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On the joint distribution of surplus before and after ruin under a Markovian regime switching model

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Abstract

We consider a Markovian regime switching insurance risk model (also called Markov-modulated risk model). The closed form solutions for the joint distribution of surplus before and after ruin when the initial surplus is zero or when the claim size distributions are phase-type distributed are obtained.

Keywords: Markovian regime switching model; Ruin theory; Phase-type distribution; Expected discounted penalty function; Coupled system of integro-differential equations

1. Introduction

In the classical insurance risk model, a compound Poisson process is often used to model the surplus process. There is a huge amount of literature devoted to the generalization of the classical model in different ways. For more detailed discussions, see Gerber [5], Grandell [8], Rolski et al. [13], Asmussen [2] and the references therein.

The Markov-modulated risk model was proposed by Asmussen [1], in which the ruin probability was studied. The model is also called Markovian regime switching model in the finance and actuarial science literature. This model can capture the feature that insurance policies may need to change if economical or political environment changes. Recently, there have been resurgent interests of using regime switching models in finance and actuarial science. Hardy [9] used monthly data from the Standard and Poor's 500 and the

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Toronto Stock Exchange 300 indices to fit a regime-switching lognormal model. The fit of the regime-switching model to the data is compared with other econometric models.

In this paper, we consider the joint distribution of the surplus before and after ruin. In particular, we assume that the claim sizes are phase-type distributed. The class of phase-type distributions is important in the analysis of insurance risk models because any positive distribution can be approximated by a sequence of phase-type distributions. If the problem can be solved in the case of phase-type distribution, the problem in a general case can be approximated by using a sequence of phase-type distributions which converges to the desired probability distribution. In the literature, many methods to find a good approximating sequence have been proposed. We shall show here that when the initial surplus is zero or the claim size distributions are phase-type, it is possible to obtain a closed form solution to the joint distribution being considered. By taking proper limits, the distribution of the surplus prior to ruin and the distribution of the deficit at ruin can be obtained.

2. The insurance risk model

Let $\{J_t\}_{t\geqslant 0}$ be a homogenous continuous-time Markov chain taking values in a finite set $\mathscr{M} = \{1, 2, \ldots, d\}$ with generator $\mathbf{\Lambda} = (\lambda_{ij})$. $\mathbf{\Lambda}$ is assumed to be irreducible with stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_d)$. J_t governs the state of economy. When the state of economy J_t is i, the claim size distribution is B_i with density b_i, Laplace transform $\hat{b}_i(s)$, moment generating function $\hat{B}_i(s)$ and mean μ_i, the arrival intensity is β_i and the premium rate is c_i. The initial surplus is $u \geqslant 0$.

Let $\{R_t^1\}, \{R_t^2\}, \ldots, \{R_t^d\}$ be d independent classical compound Poisson risk process with premium rate c_i, claim arrival rate β_i, claim size distribution B_i and zero initial surplus. The risk process $\{R_t\}$ is then given by

$$R_t = u + \sum_{i=1}^{d} \int_0^t 1(J_s = i)\, \mathrm{d}R_s^i,$$

where $1(A)$ is the indicator function of event A and the aggregate loss process $\{S_t\}$ is given by $S_t = u - R_t$. This is the same model as in Asmussen [2].

Following the proof of Theorem 12.3.2 of Rolski et al. [13], it is easy to see that

$$\lim_{t\to\infty} \frac{R_t}{t} = \sum_{i=1}^{d} \pi_i(c_i - \beta_i\mu_i). \tag{1}$$

Let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|J_0 = i)$. From the above, the condition of having a positive expected profit is

$$\sum_{i=1}^{d} \pi_i(c_i - \beta_i\mu_i) > 0. \tag{2}$$

Let $\tau(u) = \inf\{t : S_t > u\} = \inf\{t : R_t < 0\}$ be the time of ruin with initial surplus u and $\tau = \tau(0)$. For $i \in \mathscr{M}$, $u, x, y \geqslant 0$, let

$$F_i(u, x, y) = \mathbb{P}_i(\tau(u) < \infty, R_{\tau(u)^-} \leqslant x, |R_{\tau(u)}| \leqslant y | R_0 = u)$$

be the joint distribution of surplus before and after ruin with initial surplus u and

$$f_i(u,x,y)=\frac{\partial^2}{\partial x\partial y}F_i(u,x,y)$$

be the joint density of surplus before and after ruin with initial surplus u.

In order to obtain explicit formulae for the two quantities of interest above, we shall need the joint distribution and joint density of surplus before and after ruin fixing the state of economy at the time of ruin to be j. They are denoted by

$$F_{ij}(u,x,y)=\mathbb{P}_i(\tau(u)<\infty, J_{\tau(u)}=j, R_{\tau(u)^-}\leqslant x, |R_{\tau(u)}|\leqslant y|R_0=u)$$

and

$$f_{ij}(u,x,y)=\frac{\partial^2}{\partial x\partial y}F_{ij}(u,x,y).$$

The joint distribution and joint density of surplus before and after ruin can be obtained by summing $F_{ij}(u,x,y)$ and $f_{ij}(u,x,y)$ over all $j\in\mathscr{M}$.

