such that every $x \in F^c$, we have $\mathbb{E}_x(\tau_F) \leq C$. By Markov's inequality, we get that $\mathbb{P}_x(\tau_F \geq 2C) \leq \frac{1}{2}$. Now, using the strong Markov property we show that $\mathbb{P}_x(\tau_F \geq 2kC) \leq \frac{1}{2^k}$ for all $k \geq 1$, from which the existence of exponential moments $\mathbb{E}_x(\exp(\alpha\tau_F)) < \infty$ follows, for sufficiently small $\alpha > 0$. □

Proof of Proposition 1.17. Let $G(z) = \int_0^z \frac{dy}{g(y)}$. Since $G'(z) = \frac{1}{g(z)} > 0$ the function G is increasing with $G(0) = 0$ and $G(b) = B$. Since g is increasing $G' = \frac{1}{g}$ is decreasing, hence the function G is concave. Then concavity leads to the majorisation

$$m_{G\circ f}(x) \leq G'(f(x))m_f(x) = \frac{m_f(x)}{g(f(x))}.$$

The condition imposed on the statement of the proposition implies that $\Gamma G \circ f(x) \leq -1$. We conclude from Lemma 2.3. □

3. Application to some critical models

This section intends to treat some problems in order to illustrate how our methods can be applied and show their power. The problems depend of course on the choice of the family $(\gamma_x)_{x\in\mathbb{X}}$; they have the particularity that even if the parameters γ_x were constant, the problems should still have a critical behaviour. Since our γ_x are not constant and moreover are unbounded, the results of this section combine the criticality of the embedded Markov chain with the explosion/implosion phenomena.

We need first some technical conditions. Let $f \in \ell^{2+\delta_0}$ for some $\delta_0 > 0$. For every $g : \mathbb{R}_+ \to \mathbb{R}_+$ a C^3 function, we get

$$\begin{aligned} m_{g\circ f}(x) &= \mathbb{E}(g(f(\tilde{\xi}_{n+1})) - g(f(\tilde{\xi}_n))|\tilde{\xi}_n = x) \\ &= g'(f(x))m_f(x) + \frac{1}{2}g''(f(x))v_f(x) + R_g(x), \end{aligned}$$

where $R_g(x)$ is the conditional expectation of the remainder occurring in the Taylor expansion.[1]

Definition 3.1. Let $F = (F_n)_{n\in\mathbb{N}}$ be an exhaustive nested sequence of sets $F_n \uparrow \mathbb{X}$, $f \in \ell_+^{2+\delta}(\Gamma)$ and $g \in C^3(\mathbb{R}_+;\mathbb{R}_+)$. We say that the chain (or its jumps) satisfies the *remainder condition* (or shortly *condition R*) for F, f, and g, if

$$\lim_{n\to\infty} \sup_{x\in F_n^c} R_g(x) \Big/ \left[g'(f(x))m_f(x) + \frac{1}{2} g''(f(x))v_f(x) \right] = 0.$$

The quantity $D_g(f,x) := g'(f(x))m_f(x) + \frac{1}{2}g''(f(x))v_f(x)$ in the expression above is the *effective average drift* at the scale defined by the function g. If the function f is non-trivial, there exists a natural exhaustive nested sequence determined by the sub-level sets of f. When we omit to specify the nested sequence, we shall always consider the natural one.

Introduce the notation $\ln_{(0)} s = s$ and recursively, for all integers $k \geq 1$, $\ln_{(k)} s = \ln(\ln_{(k-1)} s)$ and denote $L_k(s) = \prod_{i=0}^k \ln_{(i)} s$, for $k \geq 0$, and $L_k(s) := 1$ for $k < 0$. Equivalently, we define $\exp_{(k)}$ as the inverse function of $\ln_{(k)}$. In most occasions we shall use (Lyapunov) functions $g(s) := \ln_{(l)}^{\eta} s$ with some integer $l \geq 0$ and some real $\eta \neq 0$, defined for s sufficiently large, $s \geq s_0 := \exp_{(l)}(2)$ say. It is cumbersome but straightforward to show then that for $f \in \ell^{2+\delta_0}(\Gamma)$ with some $\delta_0 > 0$, the condition R is satisfied. It will be shown in this section that condition R and Lyapunov functions of the aforementioned form play a crucial role in the study of models where the effective average drift $D_g(f,x)$ tends to 0 in some controlled way with n when $x \in F_n^c$, where (F_n) is an exhaustive nested sequence; models of this type lie in the deeply critical regime between recurrence and transience.

