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Exact parabolic asymptotics for singular n-D Burgers' random fields: Gaussian approximation

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Abstract

The rate of convergence (in the uniform Kolmogorov's distance) for probability distributions of parabolically rescaled solutions of the multidimensional Burgers' equation with random singular Gaussian initial data (with long-range dependence) to a limit Gaussian random field is discussed in this paper. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider the n-dimensional Burgers' equation

$$\frac{\partial u}{\partial t} + (u, \nabla)u = \mu \Delta u, \quad \mu > 0, \tag{1.1}$$

subject to the initial random condition

$$u(0,x) = u_0(x) = \nabla \xi(x) \tag{1.2}$$

of the gradient form. The equation describes the time evolution of the velocity field

$$u(t,x) = [u_1(t,x), \ldots, u_n(t,x)]', \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \quad n \geqslant 1.$$

The potential $\xi(x)$, $x \in \mathbb{R}^n$, is a scalar field, ∇ denotes the gradient operator on $\mathbb{R}^n$, and Δ stands for the n-dimensional Laplacian.

Eq. (1.1) is a parabolic equation with quadratic, inertial nonlinearity, which can be viewed as a simplified version of the Navier–Stokes equation, with the pressure term ∇p omitted and with the viscosity coefficient μ corresponding to the inverse of the

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Reynolds number. With random initial data, the problem (1.1)–(1.2) is also referred to as the *Burgers turbulence problem*.

Burgers' turbulence describes various physical phenomena, such as nonlinear acoustic shock waves and formation of cellular structures (sheets, filaments, nodes) in the distribution of self-gravitating matter in the late stages of the universe (see, for example, Burgers, 1974; Chorin, 1975; Gurbatov et al., 1991; Witham, 1974; Shandarin and Zeldovich, 1989; Weinberg and Gunn, 1990; Kofman et al., 1992; Woyczynski, 1993, 1998; Vergassola et al., 1994; Molchanov et al., 1997), and other types of irrotational flows. Equations related to the one-dimensional Burgers equation have been proposed in financial market models (option pricing) (see, Hodges and Carverhill, 1993).

Rosenblatt (1968, 1987) was one of the first to have considered the Burgers equation with random initial data from the rigorous perspective of probability theory and, more recently, numerous researchers studied solutions of the Burgers equation depending on different types of random initial conditions. In particular, Bulinski and Molchanov (1991), Giraitis et al. (1993), Albeverio et al. (1994) and Funaki et al. (1995) studied solutions of the Burgers equation when the intial condition was either a Gaussian random field or a shot-noise (or Gibbs–Cox) random field with weak or strong dependence. They obtained Gaussian and non-Gaussian distributions as parabolic scaling limits ($u(t, a\sqrt{t})$, $t \to \infty$) of distributions of the Burgers equation's solution random fields. Leonenko et al. (1994, 1995a, b), and Leonenko and Orsingher (1995) also obtained Gaussian and non-Gaussian limit distributions in the same context of parabolic scaling in the case when the initial condition is either a Gaussian random field or a chi-square field with long-range dependence. Analogous results under suitable non-Gaussian initial conditions with weak dependence can be found in Surgailis and Woyczynski (1994a), Hu and Woyczynski (1994), Leonenko and Deriev (1995) and Deriev and Leonenko (1997). In the Gaussian model with non-integrable oscillating correlations, the limit law of the solutions is non-Gaussian (see, Surgailis and Woyczynski, 1994a). For other results concerning limiting distributions of averaged solutions of Burgers equation see Rosenblatt (1987) and Hu and Woyczynski (1995b).

Other types of random problems for the Burgers equation have also been considered recently in the mathematical litarature. Sinai (1992), Albeverio et al. (1994), Molchanov et al. (1995), Avellaneda (1995), Fan (1995) and Hu and Woyczynski (1995a) considered the statistics of shocks in the zero-viscosity limit and related problems of hyperbolic limiting behavior ($u(t, at)$, $t \to \infty$). This type of scaling is of importance in physical applications (see, for example, Gurbatov et al., 1991; Vergassola et al., 1994). Recent results on the non-homogeneous Burgers equation taking into account external forcing can be found, e.g., in Sinai (1991), Holden et al. (1994, 1995), Saichev and Woyczynski (1997) and Molchanov et al. (1997). A general survey of the area is provided in Woyczynski (1998).

The study of important numerical simulations of Burgers turbulence (and a large number of those appear in the astrophysical literature, see, e.g., Kofman et al., 1992), questioning the rigorous estimation of the rate of convergence of the above-mentioned approximations, has begun only recently although some results on the rate of convergence to the normal law for integral functionals of homogenous isotropic random fields under strong dependence conditions were considered earlier by Leonenko (1988) (see also Ivanov and Leonenko, 1989). Bossy and Talay (1994, 1995, 1996) have studied the rates of convergence (in the L^1 norm) for the interacting particle system approximation related to the propagation of chaos result for the Burgers equation, and Leonenko et al. (1996a) found the rate of convergence to the normal law of the solutions of the one-dimensional Burgers equation with singular Gaussian data.

In this paper, we provide a rate of convergence (in the uniform Kolmogorov's distance) of probability distributions of the parabolically rescaled solutions of the n-dimensional Burgers' equation with random singular Gaussian initial data (with long-range dependence) to a limiting Gaussian random field. The singularity of the initial data is of physical importance (see Vergassola et al., 1994) but is also a source of analytic difficulties.

The paper is organized as follows. Section 2 deals with preliminaries on the Burgers equation solutions with random initial conditions, including two known results concerning the parabolic scaling limit. In Section 3, we assume that the initial velocity potential is a Gaussian, homogeneous and isotropic random field with the singular covariance function of the form $L(|x|)|x|^{-\alpha}$, $0<\alpha<n/2$, with slowly vaying L, and formulate our basic result which shows that the rate of convergence of the Kolmogorov's distance between the distribution of the rescaled solution $M(t)u(t,a\sqrt{t})$ (M is a normalizing matrix-valued function) and the standard n-dimensional normal distribution is bounded from above by $t^{-\alpha/6}L^{1/3}(\sqrt{t})$ as $t\to\infty$. The proof of this main result is given in Section 5 which is preceded by a series of auxiliary lemmas in Section 4. Finally, Section 6 contains a discussion of a possible extension of our main result to the case $0<\alpha<n$. This can be done but at the price of slower convergence rates.

2. Solutions of the Burgers' equation with random initial conditions

A crucial property of Eq. (1.1) is that it admits a linearization by the so-called (Forsyth–Florin-) Hopf–Cole substitution (see, e.g., Forsyth, 1906; Florin, 1948; Hopf, 1950)

$$u(t,x) = -2\mu\nabla\log q(t,x),$$

which reduces Eq. (1.1) to the linear diffusion equation

$$\frac{\partial q}{\partial t} = \mu\Delta q \tag{2.1}$$

subject to the initial condition

$$q(0,x) = \exp\left\{-\frac{\xi(x)}{2\mu}\right\}. \tag{2.2}$$

Using the well-known formulas for the solution of the initial-value problem (2.1)–(2.2) one obtains the following explicit solution of the Burgers' initial-value problem (1.1)–(1.2):

$$u(t,x) = -2\mu\nabla\log J(t,x) = \frac{I(t,x)}{J(t,x)}, \tag{2.3}$$

where

$$I(t,x)=\int_{\mathbb{R}^n}\frac{x-y}{t}g(t,x-y)\exp\{-\xi(y)/(2\mu)\}\,\mathrm{d}y,$$
$$J(t,x)=\int_{\mathbb{R}^n}g(t,x-y)\exp\{-\xi(y)/(2\mu)\}\,\mathrm{d}y \tag{2.4}$$

and where

$$g(t,x-y)=(4\pi\mu t)^{-n/2}\exp\left\{-\frac{|x-y|^2}{4\mu t}\right\},\quad x,y\in\mathbb{R}^n,\ \ t>0 \tag{2.5}$$

is the Gaussian (heat) kernel.

In all results presented in this paper, it is essential that the initial velocity potential $\xi(x)$ satisfies the following assumptions which we put under one umbrella as

Condition A. *The initial velocity potential $\xi(x)=\xi(x,\omega)$ is the zero-mean, measurable, almost surely differentiable, homogeneous and isotropic real Gaussian random field on $\mathbb{R}^n\times\Omega$, where $(\Omega,\mathcal{F},P)$ is the complete probability space. In addition, its variance $E\xi^2(x)=1$, and its covariance has the form*

$$B(|x|)=E\xi(0)\xi(x)=\frac{L(|x|)}{|x|^\alpha},\quad 0<\alpha<n,\ \ x\in\mathbb{R}^n,$$

where the function $L(t)$, $t>0$, is slowly varying for large values of t, and is bounded on each finite interval. Recall, that $L:(0,\infty)\mapsto(0,\infty)$ is said to be slowly varying if, for all $\lambda>0$, $\lim_{t\to\infty}L(\lambda t)/L(t)=1$.

With the initial data being random, we focus our attention on the statistical properties of solution (2.3) and, in particular, its scaling limit distribution as t tends to infinity.

