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Stochastic differential equations driven by stable processes for which pathwise uniqueness fails☆

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Abstract

Let Z_t be a one-dimensional symmetric stable process of order α with $\alpha \in (0,2)$ and consider the stochastic differential equation

$$\mathrm{d}X_t = \phi(X_{t-})\,\mathrm{d}Z_t.$$

For $\beta < (1/\alpha) \wedge 1$, we show there exists a function ϕ that is bounded above and below by positive constants and which is Hölder continuous of order β but for which pathwise uniqueness of the stochastic differential equation does not hold. This result is sharp.

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1. Introduction

Let Z_t be a one-dimensional symmetric stable process of order α with $\alpha \in (0,2)$. In this paper, we are concerned with whether or not pathwise uniqueness holds for the stochastic differential equation

$$\mathrm{d}X_t = \phi(X_{t-})\,\mathrm{d}Z_t. \tag{1.1}$$

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In integrated form this can be written as

$$X_t = X_0 + \int_0^t \phi(X_{s-})\,\mathrm{d}Z_s. \tag{1.2}$$

For details concerning the stochastic calculus of processes with jumps, see Meyer (1976).

It is relatively straightforward, using Picard iteration, to show that if ϕ is Lipschitz, then the solution to (1.1) exists and is pathwise unique. If $\alpha > 1$, it was shown in Bass (2003) that if ϕ is bounded, has modulus of continuity ρ, and ρ satisfies

$$\int_{0+} \frac{1}{\rho(x)^\alpha}\,\mathrm{d}x = \infty, \tag{1.3}$$

then (1.1) admits a strong solution and the solution is pathwise unique. As an example, if ϕ is Hölder continuous of order $1/\alpha$, then (1.3) holds. Condition (1.3) is the exact analogue of the Yamada–Watanabe condition for stochastic differential equations driven by a Brownian motion. See also Komatsu (1982, Theorem 1).

Just as in the Brownian case, one can show that condition (1.3) is sharp. That is, if the integral is finite, one can find a continuous function ϕ having ρ as its modulus of continuity for which pathwise uniqueness for (1.1) does not hold; see Bass (2003). However, just as in the Brownian case, the examples in Bass (2003) showing sharpness are a bit unsatisfying: ϕ degenerates to 0 and not only does pathwise uniqueness fail, but one does not have uniqueness in law either. In Barlow (1982), for each $\beta < \frac{1}{2}$, Barlow constructed examples of nondegenerate (i.e., bounded away from 0 and infinity) functions ϕ that were Hölder continuous of order β, but for which pathwise uniqueness did not hold for the equation

$$\mathrm{d}X_t = \phi(X_t)\,\mathrm{d}B_t,$$

driven by a one-dimensional Brownian motion B_t.

Our main result in this paper is the extension of Barlow's theorem to the stable case. We prove:

Theorem 1.1. *Let $\alpha_0 = (1/\alpha) \wedge 1$. If $\beta < \alpha_0$, there exists ϕ that is bounded above and bounded below by strictly positive finite constants and such that ϕ is Hölder continuous of order β, but for which two distinct solutions to* (1.2) *exist.*

We see from (1.3) that the result in Theorem 1.1 is sharp as far as Hölder exponents go.

See Barlow (1982) for definitions of weak, strict, and strong solutions of SDEs, weak uniqueness and pathwise uniqueness, and for information about the implications between the existence of weak solutions, strong solutions, weak uniqueness and pathwise uniqueness. We just mention here that weak uniqueness and the existence of a strong solution imply pathwise uniqueness. It is well known that when ϕ is bounded between two strictly positive constants, a weak solution to (1.2) exists and its law is unique (cf. Proposition 3.3 of Bass (2003)). So Theorem 1.1 implies that no strong solution to (1.2) exists for the ϕ in Theorem 1.1. We do not pursue this here and refer the reader to Bass (2003) for further information.

In Section 3 of Bass (2003) it is asserted that if $\alpha < 1$, there is pathwise uniqueness for (1.2) if ϕ is bounded above and below by positive constants and ϕ is continuous. There is an error in the proof of Proposition 3.2 there—the argument that the strong solution constructed there is adapted is faulty. In fact, in view of Theorem 1.1 of the present paper, ϕ being bounded between two positive constants and only continuous is not sufficient for pathwise uniqueness.

A recent paper by Williams (2001) is also concerned with pathwise solutions for SDEs driven by Lévy processes. The paper (Williams, 2001), however, involves the Stratonovich stochastic integral rather than the Itô integral considered here. The reader might notice that our Theorem 1.1 seems to contradict Theorem 3 of Komatsu (1982). We were unable to follow the last line of the proof of Theorem 3 of Komatsu (1982), which appeals to a technique of Nakao (1972). It seems to us that the Lemma of Nakao (1972), which has the hypothesis that the processes have continuous paths, uses this hypothesis in an essential way. For example, if we let M be the difference of two independent Poisson processes of rate one and let $V=M$, then (7) of Komatsu (1982) is satisfied, but M is not identically zero.

Our method owes a great deal to Barlow's paper (Barlow, 1982), but because we are working with jump processes, there are also significant differences. We give a brief outline of our proof.

