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Well-posedness of mean-field type forward–backward stochastic differential equations

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Abstract

Being motivated by a recent pioneer work Carmona and Delarue (2013), in this article, we propose a broad class of natural monotonicity conditions under which the unique existence of the solutions to Mean-Field Type (MFT) Forward–Backward Stochastic Differential Equations (FBSDE) can be established. Our conditions provided here are consistent with those normally adopted in the traditional FBSDE (without the interference of a mean-field) frameworks, and give a generic explanation on the unique existence of solutions to common MFT-FBSDEs, such as those in the linear-quadratic setting; besides, the conditions are 'optimal' in a certain sense that can elaborate on how their counter-example in Carmona and Delarue (2013) just fails to ensure its well-posedness. Finally, a stability theorem is also included.

Keywords: Mean-field type; Forward–backward stochastic differential equations; Monotonicity conditions; Well-posedness; Linear-quadratic setting

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion over the same space equipped with a natural filtration, generated by B_t, satisfying the usual continuity conditions. A fully coupled forward–backward stochastic differential equation (FBSDE) is:

$$\begin{cases} X_t = x_0 + \int_0^t b\,(s; X_s, Y_s, Z_s)\,ds + \int_0^t \sigma(s; X_s, Y_s, Z_s)dB_s \\ Y_t = g(X_T) + \int_t^T h(s; X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad \text{for any } t \in [0, T], \end{cases}$$

where X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d}$ respectively, and b, σ, g, h are functions with appropriate dimensions. Ma, Protter and Yong [9] are the first to develop the "four-step scheme" to establish the existence and uniqueness of solutions of the FBSDEs when the underlying forward equation is non-degenerate. Hu and Peng [8] later proposed certain natural monotonicity conditions (that allows the underlying forward equation to be degenerate) under which they can show the unique existence of solutions to FBSDEs when both the dimensions of X and Y are equal; Peng and Wu [11] then extended the earlier result in [8] when the dimensions of X and Y are different under the so called G-monotonicity conditions. Besides, Pardoux and Tang [10] also used a purely probabilistic approach to provide a comprehensive study, again under monotonicity conditions, on the (local) theory of FBSDEs and their connection with quasilinear parabolic partial differential equations, which includes the (local) existence and uniqueness of the solutions of those FBSDEs, their a-priori estimates, and their continuous dependence (stability) on the underlying modelling parameters. The paper of Delarue [7] is an original breakthrough in the study of the well-posedness of FBSDEs based both on probabilistic techniques and on some PDE results which contains less restrictive assumption than that posed in [9].

Mean-field type forward–backward stochastic differential equations (MFT-FBSDEs) are forward–backward stochastic differential equations in which the coefficients involved could also depend upon the distribution of the solution triple (X, Y, Z); see [2,1,3,4], and the references therein for more details of their emergence and introduction, and their subtle connections with mean-field games. The stochastic maximum principle approaches to both the solutions of mean field game problems and optimal control problems for mean field stochastic differential equations naturally reduce to the solutions of MFT-FBSDE systems (see [5,6]). In particulars, the frontier works of [3,4] provided the first probabilistic analysis of mean-field type control theory and related problems. In their interesting recent work [4], Carmona and Delarue used (Schauder) fixed-point or Picard iterative scheme together with recursive induction (over a finite partition of the time horizon) and localization argument (with respect to 'monotone' (up to a subsequence) convergence over the relaxation of bounds on the underlying coefficient functions of the dynamics) to build a coherent general theory of the existence of solutions of MFT-FBSDEs under a very mild Lipschitz condition on the coefficient functions; however, the uniqueness of solutions cannot be guaranteed in general, and a crucial counter-example (with a very simple structure) on this failure was also illustrated in [4]. In their another work [5], Carmona and Delarue implemented the continuation method as introduced in [11] to investigate the optimal control problem with McKean–Vlasov diffusion processes. More precisely, by adopting convexity assumptions on the Hamiltonian of the corresponding optimization problem, they established the unique existence of the forward–backward system via the stochastic Pontryagin (maximum) principle. It should note that the satisfaction of their proposed convexity assumption implies that of our advocated monotonicity condition (A1); nevertheless, their proposed condition may exclude some other interesting cases that could satisfy alternative Assumptions (A2), (A3), (A4). In contrast, in our present paper, we aim to establish that under the additional monotonicity conditions on the coefficient functions, the well-posedness of the solutions to the corresponding MFT-FBSDEs could be ensured. Besides, our proposed conditions seem to be 'optimal' in the sense that, on the one hand, they provide a generic explanation on the unique existence of the solutions to common MFT-FBSDEs, such as those arisen from linear-quadratic mean-field type control problems; on the other hand, they also illuminate on how the counter-example provided in [4] just critically fails to possess a unique solution in a 'continuum manner'.

The organization of our paper is as follows. Section 2 devotes to the problem formulation and settings for MFT-FBSDEs. The unique existence of their solutions is shown in Section 3. Further results on comparison principle is given in Section 4. Two motivating examples and their connections with our proposed monotonicity conditions are given in Section 5, and we finally conclude in Section 6. Q2

2. Problem formulation

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$, where $T > 0$ is an arbitrarily fixed finite number and $\mathcal{F} = \mathcal{F}_T$, be a filtered probability space satisfying the usual conditions with $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by a d-dimensional Brownian motion $\{B_t\}_{t\in[0,T]}$. In this article, we consider the following mean-field type forward–backward stochastic differential equation (MFT-FBSDE):

$$\begin{cases} dX_t = b\left(t; X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) dt + \sigma(t; X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t,Z_t)}) dB_t, \\ dY_t = -h\left(t; X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = g(X_T, \mathbb{P}_{X_T}), \end{cases} \tag{1}$$

for any $t \in [0, T]$, where X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d}$, respectively, and b, σ, g, h are functions with appropriate dimensions. $\mathbb{P}_{X_T}$ and $\mathbb{P}_{(X_s,Y_s,Z_s)} = \mathbb{P}_{(X_s,Y_s,Z_s^{(1)},\ldots,Z_s^{(d)})}$ denote the probability measures induced by X_T and (X_s, Y_s, Z_s) respectively, where $Z_s^{(i)} \in \mathbb{R}^m$ denotes the ith column of Z_s, $i = 1, \ldots, d$.

In the rest of this paper, we shall adopt $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to be the respective usual inner product and norm in Euclidean space; and for any $z_1, z_2 \in \mathbb{R}^{m\times d}$, we define $|z_1| \triangleq \left\{tr(z_1'z_1)\right\}^{\frac{1}{2}}$, $\langle z_1, z_2\rangle \triangleq tr(z_1'z_2)$ where z_1' stands for the transpose of z_1; for any $u_1 \triangleq (x_1, y_1, z_1), u_2 \triangleq (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, we define $|u_1| \triangleq \left\{tr(x_1x_1') + tr(y_1y_1') + tr(z_1z_1')\right\}^{\frac{1}{2}}$ and $(u_1, u_2) \triangleq \langle x_1, x_2\rangle + \langle y_1, y_2\rangle + \langle z_1, z_2\rangle$. We denote by $M^2(0, T; \mathbb{R}^p)$ the set of all $\mathbb{R}^p$-valued $\mathcal{F}_t$-adapted processes $v_\cdot$ such that

$$\mathbb{E}\left[\int_0^T |v_s|^2 ds\right] < \infty.$$

Definition 1. A triple process $(X, Y, Z)(\omega, t) : \Omega \times [0, T] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is called an adapted solution of (1) if (i) $(X, Y, Z) \in M^2\left(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}\right)$; and (ii) the triple also satisfies (1) $\mathbb{P}$-almost surely.

Given a $m \times n$ full-rank matrix G, in the rest of the paper, we shall adopt the following notations: $u \triangleq (x, y, z)^T$,

$$f(t; u, \mu) \triangleq \begin{pmatrix} -h(t; x, y, z, \mu) \\ b(t; x, y, z, \mu) \\ \sigma(t; x, y, z, \mu) \end{pmatrix}, \qquad A(t; u, \mu) \triangleq \begin{pmatrix} -G'h(t; x, y, z, \mu) \\ Gb(t; x, y, z, \mu) \\ G\sigma(t; x, y, z, \mu) \end{pmatrix},$$

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where μ is a probability measure on $\mathbb{R}^{n+m+md}$ and $G\sigma \triangleq (G\sigma_1, \dots, G\sigma_d)$. Let $\mathcal{W}(\cdot,\cdot)$ denote 2-Wasserstein's distance on $\mathcal{M}(\mathbb{R}^p)$ defined by

$$\mathcal{W}(\mu_1,\mu_2) \triangleq \inf\left\{\left[\int_{\mathbb{R}^p\times\mathbb{R}^p} |x-y|^2\pi(dx,dy)\right]^{\frac{1}{2}} : \pi \in \mathcal{M}(\mathbb{R}^p\times\mathbb{R}^p) \text{ with marginals } \mu_1 \text{ and } \mu_2\right\},$$

where $\mathcal{M}(\mathbb{R}^q)$ denotes the set of probability measures with finite second order moment on $\mathbb{R}^q$ for some $q \in \mathbb{N}$. It is obvious from its definition that

$$|\mathbb{E}[X_1] - \mathbb{E}[X_2]| \le \mathcal{W}(\mu_1,\mu_2) \le \left(\mathbb{E}\left[|X_1 - X_2|^2\right]\right)^{\frac{1}{2}}, \tag{2}$$

where X_1 and X_2 are q-dimensional random vectors that follow the distributions μ_1 and μ_2 respectively.

In the rest of our paper, we also consider those functions $f(t;u,\mu)$ that sometimes satisfy the following additional condition:

(H): *f is independent of u, while its dependence on μ is only through its first moment $\int v\mu(dv)$.*

Under (H), we adopt the notations $f\left(t;\int v\mu(dv)\right)$ and $A\left(t;\int v\mu(dv)\right)$ for $f(t;u,\mu)$ and $A(t;u,\mu)$ respectively.

Let $U^1 \triangleq (X^1,Y^1,Z^1)$, $U^2 \triangleq (X^2,Y^2,Z^2) \in M^2\left(0,T;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\right)$, define $(\hat{X},\hat{Y},\hat{Z}) \triangleq (X^1-X^2, Y^1-Y^2, Z^1-Z^2)$. Assume that for each $(u,\mu) \in \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{M}(\mathbb{R}^{n+m+md})$, $(x,\mu') \in \mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)$, we have $A(\cdot;u,\mu) \in M^2\left(0,T;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\right)$ and $g(x,\mu') \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. We first list the following Assumptions (A1)–(A4): there exist $L>0$, non-negative constants β_1, β_2 and α_1 with $\beta_1+\beta_2>0, \alpha_1+\beta_1>0$ such that for almost all $(t,\omega)\in[0,T]\times\Omega$, $(u^1,\mu^1),(u^2,\mu^2)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{M}(\mathbb{R}^{n+m+md})$ and $(x^3,\mu^3),(x^4,\mu^4)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)$,

- (A1)

$$\text{(i) } |f(t;u^1,\mu^1) - f(t;u^2,\mu^2)| \le L\left(|u^1-u^2| + \mathcal{W}(\mu^1,\mu^2)\right),$$

$$\text{(ii) } |g(x^3,\mu^3) - g(x^4,\mu^4)| \le L\left(|x^3-x^4| + \mathcal{W}(\mu^3,\mu^4)\right),$$

$$\text{(iii) } \mathbb{E}\left[\left(A\left(t;U_t^1,\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - A\left(t;U_t^2,\mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right), U_t^1 - U_t^2\right)\right]$$
$$\le -\beta_1\mathbb{E}[|G\hat{X}_t|^2] - \beta_2\left(\mathbb{E}[|G'\hat{Y}_t|^2] + \mathbb{E}[|G'\hat{Z}_t|^2]\right),$$

$$\text{(iv) } \mathbb{E}\left[\left\langle g(X_T^1,\mathbb{P}_{X_T^1}) - g(X_T^2,\mathbb{P}_{X_T^2}), G(X_T^1 - X_T^2)\right\rangle\right] \ge \alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right];$$

- (A2)

(i) The condition (H) holds and

$$\left|f\left(t;\mathbb{E}[V^1]\right) - f\left(t;\mathbb{E}[V^2]\right)\right| \le L\left|\mathbb{E}[V^1] - \mathbb{E}[V^2]\right|,$$

$$\text{(ii) } |g(x^3,\mu^3) - g(x^4,\mu^4)| \le L\left(|x^3-x^4| + \mathcal{W}(\mu^3,\mu^4)\right),$$

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(iii) $\mathbb{E}\left[\left(A\left(t;\mathbb{E}[U_t^1]\right)-A\left(t;\mathbb{E}[U_t^2]\right),U_t^1-U_t^2\right)\right]$

$\leq -\beta_1\left|G\mathbb{E}[\hat{X}_t]\right|^2-\beta_2\left(\left|G'\mathbb{E}[\hat{Y}_t]\right|^2+\left|G'\mathbb{E}[\hat{Z}_t]\right|^2\right),$

(iv) $\mathbb{E}\left[\left\langle g(X_T^1,\mathbb{P}_{X_T^1})-g(X_T^2,\mathbb{P}_{X_T^2}),G(X_T^1-X_T^2)\right\rangle\right]\geq \alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right];$

- (A3)

(i) $|f(t;u^1,\mu^1)-f(t;u^2,\mu^2)|\leq L\left(|u^1-u^2|+\mathcal{W}(\mu^1,\mu^2)\right),$

(ii) $|g(x^3,\mu^3)-g(x^4,\mu^4)|\leq L\left|\mathbb{E}[V^3]-\mathbb{E}[V^4]\right|,$

(iii) $\mathbb{E}\left[\left(A\left(t;U_t^1,\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right)-A\left(t;U_t^2,\mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right),U_t^1-U_t^2\right)\right]$

$\leq -\beta_1\mathbb{E}[|G\hat{X}_t|^2]-\beta_2\left(\mathbb{E}[|G'\hat{Y}_t|^2]+\mathbb{E}[|G'\hat{Z}_t|^2]\right),$

(iv) $\mathbb{E}\left[\left\langle g(X_T^1,\mathbb{P}_{X_T^1})-g(X_T^2,\mathbb{P}_{X_T^2}),G(X_T^1-X_T^2)\right\rangle\right]\geq \alpha_1\left|G\mathbb{E}[\hat{X}_T]\right|^2;$

- (A4)

(i) The condition (H) holds and

$\left|f(t;\mathbb{E}[V^1])-f(t;\mathbb{E}[V^2])\right|\leq L\left|\mathbb{E}[V^1]-\mathbb{E}[V^2]\right|,$

(ii) $|g(x^3,\mu^3)-g(x^4,\mu^4)|\leq L|\mathbb{E}[V^3]-\mathbb{E}[V^4]|,$

(iii) $\mathbb{E}\left[\left(A\left(t;\mathbb{E}[U_t^1]\right)-A\left(t;\mathbb{E}[U_t^2]\right),U_t^1-U_t^2\right)\right]$

$\leq -\beta_1\left|G\mathbb{E}[\hat{X}_t]\right|^2-\beta_2\left(\left|G'\mathbb{E}[\hat{Y}_t]\right|^2+\left|G'\mathbb{E}[\hat{Z}_t]\right|^2\right),$

(iv) $\mathbb{E}\left[\left\langle g(X_T^1,\mathbb{P}_{X_T^1})-g(X_T^2,\mathbb{P}_{X_T^2}),G(X_T^1-X_T^2)\right\rangle\right]\geq \alpha_1\left|G\mathbb{E}[\hat{X}_T]\right|^2;$

where $V^1,V^2\in\mathbb{R}^{n+m+md}$ and $V^3,V^4\in\mathbb{R}^n$ are random vectors which follow the distributions μ^1,μ^2 on $\mathbb{R}^{n+m+md}$ and μ^3,μ^4 on $\mathbb{R}^n$ respectively. Furthermore, $\beta_1>0,\ \alpha_1>0$ (resp. $\beta_2>0$) when $m>n$ (resp. $n>m$). In particular, if $m=n$, then either $\beta_1>0$ and $\alpha_1>0$, or $\beta_2>0$.