3.1. Critical models on denumerably infinite and unbounded subsets of $\mathbb{R}_+$

Consider a discrete time irreducible Markov chain $(\tilde{\xi}_n)$ on a denumerably infinite and unbounded subset $\mathbb{X}$ of $\mathbb{R}_+$. Since now $\mathbb{X}$ inherits the composition properties stemming from the field structure of $\mathbb{R}$, we can define directly $m(x) := m_{\mathrm{id}}(x)$ and $v(x) := v_{\mathrm{id}}(x)$, where id is the identity function on $\mathbb{X}$. The model is termed *critical* when the drift m tends to 0, in some precise manner, when $x \to \infty$.

In [12] Markov chains on $\mathbb{X} = \{0,1,2,\ldots\}$ were considered; it has been established that the chain is recurrent if $m(x) \leq v(x)/2x$ while is transient if $m(x) > \theta v(x)/2x$ for some $\theta > 1$. In particular, if $m(x) = \mathscr{O}(\frac{1}{x})$ and $v(x) = \mathscr{O}(1)$ the model is in the critical regime.

The case with arbitrary degree of criticality

$$m(x) = \sum_{i=0}^k \frac{\alpha_i}{L_i(x)} + o\left(\frac{1}{L_k(x)}\right) \quad \text{and} \quad v(x) = \sum_{i=0}^k \frac{x\beta_i}{L_i(x)} + o\left(\frac{x}{L_k(x)}\right),$$

with α_i, β_i constants, has been settled in [15] by using Lyapunov functions. Under the additional conditions

$$\limsup_{n\to\infty} \tilde{\xi}_n = \infty \quad \text{and} \quad \liminf_{x\in\mathbb{X}} v(x) > 0$$

[1] The most convenient form of the remainder is the Roche–Schlömlich one (see Appendix A, Table 9.IV, p. 1754 of [9] for instance).

and some technical moment conditions – guaranteeing the condition R for this model – that can straightforwardly be shown to hold if $\mathrm{id} \in \ell^{2+\delta_0}(\Gamma)$, for some $\delta_0 > 2$, it has been shown in [15] (Theorem 4) that

– if $2\alpha_0 < \beta_0$ the chain is recurrent while if $2\alpha_0 > \beta_0$ the chain is transient;
– if $2\alpha_0 = \beta_0$ and $2\alpha_i - \beta_i - \beta_0 = 0$ for all $i : 0 \le i < k$ and there exists $i : 0 \le i < k$ such that $\beta_i > 0$ then
  – if $2\alpha_k - \beta_k - \beta_0 \le 0$ the chain is recurrent,
  – if $2\alpha_k - \beta_k - \beta_0 > 0$ the chain is transient.

We assume that the moment conditions $\mathrm{id} \in \ell^{2+\delta_0}(\Gamma)$ – guaranteeing the condition R for this model – are satisfied throughout this section.

Let $(\tilde{\xi}_n)$ be a Markov chain on $\mathbb{X} = \{0, 1, \ldots\}$ satisfying for some $k \ge 0$

$$m(x) = \sum_{i=0}^{k-1} \frac{\beta_0}{2L_i(x)} + \frac{\alpha_k}{L_k(x)} + o\left(\frac{1}{L_k(x)}\right) \quad \text{and} \quad v(x) = \beta_0 + o(1),$$

with $2\alpha_k > \beta_0$. The aforementioned result guarantees the transience of the chain; we term such a chain *k-critically transient*. When $2\alpha_k < \beta_0$ the previous result guarantees the recurrence of the chain; we term such a chain *k-critically recurrent*.

In spite of its seemingly idiosyncratic character, this model proves universal as Lyapunov functions used in the study of many general models in critical regimes map those models to some k-critical models on denumerably infinite unbounded subsets of $\mathbb{R}_+$.

3.1.1. Moments of passage times

Proposition 3.2. *Let (ξ_t) be a continuous-time Markov chain on $\mathbb{X}$ and A be the finite set $A := [0, x_0] \cap \mathbb{X}$ for some sufficiently large x_0. Suppose that for some integer $k \ge 0$, its embedded chain is k-critically recurrent, i.e.*

$$m(x) = \sum_{i=0}^{k-1} \frac{\beta_0}{2L_i(x)} + \frac{\alpha_k}{L_k(x)} + o\left(\frac{1}{L_k(x)}\right) \quad \textit{and} \quad v(x) = \beta_0 + o(1),$$

with $2\alpha_k < \beta_0$. Denote $C = \alpha_k/\beta_0$ (hence $C < 1/2$) and assume there exists a constant $\kappa > 0$ such that[2] *$\gamma_x = \mathscr{O}(\frac{L_k^2(x)}{\ln_{(k)}^{\kappa}(x)})$ for large x. Define $p_0 := p_0(C, \kappa) = (1 - 2C)/\kappa$.*

1. *Assume that $v^{(\rho)}(x) = \mathscr{O}(1)$ with $\rho = \max(2, 1 - 2C) + \delta_0$. If $q < p_0$ then $\mathbb{E}_x \tau_A^q < \infty$.*
2. *Assume that $v^{(\rho)}(x) = \mathscr{O}(1)$ with $\rho = \max(2, p_0) + \delta_0$. If $q \ge p_0$ then $\mathbb{E}_x \tau_A^q = \infty$.*

Remark 3.3. We remark that when $\kappa \downarrow 0$ (for fixed k and C) then $p_0 \uparrow \infty$ implying that more and more moments exist.