For $\varepsilon > 0$, we let $X_t(\varepsilon), Y_t(\varepsilon), Z_t(\varepsilon), Z_t'(\varepsilon)$ be processes such that $Z(\varepsilon)$ and $Z'(\varepsilon)$ are independent symmetric stable processes of order α and

$$\begin{aligned}
&\mathrm{d}X_t(\varepsilon) = \phi(X_{t-}(\varepsilon))\,\mathrm{d}Z_t(\varepsilon), \qquad X_0(\varepsilon) = x_0,\\
&\mathrm{d}Y_t(\varepsilon) = [\phi(X_{t-}(\varepsilon) + Y_{t-}(\varepsilon)) - \phi(X_{t-}(\varepsilon))]\,\mathrm{d}Z_t(\varepsilon) + \varepsilon\,\mathrm{d}Z_t'(\varepsilon), \qquad Y_0(\varepsilon) = 0.
\end{aligned} \tag{1.4}$$

Suppose we can show that as $\varepsilon \downarrow 0$, the joint law of $(X_t(\varepsilon), Y_t(\varepsilon), Z_t(\varepsilon), Z_t'(\varepsilon))$ has a weak limit (X_t, Y_t, Z_t, Z_t'), where Y_t is not identically zero. Then

$$X_t = x_0 + \int_0^t \phi(X_{s-})\,\mathrm{d}Z_s,$$

$$Y_t = \int_0^t [\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})]\,\mathrm{d}Z_s,$$

and so

$$X_t + Y_t = x_0 + \int_0^t \phi(X_{s-} + Y_{s-})\,\mathrm{d}Z_s.$$

Hence X_t and X_t+Y_t are distinct solutions to (1.1) and we have pathwise nonuniqueness.

Let $T_b^\varepsilon = \inf\{t : |Y_t(\varepsilon)| \geqslant b\}$. The main goal is to show that for some $b \leqslant \frac{1}{2}$ the quantity $\mathbb{E}T_b^\varepsilon$ is bounded uniformly in ε. Once we have that, we can argue as in the first part of Barlow's paper (Barlow, 1982) to show that $Y_t(\varepsilon)$ has a nonzero limit.

For notational convenience we will omit the ε from $Y_t(\varepsilon)$ and T_b^ε. Let $I_k=[2^{-k}, 2^{-k+1}]$ and $I_k^*=[2^{-k-1}, 2^{-k+2}]$. Roughly speaking, for some $b \leqslant \frac{1}{2}$, the strong Markov property tells us that the amount of time Y_t spends in $(0,b)$ up to time T_b is bounded by

$$\sum_k [\text{expected number of crossings by } Y_t \text{ from } I_k \text{ to } (I_k^*)^c \text{ before } T_b]$$

$$\times [\text{maximum expected time to exit } I_k^* \text{ from } I_k].$$

(See Section 5 for details.)

The proof is now reduced to finding estimates for the terms in the last sum. Assuming this is done, we can obtain a similar estimate for the time spent in $(-b,0)$, then we argue that no time is spent at 0, and thus we obtain a uniform bound on $\mathbb{E}T_b$. We will now give a few more details of this strategy.

To estimate the expected number of crossings from I_k to $(I_k^*)^c$ by Y_t, we observe that Y_t is a time change of a symmetric stable process, so this is the same as the expected number of crossings from I_k to $(I_k^*)^c$ by a symmetric stable process before time T_b. We estimate this using a bound for the Green function of a symmetric stable process on an interval.

The expected time for a symmetric stable process Z_t to exit I_k^* starting from a point in I_k is of order $(2^{-k})^\alpha$, by scaling. For a constant h, the expected length of time for hZ_t to exit I_k^* starting from I_k is the same as the expected length of time for Z_t to leave $(1/h)I_k^*$ starting from $(1/h)I_k$, which is of order $(2^{-k}/h)^\alpha$. For h we want to take $h=|\phi(x+y)-\phi(x)|$, because

$$\mathrm{d}Y_t = [\phi(X_{t-}+Y_{t-})-\phi(X_{t-})]\,\mathrm{d}Z_t.$$

To complete the argument, we would like to apply the above estimates with large h, but we cannot construct ϕ so that $|\phi(x+y)-\phi(x)|$ is large for all x and y. We can, however, construct it so this expression is large enough for many x's, and that turns out to be good enough.

In Section 2 we construct ϕ, while in Section 3 we estimate the number of crossings from the set I_k to the complement of I_k^*. Section 4 is where the estimate on the expected time for Y_t to leave an interval is given, and all the parts of the proof are put together in Section 5.

We use the letter c with subscripts to denote strictly positive finite constants whose exact value is unimportant. For a process V_t that is right continuous with left limits, we denote the left limit at t by V_{t-} and the jump at time t by ΔV_t.

2. Constructing ϕ

Fix any $\gamma\in(0,1)$. Let $\bar\psi$ be the piecewise linear function on $[0,1]$ such that $\bar\psi(0)=\bar\psi(1)=0$ and $\bar\psi\left(\frac12\right)=1$; that is,

$$\bar\psi(x)=\begin{cases}2x, & 0\leqslant x\leqslant \frac12,\\ 2-2x, & \frac12\leqslant x\leqslant 1.\end{cases}$$

Define $\psi_0:\mathbb{R}\to[0,1]$ by $\psi_0(x)=\bar\psi(x-[x])$, where $[x]$ is the integer part of x. Note that ψ_0 is periodic with period 1 and agrees with $\bar\psi$ on $[0,1]$.

Set

$$\psi_n(x) = \psi_0(2^n x) \quad \text{and} \quad \phi(x) = 1 + \sum_{n=0}^{\infty} 2^{-\gamma n}\psi_n(x).$$

Note that the function ϕ is bounded and bounded away from 0 because $\psi_n(x) \geqslant 0$ and $\sum_{n=0}^{\infty} 2^{-\gamma n} < \infty$.