Remark. (1) It is known in [4] that, solely under the standard Lipschitz condition, i.e.,

$$\begin{aligned}&|b(t;x',y'z',\mu')-b(t;x,y,z,\mu)|+|h(t;x',y'z',\mu')-h(t;x,y,z,\mu)|\\&\quad+|\sigma(t;x',y'z',\mu')-\sigma(t;x,y,z,\mu)|\\&\leq L\left(|x-x'|+|y-y'|+|z-z'|+\mathcal{W}\left(\mu,\mu'\right)\right),\end{aligned}$$

where

$$L\geq 1,\qquad y,y'\in\mathbb{R}^m,\qquad x',x\in\mathbb{R}^n,\qquad z',z\in\mathbb{R}^{m\times d}\quad\text{and}$$
$$\mu',\ \mu\in\mathcal{M}\left(R^{n+m+md}\right);$$

it cannot guarantee the unique existence of the solutions of general MFT-FBSDEs. Our technical assumptions (A1)–(A4) as stated above depict trade-off between Lipschitz and monotonicity conditions, in the sense that a weaker (stronger resp.) Lipschitz condition corresponds to a stronger (weaker resp.) monotonicity condition, and both of them together play a major role in ensuring the uniqueness of the solution of the MFT-FBSDEs. The intuition behind the balance between these two conditions is hinted from the observation in the inequalities (2); indeed, the Wasserstein's distance of two probability measures is bounded below by the Euclidean norm of the difference of their respective expectations, which motivates us to consider the problem under the different influences of various Lipschitz constraints for the measure arguments in the corresponding coefficient and pay-off functions.

(2) The matrix G is used to match the dimensions of processes X_t and Y_t in monotonicity condition when they are different, and it can be arbitrarily chosen provided that it is of full rank. In particular, when X_t and Y_t are of the same dimension $n \in \mathbb{N}$, we can simply take $G = I_{n\times n}$.

(3) Assumption (A2) (i) is equivalent to

$$\text{(i}'\text{)} \ \left| f\left(t; u^1, \mu^1\right) - f\left(t; u^2, \mu^2\right)\right| \le L\left|\mathbb{E}[V^1] - \mathbb{E}[V^2]\right|;$$

indeed, (i)⇒(i′) is obvious. To show that (i′)⇒(i), note that if μ^1 and μ^2 have the same first moment, then $f(t; u^1, \mu^1) = f(t; u^2, \mu^2)$ for all μ^1 and μ^2 because of (i′), and so $f(t; u, \mu)$ is independent of u and depends solely on $(t; \int v\mu(dv))$, i.e. $f(t; u, \mu) = f(t; \int v\mu(dv))$. Similarly, by using the same argument, Assumption (A2)(iii) is equivalent to

$$\begin{aligned}&\text{(iii}'\text{)} \ \mathbb{E}\left[\left(A\left(t; U_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - A\left(t; U_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right), U_t^1 - U_t^2\right)\right]\\ &\quad \le -\beta_1\left|G\mathbb{E}[\hat{X}_t]\right|^2 - \beta_2\left(\left|G'\mathbb{E}[\hat{Y}_t]\right|^2 + \left|G'\mathbb{E}[\hat{Z}_t]\right|^2\right).\end{aligned}$$

The equivalence between (i′) (resp. (iii′)) and (A4)(i) (resp. (A4)(iii)) also holds by applying the same argument.

3. Existence and uniqueness of the solutions to the mean-field type forward–backward stochastic differential equations

Theorem 1. *Suppose that one of the above Assumptions* (A1)–(A4) *holds. There exists at most one adapted solution* (X, Y, Z) *for the MFT-FBSDE* (1).

Proof. Let $U_t^1 = (X_t^1, Y_t^1, Z_t^1)$, $U^2 = (X_t^2, Y_t^2, Z_t^2)$ be two adapted solutions of (1), we set

$$\begin{aligned}\hat{U}_t &\triangleq (\hat{X}_t, \hat{Y}_t, \hat{Z}_t) = (X_t^1 - X_t^2, Y_t^1 - Y_t^2, Z_t^1 - Z_t^2),\\ \hat{b}_t &\triangleq b\left(t; X_t^1, Y_t^1, Z_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - b\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right),\\ \hat{\sigma}_t &\triangleq \sigma\left(t; X_t^1, Y_t^1, Z_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - \sigma\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right),\\ \hat{h}_t &\triangleq h\left(t; X_t^1, Y_t^1, Z_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - h\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right),\\ \hat{f}_t &\triangleq f\left(t; X_t^1, Y_t^1, Z_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - f\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right).\end{aligned}$$

An application of Itô's formula to $\left\langle \hat{Y}_t, G\hat{X}_t\right\rangle$ yields

$$\begin{aligned}&\mathbb{E}\left[\left\langle g\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g\left(X_T^2, \mathbb{P}_{X_T^2}\right), G\hat{X}_T\right\rangle\right]\\ &\quad = \mathbb{E}\left[\int_0^T \left\langle A\left(t; U_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - A\left(t; U_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right), U_t^1 - U_t^2\right\rangle dt\right].\end{aligned}$$

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(1) If (A1) is satisfied, an immediate consequence of this assumption and a simple rearrangement gives us that

$$\alpha_1 \mathbb{E}\left[|G\hat{X}_T|^2\right] + \beta_1 \mathbb{E}\left[\int_0^T |G\hat{X}_t|^2 dt\right] \le -\beta_2 \mathbb{E}\left[\int_0^T |G'\hat{Y}_t|^2 dt + \int_0^T |G'\hat{Z}_t|^2 dt\right].$$

If $n > m$, in this case, $\beta_2 > 0$. We can have $\int_0^T |G'\hat{Y}_t|^2 dt = 0$, which implies $Y_t^1 = Y_t^2$ for almost every $t \in [0,T]$; and by the continuity of Y^1 and Y^2, it also holds that $\mathbb{P}\big(Y_t^1 = Y_t^2, \forall t \in [0,T]\big) = 1$. Apply Itô's lemma to the left hand side of the equation $|\hat{Y}_t|^2 = 0$, it follows that $\int_0^T |\hat{Z}_s|^2 ds = 0$, and thus $Z_t^1 = Z_t^2$ for almost every t. Moreover, we can see that the process $\{\hat{X}_t\}_{t\in[0,T]}$ satisfies the following mean-field type SDE:

$$d\hat{X}_t = \hat{b}_t dt + \hat{\sigma}_t dB_t, \quad \hat{X}_0 = 0,\ t \in [0,T],$$

where the coefficients satisfy

$$\begin{aligned} |\hat{b}_t| + |\hat{\sigma}_t| &\le 2\sqrt{|\hat{b}_t|^2 + |\hat{\sigma}_t|^2} \le 2|\hat{f}_t| \le 2L\left(|\hat{X}_t| + \mathcal{W}\left(\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right)\right) \\ &\le 2L\left(|\hat{X}_t| + \left(\mathbb{E}[|\hat{X}_t|^2]\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3}$$

Applying Itô's formula to $|\hat{X}_t|^2$ and by (3), we can derive that

$$\begin{aligned} \mathbb{E}\left[|\hat{X}_t|^2\right] &= 2\mathbb{E}\left[\int_0^t \left\langle \hat{b}_s, \hat{X}_s \right\rangle ds\right] + \mathbb{E}\left[\int_0^t |\hat{\sigma}_s|^2 ds\right] \\ &\le 4L\mathbb{E}\left[\int_0^t \left\{|\hat{X}_s| + \left(\mathbb{E}[|\hat{X}_s|^2]\right)^{\frac{1}{2}}\right\} \cdot |\hat{X}_s| ds\right] \\ &\quad + 4L^2\mathbb{E}\left[\int_0^t \left(|\hat{X}_s| + \left(\mathbb{E}[|\hat{X}_s|^2]\right)^{\frac{1}{2}}\right)^2 ds\right] \\ &\le 4L\left\{\mathbb{E}\left[\int_0^t |\hat{X}_s|^2 ds\right] + \int_0^t \mathbb{E}[|\hat{X}_s|]\left(\mathbb{E}[|\hat{X}_s|^2]\right)^{\frac{1}{2}} ds\right\} \\ &\quad + 8L^2\left\{\mathbb{E}\left[\int_0^t |\hat{X}_s|^2 ds\right] + \mathbb{E}\left[\int_0^t |\hat{X}_s|^2 ds\right]\right\} \\ &\le \left(8L + 16L^2\right)\int_0^t \mathbb{E}\left[|\hat{X}_s|^2\right] ds, \quad \forall t \in [0,T], \end{aligned}$$

which implies that $X_t^1 = X_t^2, \forall t \in [0,T] \cap \mathbb{Q}$ by Gronwall's inequality; and by the continuity of X^1 and X^2, it holds that $\mathbb{P}\left(X_t^1 = X_t^2, \forall t \in [0,T]\right) = 1$.

If $n < m$, in this case, $\beta_1 > 0,\ \alpha_1 > 0$. We can have $\int_0^T |G\hat{X}_t|^2 dt = 0$ and $X_T^1 = X_T^2$, which implies that $X_t^1 = X_t^2$ for almost every $t \in [0,T]$ and $g(X_T^1, \mathbb{P}_{X_T^1}) = g(X_T^2, \mathbb{P}_{X_T^2})$; and by the continuity of X^1 and X^2, it follows that $\mathbb{P}\left(X_t^1 = X_t^2, \forall t \in [0,T]\right) = 1$. Moreover, we can see that the process $\{\hat{Y}_t\}_{t\in[0,T]}$ satisfies the following mean-field type BSDE:

$$d\hat{Y}_t = -\hat{h}_t dt + \hat{Z}_t dB_t, \quad \hat{Y}_T = 0,\ t \in [0,T],$$

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by Assumption (A1) and $\hat{X}_t = 0$ for $t \in [0, T]$, the drift coefficient satisfies

$$|\hat{h}_t| \le L\left(|\hat{Y}_t| + |\hat{Z}_t| + \mathcal{W}\left(\mathbb{P}_{(X_t^1, Y_t^1, Z_t^1)}, \mathbb{P}_{(X_t^2, Y_t^2, Z_t^2)}\right)\right)$$

$$\le L\left(|\hat{Y}_t| + |\hat{Z}_t| + \left(\mathbb{E}[|\hat{Y}_t|^2]\right)^{\frac{1}{2}} + \left(\mathbb{E}[|\hat{Z}_t|^2]\right)^{\frac{1}{2}}\right). \tag{4}$$

Applying Itô's formula to $|\hat{Y}_t|^2$, by substituting (4) and note that $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2, a, b \in \mathbb{R}, \epsilon > 0$, we can derive that

$$\mathbb{E}\left[|\hat{Y}_t|^2\right] + \mathbb{E}\left[\int_t^T |\hat{Z}_s|^2 ds\right] = 2\mathbb{E}\left[\int_t^T \left\langle \hat{h}_s, \hat{Y}_s \right\rangle ds\right] \le 2\int_t^T \mathbb{E}\left[|\hat{h}_s||\hat{Y}_s|\right] ds$$

$$\le 2L\int_t^T \left\{\mathbb{E}[|\hat{Y}_s|^2] + \mathbb{E}\left[|\hat{Z}_s||\hat{Y}_s|\right] + \mathbb{E}[|\hat{Y}_s|]\left(\mathbb{E}\left[|\hat{Y}_s|^2\right]\right)^{\frac{1}{2}} + \mathbb{E}[|\hat{Y}_s|]\left(\mathbb{E}\left[|\hat{Z}_s|^2\right]\right)^{\frac{1}{2}}\right\} ds$$

$$\le 2L\int_t^T \left\{\mathbb{E}[|\hat{Y}_s|^2] + \mathbb{E}\left[|\hat{Z}_s||\hat{Y}_s|\right] + \mathbb{E}\left[|\hat{Y}_s|^2\right] + \left(\mathbb{E}[|\hat{Y}_s|^2]\right)^{\frac{1}{2}}\left(\mathbb{E}[|\hat{Z}_s|^2]\right)^{\frac{1}{2}}\right\} ds$$

$$\le 2L\int_t^T \left\{\left(2 + \frac{1}{2\epsilon} + \frac{1}{2\epsilon}\right)\mathbb{E}\left[|\hat{Y}_s|^2\right] + \left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right)\mathbb{E}\left[|\hat{Z}_s|^2\right]\right\} ds$$

$$= L\int_t^T \left\{\left(4 + \frac{2}{\epsilon}\right)\mathbb{E}\left[|\hat{Y}_s|^2\right] + 2\epsilon\mathbb{E}\left[|\hat{Z}_s|^2\right]\right\} ds,$$

then choose $\epsilon = \frac{1}{4L}$ to get that

$$\mathbb{E}\left[|\hat{Y}_t|^2\right] + \frac{1}{2}\mathbb{E}\left[\int_t^T |\hat{Z}_s|^2 ds\right] \le \left(8L^2 + 4L\right)\int_t^T \mathbb{E}\left[|\hat{Y}_s|^2\right] ds, \quad \forall t \in [0, T],$$

which implies that $Z_t^1 = Z_t^2$ for almost every t, and $Y_t^1 = Y_t^2, \forall t \in [0, T] \cap \mathbb{Q}$ by Gronwall's inequality; and by the continuity of Y^1 and Y^2, it also holds that $\mathbb{P}\left(Y_t^1 = Y_t^2, \forall t \in [0, T]\right) = 1$.