Let $u=u(t,x)$, $(t,x)\in(0,\infty)\times\mathbb{R}^n$, be the solution of the initial-value problem (1.1)–(1.2) with random initial condition satisfying Condition A. The main result of this paper concerns the so-called parabolic scaling limit for u, i.e., the limiting behavior of the random field $u(t,a\sqrt{t})$, $a\in\mathbb{R}^n$, when $t\to\infty$. Utilizing the techniques developed by Dobrushin and Major (1979) and Taqqu (1979) for analysis of general functionals of Gaussian random fields and fields with long-range dependence one can prove the following theorem (see, e.g., Albeverio et al., 1994; Surgailis and Woyczynski, 1994a; Leonenko et al., 1994; Leonenko and Orsingher, 1995):

Theorem 2.1. *Let $u(t,x)$, $(t,x)\in[0,\infty)\times\mathbb{R}^n$, $n\geqslant1$, be a solution of the initial-value problem* (1.1)–(1.2) *with random initial data satisfying Condition* A. *Then the finite-dimensional distributions of the field*

$$X_t(a)=\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}u(t,a\sqrt{t}),\quad a\in\mathbb{R}^n,$$

converge weakly, as $t\to\infty$, to the finite-dimensional distributions of the homogeneous Gaussian random field $X(a)$, $a\in\mathbb{R}^n$, with $EX(a)=0$ and the covariance function of *the form*

$$R(a,b)=R(a-b)=(R_{ij}(a-b))_{1\leqslant i,j\leqslant n}=EX(a)X(b)'$$
$$=(2\mu)^{-1-\alpha/2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{w_i z_j}{|w-z-(a-b)/\sqrt{2\mu}|^{\alpha}}\phi_n(w)\phi_n(z)\,\mathrm{d}w\,\mathrm{d}z\right)_{1\leqslant i,j\leqslant n}, \tag{2.6}$$

where $0<\alpha<n, w=(w_1,\dots,w_n)'\in\mathbb{R}^n$, $z=(z_1,\dots,z_n)'\in\mathbb{R}^n$, $a,b\in\mathbb{R}^n$,

$$\phi_n(w)=\frac{1}{(2\pi)^{n/2}}\exp\left\{-\frac{|w|^2}{2}\right\}=\prod_{j=1}^{n}\phi(w_j) \tag{2.7}$$

and

$$\phi(w)=\frac{1}{\sqrt{2\pi}}\exp\{-w^2/2\},\quad w\in\mathbb{R}^1. \tag{2.8}$$

Remark 2.1. The limiting random field $X(a)$, $a\in\mathbb{R}^n$, has singular spectral properties that are of independent interest and worth mentioning. In the paper by Leonenko et al. (1996a, b) the following spectral representation is given in terms of stochastic integrals (see, e.g., Kwapien and Woyczynski, 1992): if the random field $\xi(x)$, $x\in\mathbb{R}^n$, has the spectral density $f(\lambda)=f(|\lambda|),\lambda\in\mathbb{R}^n$, i.e.,

$$\xi(x)=\int_{\mathbb{R}^n}\mathrm{e}^{i(\lambda,x)}[f(\lambda)]^{1/2}W(\mathrm{d}\lambda), \tag{2.9}$$

where $W(.)$ is the complex Gaussian white noise and, if that density is a decreasing function for $|\lambda|\geqslant\lambda_0>0$, then

$$X(a)=-\frac{1}{i}\left[\frac{\alpha}{c_1(n,\alpha)c_2(n)}\right]^{1/2}\int_{\mathbb{R}^n}\mathrm{e}^{i(\lambda,a)}g(\lambda)\,W(\mathrm{d}\lambda), \tag{2.10}$$

where

$$g(\lambda)=\lambda\exp\{-\mu(\lambda,\lambda)\}|\lambda|^{(\alpha-n)/2},\quad \lambda\in\mathbb{R}^n, \tag{2.11}$$

$$c_1(n,\alpha)=2^{\alpha}\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n}{2}\right)\Big/\Gamma\left(\frac{n-\alpha}{2}\right),\qquad c_2(n)=2\pi^{n/2}\big/\Gamma(n/2). \tag{2.12}$$

Condition A implies, via the Tauberian theorem (see, Leonenko and Olenko, 1991), the following asymptotics for the spectral density of $\xi(x)$:

$$f(\lambda)=f(|\lambda|)\sim\alpha c_1^{-1}(n,\alpha)c_2^{-1}(n)L\left(\frac{1}{|\lambda|}\right)|\lambda|^{\alpha-n},\quad 0<\alpha<n, \tag{2.13}$$

as $|\lambda|\to 0$, so that $f(|\lambda|)\uparrow\infty$ as $|\lambda|\to 0$. From Eqs. (2.10) and (2.11) we immediately obtain the spectral density of the homogeneous random field $X(a)$, $a\in\mathbb{R}^n$, in the form

$$f_1(\lambda)=\frac{\alpha}{c_1(n,\alpha)c_2(n)}(\lambda_j\lambda_k\mathrm{e}^{-2\mu|\lambda|^2}\big/|\lambda|^{n-\alpha})_{1\leqslant j,k\leqslant n}$$

with $0<\alpha<n$, $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$. Thus, the long-memory property of $f_1(\lambda)$ follows:

$$\lim_{|\lambda|\to 0}\frac{c_1(n,\alpha)c_2(n)}{\alpha}\,\mathrm{tr}\,f_1(\lambda)=\begin{cases}0 & \text{if } \max\{0,n-2\}<\alpha<n,\\ 1 & \text{if } \alpha=n-2,\ n\geqslant 3,\\ \infty & \text{if } 0<\alpha<n-2,\ n\geqslant 3.\end{cases}$$

3. Rate of convergence to the parabolic scaling limit

Let

$$\Pi[c,d]=\{u\in\mathbb{R}^n\colon c_i\leqslant u_i\leqslant d_i,\ i=1,2,\dots,n\}$$

be a paralellepiped in $\mathbb{R}^n$, and let X and Y be arbitrary n-dimensional random vectors. Introduce a uniform (or, Kolmogorov's) distance between distributions of random vectors X and Y via the formula (see, e.g., Bhattacharia and Ranga Rao, 1976; Rachev, 1991)

$$\mathscr{K}(X,Y)=\sup_z|P(X\in\Pi[0,z])-P(Y\in\Pi[0,z])|. \tag{3.1}$$

In view of the symmetries in the initial data and the Hopf–Cole formula, it will suffice to consider only positive vectors z in definiton (3.1).

Consider the following orthogonal matrix (also called the *Helmert matrix*, see, e.g., Basilevsky, 1983):

$$O=\begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}}\\
\frac{1}{\sqrt{1\cdot 2}} & -\frac{1}{\sqrt{1\cdot 2}} & 0 & 0 & \dots & 0 & \dots & 0\\
\frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & -\frac{2}{\sqrt{2\cdot 3}} & 0 & \dots & 0 & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\frac{1}{\sqrt{i(i-1)}} & \frac{1}{\sqrt{i(i-1)}} & \frac{1}{\sqrt{i(i-1)}} & \frac{1}{\sqrt{i(i-1)}} & \dots & -\frac{i-1}{\sqrt{i(i-1)}} & \dots & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \dots & \frac{1}{\sqrt{n(n-1)}} & \dots & -\frac{n-1}{\sqrt{n(n-1)}}
\end{pmatrix}$$

$$=\begin{pmatrix}
\frac{1}{\sqrt{n}} & 0 & \dots & 0\\
0 & \frac{1}{\sqrt{1\cdot 2}} & \dots & 0\\
\dots & \dots & \dots & \dots\\
0 & 0 & \dots & \frac{1}{\sqrt{n(n-1)}}
\end{pmatrix}$$

$$\times\begin{pmatrix}
1 & 1 & 1 & 1 & \dots & 1 & 1\\
1 & -1 & 0 & 0 & \dots & 0 & 0\\
1 & 1 & -2 & 0 & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots\\
1 & 1 & 1 & 1 & \dots & 1 & -(n-1)
\end{pmatrix} \tag{3.2}$$

and the diagonal matrix

$$W_t = \mathrm{diag}\{d_1(t), \dots, d_n(t)\}, \tag{3.3}$$

where

$$d_1(t) = [A_1(t) + (n-1)B_1(t)]^{-1/2},$$

$$d_2(t) = \cdots = d_n(t) = [A_1(t) - B_1(t)]^{-1/2}$$

and for $0 < \alpha < n/2$,

$$A_1(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} w_1 z_1 \frac{\phi_n(w)\phi_n(z) L((|w-z|\sqrt{2\mu t})}{|w-z|^{\alpha} L(\sqrt{t})} \, \mathrm{d}w \, \mathrm{d}z,$$

$$B_1(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} w_1 z_2 \frac{\phi_n(w)\phi_n(z) L((|w-z|\sqrt{2\mu t})}{|w-z|^{\alpha} L(\sqrt{t})} \, \mathrm{d}w \, \mathrm{d}z,$$

where $w = (w_1, \dots, w_n)' \in \mathbb{R}^n$, $u = (u_1, \dots, u_n)' \in \mathbb{R}^n$, and $L(t)$ is a slowly varying function (see, Condition A), and, finally

$$\Delta(a,t) = \left\{ u \in \mathbb{R}^n \colon \left| u + \frac{a}{\sqrt{2\mu}} \right| < \left(\frac{t}{2\mu} \right)^{1/2} \right\}, \quad a \in \mathbb{R}^n, \ t > 0. \tag{3.4}$$

We note that $A_1(t) \neq B_1(t)$. Bearing in mind properties of slowly varying functions (see, e.g., Ivanov and Leonenko, 1989, p. 56) we have, for $0 < \alpha < n$, as $t \to \infty$,

$$A_1 = \lim_{t \to \infty} A_1(t) = (2\mu)^{-1-\alpha/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_1 z_1 \frac{\phi_n(w)\phi_n(z)}{|w-z|^{\alpha}} \, \mathrm{d}w \, \mathrm{d}z, \tag{3.5}$$

$$B_1 = \lim_{t \to \infty} B_1(t) = (2\mu)^{-1-\alpha/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_1 z_2 \frac{\phi_n(w)\phi_n(z)}{|w-z|^{\alpha}} \, \mathrm{d}w \, \mathrm{d}z. \tag{3.6}$$

We note that $A_1 \neq B_1$.