The family of functions which are Hölder continuous with exponent η will be denoted C^η. We will first show that $\phi \notin C^{\gamma+\varepsilon}$ for any $\varepsilon > 0$. We have

$$\phi(2^{-n-1}) - \phi(0) \geqslant 2^{-\gamma n}[\psi_n(2^{-n-1}) - \psi_n(0)] = 2^{-n\gamma}. \tag{2.1}$$

Then

$$\frac{|\phi(2^{-n-1}) - \phi(0)|}{(2^{-n-1})^{\gamma+\varepsilon}} \geqslant 2^{\gamma+\varepsilon+n\varepsilon},$$

which will surpass any positive constant if n is large enough.

Proposition 2.1. *If* $0 < \zeta < \gamma$, *then* $\phi \in C^\zeta$.

Proof. Since $\sum_{n=1}^{\infty} 2^{-\gamma n}$ is summable, ϕ is bounded. It is easy to see that

$$|\psi_n(x) - \psi_n(y)|/|x-y|^\zeta$$

cannot surpass the maximum value of $\psi_n(z)/z^\zeta$ for $z \in (0, 2^{-n-1}]$. Since $\psi_n'(z)$ equals 2^{n+1} for such z,

$$\frac{\psi_n(z)}{z^\zeta} = 2^{n+1} z^{1-\zeta} \leqslant 2^{(n+1)\zeta}.$$

Therefore,

$$|\psi_n(x) - \psi_n(y)| \leqslant 2^{(n+1)\zeta}|x-y|^\zeta,$$

and then

$$|\phi(x) - \phi(y)| \leqslant \sum_{n=0}^{\infty} 2^{-\gamma n} 2^{(n+1)\zeta}|x-y|^\zeta \leqslant c_1\,|x-y|^\zeta,$$

since $\gamma > \zeta$. □

Let $|A|$ denote the Lebesgue measure of a Borel set A in $\mathbb{R}$. Let $I_k^* = [2^{-k-1}, 2^{-k+2}]$ and

$$A_k(\theta) = \{x : |\phi(x+y) - \phi(x)| > \theta 2^{-k\gamma} \quad \text{for all } \ y \in I_k^*\}.$$

Proposition 2.2. *There are positive constants* k_0, θ, L, *and* δ *such that if* J *is an interval of length larger than* $L2^{-k}$, *then* $|J \cap A_k(\theta)| \geqslant \delta|J|$ *for all* $k \geqslant k_0$.

Proof. Let $k \geqslant 5$, $r = 2^{-k}$, $n = k - 5$, and j_0 a positive integer to be fixed later on. Since ψ_n has slope 2^{n+1} on $[0, 2^{-n-1}]$, we have for $y \in I_k^*$ and $0 \leqslant x \leqslant r/16$

$$2^{-\gamma n}\psi_n(x+y) - 2^{-\gamma n}\psi_n(x) = 2^{-\gamma n}2^{n+1}y \geqslant 2^{-\gamma n}2^{n+1}2^{-k-1} = c_1 r^{\gamma}, \tag{2.2}$$

where $c_1 = 2^{-5(1-\gamma)}$. If $0 \leqslant j < n$, the slope of ψ_j is positive on $[0, 2^{-n}]$, so

$$\psi_j(x+y) - \psi_j(x) \geqslant 0 \tag{2.3}$$

if $0 \leqslant x \leqslant r/16$ and $y \in I_k^*$. Next we see that

$$\left|\sum_{l=n+j_0}^{\infty} 2^{-\gamma l}\psi_l(x)\right| \leqslant \sum_{l=n+j_0}^{\infty} 2^{-l\gamma} = \frac{2^{-\gamma(n+j_0)}}{1-2^{-\gamma}} = \frac{2^{-\gamma(j_0-5)}}{1-2^{-\gamma}} r^{\gamma}.$$

Provided j_0 is chosen large enough,

$$\left|\sum_{l=n+j_0}^{\infty} 2^{-\gamma l}\psi_l(x)\right| \leqslant c_1 r^{\gamma}/4. \tag{2.4}$$

The derivative of ψ_l is bounded by 2^{l+1}, so if $y \in I_k^*$,

$$\begin{aligned} |2^{-\gamma l}\psi_l(x+y) - 2^{-\gamma l}\psi_l(x)| &\leqslant 2^{-\gamma l + l + 1}|y| \\ &\leqslant 2^{(1-\gamma)l+3} r^{1-\gamma} r^{\gamma} \leqslant 2^{l(1-\gamma)+3} 2^{-k(1-\gamma)} r^{\gamma}. \end{aligned} \tag{2.5}$$

So if j_0 is chosen to be sufficiently large and $n > j_0$,

$$\begin{aligned} \sum_{l=1}^{n-j_0-1} |2^{-\gamma l}\psi_l(x+y) - 2^{-\gamma l}\psi_l(x)| &\leqslant \frac{2^{(n-j_0-1)(1-\gamma)+3}}{2^{1-\gamma}-1} 2^{-k(1-\gamma)} r^{\gamma} \\ &\leqslant c_1 r^{\gamma}/4. \end{aligned} \tag{2.6}$$

If $0 \leqslant \varepsilon \leqslant \frac{1}{16}$ and $0 \leqslant x \leqslant \varepsilon r$, then since $|\psi_l'|$ is bounded by 2^{l+1},

$$\begin{aligned} \left|\sum_{l=n+1}^{n+j_0-1} 2^{-\gamma l}\psi_l(x)\right| &\leqslant \sum_{l=n+1}^{n+j_0-1} 2^{l(1-\gamma)+1} x \\ &= \frac{2^{(n+j_0)(1-\gamma)+1} - 2^{(n+1)(1-\gamma)+1}}{2^{1-\gamma}-1} x \\ &\leqslant (2^{1-\gamma}-1)^{-1} \varepsilon 2^{n(1-\gamma)} 2^{j_0(1-\gamma)+1} r^{\gamma} r^{1-\gamma} \\ &\leqslant c_2 \varepsilon 2^{j_0(1-\gamma)} r^{\gamma}. \end{aligned} \tag{2.7}$$