If $n = m$, a similar argument can also be applied to obtain our desired result.

(2) If (A2) is satisfied, then we can similarly deduce that

$$\alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right] + \beta_1\int_0^T \left|\mathbb{E}[G\hat{X}_t]\right|^2 dt \le -\beta_2\int_0^T \left(\left|\mathbb{E}[G'\hat{Y}_t]\right|^2 + \left|\mathbb{E}[G'\hat{Z}_t]\right|^2\right) dt.$$

If $n > m$, then $\beta_2 > 0$, we can have $\left|G'\mathbb{E}[\hat{Y}_t]\right|^2 = 0$ and $\left|G'\mathbb{E}[\hat{Z}_t]\right|^2 = 0$ for almost every $t \in [0, T]$, which implies $\mathbb{E}[\hat{Y}_t] = 0$ and $\mathbb{E}[\hat{Z}_t] = 0$, a.e. Applying Itô's formula to $|\hat{X}_t|^2$, by the Lipschitz condition imposed in (A2), $\mathbb{E}[\hat{Y}_t] = 0$ and $\mathbb{E}[\hat{Z}_t] = 0$, a.e., we can derive that

$$\mathbb{E}[|\hat{X}_t|^2] = 2\mathbb{E}\left[\int_0^t \left\langle \hat{b}_s, \hat{X}_s \right\rangle ds\right] + \int_0^t \mathbb{E}[|\hat{\sigma}_s|^2] ds$$

$$\le 2\mathbb{E}\left[\int_0^t |\hat{b}_s||\hat{X}_s| ds\right] + \int_0^t \mathbb{E}[|\hat{\sigma}_s|^2] ds$$

$$\le 2L\int_0^t \mathbb{E}[|\hat{X}_s|^2] ds + L^2\int_0^t \mathbb{E}[|\hat{X}_s|^2] ds, \quad \forall t \in [0, T].$$

An application of Gronwall's inequality yields that

$$\mathbb{E}[|\hat{X}_t|^2] = 0, \quad \forall t \in [0, T],$$

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which implies that $X_t^1 = X_t^2, \forall t \in [0, T] \cap \mathbb{Q}$; and by the continuity of X^1 and X^2, it also holds that $\mathbb{P}\left(X_t^1 = X_t^2, \forall t \in [0, T]\right) = 1$. Moreover, we can see that the process $\{\hat{Y}_t\}_{t\in[0,T]}$ satisfies the following mean-field type BSDE:

$$d\hat{Y}_t = -\hat{h}_t dt + \hat{Z}_t dB_t, \quad \hat{Y}_T = 0, \ t \in [0, T],$$

by Assumption (A2), $\mathbb{E}[\hat{X}_t] = 0$, $\mathbb{E}[\hat{Y}_t] = 0$ and $\mathbb{E}[\hat{Z}_t] = 0$ for almost every $t \in [0, T]$, the drift coefficient satisfies

$$|\hat{h}_t| \le L \left(\left|\mathbb{E}[\hat{X}_t]\right| + \left|\mathbb{E}[\hat{Y}_t]\right| + \left|\mathbb{E}[\hat{Z}_t]\right| \right) = 0, \quad \text{a.e.},$$

then in light of the continuity of Y^1 and Y^2, it follows that $\mathbb{P}\left(Y_t^1 = Y_t^2, \forall t \in [0, T]\right) = 1$, and $Z_t^1 = Z_t^2$ for almost every t by standard result of BSDE.

If $n < m$, then $\alpha_1 > 0$ and $\beta_1 > 0$, we can have $\mathbb{E}[\hat{X}_t] = 0$, for almost every t, and $\hat{X}_T = 0$. By Lipschitz's condition imposed in (A2), i.e.,

$$|\hat{h}_s| \le L \left(\left|\mathbb{E}[\hat{X}_s]\right| + \left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right| \right) = L \left(\left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right| \right), \quad \text{a.e.},$$

which can implies that $\hat{h}_s$ is independent of $(\hat{X}_s, \hat{Y}_s, \hat{Z}_s)$. Applying Itô's formula to $|\hat{Y}_t|^2$ and noting $X_T^1 = X_T^2$, we can derive that for any $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}[|\hat{Y}_t|]^2 + \mathbb{E}\left[\int_t^T |\hat{Z}_s|^2 ds\right] &= \mathbb{E}\left[\left|g\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\
&\quad + 2\mathbb{E}\left[\int_t^T \left\langle \hat{h}_s, \hat{Y}_s \right\rangle ds\right] \\
&= 0 + 2\int_t^T \left\langle \mathbb{E}[\hat{h}_s], \mathbb{E}[\hat{Y}_s] \right\rangle ds \\
&\le 2L \int_t^T \left(\left|\mathbb{E}[\hat{Y}_s]\right|^2 + \left|\mathbb{E}[\hat{Y}_s]\right| \left|\mathbb{E}[\hat{Z}_s]\right| \right) ds \\
&\le 2L \int_t^T \left(\left|\mathbb{E}[\hat{Y}_s]\right|^2 + \frac{1}{2\epsilon} \left|\mathbb{E}[\hat{Y}_s]\right|^2 + \frac{\epsilon}{2} \left|\mathbb{E}[\hat{Z}_s]\right|^2 \right) ds, \quad (5)
\end{aligned}$$

then choose $\epsilon = \frac{1}{2L}$ and apply Gronwall's inequality to get that

$$\mathbb{E}\left[\int_0^T |\hat{Z}_t|^2 dt\right] = 0, \qquad \mathbb{E}[|\hat{Y}_t|^2] = 0, \quad \forall t \in [0, T],$$

which implies that $Z_t^1 = Z_t^2$ and $Y_t^1 = Y_t^2$ for almost every t; and by the continuity of Y^1 and Y^2, it also holds that $\mathbb{P}\left(Y_t^1 = Y_t^2, \forall t \in [0, T]\right) = 1$. Moreover, we can see that the process $\{\hat{X}_t\}_{t\in[0,T]}$ satisfies the following mean-field type SDE:

$$d\hat{X}_t = \hat{b}_t dt + \hat{\sigma}_t dB_t, \quad \hat{X}_0 = 0, \ t \in [0, T],$$

where the coefficients satisfy

$$|\hat{b}_t| + |\hat{\sigma}_t| \le 2\sqrt{|\hat{b}_t|^2 + |\hat{\sigma}_t|^2} \le 2|\hat{f}_t| \le 2L \left(\left|\mathbb{E}[\hat{X}_t]\right| + \left|\mathbb{E}[\hat{Y}_t]\right| + \left|\mathbb{E}[\hat{Z}_t]\right| \right) = 0,$$

then it follows that $\mathbb{P}\left(X_t^1 = X_t^2, \forall t \in [0, T]\right) = 1$.

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Remark. To ensure the uniqueness of (1), the terminal condition on Y can barely be a function of (Y_T, Z_T) or generally $\mathbb{P}_{(Y_T,Z_T)}$; in other words, g in (1) can only depend on X_T or $\mathbb{P}_{X_T}$. In particular, the process $\{Z_t\}_{t\in[0,T]}$ is only $dt \otimes d\mathbb{P}$-a.e. defined, i.e. a prior setting for Z_T cannot be viable if there is no additional assumption imposed. g's independence of Y_T and $\mathbb{P}_{Y_T}$ can be illustrated as follows: from the derivation of (5), suppose that g is a function of $\mathbb{P}_{Y_T}$, i.e. $Y_T = g\left(X_T, \mathbb{P}_{(X_T,Y_T)}\right)$, then we can only achieve:

$$\left|g\left(X_T^1, \mathbb{P}_{(X_T^1,Y_T^1)}\right) - g\left(X_T^2, \mathbb{P}_{(X_T^2,Y_T^2)}\right)\right|^2 \le \mathcal{W}\left(\mathbb{P}_{(X_T^1,Y_T^1)}, \mathbb{P}_{(X_T^2,Y_T^2)}\right)^2 \le \mathbb{E}[|\hat{Y}_T|^2],$$

which may not lead to a usual Gronwall's inequality, without which uniqueness of the solution can hardly be concluded.

(3) If either (A3) or (A4) is satisfied, a similar argument can also be applied to obtain our desired result. □

3.1. Existence of solution under Assumption (A1)

In this section, we establish the existence of solution of the MFT-FBSDE under Assumption (A1) by the argument of continuity method first introduced in [11]. Let $\lambda \in [0, 1]$, consider the following class of mean-field type FBSDEs

$$\begin{cases} dX_t = [(1-\lambda)\beta_2(-G'Y_t) + \lambda b\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) + \phi_t]dt \\ \qquad + [(1-\lambda)\beta_2(-G'Z_t) + \lambda\sigma\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) + \psi_t]dB_t, \\ dY_t = -[(1-\lambda)\beta_1 G X_t + \lambda h\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) + \gamma_t]dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = \lambda g(X_T, \mathbb{P}_{X_T}) + (1-\lambda)G X_T + \xi, \end{cases} \tag{6}$$

where ϕ, ψ and γ are given process in $M^2(0, T)$ with values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}^m$ respectively, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Clearly, the existence of solution of (6) for $\lambda = 1$ implies that of (1).

Lemma 1. *The following system, which is that of (6) when $\lambda = 0$, has a unique solution:*

$$\begin{cases} dX_t = \left(-\beta_2 G'Y_t + \phi_t\right)dt + \left(-\beta_2 G'Z_t + \psi_t\right)dB_t, \\ -dY_t = (\beta_1 G X_t + \gamma_t)\, dt - Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = G X_T + \xi. \end{cases} \tag{7}$$

Proof. See Lemma 2.5 in [11]. □

Lemma 2. *Under Assumption (A1), we also assume that there exists a constant $\lambda_0 \in [0, 1)$ for any ϕ, ψ, γ in $M^2(0, T)$ taking values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}^m$ respectively and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, such that (6) has an adapted solution. Then there exists a $\delta_0 \in (0, 1)$ which only depends on $G, L, \alpha_1, \beta_1, \beta_2$ and T, such that for any $\lambda \in [\lambda_0, \lambda_0 + \delta_0]$, (6) has an adapted solution.*

Proof. For each $x_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and a triple $u_s \triangleq (x_s, y_s, z_s) \in M^2\left(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}\right)$, we take

$$\begin{cases} \phi_t \leftarrow \delta\left(\beta_2 G' y_t + b\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \phi_t, \\ \psi_t \leftarrow \delta\left(\beta_2 G' z_t + \sigma\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \psi_t, \\ \gamma_t \leftarrow \delta\left(-\beta_1 G x_t + h\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \gamma_t, \\ \xi \leftarrow \delta\left(g(x_T, \mathbb{P}_{x_T}) - G x_T\right) + \xi. \end{cases}$$

By assumption, there exists a constant $\lambda_0 \in [0,1)$ such that there exists a unique triple $U = (X,Y,Z) \in M^2\left(0,T;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}\right)$ satisfying the following MFT-FBSDE:

$$\begin{cases} dX_t = \big\{(1-\lambda_0)\beta_2(-G'Y_t) + \lambda_0 b\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) \\ \qquad + \delta\left(\beta_2 G' y_t + b\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \phi_t\big\} dt \\ \qquad + \big\{(1-\lambda_0)\beta_2(-G'Z_t) + \lambda_0 \sigma\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) \\ \qquad + \delta\left(\beta_2 G' z_t + \sigma\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \psi_t\big\} dB_t, \\ dY_t = -\big\{(1-\lambda_0)\beta_1 G X_t + \lambda_0 h\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) \\ \qquad + \delta\left(-\beta_1 G x_t + h\left(t; u_t, \mathbb{P}_{(x_t,y_t,z_t)}\right)\right) + \gamma_t\big\} dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = \lambda_0 g(X_T, \mathbb{P}_{X_T}) + (1-\lambda_0) G X_T + \delta\left(g(x_T, \mathbb{P}_{x_T}) - G x_T\right) + \xi. \end{cases}$$

We now proceed to prove that, if δ is sufficiently small, the mapping $I_{\lambda_0+\delta} : M^2\big(0,T;\mathbb{R}^{n+m+m\times d}\big) \times L^2(\Omega,\mathcal{F}_T,\mathbb{P}) \to M^2\left(0,T;\mathbb{R}^{n+m+m\times d}\right) \times L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ defined by $I_{\lambda_0+\delta}(u, x_T) = (U, X_T)$ is a contraction.