We assume in the following that condition (2) holds so that the ruin probability is strictly less than one starting with any non-negative surplus and any state of economy and $c_i=1$ for all $i\in\mathscr{M}$ since only events in infinite horizon are considered. Indeed, for any given sets of premium rate $\{c_i\}_{i\in\mathscr{M}}$, the transformation

$$\tilde{\lambda}_{ij}=\frac{\lambda_{ij}}{c_i},\quad \tilde{\beta}_i=\frac{\beta_i}{c_i},\quad \tilde{c}_i=1$$

yields a process $\{\tilde{S}_t\}$ such that the joint distributions of the surplus before and after ruin with initial surplus u for the corresponding $\{\tilde{R}_t\}$ and $\{R_t\}$ are the same.

3. The joint distribution of surplus before and after ruin with zero initial surplus

Assume $u=0$ and $c_i=1$ for all $i\in\mathscr{M}$. As in Asmussen [2], let $\{m_x\}_{x\geqslant 0}$ be the $\mathscr{M}$-value process obtained by observing $\{J_t\}$ only when $\{S_t\}$ is at a minimum. If $m_x=i$, then there exists a unique value of t such that $S_u>S_t$ for any $u<t$, $S_t=-x$ and $J_t=i$. One can understand the process $\{m_x\}$ as the state variable of S_t at the first time when S_t hits level $-x$. Fig. 1 below (similar to Fig. 2.1 in Chapter VI of Asmussen [2]) illustrates this when $\mathscr{M}=\{1,2,3\}$. In the figure, there are three states of $\{J_t\}$, marked by thin, thick and dashed lines, respectively, in the path of $\{S_t\}$. The corresponding values of m_x is represented by the line next to the vertical axis.

By the net profit condition (2), $S_t\to-\infty$ as $t\to\infty$. Thus $\{m_x\}$ is a non-terminating homogenous continuous-time Markov chain and its generator is denoted by $\boldsymbol{Q}$.

Consider stationary version of $\{J_t\}$ and its time-reversed version $\{\check{J}_t\}$ on a finite time interval. The generator of $\{\check{J}_t\}$ is

$$\check{\boldsymbol{\Lambda}}=[\check{\lambda}_{ij}]=\left[\frac{\pi_j}{\pi_i}\lambda_{ji}\right].$$

In matrix notation, let $\Delta=\mathrm{diag}(\pi_1,\pi_2,\ldots,\pi_d)$, then $\check{\boldsymbol{\Lambda}}=\Delta^{-1}\boldsymbol{\Lambda}'\Delta$. Let $\{\check{S}_t\}$ be defined similar to $\{S_t\}$ but with $\{J_t\}$ replaced by the time-reversed version $\{\check{J}_t\}$. The process $\{\check{m}_x\}$ is defined similarly and its generator is denoted by $\check{\boldsymbol{Q}}$. $\{\check{m}_x\}$ is also non-terminating, since $\check{S}_t\to-\infty$ as $t\to\infty$.

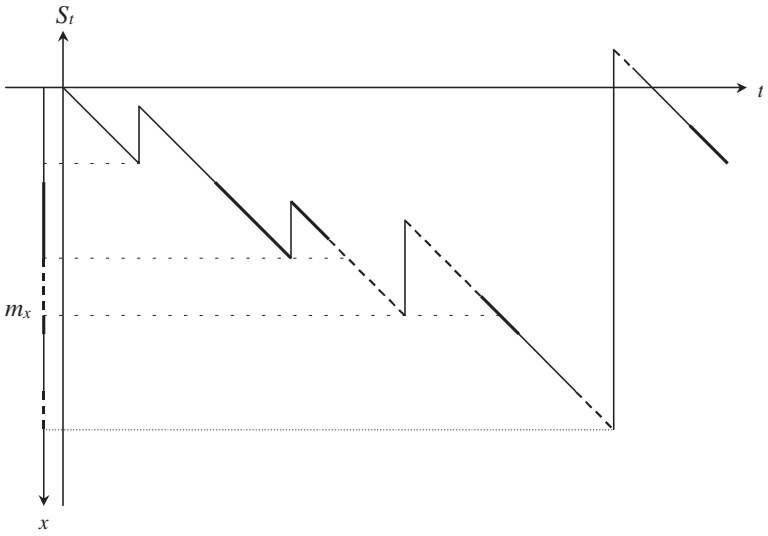

Fig. 1. The illustration of $\{m_x\}$.

Proposition VI.2.4 of Asmussen [2] (with a slight change of notations) states that $\check{\boldsymbol{Q}}$ satisfies the non-linear matrix equation $\check{\boldsymbol{Q}} = \varphi(\check{\boldsymbol{Q}})$ where

$$\varphi(\check{\boldsymbol{Q}}) = \check{\boldsymbol{\Lambda}} - \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_d) + \int_0^\infty \boldsymbol{S}(\mathrm{d}x)\mathrm{e}^{\check{\boldsymbol{Q}}x}$$

and $\boldsymbol{S}(\mathrm{d}x) = \mathrm{diag}(\beta_1 B_1(\mathrm{d}x), \beta_2 B_2(\mathrm{d}x), \ldots, \beta_d B_d(\mathrm{d}x))$. Furthermore, the sequence $\{\check{\boldsymbol{Q}}^{(n)}\}$ defined by

$$\check{\boldsymbol{Q}}^{(0)} = \check{\boldsymbol{\Lambda}} - \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_d), \quad \check{\boldsymbol{Q}}^{(n+1)} = \varphi(\check{\boldsymbol{Q}}^{(n)})$$

converges monotonically to $\check{\boldsymbol{Q}}$. The matrix $\boldsymbol{Q}$ can be found by a similar iteration scheme.