Choose j_0 so that (2.4) and (2.6) hold, and then choose $\varepsilon < \frac{1}{16}$ small so that (2.7) implies

$$\left|\sum_{l=n+1}^{n+j_0-1} 2^{-\gamma l}\psi_l(x)\right| \leqslant c_1 r^{\gamma}/4. \tag{2.8}$$

Since $2^{-\gamma l}\psi_l(x+y) \geqslant 0$ for all l, combining (2.4) and (2.8),

$$\sum_{l=n+1}^{\infty} [2^{-\gamma l}\psi_l(x+y) - 2^{-\gamma l}\psi_l(x)] \geqslant -c_1 r^{\gamma}/2 \tag{2.9}$$

if $x \in (0, \varepsilon r]$ and $y \in I_k^*$. Let

$$\tilde{\phi}(x) = \sum_{l=n-j_0}^{\infty} 2^{-\gamma l} \psi_l(x).$$

We obtain from (2.2), (2.3) and (2.9),

$$\tilde{\phi}(x+y) - \tilde{\phi}(x) \geqslant c_1 r^{\gamma}/2 \tag{2.10}$$

if $x \in (0, \varepsilon r]$ and $y \in I_k^*$.

The function $\tilde{\phi}$ is periodic with period $2^{-(n-j_0)}$. So if J is an interval of length at least $2^{-(n-j_0)+1}$, then

$$|\{x \in J : \tilde{\phi}(x+y) - \tilde{\phi}(x) \geqslant c_1 r^{\gamma}/2 \text{ for all } y \in I_k^*\}| \geqslant \varepsilon 2^{-j_0-5}|J|. \tag{2.11}$$

Using (2.6), if $|J| \geqslant 2^{-(n-j_0)+1}$, then

$$|\{x \in J : \phi(x+y) - \phi(x) \geqslant c_1 r^{\gamma}/4 \text{ for all } y \in I_k^*\}| \geqslant \varepsilon 2^{-j_0-5}|J|.$$

This implies the proposition with $k_0 = j_0 + 6$, $\theta = c_1/4$, $L = 2^{j_0+6}$, and $\delta = \varepsilon 2^{-j_0-5}$. □

3. Expected number of crossings

On a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$, a real-valued stochastic process X_t is said to be an $\{\mathscr{F}_t\}$-adapted one-dimensional symmetric stable process of order $\alpha \in (0,2)$ if for every $\lambda \in \mathbb{R}$, $t > 0$ and $s > 0$,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\lambda(X_{t+s}-X_s)}|\mathscr{F}_s] = \mathrm{e}^{-t|\lambda|^{\alpha}}.$$

In other words, for every $s > 0$, process $t \mapsto X_{t+s} - X_s$ is independent of $\mathscr{F}_s$ and is a symmetric α-stable process starting from the origin.

In this section, ϕ is a continuous function on $\mathbb{R}$ that is bounded between two strictly positive constants.

Proposition 3.1. *For each $\varepsilon > 0$, $x_0, y_0 \in \mathbb{R}$, there exists a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with processes X_t, Y_t, Z_t, Z_t', such that Z_t and Z_t' are independent one-dimensional $\{\mathscr{F}_t\}$-adapted symmetric stable processes of order α,*

$$X_t = x_0 + \int_0^t \phi(X_{s-})\,\mathrm{d}Z_s, \tag{3.1}$$

and

$$Y_t = y_0 + \int_0^t [\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})]\,\mathrm{d}Z_s + \varepsilon Z_t'. \tag{3.2}$$

Proof. Using the substitution $K_t = X_t + Y_t$, it is easy to see that the Eqs. (3.1)–(3.2) are equivalent to the following two equations:

$$\begin{aligned} X_t &= x_0 + \int_0^t \phi(X_{s-})\,\mathrm{d}Z_s, \\ K_t &= x_0 + y_0 + \int_0^t \phi(K_{s-})\,\mathrm{d}Z_s + \varepsilon Z_t'. \end{aligned} \tag{3.3}$$

The idea of the proof of weak existence for (3.3)–(3.4) is standard; cf. Bass (1988), Section 3. We take smooth ϕ_n which converge uniformly to ϕ on compact intervals and find (unique) solutions to

$$dX_t^n = \phi_n(X_{t-}^n)\,d\tilde{Z}_t, \qquad X_0^n = x_0,$$

$$dK_t^n = \phi_n(K_{t-}^n)\,d\tilde{Z}_t + \varepsilon\,d\tilde{Z}_t', \qquad K_0^n = x_0 + y_0,$$

where $\tilde{Z}_t$ and $\tilde{Z}_t'$ are independent one-dimensional symmetric α-stable processes. It is routine to show tightness and also routine to show that a weak subsequential limit (X_t, K_t, Z, Z') of $(X_t^n, K_t^n, \tilde{Z}, \tilde{Z}')$ satisfies (3.3)–(3.4), where $\tilde{Z}, \tilde{Z}'$ are independent symmetric α-stable processes. Then if we take $Y_t = K_t - X_t$, we see that (X_t, Y_t, Z, Z') solves (3.1)–(3.2). $\square$

Proposition 3.2. *Let (X_t, Y_t) be a weak solution of (3.1)–(3.2). Define*

$$A_t = \int_0^t (|\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})|^\alpha + \varepsilon^\alpha)\,ds$$

and $\sigma_t = \inf\{s \geqslant 0 : A_s > t\}$ for $t \geqslant 0$. Then $W_t = Y_{\sigma_t}$ is a symmetric α-stable process starting from y_0.