Proof of Proposition 3.2. For the function $f(x) = \ln_{(k)}^{\eta} x$ we determine

$$m_f(x) = \frac{1}{2}\beta_0 \eta (2C + \eta - 1 + o(1)) \frac{\ln_{(k)}^{\eta}(x)}{L_k^2(x)}.$$

[2] i.e. there exists a constant $c_3 > 0$ such that $\frac{1}{c_3} \le \frac{\gamma_x \ln_{(k)}^{\kappa}(x)}{L_k^2(x)} \le c_3$ for $x \ge x_0$.

1. For a $p > 0$, we remark that
$$\left[0 < p\eta < 1 - 2C \text{ and } \eta \geq \frac{\kappa}{2}\right] \Rightarrow \Gamma f^p(x) \leq -cf^{p-2}(x),$$
for some constant[3] $c > 0$. Hence $p < 2(1-2C)/\kappa$ and statement 1 of Theorem 1.5 implies for any $q \in]0, (1-2C)/\kappa[$, the q-th moment of the passage time exists. Optimising over the accessible values of q we get that $\mathbb{E}_x(\tau^q) < \infty$ for all $q < p_0$.
2. We distinguish two cases:

[$1 - 2C < \kappa$:] We verify that, choosing $\eta \in]1-2C, \kappa]$ and $p > \frac{1-2C}{\kappa}$, the three conditions of statement 2 of Theorem 1.5 (with $f = g$), namely $\Gamma f \geq -c_1$, $\Gamma f^r \leq c_2 f^{r-1}$, for some $r > 1$, and $\Gamma f^p \geq 0$, outside some finite set A.

[$\kappa \leq 1 - 2C$:] In this situation, choosing the parameters $\eta \in]0, \kappa]$ and $p > \frac{1-2C}{\kappa}$ implies simultaneous verification of the three conditions of statement 2 of Theorem 1.5.

In both situations, we conclude that for all $q > \frac{1-2C}{\kappa}$, the corresponding moment does not exist. Optimising over the accessible values of q, we get that $\mathbb{E}_x(\tau^q) = \infty$ for all $q > p_0$. □

3.1.2. Explosion and implosion

Proposition 3.4. *Let (ξ_t) be a continuous time Markov chain. Suppose that for some integer $k \geq 0$, its embedded chain is k-critically transient.*

1. *If there exist a constant $d_1 > 0$, an integer $l > k$, and a real $\kappa > 0$ (arbitrarily small) such that*
$$\gamma_x \geq d_1 L_k(x) L_l(x) (\ln_{(l)} x)^\kappa, \quad x \geq x_0,$$
then $\mathbb{P}_y(\zeta < \infty) = 1$ for all $y \in \mathbb{X}$.
2. *If there exist a constant $d_2 > 0$ and an integer $l > k$ such that*
$$\gamma_x \leq d_2 L_k(x) L_l(x), \quad x \geq x_0,$$
then the continuous-time chain is conservative.

Proof. 1. For a k-critically transient chain, chose a function f behaving at large x as $f(x) = \frac{1}{\ln^\eta_{(l)}(x)}$. Denote $C = \alpha_k/\beta_0$ (hence $C > 1/2$ for the chain to be transient). We estimate then
$$m_f(x) = -\frac{1}{2}\beta_0\eta(2C-1)\frac{1}{L_k(x)L_l(x)\ln^\eta_{(l)} x} + o\left(\frac{1}{L_k(x)L_l(x)\ln^\eta_{(l)} x}\right).$$
We conclude by Theorem 1.9.

2. Choose as the Lyapunov function the identity function $f(x) = x$ and estimate
$$m_{\ln_{(l+1)} \circ f}(x) = \frac{1}{2}\beta_0(2C-1)\frac{\ln_{(l+1)} x}{L_k(x)L_{l+1}(x)} = \frac{1}{2}\beta_0(2C-1)\frac{1}{L_k(x)L_l(x)}.$$
We conclude by Theorem 1.14 by choosing $g(s) = L_l(s)$. □

Proposition 3.5. *Let (ξ_t) be a continuous time Markov chain. Suppose that for some integer $k \geq 0$, its embedded chain is k-critically recurrent. Denote $C = \alpha_k/\beta_0$ (hence $C < 1/2$ for the chain to be recurrent). Let A be the finite set $A := [0, x_0] \cap \mathbb{X}$ for some sufficiently large x_0.*

[3] The constant c can be chosen $c \geq \frac{1}{2}\beta_0\eta(1-2C)c_3$.