Similarly, for $0 < \alpha < n/2$, we introduce

$$A_2(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} w_1 z_1 \frac{\phi_n(w)\phi_n(z) L((|w-z|\sqrt{2\mu t})}{|w-z|^{2\alpha} L(\sqrt{t})} \, \mathrm{d}w \, \mathrm{d}z, \tag{3.7}$$

$$B_2(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} w_1 z_2 \frac{\phi_n(w)\phi_n(z) L((|w-z|\sqrt{2\mu t})}{|w-z|^{2\alpha} L(\sqrt{t})} \, \mathrm{d}w \, \mathrm{d}z \tag{3.8}$$

and

$$A_2 = \lim_{t \to \infty} A_2(t) = (2\mu)^{-1-\alpha/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_1 z_1 \frac{\phi_n(w)\phi_n(z)}{|w-z|^{2\alpha}} \, \mathrm{d}w \, \mathrm{d}z, \tag{3.9}$$

$$B_2 = \lim_{t \to \infty} B_2(t) = (2\mu)^{-1-\alpha/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_1 z_2 \frac{\phi_n(w)\phi_n(z)}{|w-z|^{2\alpha}} \, \mathrm{d}w \, \mathrm{d}z, \tag{3.10}$$

where $w = (w_1, \dots, w_n)' \in \mathbb{R}^n$, $u = (u_1, \dots, u_n)' \in \mathbb{R}^n$. We note that $A_2 \neq B_2$.

Next, we introduce the following positive constants:

$$Q_1(n, \mu, \alpha) = \frac{A_2 + (n-1)B_2}{A_1 + (n-1)B_1} + (n-1)\frac{A_2 - B_2}{A_1 - B_1}, \tag{3.11}$$

where A_j, B_j, $j=1,2$, are defined by Eqs. (3.5), (3.6), (3.9) and (3.10),

$$Q_2(n,\alpha)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\phi_n(w)\phi_n(z)}{|w-z|^\alpha}\,\mathrm{d}w\,\mathrm{d}z,\quad 0<\alpha<n \tag{3.12}$$

and

$$c_3(\mu)=4\mu^2\left[\mathrm{e}^{1/(4\mu^2)}-1-\frac{1}{4\mu^2}\right].$$

The nature of these constants will become clear later on.

At last, we will also introduce the matrix

$$T_t=W_tO, \tag{3.13}$$

where the matrices O and W_t are defined by Eqs. (3.2) and (3.3), respectively.

The main result of this paper describes the rate of convergence (as $t\to\infty$) of the one-point probability distributions of the parabolically rescaled solution of the Burgers' equation to the n-dimensional standard Gaussian random distribution and is contained in the following

Theorem 3.1. *Let* $u(t,x)$, $(t,x)\in(0,\infty)\times\mathbb{R}^n$, $n\geqslant 1$, *be a solution of the initial-value problem* (2.1)–(2.2) *with random initial data satisfying Condition* A *for* $0<\alpha<n/2$, *and let* N *be an* n*-dimensional non-singular Gaussian random vector with zero mean and unit covariance matrix. Then,*

$$\limsup_{t\to\infty}\frac{t^{\alpha/6}}{L^{1/3}(\sqrt{t})}\mathscr{K}\left(\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}T_t\,u(t,a\sqrt{t}),N\right)\leqslant\frac{3}{2}v_1^{2/3}v_2^{1/3}<\infty, \tag{3.14}$$

where the matrices T_t *are defined by Eq.* (3.13),

$$v_1=v_1(n)=\begin{cases}\sqrt{2/\pi} & \text{for } n=1,\\ (n-1)\Gamma((n-1)/2)/(\sqrt{2}\Gamma(n/2)) & \text{for } n\geqslant 2\end{cases} \tag{3.15}$$

is a universal constant, depending only on the dimension n, *and*

$$v_2=v_2(n,\mu,\alpha)=\frac{2}{(2\mu)^{\alpha/2}}\left(c_3(\mu)Q_1(n,\mu,\alpha)+\frac{Q_2(n,\alpha)}{4\mu^2}\right), \tag{3.16}$$

where $Q_1(n,\mu,\alpha), Q_2(n,\alpha)$ *are defined by Eqs.* (3.11) *and* (3.12), *respectively, is a constant depending on the dimension* n, *viscosity* μ, *and the singularity type* α.

4. Auxiliary lemmas

Let $A\subset\mathbb{R}^n, n\geqslant 1$, be a closed and bounded convex set, A^δ be the set of all points at distances less than $\delta\geqslant 0$ from A, and $A^{-\rho}$ be the set of all points $u\in\mathbb{R}^n$ such that the open ball of radius $\rho\geqslant 0$ centered at u is contained in A. The following auxiliary result is due to Bhattacharia and Ranga Rao (1976), Theorem 3.1:

Lemma 4.1. *Let* g *be a nonnegative differentiable function on* $[0,\infty)$ *such that*
(i) $b=\int_0^\infty|g'(t)|t^{n-1}\,\mathrm{d}t<\infty$; *and*
(ii) $\lim_{t\to\infty}g(t)=0$.

Then, for every convex subset $A \subset \mathbb{R}^n$, *and every pair of* $\delta, \rho \geqslant 0$,

$$\int_{A^\delta \setminus A^{-\rho}} g(|u|)\, \mathrm{d}u \leqslant b(\delta+\rho) 2\pi^{n/2}/\Gamma(n/2). \tag{4.1}$$

Let X, Y be two arbitrary n-dimensional random vectors, $\mathscr{K}(.,.)$ be the Kolmogorov's distance (see Eq. (3.1)), and N be a standard n-dimensional Gaussian random vector with zero mean and unit covariance matrix. In the sequel, vector N will be called the standard Gaussian vector. The following lemma provides an estimate of the Kolmogorov's distance of a sum of random vectors from the standard Gaussian vector. It is a multidimensional version of Petrov's lemma (see, e.g., Petrov, 1960, 1995; Haeusler, 1984).

Lemma 4.2. *Let* X, Y *be two arbitrary random vectors and* N *be a standard Gaussian vector such that, for all* $a, b \in \mathbb{R}^n$,

$$|P(X \in \Pi[a,b]) - P(N \in \Pi[a,b])| \leqslant K,$$

where $K \geqslant 0$ *is a constant. Then, for any* $\varepsilon > 0$,

$$\mathscr{K}(X+Y, N) \leqslant K + P(Y \notin \Pi[-h\varepsilon, \varepsilon h]) + \varepsilon v_1(n), \tag{4.2}$$

where $v_1(n)$ *is defined in Eq.* (3.15), *and* $h = (1, \ldots, 1)' \in \mathbb{R}^n$.

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} &P(X+Y \in \Pi[0,u]) \\ &\quad = P(X+Y \in \Pi[0,u], Y \in \Pi[-\varepsilon h, \varepsilon h]) + P(X+Y \in \Pi[0,u], Y \notin \Pi[-\varepsilon h, \varepsilon h]) \end{aligned}$$

and

$$\begin{aligned} &P(X \in \Pi[\varepsilon h, u-\varepsilon h], Y \in \Pi[-\varepsilon h, \varepsilon h]) \\ &\quad \leqslant P(X+Y \in \Pi[0,u], Y \in \Pi[-\varepsilon h, \varepsilon h]) \\ &\quad \leqslant P(X \in \Pi[-\varepsilon h, u+\varepsilon h], Y \in \Pi[-\varepsilon h, \varepsilon h]). \end{aligned} \tag{4.3}$$

Note that if $u_i \leqslant 2\varepsilon$, the inequalities $\varepsilon \leqslant X_i \leqslant u_i - \varepsilon$ mean that $X_i = \varepsilon$, where $X = (X_1, \ldots, X_n)'$. Thus, we obtain

$$\begin{aligned} 0 &\leqslant P(X \in \Pi[\varepsilon h, u-\varepsilon h]) - P(X \in \Pi[\varepsilon h, u-\varepsilon h], Y \in \Pi[-\varepsilon h, \varepsilon h]) \\ &= P(X \in \Pi[\varepsilon h, u-\varepsilon h], Y \notin \Pi[-\varepsilon h, \varepsilon h]) \leqslant P(Y \notin \Pi[-\varepsilon h, \varepsilon h]). \end{aligned} \tag{4.4}$$