Let $\bar{u} \triangleq (\bar{x},\bar{y},\bar{z}) \in M^2\left(0,T;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}\right)$ and $\left(\bar{U}, \bar{X}_T\right) \triangleq I_{\lambda_0+\delta}(\bar{u},\bar{x}_T)$. We set

$$\hat{u} = (\hat{x},\hat{y},\hat{z}) = (x-\bar{x}, y-\bar{y}, z-\bar{z}), \qquad \hat{U} = (\hat{X},\hat{Y},\hat{Z}) = (X-\bar{X}, Y-\bar{Y}, Z-\bar{Z}).$$

An application of Itô's formula to $\left\langle G\hat{X}_t, \hat{Y}_t\right\rangle$ yields that

$$\begin{aligned} &\lambda_0 \mathbb{E}\left[\left\langle G\hat{X}_T, g\left(X_T,\mathbb{P}_{X_T}\right) - g\left(\bar{X}_T,\mathbb{P}_{\bar{X}_T}\right)\right\rangle\right] + (1-\lambda_0)\mathbb{E}\left[\left\langle G\hat{X}_T, G\hat{X}_T\right\rangle\right] \\ &\quad + \delta\mathbb{E}\left[\left\langle G\hat{X}_T, -G\hat{x}_T + g\left(x_T,\mathbb{P}_{x_T}\right) - g\left(\bar{x}_T,\mathbb{P}_{\bar{x}_T}\right)\right\rangle\right] \\ &= \lambda_0\mathbb{E}\left[\int_0^T \left(A\left(t;U_t,\mathbb{P}_{(X_t,Y_t,Z_t)}\right) - A\left(t;\bar{U}_t,\mathbb{P}_{(\bar{X}_t,\bar{Y}_t,\bar{Z}_t)}\right), \hat{U}_t\right) dt\right] \\ &\quad - (1-\lambda_0)\mathbb{E}\left[\int_0^T \left(\beta_1\left\langle G\hat{X}_t, G\hat{X}_t\right\rangle + \beta_2\left\langle G'\hat{Y}_t, G'\hat{Y}_t\right\rangle + \beta_2\left\langle G'\hat{Z}_t, G'\hat{Z}_t\right\rangle\right) dt\right] \\ &\quad + \delta\mathbb{E}\Bigg[\int_0^T \Big\{\beta_1\left\langle G\hat{X}_t, G\hat{x}_t\right\rangle + \beta_2\left\langle G'\hat{Y}_t, G'\hat{y}_t\right\rangle + \beta_2\left\langle G'\hat{Z}_t, G'\hat{z}_t\right\rangle \\ &\quad + \left(\hat{U}_t, A\left(t;u_t,\mathbb{P}_{(x_t,y_t,z_t)}\right) - A\left(t;\bar{u}_t,\mathbb{P}_{(\bar{x}_t,\bar{y}_t,\bar{z}_t)}\right)\right)\Big\} dt\Bigg], \end{aligned}$$

and now in light of the condition given in Assumption (A1), we have

$$\begin{aligned} &(\alpha_1\lambda_0 + (1-\lambda_0))\,\mathbb{E}\left[|G\hat{X}_T|^2\right] \\ &\quad + \mathbb{E}\left[\int_0^T \left(\beta_1\left\langle G\hat{X}_t, G\hat{X}_t\right\rangle + \beta_2\left\langle G'\hat{Y}_t, G'\hat{Y}_t\right\rangle + \beta_2\left\langle G'\hat{Z}_t, G'\hat{Z}_t\right\rangle\right) dt\right] \\ &\le \delta\mathbb{E}\Bigg[\int_0^T \Big\{\beta_1|G|^2\left(\frac{1}{2}|\hat{X}_t|^2 + \frac{1}{2}|\hat{x}_t|^2\right) + \beta_2|G|^2\left(\frac{1}{2}|\hat{Y}_t|^2 + \frac{1}{2}|\hat{y}_t|^2\right) \\ &\quad + \beta_2|G|^2\left(\frac{1}{2}|\hat{Z}_t|^2 + \frac{1}{2}|\hat{z}_t|^2\right)\Big\} dt\Bigg] + \delta\mathbb{E}\Bigg[\int_0^T |\hat{U}_t|\,\Big|A\left(t;u_t,\mathbb{P}_{(x_t,y_t,z_t)}\right) \\ &\quad - A\left(t;\bar{u}_t,\mathbb{P}_{(\bar{x}_t,\bar{y}_t,\bar{z}_t)}\right)\Big| dt\Bigg] \\ &\quad + \delta\mathbb{E}\left[\frac{1}{2}\left(|G|^2|\hat{X}_T|^2 + |G|^2\left|\hat{x}_T\right|^2\right) + |G\hat{X}_T|\left|g\left(x_T,\mathbb{P}_{x_T}\right) - g\left(\bar{x}_T,\mathbb{P}_{\bar{x}_T}\right)\right|\right] \end{aligned}$$

$$
\begin{aligned}
&\le \delta\mathbb{E}\left[\int_0^T \left\{\beta_1|G|^2\left(\frac{1}{2}|\hat{X}_t|^2+\frac{1}{2}|\hat{x}_t|^2\right)+\beta_2|G|^2\left(\frac{1}{2}|\hat{Y}_t|^2+\frac{1}{2}|\hat{y}_t|^2\right)\right.\right.\\
&\quad\left.\left.+\beta_2|G|^2\left(\frac{1}{2}|\hat{Z}_t|^2+\frac{1}{2}|\hat{z}_t|^2\right)\right\}dt\right]\\
&\quad+\delta\mathbb{E}\left[\int_0^T |\hat{U}_t||G|\left|f\left(t;u_t,\mathbb{P}_{(x_t,y_t,z_t)}\right)-f\left(t;\bar{u}_t,\mathbb{P}_{(\bar{x}_t,\bar{y}_t,\bar{z}_t)}\right)\right|dt\right]\\
&\quad+\delta\mathbb{E}\left[\frac{1}{2}\left(|G|^2|\hat{X}_T|^2+|G|^2\left|\hat{x}_T\right|^2\right)+|G\hat{X}_T|\left|g\left(x_T,\mathbb{P}_{x_T}\right)-g\left(\bar{x}_T,\mathbb{P}_{\bar{x}_T}\right)\right|\right]\\
&\le \delta\mathbb{E}\left[\int_0^T \left\{\beta_1|G|^2\left(\frac{1}{2}|\hat{X}_t|^2+\frac{1}{2}|\hat{x}_t|^2\right)+\beta_2|G|^2\left(\frac{1}{2}|\hat{Y}_t|^2+\frac{1}{2}|\hat{y}_t|^2\right)\right.\right.\\
&\quad\left.\left.+\beta_2|G|^2\left(\frac{1}{2}|\hat{Z}_t|^2+\frac{1}{2}|\hat{z}_t|^2\right)\right\}dt\right]\\
&\quad+L\delta|G|\mathbb{E}\left[\int_0^T \left(|\hat{U}_t|^2+\frac{1}{2}|\hat{u}_t|^2+\frac{1}{2}\mathcal{W}^2\left(\mathbb{P}_{(x_t,y_t,z_t)},\mathbb{P}_{(\bar{x}_t,\bar{y}_t,\bar{z}_t)}\right)\right)dt\right]\\
&\quad+\delta\mathbb{E}\left[\frac{1}{2}\left(|G|^2|\hat{X}_T|^2+|G|^2\left|\hat{x}_T\right|^2\right)+L|G|^2|\hat{X}_T|^2+\frac{L}{2}|\hat{x}_T|^2+\frac{L}{2}\mathcal{W}^2\left(\mathbb{P}_{x_T},\mathbb{P}_{\bar{x}_T}\right)\right].
\end{aligned}
$$

We also note that

$$
\begin{cases}
\mathcal{W}^2\left(\mathbb{P}_{(x_t,y_t,z_t)},\mathbb{P}_{(\bar{x}_t,\bar{y}_t,\bar{z}_t)}\right)\le\mathbb{E}\left[\left|\hat{x}_t\right|^2+\left|\hat{y}_t\right|^2+\left|\hat{z}_t\right|^2\right]=\mathbb{E}\left[\left|\hat{u}_t\right|^2\right], \quad \forall t\in[0,T],\\
\mathcal{W}^2\left(\mathbb{P}_{x_T},\mathbb{P}_{\bar{x}_T}\right)\le\mathbb{E}[|\hat{x}_T|^2].
\end{cases}
$$

Therefore, there exists a positive constant K_1 which only depends on L, G, β_1, β_2 and T such that

$$
\begin{aligned}
&(\alpha_1\lambda_0+(1-\lambda_0))\,\mathbb{E}\left[|G\hat{X}_T|^2\right]\\
&\quad+\mathbb{E}\left[\int_0^T \left(\beta_1\left\langle G\hat{X}_t,G\hat{X}_t\right\rangle+\beta_2\left\langle G'\hat{Y}_t,G'\hat{Y}_t\right\rangle+\beta_2\left\langle G'\hat{Z}_t,G'\hat{Z}_t\right\rangle\right)dt\right]\\
&\le\delta K_1\mathbb{E}\left[\int_0^T\left(\left|\hat{u}_t\right|^2+|\hat{U}_t|^2\right)dt\right]+\delta K_1\mathbb{E}\left[|\hat{X}_T|^2+|\hat{x}_T|^2\right],
\end{aligned}
$$

also note that $(\alpha_1\lambda_0+(1-\lambda_0))>\min\{1,\alpha_1\}$, we can then obtain that

$$
\begin{aligned}
&\min\{1,\alpha_1\}\times\mathbb{E}\left[|G\hat{X}_T|^2\right]\\
&\quad+\mathbb{E}\left[\int_0^T \left(\beta_1\left\langle G\hat{X}_t,G\hat{X}_t\right\rangle+\beta_2\left\langle G'\hat{Y}_t,G'\hat{Y}_t\right\rangle+\beta_2\left\langle G'\hat{Z}_t,G'\hat{Z}_t\right\rangle\right)dt\right]\\
&\le\delta K_1\mathbb{E}\left[\int_0^T\left(\left|\hat{u}_t\right|^2+|\hat{U}_t|^2\right)dt\right]+\delta K_1\mathbb{E}\left[|\hat{X}_T|^2+|\hat{x}_T|^2\right].
\end{aligned}
\tag{8}
$$

Applying Itô's formula to $|\hat{X}_t|^2$, then we can derive that

$$
\begin{aligned}
\mathbb{E}\left[|\hat{X}_t|^2\right]&=2\mathbb{E}\left[\int_0^t\left\langle\hat{X}_s,(1-\lambda_0)\beta_2(-G'\hat{Y}_s)\right\rangle ds\right]\\
&\quad+2\mathbb{E}\left[\int_0^t\left\langle\hat{X}_s,\lambda_0\left(b\left(s;U_s,\mathbb{P}_{(X_s,Y_s,Z_s)}\right)-b\left(s;\bar{U}_s,\mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right)\right\rangle ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathbb{E}\left[\int_0^t \left\langle \hat{X}_s, \delta\left(\beta_2 G'\hat{y}_s + b\left(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}\right) - b\left(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,\bar{z}_s)}\right)\right)\right\rangle ds\right] \\
&+\mathbb{E}\left[\int_0^t \left|(1-\lambda_0)\beta_2(-G'\hat{Z}_s) + \lambda_0\left(\sigma\left(s; U_s, \mathbb{P}_{(X_s,Y_s,Z_s)}\right) - \sigma\left(s; \bar{U}_s, \mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right)\right.\right. \\
&\left.\left. +\ \delta\left(\beta_2 G'\hat{z}_s + \sigma(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}) - \sigma(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,\bar{z}_s)})\right)\right|^2 ds\right] \\
\le\ &\mathbb{E}\left[\int_0^t \left(|\hat{X}_s|^2 + (1-\lambda_0)^2\beta_2^2|G|^2|\hat{Y}_s|^2\right) ds\right] \\
&+\lambda_0\mathbb{E}\left[\int_0^t \left(|\hat{X}_s|^2 + \left|b\left(s; U_s, \mathbb{P}_{(X_s,Y_s,Z_s)}\right) - b\left(s; \bar{U}_s, \mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right|^2\right) ds\right] \\
&+\mathbb{E}\left[\int_0^t \left\{\delta\beta_2\left(|\hat{X}_s|^2 + |G|^2\left|\hat{y}_s\right|^2\right)\right.\right. \\
&\left.\left. +\delta\left(|\hat{X}_s|^2 + \left|b\left(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}\right) - b\left(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,\bar{z}_s)}\right)\right|^2\right)\right\} ds\right] \\
&+\mathbb{E}\left[\int_0^t \left\{4(1-\lambda_0)^2\beta_2^2|G|^2|\hat{Z}_s|^2 + 4\lambda_0^2\left|\sigma\left(s; U_s, \mathbb{P}_{(X_s,Y_s,Z_s)}\right) - \sigma\left(s; \bar{U}_s, \mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right|^2\right.\right. \\
&\left.\left. +4\delta^2\beta_2^2|G|^2|\hat{z}_s|^2 + 4\delta^2\left|\sigma(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}) - \sigma(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,z_s)})\right|^2\right\} ds\right],
\end{aligned}
$$

by Lipschitz's assumptions imposed in (A1), (2) and Gronwall's inequality we can obtain that

$$
\sup_{0\le s\le T}\mathbb{E}\left[|\hat{X}_s|^2\right] \le \delta K_1\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + K_1\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds\right]. \tag{9}
$$

Note that

$$
\frac{1}{T}\mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right] \le \sup_{0\le s\le T}\mathbb{E}\left[|\hat{X}_s|^2\right],
$$

we can also derive that

$$
\mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right] \le \delta T K_1\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + T K_1\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds\right]. \tag{10}
$$