1. *If there exist a constant $d_1 > 0$, an integer $l > k$, and an arbitrarily small real $\kappa > 0$ such that*

$$\gamma_x \geq d_1 L_k(x) L_l(x) (\ln_{(l)} x)^\kappa, \quad x \geq x_0,$$

then there exists a constant B such that $\mathbb{E}_y \tau_A \leq B$, uniformly in $y \in A^c$, i.e. the chain implodes.

2. *If there exist a constant $d_2 > 0$ and an integer $l > k$ such that*

$$\gamma_x \leq d_2 L_k(x) L_l(x), \quad x \geq x_0,$$

then the continuous time chain does not implode.

Proof. 1. Use the function f defined for sufficiently large x by the formula $f(x) = 1 - \frac{1}{\ln^\eta_{(l)} x}$, for some $l > k$ and $\eta > 0$. We estimate then

$$m_f(x) = \frac{1}{2}\beta_0 \eta (2C-1) \frac{1}{L_k(x) L_l(x) \ln^\eta_{(l)} x} + o\left(\frac{1}{L_k(x) L_l(x) \ln^\eta_{(l)} x}\right).$$

We conclude by statement 1 of Theorem 1.15.

2. Using the function f defined for sufficiently large x by the formula $f(x) = \ln^\eta_{(l+1)} x$, for some $l \geq k$. We estimate then

$$m_f(x) = \frac{1}{2}\beta_0 \eta (2C-1) \frac{\ln^\eta_{(l+1)} x}{L_k(x) L_{l+1}(x)} + o\left(\frac{\ln^\eta_{(l+1)} x}{L_k(x) L_{l+1}(x)}\right).$$

If $\gamma_x \leq d_2 L_k(x) L_l(x)$ for large x, then, using the above estimate for the case $\eta = 1$ and the case $\eta = r$ for some small $r > 1$, we observe that the conditions $\Gamma f \geq -\epsilon$ and $\Gamma f^r \leq f^{r-1}$ are simultaneously verified. We conclude by statement 2 of Theorem 1.15. □

3.2. *Simple random walk on $\mathbb{Z}^d$ for $d = 2$ and $d \geq 3$*

Here the state space $\mathbb{X} = \mathbb{Z}^d$ and the embedded chain is a simple random walk on $\mathbb{X}$. Since in dimension 2 the simple random walk is null recurrent while in dimension $d \geq 3$ is transient, a different treatment is imposed.

3.2.1. *Dimension $d \geq 3$*

For the Lyapunov function f defined by $\mathbb{Z}^d \ni x \mapsto f(x) := \|x\|$, we can show that there exist constants $\alpha_0 > 0$ and $\beta_0 > 0$ such that $\lim_{\|x\| \to \infty} \|x\| m_f(x) = \alpha_0$ and $\lim_{\|x\| \to \infty} v_f(x) = \beta_0$ such that $C = \alpha_0/\beta_0 > 1/2$. Therefore the one dimensional process $X_t = f(\xi_t)$ has 0-critically transient Lamperti behaviour.

We get therefore that $(\xi_t)_{t \in \mathbb{R}_+}$ is a (quite unsurprisingly) transient process and that if there exist a constant $a > 0$ and

– a constant $d_1 > 0$, an integer $l > 0$, and a real $\kappa > 0$ (arbitrarily small) such that

$$\gamma_x \geq d_1 \|x\| L_l(\|x\|) (\ln_{(l)} \|x\|)^\kappa, \quad \|x\| \geq a,$$

then $\mathbb{P}_y(\zeta < \infty) = 1$ for all $y \in \mathbb{X}$;

– a constant $d_2 > 0$ and an integer $l > 0$ such that

$$\gamma_x \leq d_2 \|x\| L_l(\|x\|), \quad \|x\| \geq a,$$

then the continuous time chain is conservative.

3.2.2. Dimension 2

Using again the Lyapunov function $f(x) = \|x\|$, we show that the one dimensional process $X_t = f(\xi_t)$ is of the 1-critically recurrent Lamperti type. Hence, again using the results obtained in Section 3.1, we get that if there exist a constant $a > 0$ and

- a constant $d_1 > 0$, an integer $l > 1$, and an arbitrarily small real $\kappa > 0$ such that
$$\gamma_x \geq d_1 L_1(\|x\|) L_l(\|x\|)(\ln_{(l)} \|x\|)^\kappa, \quad \|x\| \geq a,$$
then there exists a constant C such that $\mathbb{E}_y \tau_A \leq C$, uniformly in y for $y : \|y\| \geq a$, i.e. the chain implodes;
- a constant $d_2 > 0$ and an integer $l > 1$ such that
$$\gamma_x \leq d_2 L_1(\|x\|) L_l(\|x\|), \quad \|x\| \geq a,$$
then the continuous time chain does not implode.