It follows from Eqs. (4.3) and (4.4) that

$$\begin{aligned} &P(X \in \Pi[\varepsilon h, u-\varepsilon h]) - P(Y \notin \Pi[-\varepsilon h, \varepsilon h]) \\ &\quad \leqslant P(X+Y \in \Pi[0,u], Y \in \Pi[-\varepsilon h, \varepsilon h]) \leqslant P(X \in \Pi[-\varepsilon h, u+\varepsilon h]). \end{aligned}$$

Adding the expressions we have

$$P(X+Y \in \Pi[0,u], Y \notin \Pi[-\varepsilon h, \varepsilon h]).$$

Applying the inequality

$$P(X+Y \in \Pi[0,u], Y \notin \Pi[-\varepsilon h, \varepsilon h]) \leqslant P(Y \notin \Pi[-\varepsilon h, \varepsilon h]),$$

we obtain that

$$P(X\in\Pi[\varepsilon h,u-\varepsilon h])-P(Y\notin\Pi[-\varepsilon h,\varepsilon h])\leqslant P(X+Y\in\Pi[0,u])$$
$$\leqslant P(X\in\Pi[-\varepsilon h,u+\varepsilon h])+P(Y\notin\Pi[-\varepsilon h,\varepsilon h]). \tag{4.5}$$

Using the inequality (4.1) of Lemma 4.1 with the function $g(t)=\phi_n(u)$ for any u such that $|u|=t$ (see formula (2.7) of Theorem 2.1) we obtain inequalities

$$P(N\in\Pi[-\varepsilon h,u+\varepsilon h])-P(N\in\Pi[0,u])$$
$$=\int_{\Pi[-\varepsilon h,u+\varepsilon h]\setminus\Pi[0,u]}\phi_n(|\tilde{u}|)\,\mathrm{d}\tilde{u}\leqslant\varepsilon v_1(n),$$
$$-\left(P(N\in\Pi[0,u])-P(N\in\Pi[\varepsilon h,u-\varepsilon h])\right)$$
$$\geqslant-\varepsilon v_1(n), \tag{4.6}$$

where $v_1(n)$ is defined by Eq. (3.15). So, from Eq. (4.6), we get that

$$P(X\in\Pi[-\varepsilon h,u+\varepsilon h])-P(N\in\Pi[0,u])$$
$$\leqslant P(X\in\Pi[-\varepsilon h,u+\varepsilon h])-P(N\in\Pi[-\varepsilon h,u+\varepsilon h])+\varepsilon v_1(n) \tag{4.7}$$

and

$$P(X\in\Pi[\varepsilon h,u-\varepsilon h])-P(N\in\Pi[0,u])$$
$$\geqslant P(X\in\Pi[\varepsilon h,u-\varepsilon h])-P(N\in\Pi[\varepsilon h,u-\varepsilon h])-\varepsilon v_1(n). \tag{4.8}$$

Finally, subtracting the expression $P(N\in\Pi[0,u])$ from Eqs. (4.5)–(4.8), we obtain

$$-K-\varepsilon v_1(n)-P(Y\notin\Pi[-\varepsilon h,\varepsilon h])\leqslant\mathscr{K}(X+Y,N)$$
$$\leqslant K+\varepsilon v_1(n)+P(Y\notin\Pi[-\varepsilon h,\varepsilon h]).\qquad\square$$

The next lemma estimates the Kolmogorov's distance from a standard Gaussian vector to the ratio of a random vector and a random variable. Its one-dimensional version is due to Michel and Pfanzagl (1971).

Lemma 4.3. *Let X be an n-dimensional random vector and $U>0$ be a random variable. Then, for any $\varepsilon>0$,*

$$\mathscr{K}(X/U,N)\leqslant\mathscr{K}(X,N)+P(|U-1|>\varepsilon)+\varepsilon^n. \tag{4.9}$$

Proof. Since the assertion is trivial for $\varepsilon\geqslant1$ we shall assume that $\varepsilon\in(0,1)$. The inequality $X\in\prod[0,(1-\varepsilon)z]$ implies that either $X/U\in\Pi[0,z]$ or $U<1-\varepsilon$. Hence,

$$P(X\in\Pi[0,(1-\varepsilon)z])\leqslant P(X/U\in\Pi[0,z])+P(|U-1|>\varepsilon). \tag{4.10}$$

Furthermore, using the properties of normal distribution,

$$\begin{aligned}&|P(N\in\Pi[0,z])-P(N\in\Pi[0,(1-\varepsilon)z])|\\&\quad\leqslant\min\left(2^{-n},[1-(1-\varepsilon)^n]\prod_{i=1}^n z_i(2\pi)^{-n/2}\exp\{-|z|^2(1-\varepsilon)^2/2\}\right)\\&\quad\leqslant\min(2^{-n},[(1-\varepsilon)^{-n}-1](2\pi\mathrm{e})^{-n/2})\leqslant\varepsilon^n,\end{aligned}\tag{4.11}$$

together with Eq. (4.10), implies that

$$\begin{aligned}&P(X/U\in\Pi[0,z])-P(N\in\Pi[0,z])\\&\quad\geqslant-|P(X\in\Pi[0,(1-\varepsilon)z])-P(N\in\Pi[0,(1-\varepsilon)z])|-P(|U-1|>\varepsilon)-\varepsilon^n.\end{aligned}\tag{4.12}$$

Similarly, for $0<\varepsilon<1$, the inequality $X/U\in\Pi[0,z]$ implies that $X\in\Pi[0,(1+\varepsilon)z]$ or $U>1+\varepsilon$. Hence,

$$P(X/U\in\Pi[0,z])\leqslant P(X\in\Pi[0,(1+\varepsilon)z])+P(|U-1|>\varepsilon).$$

Thus,

$$\begin{aligned}&P(X/U\in\Pi[0,z])-P(N\in\Pi[0,z])\\&\quad\leqslant\sup|P(X/U\in\Pi[0,z])-P(N\in\Pi[0,z])|+P(|U-1|>\varepsilon)+\varepsilon^n.\end{aligned}\tag{4.13}$$

Now, Eqs. (4.12) and (4.13) imply the statement of the Lemma 4.3. □

Combining the results of Lemmas 4.2 and 4.3 we immediately obtain:

Lemma 4.4. *Let X,Y be two random vectors such that for all $a,b\in\mathbb{R}^n$*

$$|P(X\in\Pi[a,b])-P(N\in\Pi[a,b])|\leqslant K,$$

where $K\geqslant0$ is a constant, and let $U>0$ be a random variable. Then, for any $\varepsilon>0$,

$$\mathscr{K}([X+Y]/U,N)\leqslant K+P(Y\notin\Pi[-\varepsilon h,\varepsilon h])+P(|U-1|>\varepsilon)+v_1(n)\varepsilon+\varepsilon^n,$$

where $v_1(n)$ is given by Eq. (3.15).

In the proof of Theorem 3.1 we will also have need for the following estimate on the tails of the maxima of a general second-order random vector's components (see, e.g., Theorem 4.1 in Karlin and Studden, 1966).

Lemma 4.5. *Let $Y=(Y_1,\dots,Y_n)'$ be a random vector with mean $EY=0$ and the covariance matrix $EYY'=\Sigma=(\sigma_{ij})_{1\leqslant i,j\leqslant n}$, and let $Z_i=Y_i/(\kappa_i\sigma_i)$, where $\sigma_i^2=\sigma_{ii}$, and $\kappa_1,\dots,\kappa_n>0$, are some constants. Then*

$$P\left(\max_{1\leqslant i\leqslant n}|Z_i|\geqslant1\right)\leqslant\frac{1}{n^2}(\sqrt{s}+\sqrt{(nq-s)(n-1)})^2,$$

where $q = \operatorname{tr} \Pi$, $s = (h, \Pi h)$, $\Pi = EZZ' = (\pi_{ij})_{1 \leqslant i,j \leqslant n}$, $\pi_{ij} = \sigma_{ij}/(\sigma_i \sigma_j \kappa_i \kappa_j)$, *and* $h = (1, \ldots, 1)'$.

The following result is obvious and is just recorded here for easy reference.

Lemma 4.6. *Let* W, T *be two arbitrary random vectors. Then, for any* $\varepsilon > 0$ *and arbitrary* $0 < \delta < 1$,

$$P(W + T \notin \Pi[-\varepsilon h, \varepsilon h])$$
$$\leqslant P(W \notin \Pi[-\delta\varepsilon h, \delta\varepsilon h]) + P(T \notin \Pi[-\varepsilon(1-\delta)h, (1-\delta)\varepsilon h]).$$

Also, we quote here another well-known result (see, e.g., Kwapien and Woyczynski, 1992, p. 177) concerning Hermite polynomials

$$H_m(u) = (-1)^m \mathrm{e}^{u^2/2} \frac{\mathrm{d}^m}{\mathrm{d}u^m} \mathrm{e}^{-u^2/2}, \quad u \in \mathbb{R}, \ m = 0, 1, \ldots \tag{4.14}$$

with leading coefficients equal to 1.

Lemma 4.7. *Let* $(\xi, \eta)'$ *be a two-dimensional Gaussian random vector with* $E\xi = E\eta = 0$, $E\xi^2 = E\eta^2 = 1$, $E\xi\eta = r$. *Then, for any* $m, q \geqslant 0$,

$$EH_m(\xi) H_q(\eta) = \delta_m^q r^m m!.$$

Finally, we provide a lemma that is quite traditional in spirit; we only sketch its proof.