Similarly, applying Itô's formula to $|\hat{Y}_t|^2$, and we can derive that

$$
\begin{aligned}
&\mathbb{E}\left[|\hat{Y}_t|^2\right] + \mathbb{E}\left[\int_t^T |\hat{Z}_s|^2 ds\right] \\
&= 2\mathbb{E}\left[\int_t^T \left\langle \hat{Y}_s, (1-\lambda_0)\beta_1 G\hat{X}_s + \lambda_0\left(h\left(s; U_s, \mathbb{P}_{(X_s,Y_s,Z_s)}\right) - h\left(s; \bar{U}_s, \mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right)\right.\right. \\
&\left.\left. -\delta\left(\beta_1 G\hat{x}_s + h\left(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}\right) - h\left(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,\bar{z}_s)}\right)\right)\right\rangle ds\right] \\
&+\mathbb{E}\left[\left|\lambda_0\left(g\left(X_T, \mathbb{P}_{X_T}\right) - g\left(\bar{X}_T, \mathbb{P}_{\bar{X}_T}\right)\right) + (1-\lambda_0)\, G\hat{X}_T\right.\right. \\
&\left.\left. +\delta\left(g\left(x_T, \mathbb{P}_{x_T}\right) - g\left(\bar{x}_T, \mathbb{P}_{\bar{x}_T}\right)\right) - \delta G\hat{x}_T\right|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_t^T \left\{\mathbb{E}[|\hat{Y}_s|^2] + (1-\lambda_0)^2\beta_1^2|G|^2\mathbb{E}[|\hat{X}_s|^2]\right\} ds \\
&\quad + \int_t^T \left\{\frac{1}{\epsilon}\mathbb{E}[|\hat{Y}_s|^2] + \epsilon\lambda_0^2\mathbb{E}\left[\left|h\left(s; U_s, \mathbb{P}_{(X_s,Y_s,Z_s)}\right) - h\left(s; \bar{U}_s, \mathbb{P}_{(\bar{X}_s,\bar{Y}_s,\bar{Z}_s)}\right)\right|^2\right]\right\} ds \\
&\quad + \int_t^T \delta\beta_1 \left\{\mathbb{E}[|\hat{Y}_s|^2] + |G|^2\mathbb{E}[|\hat{x}_s|^2]\right\} ds \\
&\quad + \int_t^T \delta \left\{\mathbb{E}[|\hat{Y}_s|^2] + \mathbb{E}\left[\left|h\left(s; u_s, \mathbb{P}_{(x_s,y_s,z_s)}\right) - h\left(s; \bar{u}_s, \mathbb{P}_{(\bar{x}_s,\bar{y}_s,\bar{z}_s)}\right)\right|^2\right]\right\} ds \\
&\quad + 4\lambda_0^2\mathbb{E}\left[\left|g\left(X_T, \mathbb{P}_{X_T}\right) - g\left(\bar{X}_T, \mathbb{P}_{\bar{X}_T}\right)\right|^2\right] + 4\left(1-\lambda_0\right)^2|G|^2\mathbb{E}[|\hat{X}_T|^2] \\
&\quad + 4\delta^2\mathbb{E}\left[\left|g\left(x_T, \mathbb{P}_{x_T}\right) - g\left(\bar{x}_T, \mathbb{P}_{\bar{x}_T}\right)\right|^2\right] + 4\delta^2|G|^2\mathbb{E}\left[|\hat{x}_T|^2\right],
\end{aligned}
$$

choosing $\epsilon = \frac{1}{8L^2}$, by Lipschitz's assumptions imposed in (A1), (2) and Gronwall's inequality to provide a bound for $\sup_{t\in[0,T]}\mathbb{E}[|Y_t|^2]$; and then substitute the latter back to the same inequality to obtain the bound for $\int_0^T \mathbb{E}[|\hat{Z}_s|^2]ds$. By combining the obtained bounds, we can deduce that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds\right] &\le \delta K_1\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + \delta K_1\mathbb{E}\left[|\hat{x}_T|^2\right] \\
&\quad + K_1\mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right] + K_1\mathbb{E}\left[|\hat{X}_T|^2\right]. \qquad (11)
\end{aligned}
$$

(i) If $n > m$, then $\beta_2 > 0$. Then GG' is a full-rank $m \times m$ matrix, we can easily derive that

$$|G'\hat{Y}_t| \ge \frac{1}{|G||(GG')^{-1}|}|\hat{Y}_t| \quad \text{and} \quad |G'\hat{Z}_t| \ge \frac{1}{|G||(GG')^{-1}|}|\hat{Z}_t|,$$

along with (8)–(10), we can have

$$\mathbb{E}\left[\int_0^T |\hat{U}_s|^2 ds\right] + \mathbb{E}\left[|\hat{X}_T|^2\right] \le \delta K\left(\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + \mathbb{E}\left[|\hat{x}_T|^2\right]\right),$$

where K is a constant which only depends on $L, \alpha_1, \beta_1, \beta_2, G$ and T.

(ii) If $m > n$, then $\alpha_1 > 0$, $\beta_1 > 0$. Then $G'G$ is a full-rank $n \times n$ matrix, we can easily derive that

$$|G\hat{X}_t| \ge \frac{1}{|G||(G'G)^{-1}|}|\hat{X}_t|,$$

along with (8) and (11), we can also have

$$\mathbb{E}\left[\int_0^T |\hat{U}_s|^2 ds\right] + \mathbb{E}\left[|\hat{X}_T|^2\right] \le \delta K\left(\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + \mathbb{E}\left[|\hat{x}_T|^2\right]\right).$$

Therefore, we always have that

$$\mathbb{E}\left[\int_0^T |\hat{U}_s|^2 ds\right] + \mathbb{E}\left[|\hat{X}_T|^2\right] \le \delta K\left(\mathbb{E}\left[\int_0^T |\hat{u}_s|^2 ds\right] + \mathbb{E}\left[|\hat{x}_T|^2\right]\right).$$

Let $\delta_0 = \frac{1}{2K}$, it is clear that the mappings

$$I_{\lambda_0+\delta}\left(u, x_T\right) = \left(U, X_T\right),$$

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are contraction for all $\delta \in (0, \delta_0)$. It follows that there is a unique fixed point which is the solution of (6) for $\lambda = \lambda_0 + \delta$, $\delta \in (0, \delta_0)$. $\square$

By applying Lemmas 1 and 2, it is easy to establish the unique existence of solution for the MFT-FBSDE (1).

Theorem 2. *Under Assumption* (A1), *there exists a unique adapted solution* (X, Y, Z) *of the MFT-FBSDE* (1).

3.2. Existence of solution under Assumption (A2), (A3) or (A4)

Similar to that in Section 3.1, we first establish the existence of solution of the MFT-FBSDE under Assumption (A2). There could be some different argument used in comparison with that in 3.1. Q3

Consider the following class of mean-field type FBSDEs

$$\begin{cases} dX_t = [(1-\lambda)\beta_2(-G'\mathbb{E}[Y_t]) + \lambda b\,(t; \mathbb{E}[U_t]) + \phi_t]dt \\ \qquad + [(1-\lambda)\beta_2(-G'\mathbb{E}[Z_t]) + \lambda\sigma\,(t; \mathbb{E}[U_t]) + \psi_t]dB_t, \\ dY_t = -[(1-\lambda)\beta_1 G\mathbb{E}[X_t] + \lambda h\,(t; \mathbb{E}[U_t]) + \gamma_t]dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = \lambda g(X_T, \mathbb{P}_{X_T}) + (1-\lambda)GX_T + \xi, \end{cases} \tag{12}$$

where ϕ, ψ and γ satisfy the same conditions as in Lemma 2 in Section 3.1. Clearly, the existence of solution of (12) for $\lambda = 1$ implies that of (1).

Lemma 3. *The following system, which is that of* (12) *when* $\lambda = 0$, *has a unique solution:*

$$\begin{cases} dX_t = \left(-\beta_2 G'\mathbb{E}[Y_t] + \phi_t\right) dt + \left(-\beta_2 G'\mathbb{E}[Z_t] + \psi_t\right) dB_t, \\ dY_t = -\left(\beta_1 G\mathbb{E}[X_t] + \gamma_t\right) dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = GX_T + \xi. \end{cases} \tag{13}$$

Proof. Taking expectations on both sides of (13) yields

$$\begin{cases} d\mathbb{E}[X_t] = \left(-\beta_2 G'\mathbb{E}[Y_t] + \mathbb{E}[\phi_t]\right) dt, \\ d\mathbb{E}[Y_t] = -\left(\beta_1 G\mathbb{E}[X_t] + \mathbb{E}[\gamma_t]\right) dt, \\ \mathbb{E}[X_0] = x_0, \qquad \mathbb{E}[Y_T] = G\mathbb{E}[X_T] + \mathbb{E}[\xi]. \end{cases} \tag{14}$$

As a special case of Lemma 1, this system (14) has a unique solution $(\mathbb{E}[X_t], \mathbb{E}[Y_t])$. In the following, we follow the method adopted in the proof of Lemma 2.5 in [11] to establish our desired result. Define

$$\check{X}_t \triangleq X_t - \mathbb{E}[X_t], \qquad \check{Y}_t \triangleq Y_t - \mathbb{E}[Y_t], \qquad \check{Z}_t \triangleq Z_t,$$
$$\check{\phi}_t \triangleq \phi_t - \mathbb{E}[\phi_t], \qquad \check{\gamma}_t \triangleq \gamma_t - \mathbb{E}[\gamma_t], \qquad \check{\xi} \triangleq \xi - \mathbb{E}[\xi], \qquad \check{\psi}_t \triangleq \psi_t.$$

To show the unique existence of solution of (13), it suffices to prove unique existence of

$$\begin{cases} d\check{X}_t = \check{\phi}_t dt + \left(-\beta_2 G'\mathbb{E}[\check{Z}_t] + \check{\psi}_t\right) dB_t, \\ d\check{Y}_t = -\check{\gamma}_t dt + \check{Z}_t dB_t, \\ \check{X}_0 = 0, \qquad \check{Y}_T = G\check{X}_T + \check{\xi}, \end{cases} \tag{15}$$

which is the difference between (13) and (14).

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(i) If $n \le m$, then $G'G$ is a strictly positive matrix, we define

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} \triangleq \begin{pmatrix} \check{X} \\ G'\check{Y} \\ G'\check{Z} \end{pmatrix}, \qquad \begin{pmatrix} Y^* \\ Z^* \end{pmatrix} \triangleq \begin{pmatrix} \left(I_m - G(G'G)^{-1}G'\right)\check{Y} \\ \left(I_m - G(G'G)^{-1}G'\right)\check{Z} \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix. Multiplying G' on both sides of the second and the third equations in (15) yields

$$\begin{cases} d\tilde{X}_t = \check{\phi}_t dt + \left(-\beta_2 \mathbb{E}[\tilde{Z}_t] + \check{\psi}_t\right) dB_t, \\ d\tilde{Y}_t = -G'\check{\gamma}_t dt + \tilde{Z}_t dB_t, \\ \tilde{X}_0 = 0, \qquad \tilde{Y}_T = G'G\tilde{X}_T + G'\check{\xi}. \end{cases} \tag{16}$$

Multiplying $\left(I_m - G(G'G)^{-1}G'\right)$ on both sides of the equations involving $\left(\check{Y}, \check{Z}\right)$ in (15) also yields

$$\begin{cases} dY_t^* = -\left(I_m - G(G'G)^{-1}G'\right)\check{\gamma}_t dt + Z_t^* dB_t, \\ Y_T^* = \left(I_m - G(G'G)^{-1}G'\right)\check{\xi}. \end{cases} \tag{17}$$

Clearly, the solution (Y^*, Z^*) is uniquely determined. Consider the following mean-field BSDE:

$$\begin{cases} -dp_t = \left(G'G\check{\phi}_t + G'\check{\gamma}_t\right) dt + \left(G'G\check{\psi}_t - q_t - \beta_2 G'G\mathbb{E}[q_t]\right) dB_t, \\ p_T = G'\check{\xi}, \end{cases} \tag{18}$$

and we claim that (18) has a unique solution $(p, q) \in M^2\left(0, T; \mathbb{R}^{n+n\times d}\right)$; indeed, define

$$w_t \triangleq q_t + \beta_2 G'G\mathbb{E}[q_t],$$

and (18) becomes a classical BSDE:

$$\begin{cases} -dp_t = \left(G'G\check{\phi}_t + G'\check{\gamma}_t\right) dt + \left(G'G\check{\psi}_t - w_t\right) dB_t, \\ p_T = G'\check{\xi}, \end{cases} \tag{19}$$

whose solution (p, w) is uniquely determined. Then by definition

$$\mathbb{E}[w_t] = \mathbb{E}\left[q_t + \beta_2 G'G\mathbb{E}[q_t]\right] = \left(I_n + \beta_2 G'G\right)\mathbb{E}[q_t],$$

thus $\mathbb{E}[q_t] = \left(I_n + \beta_2 G'G\right)^{-1} \mathbb{E}[w_t]$, and finally we can get

$$q_t = w_t - \beta_2 G'G\left(I_n + \beta_2 G'G\right)^{-1} \mathbb{E}[w_t].$$

Now, let $\tilde{X}_t$ be the solution of the SDE

$$\begin{cases} d\tilde{X}_t = \check{\phi}_t dt + \left(-\beta_2 \mathbb{E}[q_t] + \check{\psi}_t\right) dB_t, \\ \tilde{X}_0 = 0. \end{cases}$$

It is easy to check that

$$(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t) \triangleq \left(\tilde{X}_t, G'G\tilde{X}_t + p_t, q_t\right)$$

is the solution to (16). Then the triple $\left(\check{X}, \check{Y}, \check{Z}\right)$ is uniquely obtained by the substitution

$$\begin{pmatrix} \check{X} \\ \check{Y} \\ \check{Z} \end{pmatrix} = \begin{pmatrix} \tilde{X} \\ G(G'G)^{-1}\tilde{Y} + Y^* \\ G(G'G)^{-1}\tilde{Z} + Z^* \end{pmatrix};$$

indeed, its uniqueness follows immediately by working backward.