3.3. Random walk on $\mathbb{Z}_+^2$ with reflecting boundaries

3.3.1. The model in discrete time

Here $\mathbb{X} = \mathbb{Z}_+^2$. We denote by $\mathring{\mathbb{X}} = \{x \in \mathbb{Z}_+^2 : x_1 > 0, x_2 > 0\}$ the *interior* of the wedge and by $\partial_1 \mathbb{X} = \{x \in \mathbb{Z}_+^2 : x_2 = 0\}$ (and similarly for $\partial_2 \mathbb{X}$) its *boundaries*. Since $\mathbb{X}$ is a subset of a vector space, we can define directly the increment vector $D := \tilde{\xi}_{n+1} - \tilde{\xi}_n$ and the average conditional drift $m(x) := m_{\text{id}}(x) = \mathbb{E}(D|\tilde{\xi}_n = x) \in \mathbb{R}^2$. We assume that for all $x \in \mathring{\mathbb{X}}$, $m(x) = 0$ so that we are in a critical regime. For $x \in \partial_\flat \mathbb{X}$; with $\flat = 1, 2$, the drift $m^\flat(x)$ does not vanish but is a constant vector $m^\flat$ that forms angles $\phi^\flat$ with respect to the normal to $\partial_\flat \mathbb{X}$. For $x \in \mathring{\mathbb{X}}$, the conditional covariance matrix $C(x) := (C(x)_{ij})$, with $C(x)_{ij} = \mathbb{E}[D_i D_j | \tilde{\xi}_n = x]$, is the constant 2×2 matrix C, reading

$$C := \mathsf{Cov}(D, D) = \begin{pmatrix} s_1^2 & \lambda \\ \lambda & s_2^2 \end{pmatrix}.$$

There exists an isomorphism Φ on $\mathbb{R}^2$ such that $\mathsf{Cov}(\Phi D, \Phi D) = \Phi C \Phi^t = I$; it is elementary to show that

$$\Phi = \begin{pmatrix} \dfrac{s_2}{d} & -\dfrac{\lambda}{s_2 d} \\ 0 & \dfrac{1}{s_2} \end{pmatrix},$$

where $d = \sqrt{\det C}$, is a solution to the aforementioned isomorphism equation. This isomorphism maps the quadrant $\mathbb{R}_+^2$ into a squeezed wedge $\Phi(\mathbb{R}_+^2)$ having an angle ψ at its summit reading $\psi = \arctan(-d^2/\lambda)$. Obviously $\psi = \pi/2$ if $\lambda = 0$, while $\psi \in]0, \pi/2[$ if $\lambda < 0$ and $\psi \in]\pi/2, \pi[$ if $\lambda > 0$. We denote by $\mathbb{Y} = \Phi(\mathbb{X})$ the squeezed image of the lattice. The isomorphism Φ transforms the average drifts at the boundaries into $n^\flat = \Phi m^\flat$ forming new angles, $\psi_\flat$, with the normal to the boundaries of $\Phi(\mathbb{R}_+^2)$.

The discrete time model has been exhaustively treated in [1] and its extension to the case of excitable boundaries carrying internal states in [14]. Here we recall the main results of [1] under some simplifying assumptions that allow us to present them here without redefining completely the model or considering all the technicalities. The assumptions we need are that the jumps

– are bounded from below, i.e. there exists a constant $K > 0$ such that $D_1 \geq -K$ and $D_2 \geq -K$,
– satisfy a sufficient integrability condition, for instance $\mathbb{E}(\|D\|_{2+\delta_0}^{2+\delta_0}) < \infty$ for some $\delta_0 > 0$,
– are such that their covariance matrix is non degenerate.

Under these assumptions we can state the following simplified version of the results in [1].

Denote $\chi = (\psi_1 + \psi_2)/\psi$ and $A = \{x \in \mathbb{X} : \|x\| \leq a\}$.

1. If $\chi \geq 0$ the chain is recurrent.
2. If $\chi < 0$ the chain is transient.
3. If $0 < \chi < 2 + \delta_0$, then for every $p < \chi/2$ and every $x \notin A$, $\mathbb{E}_x \tau_A^p < \infty$.
4. If $0 < \chi < 2 + \delta_0$, then for every $p > \chi/2$ and every $x \notin A$, $\mathbb{E}_x \tau_A^p = \infty$.