The matrix $\check{\boldsymbol{Q}}$ is important in the calculation of the joint density of surplus before and after ruin, as illustrated in Corollary VI.2.6(a) of Asmussen [2], which states that for a measure-valued matrix $\boldsymbol{G}_+(A)$ defined by ijth element

$$G_+(i,j,A) = \mathbb{P}_i(\tau < \infty, S_\tau \in A, J_\tau = j),$$

letting $\check{\boldsymbol{K}} = \Delta^{-1}\check{\boldsymbol{Q}}'\Delta$, then

$$\boldsymbol{G}_+((z,\infty)) = \int_0^\infty \mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{S}((x+z,\infty))\,\mathrm{d}x.$$

The following theorem extends the result above.

Theorem 1. *Let* $\boldsymbol{G}(u,x,y)$ *be the matrix with ijth element*

$$\mathbb{P}_i(\tau(u) < \infty, R_{\tau(u)^-} > x, R_{\tau(u)} < -y, J_{\tau(u)} = j | R_0 = u).$$

Then

$$\boldsymbol{G}(0,x,y) = \int_x^\infty \mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z+y,\infty))\,\mathrm{d}z. \tag{3}$$

Proof. When $u=0$, the ijth element of $\boldsymbol{G}(0,x,y)$ is

$$\begin{aligned}&\mathbb{P}_i(\tau<\infty, R_{\tau^-}>x, R_\tau<-y, J_\tau=j|R_0=0)\\&\quad=\mathbb{P}_i(\tau<\infty,-S_{\tau^-}>x, S_\tau>y, J_\tau=j).\end{aligned}$$

Fix $T>0$, let $J'_t=J_{T-t^-}$ and $S'_t=S_T-S_{T-t^-}$ for all $t\in[0,T]$. The process $\{(J'_t,S'_t)\}$ has the same distribution as the time-reversed version $\{(\check{J}_t,\check{S}_t)\}$ under the stationary initial distribution. Let $\check{\tau}_z$ be the time when $\check{S}_t$ first hits level $-z$. Then

$$\begin{aligned}&\pi_i\mathbb{P}_i(J_T=j,-S_{T^-}\in[z,z+\mathrm{d}z],\tau>T)\\&\quad=\mathbb{P}_{\boldsymbol{\pi}}(J_0=i,J_T=j,-S_{T^-}\in[z,z+\mathrm{d}z],S_t<0\text{ for all }t<T)\\&\quad=\mathbb{P}_{\boldsymbol{\pi}}(J'_0=j,J'_T=i,S'_t>-z\text{ for all }t<T,-S'_{T^-}\in[z,z+\mathrm{d}z])\\&\quad=\pi_j\mathbb{P}_j(\check{J}_T=i,\check{\tau}_z>T,-\check{S}_{T^-}\in[z,z+\mathrm{d}z]).\end{aligned}\tag{4}$$

Let $g_{ij}(T)$ be the density function

$$\lim_{t\downarrow0}\frac{\mathbb{P}_i(J_T=j,\tau_z<T+t,-S_{T^-}\in[z,z+\mathrm{d}z])-\mathbb{P}_i(J_T=j,\tau_z<T,-S_{T^-}\in[z,z+\mathrm{d}z])}{t}$$

and $\check{g}_{ji}(T)$ be the density function

$$\lim_{t\downarrow0}\frac{\mathbb{P}_j(\check{J}_T=i,\check{\tau}_z<T+t,-\check{S}_{T^-}\in[z,z+\mathrm{d}z])-\mathbb{P}_j(\check{J}_T=i,\check{\tau}_z<T,-\check{S}_{T^-}\in[z,z+\mathrm{d}z])}{t}.$$

By (4),

$$\begin{aligned}\mathbb{P}_i(\tau<\infty,-S_{\tau^-}\in[z,z+\mathrm{d}z],J_\tau=j)&=\int_0^\infty\beta_j\bar{B}_j(z)g_{ij}(T)\,\mathrm{d}T\\&=\beta_j\bar{B}_j(z)\frac{\pi_j}{\pi_i}\int_0^\infty\check{g}_{ji}(T)\,\mathrm{d}T\\&=\beta_j\bar{B}_j(z)\frac{\pi_j}{\pi_i}\mathbb{P}_j(\check{m}_z=i)\,\mathrm{d}z\\&=\beta_j\bar{B}_j(z)\frac{\pi_j}{\pi_i}\boldsymbol{e}'_j\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e}_i\,\mathrm{d}z,\end{aligned}$$

where $\boldsymbol{e}_i$ is the ith unit column vector. When $\tau<\infty$ and $J_\tau=j$, denote the density function of $-S_{\tau^-}$ by $s_{ij}(z)$. By conditioning on S_{τ^-},