Lemma 4.8. *Let* $X = (X_1, \ldots, X_n)'$ *be a Gaussian random vector with zero mean and the covariance matrix* $R = EXX' = (r_{ij})_{1 \leqslant i,j \leqslant n}$ *such that the diagonal elements* $r_{ii} = A$, $i = 1, \ldots, n$ *and the off-diagonal elements* $r_{ij} = B$, $i \neq j, i, j = 1, \ldots, n$. *Then, the Gaussian random vector*

$$\tilde{X} = WOX$$

has zero mean and unit covariance matrix, where the orthogonal matrix O *is defined by Eq.* (3.2) *and the diagonal matric* $W = \operatorname{diag}\{d_1, \ldots, d_n\}$, *where* $d_1 = [A + (n-1)B]^{-1/2}$, *and* $d_2 = \cdots = d_n = [A - B]^{-1/2}$.

Proof. It is well known that the equation

$$\det |R - \mu I| = (\mu - \mu_1)(\mu - \mu_2)^{n-1} = 0$$

has two roots: $\mu_1 = A - (n-1)B$ of multiplicity 1, and $\mu_2 = A - B$ of multiplicity $n-1$. The eigenvalue μ_1 corresponds to eigenvector $(1/\sqrt{n}, \ldots, 1/\sqrt{n})'$, and the eigenvalue μ_2 corresponds to $n-1$ eigenvectors

$$\left(\frac{1}{\sqrt{i(i-1)}}, \ldots, \frac{1}{\sqrt{i(i-1)}}, -\frac{(i-1)}{\sqrt{i(i-1)}}, 0 \ldots, 0 \right), \quad 2 \leqslant i \leqslant n.$$

If we take these eigenvectors as rows of the orthogonal Helmert matrix O, then the random vector OX has zero mean and the diagonal covariance matrix diag $\{\mu_1,\dots,\mu_n\}$, where $\mu_j=\mu_2$, $2\leqslant j\leqslant n$. Now, it is obvious that the Gaussian vector $\tilde{X}=WOX$ has unit covariance matrix. □

5. Proof of the main result

This section contains a proof of Theorem 3.1, the main result of the paper. It is based on the expansion of the Hermite polynomials $\{H_m(u)\}_{m=0}^{\infty}$ (see Eq. (4.14)); recall that the latter constitute a complete orthogonal system in the Hilbert space $L_2(\mathbb{R}^1,\phi(u)\,\mathrm{d}u)$, where the function $\phi(u)$ is the Gaussian density defined by Eq. (2.8) (see, e.g., Dobrushin and Major, 1979; Taqqu, 1979; Ivanov and Leonenko, 1989; Kwapien and Woyczynski, 1992). The plan of the proof is as follows:

- The exponential $\exp(-\xi(y)/(2\mu)$ appearing in the numerator of the Hopf–Cole formula (2.5) is expanded in the Hermite series;
- The rescaled solution $T_tu(t,a\sqrt{t})$ is split into two parts, the first contains the integral of the first term of the Hermite expansion (but integrated only over a bounded set $\{y:|y|\leqslant t\}$) and the remainder;
- The first part, properly normalized, has a standard normal distribution;
- The remainder is explicitly estimated using auxiliary lemmas of Section 4.

Let $G:\mathbb{R}^1\mapsto\mathbb{R}^1$ be a function such that $EG^2(\xi(0))<\infty$, where $\xi(0)$ is a random variable with the Gaussian density (2.8) (see Condition A in Section 2). Then, in $L_2(\mathbb{R}^1,\phi(u)\,\mathrm{d}u)$, we have the following expansion:

$$G(u)=\sum_{k=0}^{\infty}\frac{C_k}{k!}H_k(u),\qquad C_k=\int_{-\infty}^{\infty}G(u)H_k(u)\phi(u)\,\mathrm{d}u. \tag{5.1}$$

By Parceval equality it follows that

$$\int_{-\infty}^{\infty}G^2(u)\phi(u)\,\mathrm{d}u=\sum_{k=0}^{\infty}\frac{C_k^2}{k!}<\infty.$$

In particular, in view of Eq. (5.1), the coefficients of the Hermite expansion of the function

$$G(u)=\exp\left\{-\frac{u}{2\mu}\right\},\qquad u\in\mathbb{R}^1$$

are given by the formulae

$$C_0=\exp\left\{\frac{1}{8\mu^2}\right\},\quad C_1=-\frac{1}{2\mu}\exp\left\{\frac{1}{8\mu^2}\right\},\quad C_2=\frac{1}{4\mu^2}\exp\left\{\frac{1}{8\mu^2}\right\},$$

$$C_k=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{u+u^2\mu}{2\mu}\right\}H_k(u)\,\mathrm{d}u,\quad k=3,4,\dots\ . \tag{5.2}$$

In turn, these classical results imply the following expansion in the Hilbert space $L^2(\Omega)$ of random variables with finite second moments:

$$\exp\left\{-\frac{\xi(y)}{2\mu}\right\} = \sum_{k=0}^{\infty} C_k \frac{H_k(\xi(y))}{k!}, \tag{5.3}$$

where the C_k's are defined by Eq. (5.2).

Now, consider the random vectors

$$\eta_k(t,a) = \int_{\mathscr{D}_n(t)} \frac{a\sqrt{t}-y}{t} g(t,a\sqrt{t}-y) H_k(\xi(y))\,\mathrm{d}y, \quad k=0,1,\ldots, \tag{5.4a}$$

where $\mathscr{D}_n(t)=\{x\in\mathbb{R}^n\colon\ |y|\leqslant t\}$, $a,y\in\mathbb{R}^n$. It follows from Lemma 4.7 that

$$E\eta_k(t,a)\eta_j(t,a)' = \delta_k^j E\eta_k(t,a)\eta_k(t,a)', \quad k\geqslant 1,\ j\geqslant 1,\ a\in\mathbb{R}^n,$$

where

$$E\eta_k(t,a)\eta_k(t,a)' = \left(\psi^2_{k,i,j}(t)\right)_{1\leqslant i,j\leqslant n}$$

and

$$\begin{aligned}\psi^2_{k,i,j}(t) &= k!\int_{\mathscr{D}_n(t)}\int_{\mathscr{D}_n(t)} \frac{a_i\sqrt{t}-y_{1i}}{t}\cdot\frac{a_j\sqrt{t}-y_{2j}}{t}\\ &\quad\times g(t,a\sqrt{t}-y_1)g(t,a\sqrt{t}-y_2)B^k(|y_1-y_2|)\,\mathrm{d}y_1\,\mathrm{d}y_2,\end{aligned}$$

where $y_1=(y_{11},\ldots,y_{1n})'$, $y_2=(y_{21},\ldots,y_{2n})'$, and B is the correlation function from Condition A. Changing variables via the transformation

$$\frac{w_i^2}{2} = \frac{(a_i\sqrt{t}-y_{1i})^2}{4\mu t}, \qquad \frac{z_i^2}{2} = \frac{(a_i\sqrt{t}-y_{2i})^2}{4\mu t}, \quad i=1,\ldots,n$$

and utilizing basic properties of the slowly varying function L (see, e.g., Ivanov and Leonenko, 1989, p. 56), we have, for $0<\alpha<n/k, k\geqslant 1$, and $t\to\infty$,

$$\begin{aligned}\psi^2_{k,i,j}(t) &= \frac{2\mu k!}{t}\int_{\Delta(a,t)}\int_{\Delta(a,t)} w_i z_j \phi_n(w)\phi_n(z) B^k(\sqrt{2\mu t}|w-z|)\,\mathrm{d}z\,\mathrm{d}w\\ &= \frac{2\mu k! L^k(\sqrt{t})}{(2\mu)^{k\alpha/2} t^{1+(k\alpha)/2}}\int_{\Delta(a,t)}\int_{\Delta(a,t)} w_i z_j\phi_n(w)\phi_n(z)\frac{L^k(\sqrt{2\mu t}|w-z|)}{|w-z|^{k\alpha}L^k(\sqrt{t})}\,\mathrm{d}z\,\mathrm{d}w\\ &= (2\mu)^{1-k\alpha/2} k! c_{1,i,j}(k,\alpha)\frac{L^k(\sqrt{t})}{t^{1+(k\alpha)/2}}(1+\mathrm{o}(1)), \quad k=1,2,\ldots, t\to\infty,\end{aligned} \tag{5.4b}$$

where $\Delta(a,t)$ is defined by Eq. (3.4), and

$$c_{1,i,j}(k,\alpha) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{w_i z_j\phi_n(w)\phi_n(z)}{|w-z|^{k\alpha}}\,\mathrm{d}w\,\mathrm{d}z, \quad i,j,=1,\ldots,n.$$

In order to apply Lemma 4.4, we shall represent the rescaled solution appearing in Theorem 3.1 (see Eq. (3.14)) in the following form:

$$\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} T_t u(t,a\sqrt{t}) = \frac{X_t+Y_t}{U_t}. \tag{5.5}$$

The random vectors X_t, Y_t and the random variable U_t are defined via the formulae

$$X_t = C_1 \exp\left\{-\frac{1}{8\mu^2}\right\} \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} T_t \int_{\mathscr{D}_n(t)} \frac{a\sqrt{t}-y}{t} g(t, a\sqrt{t}-y) H_1(\xi(y))\,\mathrm{d}y, \quad (5.6)$$

$$Y_t = \exp\left\{-\frac{1}{8\mu^2}\right\} \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} T_t [S_t + V_t], \quad (5.7)$$

where

$$S_t = \sum_{k\geqslant 2} \frac{C_k}{k!} \eta_k(t,a),$$

$$V_t = \int_{\mathbb{R}^n \setminus \mathscr{D}_n(t)} \frac{a\sqrt{t}-y}{t} g(t, a\sqrt{t}-y) \exp\left\{-\frac{\xi(y)}{2\mu}\right\} \mathrm{d}y$$

and

$$U_t = J(t, a\sqrt{t}) \exp\left\{-\frac{1}{8\mu^2}\right\}, \quad (5.8)$$

where $J(t)$ is defined in Eq. (2.5).