Proof. The proof is a straightforward modification of arguments used in Proposition 3.1 and Theorem 3.1 of Rosiński and Woyczyński (1986). $\square$

Recall that

$$I_k = [2^{-k}, 2^{-k+1}], \qquad I_k^* = [2^{-k-1}, 2^{-k+2}].$$

Let $R_1^Y = \inf\{t : Y_t \in I_k\}$,

$$S_i^Y = \inf\{t > R_i^Y : Y_t \notin I_k^*\} \quad \text{and} \quad R_{i+1}^Y = \inf\{t > S_i^Y : Y_t \in I_k\} \quad \text{for } i \geqslant 1.$$

Let

$$N_k^Y(t) = \sup\{j : R_j^Y \leqslant t\},$$

the number of crossings from I_k to $(I_k^*)^c$. Let $T_b^Y = \inf\{t : |Y_t| \geqslant b\}$. Recall that $W_t = Y_{\sigma_t}$ and define R_i^W, S_i^W, N_k^W, and T_b^W analogously, but in terms of W instead of Y.

Proposition 3.3. *For $b > 0$, there exists $c_1 = c_1(b) > 0$ such that*

$$\mathbb{E}N_k^W(T_b^W) \leqslant \begin{cases} c_1 2^{k(\alpha-1)}, & \alpha > 1, \\ c_1 k, & \alpha = 1, \\ c_1, & \alpha < 1. \end{cases} \tag{3.4}$$

Proof. We drop the superscripts W from the notation. Let τ_k be the first exit from I_k^* by W_t. Since $W_t \in I_k^*$ when $R_i < t < S_i$, by the strong Markov property,

$$\mathbb{E}\int_0^{T_b} 1_{I_k^*}(W_s)\,ds \geqslant \sum_{i=1}^{\infty} \mathbb{E}(S_i \wedge T_b - R_i \wedge T_b)$$

$$= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{E}^{W(R_i \wedge T_b)} S_1; R_i < T_b]$$

$$\geqslant [\mathbb{E}(N_k(T_b) - 1] \left[\inf_{x \in I_k} \mathbb{E}^x \tau_k \right]. \tag{3.5}$$

Let $U_k = \inf\{t : |W_t - W_0| \geqslant 2^{-k-3}\}$, the time for W_t to move a distance at least 2^{-k-3}. If $x \in I_k$, $\mathbb{E}^x \tau_k \geqslant \mathbb{E}^x U_k = \mathbb{E}^0 U_k$. By scaling,

$$\mathbb{E}^0 U_k = c_2 (2^{-k})^\alpha \mathbb{E}^0 U_0 = c_3 2^{-k\alpha}.$$

Combining with (3.6) we have

$$\mathbb{E} N_k(T_b) \leqslant 1 + c_3 2^{k\alpha} \mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s)\, \mathrm{d}s. \tag{3.6}$$

Suppose $\alpha > 1$. The Green function for W_t killed on exiting $[-b, b]$ is bounded (see Corollary 4 of Blumenthal et al., 1961), so

$$\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s)\, \mathrm{d}s \leqslant c_4 |I_k^*| \leqslant c_5 2^{-k}.$$

If $\alpha = 1$, the Green function is bounded by $c_6 \log(1/|x|)$ (again see Blumenthal et al., 1961), and then

$$\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s)\, \mathrm{d}s \leqslant c_6 \int_{I_k^*} \log(1/|x|)\, \mathrm{d}x$$

$$= c_6 \int_{2^{-k-1}}^{2^{-k+2}} \log(1/|x|)\, \mathrm{d}x \leqslant c_7 k 2^{-k}.$$

Finally, if $\alpha < 1$, the Green function is bounded by $c_8 |x|^{\alpha-1}$; see Blumenthal et al. (1961). In this case

$$\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s)\, \mathrm{d}s \leqslant c_9 \int_{I_k^*} |x|^{\alpha-1}\, \mathrm{d}x \leqslant c_{10} 2^{-k\alpha}.$$

If we substitute the appropriate estimate for $\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s)\, \mathrm{d}s$ into (3.7), we obtain the proposition. □

Corollary 3.4. *For $b > 0$, there exists $c_1 = c_1(b) > 0$ such that*

$$\mathbb{E} N_k^Y(T_b^Y) \leqslant \begin{cases} c_1 2^{k(\alpha-1)}, & \alpha > 1, \\ c_1 k, & \alpha = 1, \\ c_1, & \alpha < 1. \end{cases} \tag{3.7}$$

Proof. This follows from Proposition 3.3 and the fact that Y is a nondegenerate time change of W (see Proposition 3.2). □

4. Expected time to leave an interval

Let (X_t, Y_t) be a weak solution of (3.1)–(3.2). We want an estimate on $\mathbb{E}\tau_k$, where $\tau_k = \inf\{t : Y_t \notin I_k^*\}$ (note that here τ_k is defined in terms of Y_t). Let $\alpha_0 = \frac{1}{\alpha} \wedge 1$, choose any $\beta < \alpha_0$, and then fix any $\gamma \in (\beta, \alpha_0)$. Construct ϕ as in Section 2, and let k_0, θ, L, and δ be as in the statement of Proposition 2.2.