$$\begin{aligned}\mathbb{P}_i(\tau<\infty,-S_{\tau^-}>x,S_\tau>y,J_\tau=j)&=\int_x^\infty\mathbb{P}(Y>y+z|\,Y>z,\,Y\sim B_j)s_{ij}(z)\,\mathrm{d}z\\&=\int_x^\infty\frac{\bar{B}_j(y+z)}{\bar{B}_j(z)}\frac{\pi_j}{\pi_i}\beta_j\bar{B}_j(z)\boldsymbol{e}'_j\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e}_i\,\mathrm{d}z\\&=\int_x^\infty\frac{\pi_j}{\pi_i}\beta_j\bar{B}_j(y+z)\boldsymbol{e}'_j\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e}_i\,\mathrm{d}z.\end{aligned}\tag{5}$$

Rewriting in matrix form, the result is obtained. □

Assuming all claim size distributions are absolutely continuous, the joint density of surplus before and after ruin starting with zero initial surplus can now be obtained.

Theorem 2. *The joint density of surplus before and after ruin starting with zero initial surplus and state of economy i is given by*

$$f_i(0,x,y)=\sum_{j=1}^{d}\frac{\pi_j}{\pi_i}\beta_j b_j(y+x)\boldsymbol{e}_j'\mathrm{e}^{\check{\boldsymbol{Q}}x}\boldsymbol{e}_i=\boldsymbol{e}_i'\mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{s}(x+y)\boldsymbol{e}, \tag{6}$$

where $\boldsymbol{s}(x)=\mathrm{diag}(\beta_1 b_1(x),\beta_2 b_2(x),\ldots,\beta_d b_d(x))$.

Proof. Since all the claim size distributions are absolutely continuous, (5) can be differentiated twice to yield

$$f_{ij}(0,x,y)=\frac{\pi_j}{\pi_i}\beta_j b_j(y+x)\boldsymbol{e}_j'\mathrm{e}^{\check{\boldsymbol{Q}}x}\boldsymbol{e}_i$$

and hence a closed form solution of $f_i(0,x,y)$ can be obtained from

$$f_i(0,x,y)=\sum_{j=1}^{d}f_{ij}(0,x,y)=\sum_{j=1}^{d}\frac{\pi_j}{\pi_i}\beta_j b_j(y+x)\boldsymbol{e}_j'\mathrm{e}^{\check{\boldsymbol{Q}}x}\boldsymbol{e}_i=\boldsymbol{e}_i'\mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{s}(x+y)\boldsymbol{e}. \qquad \square$$

By integrating the joint density, the joint distribution of surplus before and after ruin starting with zero initial surplus and $J_0=i$ is

$$F_i(0,x,y)=\int_0^x\int_0^y\boldsymbol{e}_i'\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{s}(z+v)\,\mathrm{d}v\,\mathrm{d}z\boldsymbol{e}=\boldsymbol{e}_i'\int_0^x\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z,z+y))\,\mathrm{d}z\boldsymbol{e}.$$

Thus the distributions of surplus before ruin and the deficit at ruin, starting with zero initial surplus and $J_0=i$, are given by

$$F_i(0,x,\infty)=\boldsymbol{e}_i'\int_0^x\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z,\infty))\,\mathrm{d}z\boldsymbol{e}$$

and

$$F_i(0,\infty,y)=\boldsymbol{e}_i'\int_0^\infty\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z,z+y))\,\mathrm{d}z\boldsymbol{e}.$$

Let $\psi_{ij}(0)=\mathbb{P}_i(\tau<\infty,J_\tau=j|R_0=0)$ and $\psi_i(0)=\mathbb{P}_i(\tau<\infty|R_0=0)=\sum_{j=1}^{d}\psi_{ij}(0)$ be the infinite-horizon ruin probability starting with zero initial surplus and $J_0=i$, then

$$\psi_{ij}(0)=\boldsymbol{e}_i'\boldsymbol{G}(0,0,0)\boldsymbol{e}_j=\boldsymbol{e}_i'\int_0^\infty\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z,\infty))\,\mathrm{d}z\boldsymbol{e}_j$$

and the closed form solution of $\psi_i(0)$ is

$$\boldsymbol{e}_i'\int_0^\infty\mathrm{e}^{\check{\boldsymbol{K}}z}\boldsymbol{S}((z,\infty))\,\mathrm{d}z\boldsymbol{e}, \tag{7}$$

where $\boldsymbol{e}$ is the column vector with all entries equal to 1.