Note, that $C_0\eta_0(t,a)\to 0$, $t\to\infty$ (the random vectors η_k were introduced in Eq. (5.4a)), that $H_1(u)=u$, $u\in\mathbb{R}^1$, and that $\xi(y), y\in R^n$, is a Gaussian field (see Condition A). So, the random field

$$\tilde{\eta}_1 = \tilde{\eta}_1(t,a) = C_1 \mathrm{e}^{-1/8\mu^2} \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} \eta_1(t,a)$$

is also Gaussian, and in view of Eqs. (5.4a) and (5.4b) with $k=1$,

$$E\tilde{\eta}_1\tilde{\eta}_1' = (\sigma_{ij}^2(t))_{1\leqslant i,j\leqslant n},$$

where

$$\sigma_{i,j}^2(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} \frac{w_i z_j \phi_n(w)\phi_n(z) L(|w-z|\sqrt{2\mu t})}{|w-z|^\alpha L(\sqrt{t})} \mathrm{d}w\,\mathrm{d}z, \quad 0<\alpha<n. \quad (5.9)$$

We note that, from Eqs. (3.4)–(3.5) and (5.9),

$$\sigma_{ii}^2(t) = \sigma_{11}^2(t) = A_1(t), \quad i=1,\ldots,n$$

and

$$\sigma_{ij}^2(t) = \sigma_{12}^2(t) = B_1(t), \quad i,j=1,\ldots,n, \; i\neq j.$$

By Lemma 4.8, for every $t>0$, the random field

$$X_t = T_t\tilde{\eta}_1(t,a),$$

where $T_t = W_t O$, the matrices W_t and O defined by Eqs. (3.2) and (3.3), respectively, has the standard Gaussian n-dimensional distribution. Thus, for all $a,b\in\mathbb{R}^n$,

$$|P(X_t\in\Pi[a,b]) - P(N_t\in\Pi[a,b])| = 0, \quad (5.10a)$$

so that we may choose $K=0$ in Lemma 4.4.

From Lemma 4.6 and Eq. (5.7), for any $\varepsilon > 0$ and an arbitrary $\delta \in (0,1)$,

$$P(Y_t \notin \Pi[-\varepsilon h, \varepsilon h]) \leqslant P(A_t T_t S_t \notin \prod[-\varepsilon\delta h, \varepsilon\delta h]) + P(A_t T_t V_t \notin \prod[-\varepsilon(1-\delta)h, \varepsilon(1-\delta)h]), \tag{5.10b}$$

where

$$A_t = \mathrm{e}^{-1/8\mu^2} \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} = -\frac{1}{2\mu C_1} \frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}$$

and the matrix $T_t = W_t O$ is defined by Eq. (3.13).

For $r \leqslant k$, the function

$$A(w,z) = B^r(\sqrt{2\mu t}|w-z|) - B^k(\sqrt{2\mu t}|w-z|) \geqslant 0.$$

Therefore,

$$\frac{\psi^2_{r,i,j}(t)}{r!} - \frac{\psi^2_{k,i,j}(t)}{k!} \equiv \frac{2\mu}{t} \int_{\Delta(a,t)} \int_{\Delta(a,t)} \phi_n(w)\phi_n(z) A(w,z) \times \mathrm{d}w_1 \cdots \mathrm{d}w_{k-1} \mathrm{d}(w_k^2/2)\, \mathrm{d}w_{k+1} \cdots \mathrm{d}w_n\, \mathrm{d}z_1 \cdots \mathrm{d}z_{r-1} \times \mathrm{d}(z_r^2/2)\, \mathrm{d}z_{r+1} \cdots \mathrm{d}z_n \geqslant 0$$

and for $r \leqslant k$,

$$\frac{1}{k!}\psi^2_{k,i,j}(t) \leqslant \frac{1}{r!}\psi^2_{r,i,j}(t),$$

so that, for $0 < \alpha < n/2$,

$$q = \mathrm{tr}\, E[A_t T_t S_t][A_t T_t S_t]' = A_t^2\, \mathrm{tr}\, W_t O \sum_{k \geqslant 2} (C_k/k!)^2 \Psi_t O' W_T' \leqslant \frac{L(t)}{t^{\alpha/2}} (2\mu)^{-\alpha/2} \left(\frac{1}{C_1^2} \sum_{k\geqslant 2} \frac{C_k^2}{k!} \right) \mathrm{tr}\, W_t O P_t O' W_t',$$

where $\Psi_{t,k} = (\psi_{k,i,j}(t))_{1\leqslant i,j \leqslant n}$, $P_t = (p_{ij}(t))_{1 \leqslant i,j \leqslant n}$, with $\psi_{k,i,j}(t)$ defined by Eqs. (5.4a) and (5.4b), and

$$p_{ij}(t) = (2\mu)^{-1-\alpha/2} \int_{\Delta(a,t)} \int_{\Delta(a,t)} \frac{w_i z_j \phi_n(w)\phi_n(z) L(|w-z|\sqrt{2\mu t})}{|w-z|^{2\alpha} L(\sqrt{t})}\, \mathrm{d}w\, \mathrm{d}z, \quad 0 < \alpha < n/2,$$

where $\Delta(a,t)$ is defined by Eq. (2.4) and $w = (w_1, \ldots, w_n)'$, $z = (z_1, \ldots, z_n)' \in \mathbb{R}^n$. We note that $p_{11}(t) = A_2(t)$, $p_{12}(t) = B_2(t)$, where $A_2(t)$ and $B_2(t)$ are defined by Eqs. (3.7) and (3.8), respectively. We used the fact that the matrix O has negative elements only on the diagonal.

From Lemma 4.7, we obtain

$$OP_tO' = \mathrm{diag}\{A_2(t) + (n-1)B_2(t), A_2(t) - B_2(t), \ldots, A_2(t) - B_2(t)\},$$

so that

$$q \leqslant \frac{L(\sqrt{t})}{t^{\alpha/2}}(2\mu)^{-\alpha/2}\left(\frac{1}{C_1^2}\sum_{k\geqslant 2}\frac{C_k^2}{k!}\right) \\ \times\left[\frac{A_2(t)+(n-1)B_2(t)}{A_1(t)+(n-1)B_1(t)}+(n-1)\frac{A_2(t)-B_2(t)}{A_1(t)-B_1(t)}\right]. \tag{5.11}$$

In view of Lemma 4.5 (in our case, $s=q$, and $Z=(\varepsilon\delta)^{-1}A_tT_tS_t$),

$$P(A_tT_tS_t\notin \Pi[-\varepsilon\delta h,\varepsilon\delta h]) \\ =P((\varepsilon\delta)^{-1}A_tT_tS_t\notin\Pi[-h,h]) \\ \leqslant P\left(\max_{1\leqslant i\leqslant n}|Z_i|\geqslant 1\right)\leqslant(\varepsilon\delta)^{-2}\frac{L(t)}{t^{\alpha/2}}(2\mu)^{-\alpha/2}c_3(\mu)Q_{1t}(n,\mu,\alpha), \tag{5.12}$$

where

$$c_3(\mu)=\frac{1}{C_1^2}\sum_{k\geqslant 2}\frac{C_k^2}{k!}=\frac{1}{C_1^2}\left(\int_{-\infty}^{\infty}(\mathrm{e}^{-u/2\mu})^2\phi(u)\,\mathrm{d}u-C_0^2-C_1^2\right) \\ =[\mathrm{e}^{1/2\mu^2}-(1-(2\mu)^{-2})\mathrm{e}^{1/4\mu^2}][(2\mu)^{-2}\mathrm{e}^{1/4\mu^2}]^{-1} \\ =4\mu^2[\mathrm{e}^{1/4\mu^2}-1-1/4\mu^2]$$

and

$$Q_{1t}(n,\mu,\alpha)=\frac{A_2(t)+(n-1)B_2(t)}{A_1(t)+(n-1)B_1(t)}+(n-1)\frac{A_2(t)-B_2(t)}{A_1(t)-B_1(t)}. \tag{5.13}$$

It is obvious that

$$\lim_{t\to\infty}Q_{1t}(n,\mu,\alpha)=Q_1(n,\mu,\alpha),$$

where $Q_1(n,\mu,\alpha)$ is defined by Eq. (3.11).