Let $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a C^2 function. Define $\tilde{f}(y) = f(\Phi^{-1} y)$. Although the Hessian operator does not in general transform as a tensor, the linearity of Φ allows however to write $\mathsf{Hess}_f(x) = \Phi^t \mathsf{Hess}_{\tilde{f}}(\Phi x)\Phi$. For every $x \in \overset{\circ}{\mathbb{X}}$ we establish then[4] the identity:

$$\mathbb{E}(\langle D, \mathsf{Hess}_{f\circ\Phi}(x) D\rangle | \tilde{\xi}_n = x) = \mathsf{Lap}_{f\circ\Phi}(x).$$

We denote $h_{\beta,\beta_1}(x) = \|x\|^\beta \cos(\beta \arctan(\frac{x_2}{x_1}) - \beta_1)$. Then this function is harmonic, i.e. $\mathsf{Lap}_{h_{\beta,\beta_1}} = 0$. We are interested in harmonic functions that are positive on $\Phi(\mathbb{R}_+^2)$; positivity and geometry impose then conditions on β and β_1. In fact, $\operatorname{sign}(\beta)\beta_1$ is the angle of $\nabla h_{\beta,\beta_1}(x)$ at $x \in \partial_1 \mathbb{X}$, with the normal to $\partial_1 \mathbb{X}$. Similarly, if $\beta_2 = \beta\psi - \beta_1$, then $\operatorname{sign}(\beta)\beta_2$ is the angle of the gradient with the normal to $\partial_2 \mathbb{X}$. Now, it becomes evident that β_i, $i = 1, 2$, must lie in the interval $]-\pi/2, \pi/2[$ and subsequently $\beta = \frac{\beta_1+\beta_2}{\psi}$. Notice also that the datum of two admissible angles β_1 and β_2 uniquely determines the harmonic function whose gradient at the boundaries forms angles as above. Hence, $\langle \nabla h_{\beta,\beta_1}(y), n^\flat \rangle = \|y\|^\beta \beta \sin(\psi_\flat - \beta_\flat)$, for $y \in \partial_\flat \mathbb{Y}$ and $\flat = 1, 2$.

Let now $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^3 function and $h = h_{\beta,\beta_1}$ a harmonic function that remains positive in $\Phi(\mathbb{R}_+^2)$. On denoting $y = \Phi x$, abbreviating $\Xi := g(h(\Phi\tilde{\xi}_{n+1})) - g(h(\Phi\tilde{\xi}_n))$, and using the fact that h is harmonic, we get

$$\begin{aligned}
\mathbb{E}(\Xi | \tilde{\xi}_n = x) &= g'(h(y))\mathbb{E}(\langle \nabla h(y), \Phi D\rangle | \tilde{\xi}_n = x) \\
&\quad + \frac{g''(h(y))}{2}\mathbb{E}(\langle \nabla h(y), \Phi D\rangle^2 | \tilde{\xi}_n = x) \\
&\quad + \frac{[g'(h(y))]^2}{2}\mathbb{E}(\langle \Phi D, \mathsf{Hess}_h(y)\Phi D\rangle | \tilde{\xi}_n = x) + R_3 \\
&= g'(h(y))\langle \nabla h(y), n(y)\rangle + \frac{g''(h(y))}{2}\|\nabla h(y)\|^2 + R_3(y),
\end{aligned}$$

where R_3 is the remainder of the Taylor expansion. The value of the conditional increment depends on the position of x. If $x \in \partial_\flat \mathbb{X}$ the dominant term of the right hand side is $g'(h(y)) \langle \nabla h(y), n^\flat \rangle$, while in the interior of the space, that term strictly vanishes because there $n(y) = 0$; hence the dominant term becomes the term $\frac{g''(h(y))}{2}\|\nabla h(y)\|^2$.

[4] Since we have used the symbol Δ to denote the jumps of the process, we introduce the symbol Lap to denote the Laplacian.

3.3.2. The model in continuous-time

Proposition 3.6. *Let* $0 < \chi = (\psi_1 + \psi_2)/\psi$ *(hence the chain is recurrent) and* $A := A_a = \{x \in \mathbb{X} : \|x\| \leq a\}$ *for some* $a > 0$*, and* $\gamma_x = \mathcal{O}(\|x\|^{2-\kappa})$*; denote* $p_0 = \chi/\kappa$*. Suppose further that* $\mathrm{id} \in \ell^\rho(\Gamma)$ *for some* $\rho > 2$*.*

1. *If* $q < p_0$*, then* $\mathbb{E}_x(\tau_A^q) < \infty$*.*
2. *If* $q > p_0$*, then* $\mathbb{E}_x(\tau_A^q) = \infty$*.*

Proof. Consider the Lyapunov function $f(x) = h_{\beta,\beta_1}(x)^\eta$.