Fix $k \geqslant k_0$. For simplicity write r for 2^{-k} and set $t_0 = r^{\alpha(1-\gamma)}$. Recall the definition of $A_k(\theta)$ from Section 2. Let

$$C_t = \sum_{s \leqslant t} 1_{\{|\Delta Z_s| > 8\theta^{-1} r^{1-\gamma}\}} 1_{\{X_{s-} \in A_k(\theta)\}}. \tag{4.1}$$

Lemma 4.1. *There is a constant $c_1 > 0$, independent of $k \geqslant k_0$ and such that $\mathbb{E}C_{t_0} \geqslant c_1$.*

Proof. Recall that the symmetric α-stable process Z has Lévy kernel $c(\alpha)/|z|^{1+\alpha}$ for some $c(\alpha) > 0$; see (Bertoin, 1996, p. 13). The process

$$V_t = \sum_{s \leqslant t} 1_{(|\Delta Z_s| > 8\theta^{-1} r^{1-\gamma})} \tag{4.2}$$

is a Poisson process with parameter $c(\alpha)\alpha^{-1}(8\theta^{-1}r^{1-\gamma})^{-\alpha}$ (cf. Bertoin, 1996). Since Z_t is an $\{\mathscr{F}_t\}$-adapted symmetric α-stable process, it follows that $M_t = V_t - c(\alpha)\alpha^{-1}(8\theta^{-1}r^{1-\gamma})^{-\alpha}t$ is a purely discontinuous square integrable martingale with respect to $\{\mathscr{F}_t\}$ (note that this filtration is larger than the natural filtration generated by M_t). Hence the stochastic integral $\int_0^t 1_{A_k(\theta)}(X_{s-})\,\mathrm{d}M_s$ is also a square integrable martingale with respect to $\{\mathscr{F}_t\}$. It follows that

$$\begin{aligned}\mathbb{E}C_t &= \mathbb{E}\int_0^t 1_{A_k(\theta)}(X_{s-})\,\mathrm{d}V_s \\ &= c_2\mathbb{E}\int_0^t 1_{A_k(\theta)}(X_{s-})r^{-(1-\gamma)\alpha}\,\mathrm{d}s \\ &= c_2 r^{-\alpha(1-\gamma)}\mathbb{E}\int_0^t 1_{A_k(\theta)}(X_s)\,\mathrm{d}s.\end{aligned} \tag{4.3}$$

In the last equality we used the fact that $X_{s-} = X_s$ for all but countably many s's.

Since $X_t = x_0 + \int_0^t \phi(X_{s-})\,\mathrm{d}Z_s$ and ϕ is bounded between two positive numbers, by Theorem 3.1 of Rosiński and Woyczyński (1986), $W_t = X_{\sigma_t}$ is a symmetric α-stable process starting from x_0, where

$$\sigma_t = \inf\left\{s \geqslant 0 : \int_0^s \phi(X_{u-})^\alpha\,\mathrm{d}u > t\right\}.$$

Note that $\mathrm{d}\sigma_t/\mathrm{d}t$ is bounded between two positive constants since ϕ is. Therefore, for some c_3 and c_4,

$$\mathbb{E}\int_0^{t_0} 1_{A_k(\theta)}(X_s)\,\mathrm{d}s = \mathbb{E}\int_0^{\sigma^{-1}(t_0)} 1_{A_k(\theta)}(W_t)\frac{\mathrm{d}\sigma_t}{\mathrm{d}t}\,\mathrm{d}t$$

$$\geqslant c_3 \mathbb{E} \int_0^{c_4 t_0} 1_{A_k(\theta)}(W_t)\, dt$$

$$\geqslant c_3 \mathbb{E} \int_{c_4 t_0/2}^{c_4 t_0} 1_{A_k(\theta)}(W_t)\, dt, \tag{4.4}$$

where we used the change of variables $s = \sigma_t$ in the first line. If $p_s(x, y)$ is the transition density for a symmetric stable process of order α, then there exists (see Proposition 3.1 of Kolokoltsov, 2000) $c_5 > 0$ such that

$$p_s(x, y) \geqslant c_5 t_0^{-1/\alpha} \quad \text{for } s \in [c_4 t_0/2, c_4 t_0] \quad \text{and} \quad |y - x| \in [-L t_0^{1/\alpha}, L t_0^{1/\alpha}]. \tag{4.5}$$

Let $J = [x_0 - L t_0^{1/\alpha}, x_0 + L t_0^{1/\alpha}]$. Putting (4.4) and (4.5) together and using Proposition 2.2,

$$\mathbb{E} \int_0^{t_0} 1_{A_k(\theta)}(X_s)\, ds \geqslant c_6 t_0^{1-(1/\alpha)} |A_k(\theta) \cap J|$$

$$\geqslant c_7 t_0^{1-(1/\alpha)} \delta |J| = 2 c_7 t_0^{1-(1/\alpha)} \delta L t_0^{1/\alpha}$$

$$\geqslant c_8 t_0.$$

Therefore, using (4.3),

$$\mathbb{E} C_{t_0} \geqslant c_9\, t_0 r^{-(1-\gamma)\alpha} = c_9. \qquad \square$$

Proposition 4.2. *There exists $c_1 \geqslant 0$ not depending on $x_0, y_0 \in \mathbb{R}$, $k \geqslant k_0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(C_{t_0} \geqslant 1) \geqslant c_1$.*

Proof. With V_t defined as in (4.2), we have $C_t \leqslant V_t$, and as V_t is a Poisson process with parameter $c_2 r^{-\alpha(1-\gamma)}$,

$$\mathbb{E} V_{t_0}^2 = (c_2 t_0 r^{-\alpha(1-\gamma)})^2 + c_2 t_0 r^{-\alpha(1-\gamma)} = c_2^2 + c_2 = c_3.$$