(ii) If $n > m$, GG' is a positive definite matrix. Similar approach as in (i) can be adopted, we now set

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} \triangleq \begin{pmatrix} G\check{X} \\ \check{Y} \\ \check{Z} \end{pmatrix}, \qquad X^* \triangleq \left(I_n - G'(GG')^{-1}G\right)\check{X}.$$

Then X^* is the unique solution of the following SDE:

$$\begin{cases} dX_t^* = \left(I_n - G'(GG')^{-1}G\right)\check{\phi}_t dt + \left(I_n - G'(GG')^{-1}G\right)\check{\psi}_t dB_t, \\ X_0^* = 0. \end{cases}$$

Multiplying G on both sides of the equation governing $\check{X}$ in (15) yields:

$$\begin{cases} d\tilde{X}_t = G\check{\phi}_t dt + \left(-\beta_2 GG'\mathbb{E}[\tilde{Z}_t] + G\check{\psi}_t\right) dB_t, \\ d\tilde{Y}_t = -\check{\gamma}_t dt + \tilde{Z}_t dB_t, \\ \tilde{X}_0 = 0, \qquad \tilde{Y}_T = \tilde{X}_T + \check{\xi}. \end{cases} \tag{20}$$

To solve (20), we consider the following mean-field BSDE:

$$\begin{cases} -dp_t = \left(G\check{\phi}_t + \check{\gamma}_t\right) dt + \left(G\check{\psi}_t - q_t - \beta_2 GG'\mathbb{E}[q_t]\right) dB_t, \\ p_T = \check{\xi}, \end{cases} \tag{21}$$

and it is easy to show that (21) has a unique solution (p, q) by using a similar argument that tackles (18). Let $\tilde{X}$ be the solution of the SDE:

$$\begin{cases} d\tilde{X}_t = G\check{\phi}_t dt + \left(G\check{\psi}_t - \beta_2 GG'\mathbb{E}[q_t]\right) dB_t, \\ \tilde{X}_0 = 0. \end{cases}$$

Then it is easy to check that

$$(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t) \triangleq \left(\tilde{X}_t, \tilde{X}_t + p_t, q_t\right)$$

is the solution of (20). The unique existence $(\check{X}, \check{Y}, \check{Z})$ is now evident via the following relation:

$$\begin{pmatrix} \check{X} \\ \check{Y} \\ \check{Z} \end{pmatrix} = \begin{pmatrix} G'(GG')^{-1}\tilde{X} + X^* \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix};$$

again its uniqueness follows from working backward. □

Lemma 4. *Under Assumption* (A2)*, we also assume that there exists a constant* $\lambda_0 \in [0, 1)$ *for any* ϕ, ψ, γ *in* $M^2(0, T)$ *taking values in* $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ *and* $\mathbb{R}^m$ *respectively and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, *such that* (12) *has an adapted solution. Then there exists a* $\delta_0 \in (0, 1)$ *which only depends on* $G, L, \alpha_1, \beta_1, \beta_2$ *and* T, *such that for any* $\lambda \in [\lambda_0, \lambda_0 + \delta_0]$, (12) *has an adapted solution.*

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Proof. We use similar argument as that in the proof for Lemma 2. Taking

$$\begin{cases} \phi_t \leftarrow \delta\left(\beta_2 G'\mathbb{E}[y_t] + b\left(t;\mathbb{E}[u_t]\right)\right) + \phi_t, \\ \psi_t \leftarrow \delta\left(\beta_2 G'\mathbb{E}[z_t] + \sigma\left(t;\mathbb{E}[u_t]\right)\right) + \psi_t, \\ \gamma_t \leftarrow \delta\left(-\beta_1 G\mathbb{E}[x_t] + h\left(t;\mathbb{E}[u_t]\right)\right) + \gamma_t, \\ \xi \leftarrow \delta\left(g(x_T, \mathbb{P}_{x_T}) - Gx_T\right) + \xi. \end{cases}$$

In light of the statement assumption, there exists a constant $\lambda_0 \in [0, 1)$ such that there exists a unique triple $U = (X, Y, Z) \in M^2\left(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}\right)$ satisfying the following MFT-FBSDE:

$$\begin{cases} dX_t = \left\{(1-\lambda_0)\beta_2(-G'\mathbb{E}[Y_t]) + \lambda_0 b\left(t;\mathbb{E}[U_t]\right)\right. \\ \qquad\qquad \left. + \delta\left(\beta_2 G'\mathbb{E}[y_t] + b\left(t;\mathbb{E}[u_t]\right)\right) + \phi_t\right\}dt \\ \qquad\qquad + \left\{(1-\lambda_0)\beta_2(-G'\mathbb{E}[Z_t]) + \lambda_0\sigma\left(t;\mathbb{E}[U_t]\right)\right. \\ \qquad\qquad \left. + \delta\left(\beta_2 G'\mathbb{E}[z_t] + \sigma\left(t;\mathbb{E}[u_t]\right)\right) + \psi_t\right\}dB_t, \\ dY_t = -\left\{(1-\lambda_0)\beta_1 G\mathbb{E}[X_t] + \lambda_0 h\left(t;\mathbb{E}[U_t]\right)\right. \\ \qquad\qquad \left. + \delta\left(-\beta_1 G\mathbb{E}[x_t] + h\left(t;\mathbb{E}[u_t]\right)\right) + \gamma_t\right\}dt + Z_t dB_t, \\ X_0 = x_0, \qquad Y_T = \lambda_0 g(X_T, \mathbb{P}_{X_T}) + (1-\lambda_0)GX_T + \delta\left(g(x_T, \mathbb{P}_{x_T}) - Gx_T\right) + \xi. \end{cases}$$

We aim to show that when δ is sufficiently small, the mapping $I_{\lambda_0+\delta}\left(u, x_T\right) = (U, X_T)$ is a contraction. We first apply Itô's formula to $\left\langle G\hat{X}_t, \hat{Y}_t\right\rangle$ and we obtain

$$\begin{aligned} &\lambda_0\mathbb{E}\left[\left\langle G\hat{X}_T, g\left(X_T, \mathbb{P}_{X_T}\right) - g\left(\bar{X}_T, \mathbb{P}_{\bar{X}_T}\right)\right\rangle\right] + (1-\lambda_0)\mathbb{E}\left[\left\langle G\hat{X}_T, G\hat{X}_T\right\rangle\right] \\ &\quad + \delta\mathbb{E}\left[\left\langle G\hat{X}_T, -G\hat{x}_T + g\left(x_T, \mathbb{P}_{x_T}\right) - g\left(\bar{x}_T, \mathbb{P}_{\bar{x}_T}\right)\right\rangle\right] \\ &= \lambda_0\mathbb{E}\left[\int_0^T \left(A\left(t;\mathbb{E}[U_t]\right) - A\left(t;\mathbb{E}[\bar{U}_t]\right), \hat{U}_t\right) dt\right] \\ &\quad - (1-\lambda_0)\int_0^T \left\{\beta_1\left\langle\mathbb{E}[G\hat{X}_t], \mathbb{E}[G\hat{X}_t]\right\rangle + \beta_2\left\langle\mathbb{E}[G'\hat{Y}_t], \mathbb{E}[G'\hat{Y}_t]\right\rangle\right. \\ &\quad \left. + \beta_2\left\langle\mathbb{E}[G'\hat{Z}_t], \mathbb{E}[G'\hat{Z}_t]\right\rangle\right\} dt \\ &\quad + \delta\int_0^T \left\{\beta_1\left\langle\mathbb{E}[G\hat{X}_t], \mathbb{E}[G\hat{x}_t]\right\rangle + \beta_2\left\langle\mathbb{E}[G'\hat{Y}_t], \mathbb{E}[G'\hat{y}_t]\right\rangle + \beta_2\left\langle\mathbb{E}[G'\hat{Z}_t], \mathbb{E}[G'\hat{z}_t]\right\rangle\right\} dt \\ &\quad + \delta\int_0^T \mathbb{E}\left[\left(\hat{U}_t, A\left(t;\mathbb{E}[u_t]\right) - A\left(t;\mathbb{E}[\bar{u}_t]\right)\right)\right] dt. \end{aligned}$$

According to the condition specified in Assumption (A2), we follow the same lines of argument as in the proof for Lemma 2, there exists a positive constant K_1 which only depends on L, G, β_1, β_2 and T such that

$$\begin{aligned} &\min\{1, \alpha_1\}\cdot\mathbb{E}\left[|G\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T \left(\beta_1\left|G\mathbb{E}[\hat{X}_t]\right|^2 + \beta_2\left|G'\mathbb{E}[\hat{Y}_t]\right|^2 + \beta_2\left|G'\mathbb{E}[\hat{Z}_t]\right|^2\right) dt\right] \\ &\leq \delta K_1\left[\int_0^T \left(\mathbb{E}\left[|\hat{u}_t|^2\right] + \mathbb{E}\left[|\hat{U}_t|^2\right]\right) dt\right] + \delta K_1\mathbb{E}\left[|\hat{X}_T|^2 + |\hat{x}_T|^2\right]. \end{aligned} \tag{22}$$

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Next we apply Itô's formula to both $|\hat{X}_t|^2$ and $|\hat{Y}_t|^2$, and by (iii) and (iv) imposed in (A2) and Gronwall's inequality we can have

$$\begin{aligned}\sup_{0\le s\le T}\mathbb{E}\left[|\hat{X}_s|^2\right] \le\ & \delta K_1\int_0^T\left(\left|[\mathbb{E}\hat{x}_s]\right|^2+\left|\mathbb{E}[\hat{y}_s]\right|^2+\left|\mathbb{E}[\hat{z}_s]\right|^2\right)ds\\ & +K_1\left\{\int_0^T\left|\mathbb{E}[\hat{Y}_s]\right|^2ds+\int_0^T\left|\mathbb{E}[\hat{Z}_s]\right|^2ds\right\},\end{aligned}\tag{23}$$

$$\begin{aligned}\mathbb{E}\left[\int_0^T|\hat{X}_s|^2ds\right] \le\ & \delta TK_1\int_0^T\left(\left|[\mathbb{E}\hat{x}_s]\right|^2+\left|\mathbb{E}[\hat{y}_s]\right|^2+\left|\mathbb{E}[\hat{z}_s]\right|^2\right)ds\\ & +TK_1\left\{\int_0^T\left|\mathbb{E}[\hat{Y}_s]\right|^2ds+\int_0^T\left|\mathbb{E}[\hat{Z}_s]\right|^2ds\right\},\end{aligned}\tag{24}$$

$$\begin{aligned}\mathbb{E}\left[\int_0^T\left(|\hat{Y}_s|^2+|\hat{Z}_s|^2\right)ds\right] \le\ & \delta K_1\int_0^T\left(\left|[\mathbb{E}\hat{x}_s]\right|^2+\left|\mathbb{E}[\hat{y}_s]\right|^2+\left|\mathbb{E}[\hat{z}_s]\right|^2\right)ds\\ & +\delta K_1\left|\mathbb{E}\left[\hat{x}_T\right]\right|^2+K_1\int_0^T\left|\mathbb{E}[\hat{X}_s]\right|^2ds+K_1\mathbb{E}\left[|\hat{X}_T|^2\right].\end{aligned}\tag{25}$$

As in Lemma 2, by considering the cases $n>m$ and $m>n$ separately, we still have

$$\mathbb{E}\left[\int_0^T|\hat{U}_s|^2ds\right]+\mathbb{E}\left[|\hat{X}_T|^2\right]\le\delta K\left(\mathbb{E}\left[\int_0^T|\hat{u}_s|^2ds\right]+\mathbb{E}\left[|\hat{x}_T|^2\right]\right),$$

where K only depends on $L, G, \alpha_1, \beta_1, \beta_2$ and T. Let $\delta_0=\frac{1}{2K}$, it is then clear that the mappings

$$I_{\lambda_0+\delta}(u,x_T)=(U,X_T),$$

are contraction for all $\delta\in(0,\delta_0)$, therefore there is a unique fixed point which is the solution of (12) for $\lambda=\lambda_0+\delta$, $\delta\in(0,\delta_0)$. □

Similar to Theorem 2, by applying Lemmas 3 and 4, it is easy to establish the unique existence of solution for the MFT-FBSDE (1) under Assumption (A2).

Theorem 3. *Under Assumption* (A2), *there exists a unique adapted solution* (X,Y,Z) *of the MFT-FBSDE* (1).

Similarly, we can also establish the existence result under Assumption (A3) or (A4).

Theorem 4. *Under either Assumption* (A3) *or* (A4), *there exists a unique adapted solution* (X,Y,Z) *of the MFT-FBSDE* (1).

4. Stability theorem

In this section we provide a stability theorem for the solutions of the mean-field type forward–backward stochastic differential equations when the initial and terminal conditions are different. We consider the following two MFT-FBSDEs with different initial and terminal conditions:

$$\begin{cases}dX_t^1=b\left(t;X_t^1,Y_t^1,Z_t^1,\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right)dt+\sigma\left(t;X_t^1,Y_t^1,Z_t^1,\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right)dB_t,\\ dY_t^1=-h\left(t;X_t^1,Y_t^1,Z_t^1,\mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right)dt+Z_t^1dB_t,\\ X_0^1=x_0^1,\qquad Y_T^1=g^1\left(X_T^1,\mathbb{P}_{X_T^1}\right),\end{cases}\tag{26}$$

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and

$$\begin{cases} dX_t^2 = b\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right)dt + \sigma\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right)dB_t, \\ dY_t^2 = -h\left(t; X_t^2, Y_t^2, Z_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right)dt + Z_t^2 dB_t, \\ X_0^2 = x_0^2, \qquad Y_T^2 = g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right). \end{cases} \tag{27}$$

Suppose that the common coefficient functions and the corresponding terminal function g's of (26) and (27) satisfy the common one of the Assumptions (A1)–(A4). In accordance with Theorems 2–4, the solutions of the above systems uniquely exist and we denote them by (X^1, Y^1, Z^1) and (X^2, Y^2, Z^2) respectively.

Theorem 5. *Suppose that one of above Assumptions* (A1)–(A4) *holds. Then there exists a positive constant C which only depends on $T, \alpha_1, \beta_1, \beta_2, G$ and L such that*

$$\begin{aligned} &\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{U}_t|^2 dt\right] \\ &\leq C\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + |\hat{x}_0|^2\right), \end{aligned}$$

where $\hat{x}_0 \triangleq x_0^1 - x_0^2$.