1. If $0 < p\eta < 1$ then $m_{f^p}(x) < 0$. The condition $\Gamma f^p \leq -cf^{p-2}$ reads then $\gamma_x \geq C\frac{h_\beta^{p\eta-2\eta}(x)}{h_\beta^{p\eta-2}\|\nabla h_\beta\|^2} = C'\frac{\|x\|^{2\beta-2\beta\eta}}{\|x\|^{2\beta-2}} = C'\|x\|^{2-2\beta\eta}$ from which it follows that $2\beta\eta > \kappa$. This inequality, combined with $0 < p\eta < 1$, yields that for all $q < \frac{\beta}{\kappa} < \frac{\chi}{\kappa}$, $\mathbb{E}_x(\tau_A^q) < \infty$. Hence, on optimising on the accessible values of q we obtain the value of p_0.
2. We proceed similarly; we need however to use the full-fledged version of statement 2 of Theorem 1.5, with both the function f and $g = h_{\chi,\psi_1}^\eta$. Then $f(x) \leq Cg(x)$ and

$$[\eta\beta < \kappa \text{ and } \eta p > 1] \Rightarrow [\Gamma g \geq -\epsilon \text{ and } \Gamma g^r \leq cg^{r-1} \text{for } r > 1, \text{ and } \Gamma f^p \geq 0].$$

Simultaneous verification of these inequalities yields $p_0 = \chi/\kappa$. □

Using again Lyapunov functions of the form $f = h_\beta^\eta$ we can show that the drift of the chain in the transient case can be controlled by two 0-critically transient Lamperti processes in the variable $\|x\|$ that are uniformly comparable. We can thus show, using methods developed in Section 3.1 the following

Proposition 3.7. *Let* $\chi < 0$ *(hence the chain is transient).*

1. *If there exist a constant* $d_1 > 0$ *and an arbitrary integer* $l > 0$ *such that* $\gamma_x \geq d_1\|x\|L_l(\|x\|) \ln_{(l)}^\kappa \|x\|$*, for some arbitrarily small* $\kappa > 0$*, then the chain explodes.*
2. *If there exist a constant* $d_2 > 0$ *and an arbitrary integer* $l > 0$ *such that* $\gamma_x \leq d_2\|x\|L_l(\|x\|)$*, then the chain does not explode.*

With the help of similar arguments we can show the following

Proposition 3.8. *Let* $0 < \chi$ *(hence the chain is recurrent).*

1. *If there exist a constant* $d_1 > 0$ *and an arbitrary integer* $l > 0$ *such that* $\gamma_x \geq d_1\|x\|L_l(\|x\|) \ln_{(l)}^\kappa \|x\|$*, for some arbitrarily small* $\kappa > 0$*, then the chain implodes.*
2. *If there exist a constant* $d_2 > 0$ *and an arbitrary integer* $l > 0$ *such that* $\gamma_x \leq d_2\|x\|L_l(\|x\|)$*, then the chain does not implode.*

3.4. Collection of one-dimensional complexes

We introduce some simple models to illustrate two phenomena:

– it is possible to have $0 < \mathbb{P}_x(\zeta < \infty) < 1$,
– it is possible to have $\mathbb{P}_x(\zeta = \infty) = 0$ and $\mathbb{E}_x\zeta = \infty$.

The simplest situation corresponds to a continuous-time Markov chain whose embedded chain is a simple transient random walk on $\mathbb{X} = \mathbb{Z}$ with non trivial tail boundary. For instance, choose some $p \in]1/2, 1[$ and transition matrix

$$P_{xy} = \begin{cases} 1/2 & \text{if } x = 0, y = x \pm 1; \\ p & \text{if } x > 0, y = x + 1 \text{ or } x < 0, y = x - 1; \\ 1 - p & \text{if } x > 0, y = x - 1 \text{ or } x < 0, y = x + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $x \neq 0$, $0 < \mathbb{P}_x(\tau_0 = \infty) < 1$. Suppose now that $\gamma_x = c$ for $x < 0$ while there exist a sufficiently large integer $l \geq 0$ and an arbitrarily small $\delta > 0$ such that $\gamma_x \geq c L_l(x) \ln_{(l)}^{\delta} x$ for $x \geq x_0$. Then, using Theorem 1.12, we establish that $\mathbb{E}_x(\zeta | \tau_{\mathbb{Z}_-} > \zeta) < \infty$ for all $x > 0$ while $\mathbb{P}_x(\zeta = \infty | \tau_{\mathbb{Z}_+} = \infty) = 1$ for all $x < 0$. This result combined with irreducibility of the chain leads to the conclusion: $0 < \mathbb{P}_x(\zeta < \infty) < 1$ for all $x \in \mathbb{X}$.