It is also easy to see that

$$E[V_t][V_t]'=(\gamma_{ij}(t))_{1\leqslant i,j\leqslant n},$$

where

$$|\gamma_{ij}(t)|\leqslant\frac{\mathrm{e}^{1/2\mu}}{t}\int_{\mathbb{R}^n\setminus\Delta(a,t)}\int_{\mathbb{R}^n\setminus\Delta(a,t)}|w_iu_j|\phi_n(w)\phi_n(u)\,\mathrm{d}w\,\mathrm{d}u\leqslant\mathrm{const}\frac{\mathrm{e}^{-t/\mu}}{t}. \tag{5.15}$$

From Eqs. (5.10b), (5.12) and (5.15),

$$P(Y_t\notin\Pi[-\varepsilon h,\varepsilon h])\leqslant\frac{1}{\varepsilon^2}\left[\frac{L(\sqrt{t})}{t^{\alpha/2}}\frac{1}{\delta^2}(2\mu)^{-\alpha/2}\cdot c_3(\mu)Q_{1t}(n,\mu,\alpha)+(1-\delta)^{-2}R_t\right], \tag{5.16}$$

where, as $t\to\infty$,

$$R_t\leqslant\mathrm{const}\frac{\mathrm{e}^{-t/\mu}t^{\alpha/2}}{L(\sqrt{t})}. \tag{5.17}$$

We also note that

$$U_t - 1 = \mathrm{e}^{-1/8\mu^2} \int_{\mathbb{R}^n} g(t, a\sqrt{t} - y)(\mathrm{e}^{-\xi(y)/2\mu} - \mathrm{e}^{1/8\mu^2})\,\mathrm{d}y = \Sigma_1(t) + \Sigma_2(t) + \Sigma_3(t), \tag{5.18}$$

where

$$\Sigma_1(t) = \mathrm{e}^{-1/8\mu^2} C_1 \int_{\mathscr{D}_n(t)} g(t, a\sqrt{t} - y)\xi(y)\,\mathrm{d}y,$$

$$\Sigma_2(t) = \mathrm{e}^{-1/8\mu^2} \sum_{k=2}^{\infty} \frac{C_k}{k!} \int_{\mathscr{D}_n(t)} g(t, a\sqrt{t} - y) H_k(\xi(y))\,\mathrm{d}y,$$

$$\Sigma_3(t) = \mathrm{e}^{-1/8\mu^2} C_1 \int_{\mathbb{R}^n \setminus \mathscr{D}_n(t)} g(t, a\sqrt{t} - y)(\mathrm{e}^{-\xi(y)/2\mu} - \mathrm{e}^{1/8\mu^2})\,\mathrm{d}y.$$

Estimating and analyzing the limiting behavior of the above integrals shows that

$$\operatorname{Var} \Sigma_1(t) \leqslant (2\mu)^{-\alpha/2-2} Q_{2t}(n, \alpha) \frac{L(t)}{t^{\alpha/2}}, \tag{5.19}$$

where

$$\lim_{t\to\infty} Q_{2t}(n, \alpha) = Q_2(n, \alpha) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\phi_n(w)\phi_n(z)}{|w - z|^\alpha}\,\mathrm{d}w\,\mathrm{d}z, \quad 0 < \alpha < n,$$

$Q_2(n, \alpha)$ is defined by Eq. (3.12), and

$$\operatorname{Var} \Sigma_2(t) \leqslant (2\mu)^{-\alpha} Q_{2t}(n, 2\alpha) \frac{L^2(\sqrt{t})}{t^\alpha} c_4, \quad c_4 > 0, \tag{5.20}$$

$$\operatorname{Var} \Sigma_3(t) = \mathrm{o}\left(\frac{L(\sqrt{t})}{t^\alpha}\right). \tag{5.21}$$

So, from Eqs. (5.18)–(5.21),

$$P\{|U_t - 1| > \varepsilon\} \leqslant \frac{1}{\varepsilon^2} \left[\frac{L(\sqrt{t})}{t^{\alpha/2}} (2\mu)^{-\alpha/2-2} \delta^{-2} Q_{2t}(n, \alpha) + W_t\right], \tag{5.22}$$

where

$$W_t = \mathrm{o}(t^{-\alpha/2} L(\sqrt{t})), \quad t \to \infty. \tag{5.23}$$

Applying Lemma 4.4 to Eq. (5.5), and using Eqs. (5.9), (5.16), and (5.23), we obtain, for any $\varepsilon > 0$, that

$$\mathscr{K}\left(\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})} T_t u(t, a\sqrt{t}), N\right)$$
$$\leqslant K + P(Y_t \notin \Pi[-\varepsilon h, \varepsilon h]) + P(|U_t - 1| > \varepsilon) + \varepsilon v_1(n) + \varepsilon^n$$

$$\leqslant v_1(n)\varepsilon + \frac{1}{\varepsilon^2}\left[\frac{L(\sqrt{t})}{(2t\mu)^{\alpha/2}}\left(\frac{c_3(\mu)Q_{1t}(n,\mu,\alpha)}{\delta^2} + \frac{Q_{2t}(n,\alpha)}{4\mu^2\delta^2}\right)\right.$$
$$\left. + (1-\delta)^{-2}R_t + W_t\right] + \varepsilon^n, \tag{5.24}$$

because we may choose $K=0$ (see Eq. (5.10a)). In order to minimize the first two terms of the right-hand side of inequality (5.24), set

$$\varepsilon = \frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}} v_1^{-1/3}(2\mu)^{-\alpha/6}\frac{1}{\delta^2}\left(2c_3(\mu)Q_{1t}(n,\mu,\alpha) + \frac{Q_{2t}(n,\alpha)}{2\mu^2}\right)^{1/3}.$$

Thus, we arrive at the following inequality:

$$\mathscr{K}\left(\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}T_t u(t,a\sqrt{t}),N\right)$$
$$\leqslant \frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}}\left\{v_1^{2/3}\left[\frac{2}{\delta^2(2\mu)^{\alpha/2}}\left(c_3(\mu)Q_{1t}(n,\mu,\alpha) + \frac{Q_{2t}(n,\alpha)}{4\mu^2}\right)\right]^{1/3}\right\}$$
$$+\frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}}\left\{v_1^{2/3}\frac{1}{2}\left[2\frac{1}{\delta^2(4\mu)^{\alpha/2}}\left(c_3(\mu)Q_{1t}(n,\mu,\alpha) + \frac{Q_{2t}(n,\alpha)}{4\mu^2}\right)\right]^{1/3}\right.$$
$$\left.+\frac{t^{\alpha/2}}{L(\sqrt{t})}(c_5R_t + c_6W_t)\right\} + \mathrm{o}\left(\frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}}\right),$$

where $c_5, c_6 > 0$.

Now, the above inequality and the relationships (5.14), (5.17), (5.22) and (5.24) imply Theorem 3.1. □

6. Extensions and generalizations

Theorem 3.1 gives the convergence rate to zero of the Kolmogorov distance between the rescaled solutions and the multivariate Gaussian law only for $\alpha \in (0, n/2)$. On the other hand, it follows from Theorems 2.1 and 2.2 (see also Albeverio et al., 1994; Surgailis and Woyczynski, 1994a, b; Leonenko et al., 1994, 1995a) that the asymptotic normality of the rescaled solutions of the Burgers' equation is obtained for all $\alpha \in (0, n)$ (see, Condition A). As it turns out, our method is also applicable in the broader interval $\alpha \in (0, n)$ but at the price of slower convergence rates.

Theorem 6.1. *Let* $u(t,x)$, $(t,x) \in (0,\infty)\times\mathbb{R}^n$, $n \geqslant 1$, *be a solution of the initial-value problem* (2.1)–(2.2) *with random initial data satisfying Condition* A, *and let* N *be an* n*-dimensional Gaussian vector with zero mean and unit covariance matrix. Then, there exists a* $\rho \in (0, \frac{1}{2})$ *such that*

$$\limsup_{t\to\infty}\left[\frac{t^{\alpha/2-\rho\alpha}}{L(t^{1/2-\rho})}\right]^{1/3}\mathscr{K}\left(\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}T_t u(t,a\sqrt{t}),\ N\right) \leqslant 32 v_1^{2/3} v_3^{1/3}, \tag{6.1}$$

where v_1 *is defined by* Eq. (3.15),

$$v_3 = v_3(n, \mu, \alpha) = 2c_3(\mu)(2\mu)^{-\alpha/2} Q_3(n, \mu, \alpha)$$

and

$$Q_3(n, \mu, \alpha) = \frac{F_1 + (n-1)F_2}{A_1 + (n-1)B_1} + (n-1)\frac{F_1 - F_2}{A_1 - B_1} \tag{6.2}$$

with A_1 *and* B_1 *defined by Eqs.* (3.5) *and* (3.6), *and*

$$F_i = (2\mu)^{-1-\alpha/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |w_1 z_i| \frac{\phi_n(z)\phi_n(w)}{|w-z|^\alpha} \, dz \, dw, \quad 0 < \alpha < n, \ \ i = 1, 2, \tag{6.3}$$

and $w = (w_1, \ldots, w_n)'$, $z = (z_1, \ldots, z_n)' \in \mathbb{R}^n$.