By Lemma 4.1,

$$c_4 \leqslant \mathbb{E} C_{t_0} = \mathbb{E}[C_{t_0}; C_{t_0} \geqslant 1] \leqslant \mathbb{E}[V_{t_0}; C_{t_0} \geqslant 1]$$

$$\leqslant (\mathbb{E} V_{t_0}^2)^{1/2} (\mathbb{P}(C_{t_0} \geqslant 1))^{1/2} = c_3^{1/2} (\mathbb{P}(C_{t_0} \geqslant 1))^{1/2}.$$

Rearranging yields the result. $\square$

Recall that $k \geqslant k_0$ and $\tau_k = \inf\{t : Y_t \notin I_k^*\}$.

Proposition 4.3. *There exists $c_1 \geqslant 0$, not depending on k and $\varepsilon \in (0, 1)$, such that for every starting point (x_0, y_0) for (X_t, Y_t) in (3.1)–(3.2), $\mathbb{E}\tau_k \leqslant c_1 (2^{-k})^{\alpha(1-\gamma)}$.*

Proof. Let $U_s = \inf\{t > s : C_t \geqslant 1\}$. Then

$$\mathbb{P}(U_0 > m t_0) \leqslant \mathbb{P}(U_{(m-1)t_0} > m t_0, U_0 > (m-1) t_0)$$

$$= \mathbb{E}[\mathbb{P}(U_{(m-1)t_0} > m t_0 | \mathscr{F}_{(m-1)t_0}); U_0 > (m-1) t_0]. \tag{4.6}$$

The conditional law of $(X_{t+(m-1)t_0}, Y_{t+(m-1)t_0})$ given $\mathscr{F}_{(m-1)t_0}$ solves an SDE of the same form as (3.1) and (3.2); cf. Bass (1988), proof of Proposition 3.2. This and Proposition 4.2 give

$$\mathbb{P}(C_{mt_0} - C_{(m-1)t_0} \geqslant 1 | \mathscr{F}_{(m-1)t_0}) \geqslant c_2. \tag{4.7}$$

Inequality (4.7) implies

$$\mathbb{P}(U_{(m-1)t_0} > mt_0 | \mathscr{F}_{(m-1)t_0}) \leqslant 1 - c_2.$$

Substituting this in (4.6),

$$\mathbb{P}(U_0 > mt_0) \leqslant (1 - c_2)\mathbb{P}(U_0 > (m-1)t_0).$$

Using induction, $\mathbb{P}(U_0 > mt_0) \leqslant (1 - c_2)^m$, and from this it follows that

$$\mathbb{E}U_0 \leqslant c_3 t_0 = c_4(2^{-k})^{\alpha(1-\gamma)}. \tag{4.8}$$

Recall that the probability that Z and Z' jump at the same time is 0. At time U_0 the process C has a jump so $\Delta Z_{U_0} > 8\theta^{-1} r^{1-\gamma}$ and $X_{U_0-} \in A_k(\theta)$. Had Y_t not exited I_k^* by that time, then $\phi(X_{U_0-} + Y_{U_0-}) - \phi(X_{U_0-}) > \theta r^{\gamma}$ (by the definition of $A_k(\theta)$), and therefore Y would have had a jump of size at least $(8\theta^{-1} r^{1-\gamma})(\theta r^{\gamma}) = 8r$. This would have meant that $Y_{U_0} \notin I_k^*$. We have thus shown that $\tau_k \leqslant U_0$. This combined with (4.7) completes the proof. □

5. Pathwise nonuniqueness

It follows from Proposition 3.1 that for each $i \geqslant 1$, there exists a filtered probability space $(\Omega^{(i)}, \mathscr{F}^{(i)}, \{\mathscr{F}_t^{(i)}\}, \mathbb{P}^{(i)})$ and processes $X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}, Z_{t'}^{(i)}$ such that $Z_t^{(i)}$ and $Z_{t'}^{(i)}$ are independent $\{\mathscr{F}_t^{(i)}\}$-adapted symmetric stable processes of order α,

$$X_t^{(i)} = x_0 + \int_0^t \phi(X_{s-}^{(i)})\,\mathrm{d}Z_s^{(i)}, \tag{5.1}$$

and

$$Y_t^{(i)} = \int_0^t [\phi(X_{s-}^{(i)} + Y_{s-}^{(i)}) - \phi(X_{s-}^{(i)})]\,\mathrm{d}Z_s^{(i)} + \frac{1}{i} Z_{t'}^{(i)}. \tag{5.2}$$

Let $T_b^i = \inf\{t : |Y_t^{(i)}| \geqslant b\}$ and define $N_k^{Y^{(i)}}(t)$ analogously to $N_k^Y(t)$ in Section 3.

Proposition 5.1. *Let k_0 be as in Proposition 2.2 and $b = 2^{-k_0}$. If $k \geqslant k_0$, then*

$$\mathbb{E}\int_0^{T_b^i} 1_{I_k}(Y_s^{(i)})\,\mathrm{d}s \leqslant c_1(2^{-k})^{\alpha(1-\gamma)}\mathbb{E}N_k^{Y^{(i)}}(T_b^i), \tag{5.3}$$

where c_1 is independent of k.