Comparing with the result in the classical compound Poisson risk model

$$\mathbb{P}(\tau<\infty,-S_{\tau^-}>x,S_\tau>y)=\beta\int_{x+y}^\infty\bar{B}(z)\,\mathrm{d}z$$

and

$$f(0,x,y)=\beta b(x+y),$$

the symmetry between x and y is lost in the Markov-modulated risk model because of the presence of $\check{\boldsymbol{Q}}$ and $\check{\boldsymbol{K}}$. But if one starts with the stationary initial distribution $\boldsymbol{\pi}$, using the fact that $\mathrm{e}^{\check{\boldsymbol{Q}}z}$ is a stochastic matrix,

$$\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e} = \boldsymbol{e}$$

and on combining with (5),

$$\begin{aligned}
&\mathbb{P}_{\boldsymbol{\pi}}(\tau<\infty, -S_{\tau^-}>x, S_\tau>y)\\
&\quad=\sum_{i=1}^{d}\sum_{j=1}^{d}\pi_i\int_x^\infty \frac{\pi_j}{\pi_i}\beta_j\bar{B}_j(y+z)\boldsymbol{e}_j'\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e}_i\,\mathrm{d}z\\
&\quad=\sum_{j=1}^{d}\int_x^\infty \pi_j\beta_j\bar{B}_j(y+z)\boldsymbol{e}_j'\mathrm{e}^{\check{\boldsymbol{Q}}z}\boldsymbol{e}\,\mathrm{d}z=\sum_{j=1}^{d}\int_x^\infty \pi_j\beta_j\bar{B}_j(y+z)\boldsymbol{e}_j'\boldsymbol{e}\,\mathrm{d}z\\
&\quad=\sum_{j=1}^{d}\int_x^\infty \pi_j\beta_j\bar{B}_j(y+z)\,\mathrm{d}z=\beta\int_x^\infty \bar{B}(y+z)\,\mathrm{d}z=\beta\int_{x+y}^\infty \bar{B}(z)\,\mathrm{d}z,
\end{aligned}$$

where $\beta=\sum_{j=1}^{d}\pi_j\beta_j$ and $B(x)=\frac{1}{\beta}\sum_{j=1}^{d}\pi_j\beta_j B_j(x)$ are the average claim arrival rate and average claim size distribution. In this case, the symmetry between x and y is preserved.

4. A coupled system of integro-differential equations for the expected discounted penalty function

Gerber and Shiu [6] introduced the function

$$\phi(u)=\mathbb{E}[\mathrm{e}^{-\delta\tau(u)}w(R_{\tau(u)^-},|R_{\tau(u)}|)1(\tau(u)<\infty)|R_0=u]$$

for $\delta\geqslant 0$ and bivariate non-negative function w for the classical compound Poisson risk model. This function is called the expected discounted penalty function because if one treat δ as the constant force of interest and w as the benefit of an insurance payable at the time of ruin, with the benefit amount varying according to the surplus before and after ruin, $\phi(u)$ is the actuarial present value of the insurance. The expected discounted penalty function unifies the study of ruin probability, joint distribution of surplus before and after ruin, moments of the surplus at ruin and the time of ruin. For example, to study the distribution of the time of ruin, one can set $w(x,y)=1$ for all $x,y\geqslant 0$ and $\phi(u)$ is the Laplace transform of the time of ruin. For a detailed study of the expected discounted penalty function in the classical model, one can refer to Gerber and Shiu [6].

In this section we shall derive a set of integro-differential equations satisfied by the Gerber–Shiu expected discounted penalty function in the Markov-modulated risk model defined by

$$\phi_i(u)=\mathbb{E}_i[\mathrm{e}^{-\delta\tau(u)}w(R_{\tau(u)^-},|R_{\tau(u)}|)1(\tau(u)<\infty)|R_0=u],$$

where w is a bivariate non-negative function and $u\geqslant 0$. This function is useful in obtaining quantities regarding the time of ruin.

Theorem 3. *Let* $w_i(u) = \int_u^\infty w(u, z-u)B_i(\mathrm{d}z)$ *and* $\hat{w}_i(s) = \int_0^\infty \mathrm{e}^{-su} w_i(u)\,\mathrm{d}u$. *Then* $\phi_i(u)$ *satisfies*

$$(\beta_i + \delta)\phi_i(u) - \sum_{j=1}^{d} \lambda_{ij}\phi_j(u) = \phi_i'(u) + \beta_i \left[\int_0^u \phi_i(u-z)B_i(\mathrm{d}z) + w_i(u) \right]$$

and the Laplace transform of $\phi_i(u)$, *denoted by* $\hat{\phi}_i(s)$, *satisfies*

$$\left[\beta_i + \delta - s - \beta_i \hat{b}_i(s)\right]\hat{\phi}_i(s) - \sum_{j=1}^{d} \lambda_{ij}\hat{\phi}_j(s) = \beta_i \hat{w}_i(s) - \phi_i(0).$$

Proof. By the property of Markov process,

$$\begin{aligned}\phi_i(u) = \mathrm{e}^{-\delta\,\mathrm{d}t}\Bigg\{ & (1-\beta_i\,\mathrm{d}t)(1+\lambda_{ii}\,\mathrm{d}t)\phi_i(u+\mathrm{d}t) + (1-\beta_i\,\mathrm{d}t) \\ & \times \sum_{j\neq i} \lambda_{ij}\,\mathrm{d}t\phi_j(u+\mathrm{d}t) + \beta_i\,\mathrm{d}t(1+\lambda_{ii}\,\mathrm{d}t) \\ & \times \left[\int_0^u \phi_i(u-z)B_i(\mathrm{d}z) + \int_u^\infty w(u,z-u)B_i(\mathrm{d}z) \right] + \mathrm{o}(\mathrm{d}t) \Bigg\} \end{aligned} \tag{8}$$

where the four terms correspond to

(1) no change of state and no claim in $\mathrm{d}t$,
(2) a change of state but no claim in $\mathrm{d}t$,
(3) no change of state but a claim arrives in $\mathrm{d}t$, and
(4) all other events with total probability $\mathrm{o}(\mathrm{d}t)$.