Proof. We follow the scheme of the proof of Theorem 3.1, including necessary modifications. In particular, we have Eqs. (5.5), (5.6) and (5.9). Let us introduce the sets

$$Y_1 = \{w \in \Delta(a,t),\ z \in \Delta(a,t);\ |w-z|\sqrt{2\mu t} \leqslant t^{1/2-\rho}\},$$
$$Y_2 = \{w \in \Delta(a,t),\ z \in \Delta(a,t);\ |w-z|\sqrt{2\mu t} > t^{1/2-\rho}\},$$

where $\rho \in (0, \frac{1}{2})$ is fixed.

We note that, for all $k \geqslant 2$,

$$\psi^2_{k,i,j}(t) \leqslant \tilde{\psi}^2_{k,i,j}(t), \qquad \frac{1}{k!}\tilde{\psi}^2_{k,i,j}(t) \leqslant \frac{1}{2}\tilde{\psi}^2_{2,i,j}(t),$$

where, for $k = 1, 2, \ldots,$

$$\psi^2_{k,i,j}(t) = \frac{2\mu k!}{t} \int_{\Delta(a,t)} \int_{\Delta(a,t)} w_i z_j \phi_n(w)\phi_n(z)\, B^k(|w-z|\sqrt{2\mu t})\, dw \, dz$$

and

$$\tilde{\psi}^2_{k,i,j}(t) = \frac{2\mu k!}{t} \int_{\Delta(a,t)} \int_{\Delta(a,t)} |w_i z_j| \phi_n(w)\phi_n(z)\, B^k(|w-z|\sqrt{2\mu t})\, dw \, dz.$$

Hence,

$$\begin{aligned}\tilde{\psi}^2_{2,i,j}(t) &= \frac{2\mu}{t}\left(\int\int_{Y_1} + \int\int_{Y_2}\right) |w_i z_j| \phi_n(w)\phi_n(z)\, B^2(|w-z|\sqrt{2\mu t})\, dw \, dz,\\ &= U_1(t) + U_2(t).\end{aligned}$$

On the set Y_1, we have $B^2(\,.\,) \leqslant 1$, and on the set Y_2,

$$B^2(|w-z|\sqrt{2\mu t}) \leqslant B(t^{1/2-\rho}) B(|w-z|\sqrt{2\mu t}),$$

such that

$$U_2(t) \leqslant \frac{L(\sqrt{t})}{t^{1+\alpha/2}} B(t^{1/2-\rho}) F_{i,j}(t), \tag{6.4}$$

where

$$F_{i,j}(t)=(2\mu)^{1-\alpha/2}\int\int_{Y_2}|w_iz_j|\phi_n(w)\phi_n(z)\,\frac{L(|w-z|\sqrt{2\mu t})}{|w-z|^{\alpha}L(\sqrt{t})}\,\mathrm{d}w\,\mathrm{d}z,\quad 0<\alpha<n. \tag{6.5}$$

Note, that

$$(2\mu)^{-2}\lim_{t\to\infty}F_{i,j}(t)=F_{i,j}=(2\mu)^{-1-\alpha/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|w_iz_j|\phi_n(w)\phi_n(z)\,\frac{1}{|w-z|^{\alpha}}\,\mathrm{d}w\,\mathrm{d}z$$

for $0<\alpha<n$, and $F_{1,1}(t)\neq F_{1,2}(t)$, $F_{1,1}(t)=F_{i,i}(t)$, $i=2,\dots,n$, $F_{1,2}(t)=F_{i,j}(t)$, $i,j=1,\dots,n,i\neq j$.

In view of Eq. (6.3), it is also clear that

$$F_1=F_{i,i},\ i=1,\dots,n,\quad F_2=F_{i,j},\ i,j=1,\dots,n,i\neq j,\quad F_1\neq F_2.$$

For $U_1(t)$, we have the estimate

$$U_1(t)\leqslant c_7\mathrm{e}^{-t/\mu}t^{n(1/2-\rho)},\quad c_7>0, \tag{6.6}$$

so that, for $0<\alpha<n$, instead of Eq. (5.11), we obtain

$$\begin{aligned} q &= \operatorname{tr}E[A_tT_tS_t]\,[A_tT_tS_t]' \\ &\leqslant B(t^{1/2-\rho})\left(\frac{1}{C_1^2}\sum_{k=2}^{\infty}\frac{C_k^2}{k!}\right)Q_{3,t}(n,\mu,\alpha)+\mathrm{o}(B(t^{1/2-\rho})), \end{aligned}$$

where

$$Q_{3,t}(n,\mu,\alpha)=(2\mu)^{-2}\left[\frac{F_{1,1}(t)+(n-1)F_{1,2}(t)}{A_1(t)+(n-1)B_1(t)}+(n-1)\frac{F_{1,1}(t)+F_{1,2}(t)}{A_1(t)+B_1(t)}\right],$$

$F_{1,1}(t)$ and $F_{1,2}(t)$ are defined by Eq. (6.5). It is clear that

$$\lim_{t\to\infty}Q_{3,t}(n,\mu,\alpha)=Q_3(n,\mu,\alpha),$$

where $Q_3(n,\mu,\alpha)$ is defined by Eq. (6.1). Now, instead of Eq. (5.12), Lemma 4.5 implies that, for $0<\alpha<n$,

$$\begin{aligned} &P\,(A_tT_tS_t\notin\Pi[-\varepsilon\delta h,\varepsilon\delta h]) \\ &\quad\leqslant\frac{1}{\varepsilon^2\delta^2}\left[B(t^{1/2-\rho})\frac{1}{(2\mu)^{\alpha/2}}c_3(\mu)Q_{3,t}(n,\mu,\alpha)+\mathrm{o}(B(t^{1/2-\rho}))\right]. \end{aligned} \tag{6.7}$$

Hence,

$$\begin{aligned} &P\,(Y_t\notin\Pi[-\varepsilon h,\varepsilon h]) \\ &\quad\leqslant\frac{1}{\varepsilon^2}\left[B(t^{1/2-\rho})\frac{1}{(2\mu)^{\alpha/2}}c_3(\mu)Q_{3,t}(n,\mu,\alpha)+(1-\delta)^{-2}R_t\right] \end{aligned}$$

with

$$R_t = \mathrm{o}(B(t^{1/2-\rho})) \tag{6.8}$$

and from Eq. (5.18) we have

$$P(|U_t - 1| > \varepsilon) \leqslant \frac{1}{\varepsilon^2}\tilde{R}_t, \quad \tilde{R}_t = \mathrm{o}(B(t^{1/2-\rho})). \tag{6.9}$$

To apply Lemma 4.4 to Eq. (5.5) in the case $0<\alpha<n$, and utilizing Eqs. (6.7)–(6.9), we obtain, instead of Eq. (5.24), the following estimate: for any $\varepsilon>0$,

$$\begin{aligned}&\mathscr{K}\left(\frac{t^{1/2-\alpha/4}}{L^{1/2}(\sqrt{t})}T_t u(t, a\sqrt{t}),\ N\right)\\ &\quad\leqslant v_1\varepsilon + \frac{1}{\varepsilon^2}B(t^{1/2-\rho})\left[\frac{c_3(\mu)Q_{3,t}(n,\mu,\alpha)}{\delta^2(2\mu)^{\alpha/2}} + \mathrm{o}(B(t^{1/2-\rho}))\right] + \varepsilon^n.\end{aligned} \tag{6.10}$$

In order to minimize the right-hand side of Eq. (6.10), set

$$\varepsilon = B^{1/3}(t^{1/2-\rho})v_1^{-1/3}(2\mu)^{-\alpha/6}\delta^{-2}(2c_3(\mu)Q_{3,t}(n,\mu,\alpha))^{1/3}.$$

Thus, we derive the following inequality:

$$\mathscr{K}\left(\frac{t^{1/2+\alpha/4}}{L^{1/2}(\sqrt{t})}T_t u(t, a\sqrt{t}),\ N\right) \leqslant \frac{L^{1/3}(t^{1/2-\rho})}{t^{\alpha/6-(\rho\alpha)/3}}\cdot\frac{3}{2}\cdot v_1^{2/3}[v_3(t)]^{1/3} + \mathrm{o}(B^{1/3}(t^{1/2-\rho})),$$

where

$$v_3(t) = \frac{2c_3(\mu)}{(2\mu)^{\alpha/2}\delta^2}Q_{3,t}(n,\mu,\alpha) \to v_3, \quad t\to\infty.$$

Theorem 6.1 follows directly from the last relationship. □

Remark 6.1. All the results of this paper can be extended, without much additional effort, to the more general parabolic limit (see, e.g., Albeverio et al., 1994; Surgailis and Woyczynski, 1994a, b)

$$\lim_{\beta\to\infty} M(\beta)u(t\beta, x\sqrt{\beta}), \quad t>0,\ x\in\mathbb{R}^n,$$

where $M(\beta)$ is an appropriate normalizing matrix-valued function. For $t=1$, this limit reduces to the limit considered in this paper, and we deliberately considered only the simpler case to make our presentation more readable.

The proposed method is also applicable in the case of non-Gaussian limit distributions (see, Albeverio et al., 1994; Surgailis and Woyczynski, 1994a, 1994b; Leonenko and Orsingher, 1995; Leonenko et al., 1994, and others), but additional modifications are needed (see, Remark 4.1). We will return to that problem in another paper.

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