Proof. We drop the (i)'s from the notation. Suppose R is any finite stopping time. The conditional law of (X_t, Y_t) given $\mathscr{F}_R$ is again a solution to

$$\mathrm{d}X_t = \phi(X_{t-})\,\mathrm{d}Z_t,$$

$$\mathrm{d}Y_t = [\phi(X_{t-} + Y_{t-}) - \phi(X_{t-})]\,\mathrm{d}Z_t + \frac{1}{i}\,\mathrm{d}Z_t', \tag{5.4}$$

starting from (X_R, Y_R). So the argument of Section 4 shows that the expected amount of time for Y_t to leave I_k^* after time R is again bounded by $c_2(2^{-k})^{\alpha(1-\gamma)}$ (see Proposition 4.3). Let $R_j = R_j^Y$, $S_j = S_j^Y$ be defined as in Section 3. Then

$$\begin{aligned}
\mathbb{E}\int_0^{T_b} 1_{I_k}(Y_s)\,\mathrm{d}s &\leqslant \sum_{j=1}^{\infty} \mathbb{E}(S_j \wedge T_b - R_j \wedge T_b) \\
&= \sum_{j=1}^{\infty} \mathbb{E}[\mathbb{E}[(S_j \wedge T_b - R_j \wedge T_b|\mathscr{F}_{R_j\wedge T_b}]; R_j < T_b] \\
&\leqslant \sum_{j=1}^{\infty} c_2(2^{-k})^{\alpha(1-\gamma)}\mathbb{E}1_{(R_j<T_b)} \\
&= c_2(2^{-k})^{\alpha(1-\gamma)}\mathbb{E}\sum_{j=1}^{\infty} 1_{(R_j<T_b)} \\
&= c_2(2^{-k})^{\alpha(1-\gamma)}\mathbb{E}N_k^Y(T_b). \qquad \square
\end{aligned}$$

Recall that $\alpha_0 = (1/\alpha) \wedge 1$, $\beta < \alpha_0$, $\gamma \in (\beta, \alpha_0)$, and $b = 2^{-k_0}$, where k_0 is given in Proposition 2.2.

Theorem 5.2. *There exists c_1 such that $\mathbb{E}T_b^i \leqslant c_1$ for all $i \geqslant 1$.*

Proof. By Proposition 5.1,

$$\begin{aligned}
\mathbb{E}\int_0^{T_b^i} 1_{(0,b)}(Y_s^{(i)})\,\mathrm{d}s &\leqslant \sum_{k=1}^{\infty} \mathbb{E}\int_0^{T_b^i} 1_{I_k}(Y_s^{(i)})\,\mathrm{d}s \\
&\leqslant \sum_{k=1}^{\infty} c_2(2^{-k})^{\alpha(1-\gamma)}\mathbb{E}N_k^{Y^{(i)}}(T_b^i). \qquad (5.5)
\end{aligned}$$

If $\alpha > 1$, then by Corollary 3.4, the right-hand side is bounded by

$$\sum_{k=1}^{\infty} c_2(2^{-k})^{\alpha(1-\gamma)} c_3(2^{-k})^{1-\alpha}.$$

As $\alpha(1-\gamma) + (1-\alpha) = 1 - \alpha\gamma > 0$, this is summable, and we have

$$\mathbb{E}\int_0^{T_b^i} 1_{(0,b)}(Y_s^{(i)})\,\mathrm{d}s \leqslant c_4.$$

If $\alpha \leqslant 1$, then by Corollary 3.4 the right-hand side of (5.5) is bounded by

$$\sum_{k=1}^{\infty} c_2(2^{-k})^{\alpha(1-\gamma)} c_5 k \quad \text{or} \quad \sum_{k=1}^{\infty} c_2(2^{-k})^{\alpha(1-\gamma)} c_5.$$

In either case, as $\gamma < 1$, we have $\alpha(1-\gamma) > 0$, and both series are summable.

The same arguments with only cosmetic changes imply that

$$\mathbb{E}\int_0^{T_b^i} 1_{(-b,0)}(Y_s^{(i)})\,\mathrm{d}s \leqslant c_4$$

with c_4 independent of i. Since the expected amount of time a symmetric stable process of order α spends at 0 is 0 and $Y_t^{(i)}$ is a nondegenerate time change of a symmetric stable process, then $Y_t^{(i)}$ spends 0 time at 0. That is,

$$\mathbb{E}\int_0^{T_b^i} 1_{\{0\}}(Y_s^{(i)})\,\mathrm{d}s = 0.$$

Combining, we have our theorem. □

Proof Theorem 1.1. It is routine (Bass, 1988, Section 3) to see that the quadruples of processes $(X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}, Z_{t'}^{(i)})$ are tight and any subsequential limit point (X_t, Y_t, Z_t, Z_t') under weak convergence will satisfy (3.1)–(3.2) with $y_0 = 0$ and $\varepsilon = 0$ there. By Theorem 5.2, $\mathbb{E}T_b^i \leqslant c_1$. We have that X_t satisfies (1.2) and so does $X_t + Y_t$. We have for $t_1 > 0$,

$$\mathbb{P}\left(\sup_{s\leqslant t_1}|Y_s| \leqslant b\right) \leqslant \limsup_i \mathbb{P}\left(\sup_{s\leqslant t_1}|Y_s^{(i)}| \leqslant b\right)$$
$$\leqslant \limsup_i \mathbb{P}(T_b^i \geqslant t_1) \leqslant \limsup_i \frac{\mathbb{E}T_b^i}{t_1}.$$

If we set $t_1 = 2c_1$, then the right side is less than $\frac{1}{2}$, which proves that with probability at least $\frac{1}{2}$, we have $\sup_{s\leqslant t_1}|Y_s| \geqslant b$. Therefore, our two solutions X_t and $X_t + Y_t$ are not identically equal and pathwise uniqueness fails. □

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