Eq. (8) can be simplified to

$$\begin{aligned}\phi_i(u) &= (1-\delta\,\mathrm{d}t)\Bigg\{ [1 + (\lambda_{ii} - \beta_i)\,\mathrm{d}t]\phi_i(u) + \phi_i'(u)\,\mathrm{d}t + \sum_{j\neq i}\lambda_{ij}\phi_j(u)\,\mathrm{d}t \\ &\quad + \beta_i\,\mathrm{d}t\left[\int_0^u \phi_i(u-z)B_i(\mathrm{d}z) + \int_u^\infty w(u,z-u)B_i(\mathrm{d}z) \right] + \mathrm{o}(\mathrm{d}t) \Bigg\} \\ &= \phi_i(u) + \phi_i'(u)\,\mathrm{d}t + (\lambda_{ii} - \beta_i - \delta)\phi_i(u)\,\mathrm{d}t + \sum_{j\neq i}\lambda_{ij}\phi_j(u)\,\mathrm{d}t \\ &\quad + \beta_i\,\mathrm{d}t\left[\int_0^u \phi_i(u-z)B_i(\mathrm{d}z) + \int_u^\infty w(u,z-u)B_i(\mathrm{d}z) \right] + \mathrm{o}(\mathrm{d}t). \end{aligned}$$

Cancelling $\phi_i(u)$, dividing both sides by $\mathrm{d}t$ and taking limit, the equation above reduces to

$$(\beta_i + \delta)\phi_i(u) - \sum_{j=1}^{d} \lambda_{ij}\phi_j(u) - \phi_i'(u) = \beta_i \left[\int_0^u \phi_i(u-z)B_i(\mathrm{d}z) + w_i(u) \right],$$

which is an integro-differential equation corresponding to (2.16) of Gerber and Shiu [6].

Multiplying both sides by e^{-su} and integrating with respect to u, the above becomes

$$\left[\beta_i + \delta - s - \beta_i \hat{b}_i(s)\right]\hat{\phi}_i(s) - \sum_{j=1}^{d} \lambda_{ij}\hat{\phi}_j(s) = \beta_i \hat{w}_i(s) - \phi_i(0). \qquad \square$$

From the above system of linear equations, if the values of $\phi_i(0)$ for all $i \in \mathscr{M}$ are known and the matrix

$$\boldsymbol{A}(s) = \operatorname{diag}(\beta_1(1-\hat{b}_1(s)), \beta_2(1-\hat{b}_2(s)), \ldots, \beta_d(1-\hat{b}_d(s))) + (\delta - s)\boldsymbol{I}_{d\times d} - \boldsymbol{\Lambda}$$

is invertible, the Laplace transform of the expected discounted penalty function $\hat{\phi}_i(s)$ can be obtained. The difficulty and limitation of the use of the coupled system lies in the determination of $\phi_i(0)$, since, unlike the classical compound Poisson risk model, the boundary condition when s tends to infinity does not lead to a system of equations that can be used to solve $\phi_i(0)$. But in virtue of Theorem 2, if the discount rate δ is equal to 0, then $\phi_i(0)$ can be readily obtained. Thus, it may be possible to obtain the Laplace transform of the joint distribution and the joint survival function of the surplus before and after ruin. The Laplace transform of the marginal distributions $F_i(u, x, \infty)$ and $F_i(u, \infty, y)$ and ruin probability can also be obtained by taking proper limits.

To obtain the Laplace transform of $\psi_i(u)$, put $w(s,t) = 1$ for all s, t and $\delta = 0$. Then

$$w_i(u) = \int_u^\infty B_i(\mathrm{d}z) = \bar{B}_i(u)$$

and hence

$$\hat{w}_i(s) = \int_0^\infty \mathrm{e}^{-sz}\bar{B}_i(z)\,\mathrm{d}z = \frac{1-\hat{b}_i(s)}{s}.$$

The corresponding $\psi_i(0)$ is given by (7).

To obtain the Laplace transform of $\mathbb{P}_i(\tau(u)<\infty, R_{\tau(u)^-} > x, |R_{\tau(u)}| > y | R_0 = u)$, put $w(s,t) = 1(s>x, t>y)$ and $\delta = 0$. Then

$$w_i(u) = \int_u^\infty 1(u>x, z-u>y)B_i(\mathrm{d}z) = 1(u>x)\bar{B}_i(u+y)$$

and hence

$$\hat{w}_i(s) = \int_x^\infty \mathrm{e}^{-sz}\bar{B}_i(z+y)\,\mathrm{d}z.$$

The corresponding initial condition is $\phi_i(0) = \boldsymbol{e}_i'\boldsymbol{G}(0,x,y)\boldsymbol{e}$.

To illustrate the use of the coupled system of integro-differential equations, we consider a simple example which leads to an explicit formula for the expected discounted penalty function.