Proof. Applying Itô's formula to $\left\langle G\hat{X}_t, \hat{Y}_t\right\rangle$ which yields that

$$\begin{aligned} &\mathbb{E}\left[\left\langle G\hat{X}_T, g^1\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right\rangle\right] - \left\langle G\hat{x}_0, \mathbb{E}[\hat{Y}_0]\right\rangle \\ &= \mathbb{E}\left[\int_0^T \left\langle A\left(t; U_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - A\left(t; U_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right), U_t^1 - U_t^2\right\rangle dt\right]. \end{aligned}$$

(1) If (A1) is satisfied, then we can derive that

$$\begin{aligned} &\alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right] + \mathbb{E}\left[\left\langle G\hat{X}_T, g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right\rangle\right] - \left\langle \hat{x}_0, G'\mathbb{E}[\hat{Y}_0]\right\rangle \\ &\leq -\beta_1\mathbb{E}\left[\int_0^T |G\hat{X}_t|^2 dt\right] - \beta_2\mathbb{E}\left[\int_0^T \left(|G'\hat{Y}_t|^2 + |G'\hat{Z}_t|^2\right)dt\right], \end{aligned} \tag{28}$$

by Cauchy–Schwarz inequality and $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$ $(a, b \in \mathbb{R}, \epsilon > 0)$, we can get that

$$\begin{aligned} &\alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right] + \beta_1\mathbb{E}\left[\int_0^T |G\hat{X}_t|^2 dt\right] + \beta_2\mathbb{E}\left[\int_0^T \left(|G'\hat{Y}_t|^2 + |G'\hat{Z}_t|^2\right)dt\right] \\ &\leq \epsilon\mathbb{E}\left[|G\hat{X}_T|^2\right] + \frac{1}{4\epsilon}\mathbb{E}\left[\left|g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\ &\quad + \epsilon|G|^2\mathbb{E}[|\hat{Y}_0|^2] + \frac{1}{4\epsilon}|\hat{x}_0|^2 \\ &\leq \epsilon\mathbb{E}\left[|G\hat{X}_T|^2\right] + \frac{1}{4\epsilon}\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] \\ &\quad + \epsilon|G|^2\mathbb{E}[|\hat{Y}_0|^2] + \frac{1}{4\epsilon}|\hat{x}_0|^2. \end{aligned} \tag{29}$$

Applying the similar technique as in Lemma 2 to $|\hat{X}_t|^2$ and $|\hat{Y}_t|^2$ and we can obtain that

$$\sup_{t\in[0,T]} \mathbb{E}[|\hat{X}_t|^2] \leq K_2 \left(\mathbb{E}\left[\int_0^T \left(|\hat{Y}_t|^2 + |\hat{Z}_t|^2\right) dt\right] + |\hat{x}_0|^2\right), \tag{30}$$

$$\begin{aligned}&\sup_{t\in[0,T]} \mathbb{E}\left[|\hat{Y}_t|^2\right] + \mathbb{E}\left[\int_0^T |\hat{Z}_s|^2 ds\right]\\ &\leq K_2 \left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + \mathbb{E}[|\hat{X}_T|^2] + \mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right]\right),\end{aligned} \tag{31}$$

where K_2 is a positive constant depending only on T and L. Moreover, we can have that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds\right] \leq \frac{TK_2}{T\wedge 1}&\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right]\right.\\ &\left.+ \mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right]\right),\end{aligned} \tag{32}$$

$$\begin{aligned}\mathbb{E}\left[|\hat{Y}_0|^2\right] \leq K_2 &\left\{\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right]\right.\\ &\left.+ \mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{X}_s|^2 ds\right]\right\},\end{aligned} \tag{33}$$

$$\mathbb{E}\left[\int_0^T |\hat{X}_t|^2 dt\right] \leq TK_2 \left(\mathbb{E}\left[\int_0^T \left(|\hat{Y}_t|^2 + |\hat{Z}_t|^2\right) dt\right] + |\hat{x}_0|^2\right). \tag{34}$$

(i) If $m > n$, then $\beta_1 > 0$, $\alpha_1 > 0$. Choose a small enough ϵ, combining (29), (32) and (33) we can deduce that

$$\begin{aligned}&\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{U}_t|^2 dt\right]\\ &\leq C\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + |\hat{x}_0|^2\right),\end{aligned}$$

where C is a positive constant depending only on T, α_1, β_1, β_2, G and L.

(ii) If $n > m$, then $\beta_2 > 0$. Choose a small enough ϵ, combining (29), (30), (33) and (34), we can also deduce that

$$\begin{aligned}&\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{U}_t|^2 dt\right]\\ &\leq C\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + |\hat{x}_0|^2\right).\end{aligned}$$

(2) If (A2) is satisfied, then we can derive that

$$\begin{aligned}&\alpha_1\mathbb{E}\left[|G\hat{X}_T|^2\right] + \mathbb{E}\left[\left\langle G\hat{X}_T, g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right\rangle\right] - \left\langle \hat{x}_0, G'\mathbb{E}[\hat{Y}_0]\right\rangle\\ &\leq -\beta_1\int_0^T \left|G\mathbb{E}[\hat{X}_t]\right|^2 dt - \beta_2\int_0^T \left|G'\mathbb{E}[\hat{Y}_t]\right|^2 dt - \beta_2\int_0^T \left|G'\mathbb{E}[\hat{Z}_t]\right|^2 dt,\end{aligned}$$

by Cauchy–Schwarz inequality and $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ ($a, b \in \mathbb{R}, \epsilon > 0$), we can get

$$\begin{aligned}
&\alpha_1 \mathbb{E}\left[|G\hat{X}_T|^2\right] + \beta_1 \int_0^T \left|G\mathbb{E}[\hat{X}_t]\right|^2 dt + \beta_2 \int_0^T \left(\left|G'\mathbb{E}[\hat{Y}_t]\right|^2 + \left|G'\mathbb{E}[\hat{Z}_t]\right|^2\right) dt \\
&\quad \le \epsilon \mathbb{E}\left[|G\hat{X}_T|^2\right] + \frac{1}{4\epsilon}\mathbb{E}\left[\left|g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\
&\qquad + \epsilon |G|^2 \mathbb{E}[|\hat{Y}_0|^2] + \frac{1}{4\epsilon}|\hat{x}_0|^2 \\
&\quad \le \epsilon \mathbb{E}\left[|G\hat{X}_T|^2\right] + \frac{1}{4\epsilon}\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)} \left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] \\
&\qquad + \epsilon |G|^2 \mathbb{E}[|\hat{Y}_0|^2] + \frac{1}{4\epsilon}|\hat{x}_0|^2.
\end{aligned} \tag{35}$$

By applying Itô's formula to $|\hat{X}_t|^2$, (2) and Lipschitz's conditions imposed in (A2), for any $t \in [0, T]$ we can obtain that

$$\begin{aligned}
\mathbb{E}[|\hat{X}_t|^2] - |\hat{x}_0|^2 &= 2\mathbb{E}\left[\int_0^t \left\langle b\left(s; \mathbb{E}[U_s^1]\right) - b\left(s; \mathbb{E}[U_s^2]\right), \hat{X}_s\right\rangle ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \left|\sigma\left(s; \mathbb{E}[U_s^1]\right) - \sigma\left(s; \mathbb{E}[U_s^2]\right)\right|^2\right] \\
&\le 2L \int_0^t \left|\mathbb{E}[\hat{X}_s]\right| \left(\left|\mathbb{E}[\hat{X}_s]\right| + \left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right|\right) ds \\
&\quad + L^2 \int_0^t \left(\left|\mathbb{E}[\hat{X}_s]\right| + \left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right|\right)^2 ds \\
&\le \left(4L + 3L^2\right) \int_0^t \left|\mathbb{E}[\hat{X}_s]\right|^2 ds + \left(L + 3L^2\right) \int_0^t \left|\mathbb{E}[\hat{Y}_s]\right|^2 ds \\
&\quad + \left(L + 3L^2\right) \int_0^t \left|\mathbb{E}[\hat{Z}_s]\right|^2 ds \\
&\le \left(4L + 3L^2\right) \int_0^t \mathbb{E}[|\hat{X}_s|^2] ds + \left(L + 3L^2\right) \int_0^t \left|\mathbb{E}[\hat{Y}_s]\right|^2 ds \\
&\quad + \left(L + 3L^2\right) \int_0^t \left|\mathbb{E}[\hat{Z}_s]\right|^2 ds,
\end{aligned}$$

by using Gronwall's inequality, we can get that

$$\sup_{t\in[0,T]} \mathbb{E}[|\hat{X}_t|^2] \le K_2 \left(\int_0^T \left|\mathbb{E}[\hat{Y}_t]\right|^2 dt + \int_0^T \left|\mathbb{E}[\hat{Z}_t]\right|^2 dt + |\hat{x}_0|^2\right), \tag{36}$$

which implies that

$$\mathbb{E}\left[\int_0^T |\hat{X}_t|^2 dt\right] \le T K_2 \left(\int_0^T \left|\mathbb{E}[\hat{Y}_t]\right|^2 dt + \int_0^T \left|\mathbb{E}[\hat{Z}_t]\right|^2 dt + |\hat{x}_0|^2\right). \tag{37}$$

Similarly, applying Itô's formula to $|\hat{Y}_t|^2$, by (2) and Lipschitz's conditions imposed in (A2), we can obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T |\hat{Z}_s|^2 ds\right] + \mathbb{E}\left[|\hat{Y}_t|^2\right] \\
&= 2\mathbb{E}\left[\int_t^T \left\langle \hat{Y}_s, h\left(s; \mathbb{E}\left[U_s^1\right]\right) - h\left(s; \mathbb{E}\left[U_s^2\right]\right)\right\rangle ds\right] \\
&\quad + \mathbb{E}\left[\left|g^1\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\
&\le 2L\mathbb{E}\left[\int_t^T |\hat{Y}_s|\left(\left|\mathbb{E}[\hat{X}_s]\right| + \left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right|\right) ds\right] \\
&\quad + 2\mathbb{E}\left[\left|g^1\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\
&\quad + 2\mathbb{E}\left[\left|g^1\left(X_T^2, \mathbb{P}_{X_T^2}\right) - g^2\left(X_T^2, \mathbb{P}_{X_T^2}\right)\right|^2\right] \\
&\le 2L\int_t^T \mathbb{E}\left[|\hat{Y}_s|^2\right]^{\frac{1}{2}}\left(\left|\mathbb{E}[\hat{X}_s]\right| + \left|\mathbb{E}[\hat{Y}_s]\right| + \left|\mathbb{E}[\hat{Z}_s]\right|\right) ds \\
&\quad + 2L^2\mathbb{E}\left[\left(|\hat{X}_T| + \mathcal{W}\left(\mathbb{P}_{X_T^1}, \mathbb{P}_{X_T^2}\right)\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] \\
&\le 2L\int_t^T \left(\mathbb{E}\left[|\hat{Y}_s|^2\right]^{\frac{1}{2}}\left|\mathbb{E}[\hat{X}_s]\right| + \mathbb{E}\left[|\hat{Y}_s|^2\right] + \mathbb{E}\left[|\hat{Y}_s|^2\right]^{\frac{1}{2}}\left|\mathbb{E}[\hat{Z}_s]\right|\right) ds \\
&\quad + 4L^2\left\{\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathcal{W}\left(\mathbb{P}_{X_T^1}, \mathbb{P}_{X_T^2}\right)^2\right\} \\
&\quad + 2\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] \\
&\le L\int_t^T\left\{\mathbb{E}\left[|\hat{Y}_s|^2\right] + \left|\mathbb{E}[\hat{X}_s]\right|^2\right\} ds + 2L\int_t^T \mathbb{E}[|\hat{Y}_s|^2] ds \\
&\quad + L\int_t^T\left\{\frac{1}{\epsilon'}\mathbb{E}\left[|\hat{Y}_s|^2\right] + \epsilon'\left|\mathbb{E}[\hat{Z}_s]\right|^2\right\} ds \\
&\quad + 8L^2\mathbb{E}\left[|\hat{X}_T|^2\right] + 2\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right],
\end{aligned}$$

then we choose $\epsilon' = \frac{1}{2L}$ and by Gronwall's inequality to get that

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[|\hat{Y}_t|^2\right] + \mathbb{E}\left[\int_0^T |\hat{Z}_s|^2 ds\right] \\
&\le K_2\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + \mathbb{E}[|\hat{X}_T|^2] + \int_0^T\left|\mathbb{E}[\hat{X}_s]\right|^2 ds\right), \quad (38)
\end{aligned}$$

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which can imply that

$$\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s|^2 + |\hat{Z}_s|^2\right) ds\right] \le \frac{TK_2}{T\wedge 1}\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + \mathbb{E}\left[|\hat{X}_T|^2\right] + \int_0^T \left|\mathbb{E}[\hat{X}_s]\right|^2 ds\right), \tag{39}$$

$$\mathbb{E}\left[|\hat{Y}_0|^2\right] \le K_2\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + \mathbb{E}\left[|\hat{X}_T|^2\right] + \int_0^T \left|\mathbb{E}[\hat{X}_s]\right|^2 ds\right). \tag{40}$$

(i) If $m > n$, then $\beta_1 > 0, \alpha_1 > 0$. Choosing a small enough ϵ, by combining (35), (37), (39) and (40), we can deduce that

$$\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{U}_t|^2 dt\right] \le C\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + |\hat{x}_0|^2\right),$$

Q4 where C is a positive constant, which depends on $T, \alpha_1, \beta_1, \beta_2, G$ and L.

(ii) If $n > m$, then $\beta_2 > 0$. Choosing a small enough ϵ, by combining (35), (37), (39) and (40), we can also deduce that

$$\mathbb{E}\left[|\hat{X}_T|^2\right] + \mathbb{E}\left[\int_0^T |\hat{U}_t|^2 dt\right] \le C\left(\mathbb{E}\left[\sup_{(x,\mu)\in\mathbb{R}^n\times\mathcal{M}(\mathbb{R}^n)}\left|g^1(x,\mu) - g^2(x,\mu)\right|^2\right] + |\hat{x}_0|^2\right).$$

(iii) Finally, if either Assumption (A3) or (A4) holds, a similar argument can be applied to obtain our desired result. □

5. Examples

In this section, we show how our previous theorems can be applied to study two representative examples as follows. In particular, our Example 1 is motivated from [4]; while Example 2 is arisen from [2,1]. Though these examples could be treated from the first principle, they still illustrate that our theorems are 'optimal' in the sense that how their counter-examples in [4] just fail to ensure its well-posedness in a continuous manner.