It is worth noting that bending the axis $\mathbb{Z}$ at 0 allows considering the state space as the gluing of two one-dimensional complexes $\mathbb{X}_2 = \{0\} \cup \mathbb{N} \times \{-, +\}$; every point $x \in \mathbb{Z} \setminus \{0\}$ is now represented as $x = (|x|, \mathsf{sgn}(x))$. This construction can be generalised by gluing a denumerably infinite family of one-dimensional complexes through a particular point o and introducing the state space $\mathbb{X}_\infty = \{o\} \cup \mathbb{N} \times \mathbb{N}$; every point $x \in \mathbb{X}_\infty \setminus \{o\}$ can be represented as $x = (x_1, x_2)$ with $x_1, x_2 \in \mathbb{N}$.

Let $(\xi_t)_{t \in [0,\infty[}$ be a continuous time Markov chain evolving on the state space $\mathbb{X}_\infty$. Its embedded (at the moments of jumps) chain $(\tilde{\xi}_n)_{n \in \mathbb{N}}$ has transition matrix given by

$$P_{xy} = \begin{cases} \pi_{y_2} & \text{if } x = o, y = (1, y_2), y_2 \in \mathbb{N}, \\ p & \text{if } x = (x_1, x_2), y = (x_1 + 1, x_2), x_1 \geq 1, x_2 \in \mathbb{N}, \\ 1 - p & \text{if } x = (x_1, x_2), y = (x_1 - 1, x_2), x_1 > 1, x_2 \in \mathbb{N}, \\ 1 - p & \text{if } x = (1, x_2), n \in \mathbb{N}, y = o, \\ 0 & \text{otherwise,} \end{cases}$$

where $1/2 < p < 1$ and $\pi = (\pi_n)_{n \in \mathbb{N}}$ is a probability vector on $\mathbb{N}$, satisfying $\pi_n > 0$ for all n. The chain is obviously irreducible and transient.

The space $\mathbb{X}_\infty$ must be thought as a "mock-tree" since, for transient Markov chains, it has a sufficiently rich boundary structure without any of the complications of the homogeneous tree (the study on full-fledged trees is postponed in a subsequent publication). Suppose that for every $n \in \mathbb{N}$ there exist an integer $l_n \geq 0$, a real $\delta_n > 0$, and a $K_n > 0$ such that for $x = (x_1, x_2) \in \mathbb{N} \times \mathbb{N}$ for x_1 large enough, γ_x satisfies $\gamma_{(x_1,x_2)} = K_{x_2} \mathcal{O}(L_{l_{x_2}}(x_1) \ln_{(l_{x_2})}^{\delta_{x_2}} x_1)$.

By applying Theorem 1.12, we establish that $\mathcal{Z}_{x_2} := \mathbb{E}_{(x_1,x_2)}(\zeta | \tau_o = \infty) < \infty$ for all $x_1 > 0$ and all $x_2 \in \mathbb{N}$, hence $\mathbb{P}_{(x_1,x_2)}(\zeta = \infty | \tau_o = \infty) = 0$. Irreducibility implies then that $\mathbb{P}_o(\zeta = \infty) = 0$. However,

$$\mathbb{E}_o(\zeta) \geq \sum_{x_2 \in \mathbb{N}} \pi_{x_2} \mathbb{E}_{(1,x_2)}(\zeta | \tau_o = \infty) \mathbb{P}_{(1,x_2)}(\tau_o = \infty) = \frac{2p - 1}{p} \sum_{x_2 \in \mathbb{N}} \pi_{x_2} \mathcal{Z}_{x_2}.$$

Since the sequences $(l_n)_n$, $(\delta_n)_n$, and $(K_n)_n$ are totally arbitrary, while the positive sequence $(\pi_n)_n$ must solely satisfy the probability constraint $\sum_{n \in \mathbb{N}} \pi_n = 1$, all possible behaviour for $\mathbb{E}_o \zeta$ can occur. In particular, we can choose, for all $n \in \mathbb{N}$, $l_n = 0$ and $\delta_n = 1$; this choice gives $\gamma_{(x_1,n)} = K_n \mathcal{O}(\frac{1}{x_1^2})$ for every n and for large x_1, leading to the estimate $\mathcal{Z}_n \geq C K_n$, for all n.

Choosing now, for instance, $\pi_n = \mathcal{O}(1/n^2)$ and $K_n = \mathcal{O}(n)$ for large n, we get

$$\mathbb{P}_o(\zeta = \infty) = 0 \quad \text{and} \quad \mathbb{E}_o \zeta = \infty.$$

This remark leads naturally to the question whether for transient exploding chains with non-trivial tail boundary, there exists some critical $q > 0$ such that $\mathbb{E}(\zeta^p) < \infty$ for $p < q$ while $\mathbb{E}(\zeta^p) = \infty$ for $p > q$. Such models include continuous time random walks on the homogeneous tree and more generally on non-amenable groups. These questions are currently under investigation and are postponed to a subsequent publication.

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