Consider a Markov-modulated risk model with two states of economy,

$$\boldsymbol{\Lambda} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \frac{9}{2} \\ \frac{3}{2} \end{bmatrix}.$$

In state of economy 1, claims are exponentially distributed with mean $\mu_1 = \frac{1}{3}$ whereas in state of economy 2, claims are exponentially distributed with mean $\mu_2 = \frac{1}{4}$. Thus $\sum_{j=1}^2 \pi_j\beta_j\mu_j = \frac{15}{16} < 1$ and condition (2) holds. We are interested in finding the probability that starting in state 1 and initial surplus u, ruin occurs and the deficit at ruin exceeds y. The proper choice of the expected discount penalty function is $w(s,t) = 1(t>y)$ and the discount rate δ is zero.

First we shall obtain the initial value $\phi_1(0)$ by Eq. (3), which involves the calculation of the matrix $\check{\boldsymbol{K}}$. The stationary distribution of the continuous-time Markov chain is $[0.5, 0.5]$.

By Proposition VI.2.4 of Asmussen [2] stated in Section 3, it can be found that

$$\check{\boldsymbol{Q}} = \begin{bmatrix} -2.78743178 & 2.78743178 \\ 1.23014682 & -1.23014682 \end{bmatrix} \text{ and } \check{\boldsymbol{K}} = \begin{bmatrix} -2.78743178 & 1.23014682 \\ 2.78743178 & -1.23014682 \end{bmatrix}.$$

Let

$$\boldsymbol{P} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{W}(y) = \begin{bmatrix} \frac{9}{2}\mathrm{e}^{-3y} \\ \frac{3}{2}\mathrm{e}^{-4y} \end{bmatrix}.$$

Then by the mixed product rule of Kronecker product, the initial value $\phi_1(0)$ can be obtained from

$$\begin{aligned} \phi_1(0) &= \boldsymbol{e}_1' \boldsymbol{G}(0,0,y)\boldsymbol{e} = \boldsymbol{e}_1' \int_0^\infty \mathrm{e}^{\check{\boldsymbol{K}}z} \begin{bmatrix} \frac{9}{2}\mathrm{e}^{-3(z+y)} \\ \frac{3}{2}\mathrm{e}^{-4(z+y)} \end{bmatrix} \mathrm{d}z \\ &= \sum_{i=1}^{2} \boldsymbol{e}_1' \int_0^\infty \mathrm{e}^{\check{\boldsymbol{K}}z} \boldsymbol{e}_i \boldsymbol{e}_i' \mathrm{e}^{\boldsymbol{P}z} \,\mathrm{d}z \boldsymbol{W}(y) = \sum_{i=1}^{2} (\boldsymbol{e}_1' \otimes \boldsymbol{e}_i')(-\check{\boldsymbol{K}} \oplus \boldsymbol{P})^{-1}(\boldsymbol{e}_i \otimes \boldsymbol{W}(y)), \end{aligned}$$

where

$$(-\check{\boldsymbol{K}} \oplus \boldsymbol{P})^{-1} = \begin{bmatrix} 0.200931 & 0 & 0.058432 & 0 \\ 0 & 0.163084 & 0 & 0.038358 \\ 0.132402 & 0 & 0.274902 & 0 \\ 0 & 0.086916 & 0 & 0.211642 \end{bmatrix}.$$

After some simplifications, $\phi_1(0) = 0.904189\mathrm{e}^{-3y} + 0.057537\mathrm{e}^{-4y}$.

Then we shall solve the system of integro-differential equations in Theorem 3:

$$\frac{9}{2}\phi_1(u) - [-\phi_1(u) + \phi_2(u)] - \phi_1'(u) = \frac{9}{2}\left[\int_0^u \phi_1(u-z)3\mathrm{e}^{-3z}\,\mathrm{d}z + \mathrm{e}^{-3(u+y)}\right]$$

and

$$\frac{3}{2}\phi_2(u) - [\phi_1(u) - \phi_2(u)] - \phi_2'(u) = \frac{3}{2}\left[\int_0^u \phi_2(u-z)4\mathrm{e}^{-4z}\,\mathrm{d}z + \mathrm{e}^{-4(u+y)}\right].$$

Letting $u = 0$, the first integro-differential equation leads to

$$\phi_1'(0) = \frac{11\phi_1(0) - 2\phi_2(0) - 9\mathrm{e}^{-3y}}{2}.$$

To eliminate $\phi_2(0)$, note that the effect of $\check{\boldsymbol{Q}}$ or $\check{\boldsymbol{K}}$ disappears when the risk process starts with the stationary initial distribution $\boldsymbol{\pi}$. Mathematically, $\phi_{\boldsymbol{\pi}}(0) = \boldsymbol{\pi}\boldsymbol{G}(0,0,y)\boldsymbol{e}$ can be simplified into

$$\frac{1}{2}\phi_1(0) + \frac{1}{2}\phi_2(0) = \int_0^\infty \left[\frac{1}{2}\left(\frac{9}{2}\mathrm{e}^{-3(z+y)}\right) + \frac{1}{2}\left(\frac{3}{2}\mathrm{e}^{-4(z+y)}\right)\right]\mathrm{d}z$$

from which we can obtain

$$\phi_2(0) = \tfrac{3}{2}\mathrm{e}^{-3y} + \tfrac{3}{8}\mathrm{e}^{-4y} - \phi_1(0).$$

Thus

$$\phi_1'(0) = \tfrac{13}{2}\phi_1(0) - 6\mathrm{e}^{-3y} - \tfrac{3}{8}\mathrm{e}^{-4y}.$$

Differentiating the equations with respect to u and eliminating the two integral terms by making use of the two original integro-differential equations, we arrive at

$$(2D^2 - 5D - 6)\phi_1(u) + (2D + 6)\phi_2(u) = 0,$$
$$(2D + 8)\phi_1(u) + (2D^2 + 3D - 8)\phi_2(u) = 0,$$

where the symbol D is the differential operator.