Example 1. Let us consider the following MFT-FBSDE with $m = n = d = 1, \alpha \in \mathbb{R}, \lambda \in [0, 1]$ and $\theta \in [0, 1]$, which is motivated by [4] (in which the authors considered the case when $\alpha = 1, \lambda = 1$ and $\theta = 1$),

$$\begin{cases} dX_t = \alpha\mathbb{E}[Y_t]\,dt \\ dY_t = -(\lambda\mathbb{E}[X_t] + (1-\lambda)X_t)\,dt + Z_t dB_t \\ X_0 = 0, \qquad Y_T = \theta\mathbb{E}[X_T] + (1-\theta)X_T, \end{cases} \tag{41}$$

where $T \in \mathbb{R}^+$ and satisfies

$$\alpha \sin\left(\sqrt{\alpha}T\right) = \sqrt{\alpha}\cos\left(\sqrt{\alpha}T\right), \tag{42}$$

when $\alpha \geq 0$. Therefore, in this case the matrix G in our previous theorem is just 1 and

$$U_t = \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix}, \qquad A\left(t; U_t, \mathbb{P}_{(X_t,Y_t,Z_t)}\right) = \begin{pmatrix} -\lambda\mathbb{E}[X_t] - (1-\lambda)X_t \\ \alpha\mathbb{E}[Y_t] \\ 0 \end{pmatrix},$$

$$g(X, \mathbb{P}_{X_T}) = \theta\mathbb{E}[X_T] + (1-\theta)X_T.$$

Then

$$\mathbb{E}\left[\left(A\left(t; U_t^1, \mathbb{P}_{(X_t^1,Y_t^1,Z_t^1)}\right) - A\left(t; U_t^2, \mathbb{P}_{(X_t^2,Y_t^2,Z_t^2)}\right), U_t^1 - U_t^2\right)\right]$$

$$= -\lambda\left(\mathbb{E}\left[X_t^1 - X_t^2\right]\right)^2 - (1-\lambda)\mathbb{E}\left[(X_t^1 - X_t^2)^2\right] + \alpha\left(\mathbb{E}\left[Y_t^1 - Y_t^2\right]\right)^2,$$

$$\mathbb{E}\left[\left\langle g\left(X_T^1, \mathbb{P}_{X_T^1}\right) - g\left(X_T^2, \mathbb{P}_{X_T^2}\right), X_T^1 - X_T^2\right\rangle\right]$$

$$= \theta\left(\mathbb{E}[X_T^1 - X_T^2]\right)^2 + (1-\theta)\mathbb{E}\left[\left(X_T^1 - X_T^2\right)^2\right].$$

(a) If $\alpha > 0$, there does not exist an apparent non-negative couple (β_1, β_2) with $\beta_1 + \beta_2 > 0$ such that

$$-\beta_1\left(\mathbb{E}\left[X_t^1 - X_t^2\right]\right)^2 - \beta_2\left\{\left(\mathbb{E}\left[Y_t^1 - Y_t^2\right]\right)^2 + \left(\mathbb{E}\left[Z_t^1 - Z_t^2\right]\right)^2\right\}$$

$$\geq -\lambda\left(\mathbb{E}\left[X_t^1 - X_t^2\right]\right)^2 - (1-\lambda)\mathbb{E}\left[(X_t^1 - X_t^2)^2\right] + \alpha\left(\mathbb{E}\left[Y_t^1 - Y_t^2\right]\right)^2,$$

and this observation illuminate that it is probably that none of Assumptions (A1)–(A4) could be satisfied, which hints perhaps that (41) might possess no unique solution! Indeed, for $\alpha > 0$, taking expectation on both sides of (41) yields that

$$\begin{cases} d\mathbb{E}[X_t] = \alpha\mathbb{E}[Y_t]\,dt, \\ d\mathbb{E}[Y_t] = -\mathbb{E}[X_t]\,dt, \\ \mathbb{E}[X_0] = 0, \qquad \mathbb{E}[Y_T] = \mathbb{E}[X_T], \end{cases} \tag{43}$$

then we can get that

$$\frac{d^2}{dt^2}\mathbb{E}[X_t] + \alpha\mathbb{E}[X_t] = 0, \quad \mathbb{E}[X_0] = 0,$$

a simple calculation yields that

$$\mathbb{E}[X_t] = K_1 \sin\sqrt{\alpha}t, \quad K_1 \in \mathbb{R},$$

by comparing its derivative with the expectation $\mathbb{E}[Y_t]$, we can deduce that

$$\mathbb{E}[Y_t] = \frac{K_1}{\sqrt{\alpha}}\cos\sqrt{\alpha}t, \quad K_1 \in \mathbb{R}.$$

We can check that $\mathbb{E}[X_T] = \mathbb{E}[Y_T]$ holds automatically by condition (42) for any $K_1 \in \mathbb{R}$. Therefore, the following expressions could serve as solutions for this system:

$$X_t = K_1 \sin\sqrt{\alpha}t, \qquad Y_t = \frac{K_1}{\sqrt{\alpha}}\cos\sqrt{\alpha}t, \qquad Z_t = 0, \quad K_1 \in \mathbb{R}.$$

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(b) If $\alpha \le 0$, we analyse the solution of system (41) as follows:

(1) For $\lambda = 1$ and $\theta = 1$, then Assumption (A4) is clearly satisfied (by taking $\beta_1 = 1$ and $\beta_2 = 0$). Therefore, there exists a unique solution for the system (41); indeed, we can solve it out explicitly as shown below:

(i) For $\alpha < 0$, taking expectation on both sides of (41), we can get

$$\begin{cases} d\mathbb{E}[X_t] = \alpha\mathbb{E}[Y_t]\,dt, \\ d\mathbb{E}[Y_t] = -\mathbb{E}[X_t]\,dt, \\ \mathbb{E}[X_0] = 0, \qquad \mathbb{E}[Y_T] = \mathbb{E}[X_T]. \end{cases} \tag{44}$$

By simple calculations, we can get

$$\mathbb{E}[X_t] = Ae^{-\sqrt{-\alpha}t} + Be^{\sqrt{-\alpha}t}, \tag{45}$$

where A, B are two constants to be determined. Since $\mathbb{E}[X_0] = 0$, we can deduce that $A + B = 0$,

$$\mathbb{E}[X_t] = Ae^{-\sqrt{-\alpha}t} - Ae^{\sqrt{-\alpha}t}. \tag{46}$$

Substituting (46) into (44), we can obtain that

$$\mathbb{E}[Y_t] = \frac{A}{\sqrt{-\alpha}}e^{-\sqrt{-\alpha}t} + \frac{A}{\sqrt{-\alpha}}e^{\sqrt{-\alpha}t}. \tag{47}$$

By $\mathbb{E}[Y_T] = \mathbb{E}[X_T]$, we can derive that

$$\frac{A}{\sqrt{-\alpha}}e^{-\sqrt{-\alpha}T} + \frac{A}{\sqrt{-\alpha}}e^{\sqrt{-\alpha}T} = Ae^{-\sqrt{-\alpha}T} - Ae^{\sqrt{-\alpha}T},$$

and then

$$A\left[\left(\frac{1}{\sqrt{-\alpha}} - 1\right) + \left(\frac{1}{\sqrt{-\alpha}} + 1\right)e^{2\sqrt{-\alpha}T}\right] = 0,$$

which implies that $A = 0$, and we can obtain that $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$. Therefore, we first see that the solution X is

$$X_t = 0.$$

Note that Y satisfies the following standard BSDE:

$$dY_t = Z_t dB_t, \quad Y_T = 0,$$

by the uniqueness of the BSDE, we can conclude that $Y_t = 0$ and $Z_t = 0$.

(ii) For $\alpha = 0$, it is obvious that $X_t = 0$, $Y_t = 0$ and $Z_t = 0$ is the unique solution of the system.

(2) For $0 \le \lambda < 1$ and $\theta = 1$, we can verify that Assumption (A3) is satisfied (by taking $\beta_1 = 1 - \lambda$ and $\beta_2 = 0$). Therefore, there exists a unique solution for the system (41); indeed, essentially the same calculation as in the case (b) (1) can be carried out to show that

$$X_t = Y_t = Z_t = 0,$$

is the unique solution of (41).

(3) For $\lambda = 1$ and $0 \le \theta < 1$, it is easy to verify that Assumption (A2) is satisfied (by taking $\beta_1 = 1$ and $\beta_2 = 0$), and there exists a unique solution $X_t = Y_t = Z_t = 0$.

(4) For $0 \le \lambda < 1$ and $0 \le \theta < 1$, it is easy to verify that Assumption (A1) is satisfied (by taking $\beta_1 = 1 - \lambda$ and $\beta_2 = 0$), and there exists a unique solution $X_t = Y_t = Z_t = 0$ for the system (41).

Remark. 1. The case $\lambda = 1$ and $\theta = 1$ can be viewed as a filtering problem in engineering. X_t is unobservable process while Y_t is the observable process, and the observation Y_t is affected directly by the mean of X_t, which cannot be directly observed.

2. The Lipschitz condition provided in Assumptions (A4) cannot involve space variables; indeed, any appearance of space variables in f and g would lead to a stronger monotonicity condition which is classified as a case under (A1).

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Example 2. Consider the following MFT-FBSDE arisen from solving the mean-field type linear-quadratic stochastic control problem as introduced in [2,1] with X and Y are both n-dimensional stochastic processes,

$$\begin{cases} dX_t = \left(A_t X_t - B_t R_t^{-1} B_t' Y_t + \bar{A}_t \mathbb{E}[X_t]\right) dt + \sigma_t dB_t \\ -dY_t = \left\{\left(Q_t + \bar{Q}_t\right) X_t + \bar{A}_t Y_t - \bar{Q}_t S_t \mathbb{E}[Y_t] - S_t' \bar{Q}_t (I_n - S_t) \mathbb{E}[Y_t] \right. \\ \qquad \left. + \bar{A}_t \mathbb{E}[Y_t]\right\} dt + Z_t dB_t \\ X_0 = x_0, \qquad Y_T = \left(Q_T + \bar{Q}_T\right) X_T - \bar{Q}_T S_T \mathbb{E}[X_T] - S_T' \bar{Q}_T \mathbb{E}[X_T], \end{cases}$$

where A, B, $\bar{A}$, S are bounded deterministic matrix-valued functions in time of suitable size, Z and σ are L^2-function in time of suitable size. Q and $\bar{Q}$ are non-negative definite matrix-valued functions in time of suitable size. We also assume that $R > \delta I_n$ for some $\delta > 0$. Note that since both X_t and Y_t are of dimension n, we can choose $G = I_n$, then $U_t = (X_t, Y_t, Z_t)$ and

$$\begin{aligned} & A\left(t; U_t, \mathbb{P}_{(X_t, Y_t, Z_t)}\right) \\ &= \begin{pmatrix} -\left(\left(Q_t + \bar{Q}_t\right) X_t - \bar{A}_t Y_t + \bar{Q}_t S_t \mathbb{E}[X_t] + S_t' \bar{Q}_t (I_n - S_t) \mathbb{E}[Y_t] - \bar{A}_t \mathbb{E}[Y_t]\right) \\ A_t X_t - B_t R_t^{-1} B_t' Y_t + \bar{A}_t \mathbb{E}[X_t] \\ \sigma_t \end{pmatrix}. \end{aligned}$$

Then we can derive that

$$\begin{aligned} & \mathbb{E}\left[\left\langle A\left(t; U_t^1, \mathbb{P}_{(X_t^1, Y_t^1, Z_t^1)}\right) - A\left(t; U_t^2, \mathbb{P}_{(X_t^2, Y_t^2, Z_t^2)}\right), U_t^1 - U_t^2 \right\rangle\right] \\ &= -\mathbb{E}\left[\left\langle \left(Q_t + \bar{Q}_t\right)\left(X_t^1 - X_t^2\right), X_t^1 - X_t^2 \right\rangle\right] - \mathbb{E}\left[\left\langle A_t'\left(Y_t^1 - Y_t^2\right), X_t^1 - X_t^2 \right\rangle\right] \\ &\quad + \left\langle \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle + \left\langle S_t' \bar{Q}_t (I_n - S_t) \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle \\ &\quad - \left\langle \bar{A}_t \mathbb{E}[Y_t^1 - Y_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle + \mathbb{E}\left[\left\langle A_t (X_t^1 - X_t^2), Y_t^1 - Y_t^2 \right\rangle\right] \\ &\quad - \mathbb{E}\left[\left\langle B_t R_t^{-1} B_t' (Y_t^1 - Y_t^2), Y_t^1 - Y_t^2 \right\rangle\right] + \left\langle \bar{A}_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[Y_t^1 - Y_t^2] \right\rangle \\ &= -\mathbb{E}\left[\left\langle \left(Q_t + \bar{Q}_t\right)\left(X_t^1 - X_t^2\right), X_t^1 - X_t^2 \right\rangle\right] + 2\left\langle \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle \\ &\quad - \left\langle S_t' \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle - \mathbb{E}\left[\left\langle B_t R_t^{-1} B_t' (Y_t^1 - Y_t^2), Y_t^1 - Y_t^2 \right\rangle\right] \\ &\le -\left\langle \bar{Q}_t \mathbb{E}\left[X_t^1 - X_t^2\right], \mathbb{E}[X_t^1 - X_t^2] \right\rangle + 2\left\langle \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle \\ &\quad - \left\langle S_t' \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle - \mathbb{E}\left[\left\langle B_t R_t^{-1} B_t' (Y_t^1 - Y_t^2), Y_t^1 - Y_t^2 \right\rangle\right] \\ &\quad - \mathbb{E}\left[\left\langle Q_t (X_t^1 - X_t^2), X_t^1 - X_t^2 \right\rangle\right] \\ &\le -\mathbb{E}\left[\left\langle B_t R_t^{-1} B_t' (Y_t^1 - Y_t^2), Y_t^1 - Y_t^2 \right\rangle\right] - \mathbb{E}\left[\left\langle Q_t (X_t^1 - X_t^2), X_t^1 - X_t^2 \right\rangle\right], \end{aligned}$$

where the last inequality holds by noting that

$$\begin{aligned} & -\left\langle \bar{Q}_t \mathbb{E}\left[X_t^1 - X_t^2\right], \mathbb{E}[X_t^1 - X_t^2] \right\rangle + 2\left\langle \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle \\ &\quad - \left\langle S_t' \bar{Q}_t S_t \mathbb{E}[X_t^1 - X_t^2], \mathbb{E}[X_t^1 - X_t^2] \right\rangle \\ &= -\mathbb{E}[\left(X_t^1 - X_t^2\right)'] \left(\bar{Q}_t' - 2S_t' \bar{Q}_t' + S_t' \bar{Q}_t' S_t\right) \mathbb{E}[X_t^1 - X_t^2] \end{aligned}$$

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$$= -\mathbb{E}[\left(X_t^1 - X_t^2\right)'] \left(\bar{q}_t\bar{q}_t' - 2S_t'\bar{q}_t\bar{q}_t' + S_t'\bar{q}_t\bar{q}_t'S_t\right) \mathbb{E}[X_t^1 - X_t^2]$$

$$= -\mathbb{E}[\left(X_t^1 - X_t^2\right)'] \left(\bar{q}_t\bar{q}_t' - S_t'\bar{q}_t\bar{q}_t' - \bar{q}_t\bar{q}_t'S_t + S_t'\bar{q}_t\bar{q}_t'S_t\right) \mathbb{E}[X_t^1 - X_t^2]$$

$$= -\mathbb{E}[\left(X_t^1 - X_t^2\right)'] \left(S_t'\bar{q}_t - \bar{q}_t\right) \left(S_t'\bar{q}_t - \bar{q}_t\right)' \mathbb{E}[X_t^1 - X_t^2] \leq 0,$$

where $\bar{Q}_t = \bar{q}_t\bar{q}_t'$ for some $\bar{q}_t \in \mathbb{R}^{n\times n}$ by its non-negative definite nature. Therefore, Assumption (A1) is satisfied in the present case, and there exists a unique solution to this system in light of our general theorem, which provides a generic explanation from a higher standpoint.

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