By eliminating $\phi_2(u)$ from the system of linear differential equations, one can obtain a fourth order linear differential equation of $\phi_1(u)$. The roots of the resulting characteristic equation are $4.017579, 0, -0.129265$ and -2.888313. Noting that $\phi_1(u) \to 0$ as $u \to \infty$,

$$\phi_1(u) = A\mathrm{e}^{-0.129265u} + B\mathrm{e}^{-2.888313u}$$

where A and B (which are functions of y only) can be determined from $\phi_1(0)$ and $\phi_1'(0)$. By making use of

$$\phi_1(0) = A + B,$$
$$\phi_1'(0) = \tfrac{13}{2}\phi_1(0) - 6\mathrm{e}^{-3y} - \tfrac{3}{8}\mathrm{e}^{-4y} = -0.129265A - 2.888313B,$$

it can be seen that

$$A = 3.40274\phi_1(0) - 2.17466\mathrm{e}^{-3y} - 0.13592\mathrm{e}^{-4y},$$
$$B = -2.40274\phi_1(0) + 2.17466\mathrm{e}^{-3y} + 0.13592\mathrm{e}^{-4y}.$$

Finally, on combining with the initial condition $\phi_1(0)$, we obtain

$$\phi_1(u) = [0.902055\mathrm{e}^{-3y} + 0.059866\mathrm{e}^{-4y}]\mathrm{e}^{-0.129265u},$$
$$+ [0.0021342\mathrm{e}^{-3y} - 0.0023291\mathrm{e}^{-4y}]\mathrm{e}^{-2.888313u}.$$

5. Barrier probabilities of $\{S_t\}$ in the case of phase-type claims

A phase-type distribution F is the distribution of the life time of a terminating continuous-time Markov chain $\{M_t\}_{t\geqslant 0}$ with finitely many states, one of which is absorbing and all others are transient. Let the state space of $\{M_t\}$ be $\{1, \ldots, d, 0\} = E \cup \{0\}$ and 0 be the absorbing state. Then the generator of $\{M_t\}$ admits the structure

$$\begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \mathbf{0}' & 0 \end{bmatrix},$$

where $\boldsymbol{T}$ is a sub-intensity matrix, $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{e}$ and $\mathbf{0}$ is a zero column vector. Let $(\alpha_1, \alpha_2, \ldots, \alpha_d, 0) = (\boldsymbol{\alpha}, 0)$ be the initial distribution of $\{M_t\}$ so that the continuous-time Markov chain will not start at the absorbing state. We denote the distribution of F by $\mathrm{PH}(E, \boldsymbol{\alpha}, \boldsymbol{T})$.

Two important characteristics of phase-type distribution are

(1) it is closed under mixture and convolution, and

(2) it is dense in the set of all distributions with positive support, that is, for any given distribution F on $(0,\infty)$, there exists a sequence $\{F_n\}$ of phase-type distributions which converges in distribution to F.

Erlang(n) distribution and mixture of exponential distributions are all in the family of phase-type distribution. For more information about phase-type distribution, one can refer to Neuts [12].

Li and Garrido [10] obtained the closed form solution of the infinite-horizon ruin probability and the joint distribution of the surplus before and after ruin for the Sparre Andersen models when the initial surplus is zero or the claim sizes belong to the rational family which includes the phase-type distribution as a special case. See also Li and Garrido [11]. We will show in the following that, when the claim size distribution in each of the states is phase-type, it is possible to obtain a closed form solution of the infinite-horizon ruin probability and the joint distribution of the surplus before and after ruin under the regime switching model.

Let the claim size distribution in state i be PH($E^{(i)}, \boldsymbol{\alpha}^{(i)}, \boldsymbol{T}^{(i)}$). From Proposition VIII.5.5 and Theorem VIII.5.6 of Asmussen [2], it is immediate that the distribution of the first overshoot of $\{S_t\}$ above zero given $J_0 = i$ follows the distribution PH($E, \boldsymbol{\theta}^{(i)}, \boldsymbol{T}$) where

$$E = \bigcup_{j\in\mathscr{M}} E^{(j)} \quad \text{and} \quad \boldsymbol{\theta}^{(i)} = (\boldsymbol{\theta}^{(i)}_{1\cdot}, \boldsymbol{\theta}^{(i)}_{2\cdot}, \ldots, \boldsymbol{\theta}^{(i)}_{d\cdot})$$

with $\boldsymbol{\theta}^{(i)}_{j\cdot} = \beta_j(\boldsymbol{e}_i' \otimes \boldsymbol{\alpha}^{(j)})(-\check{\boldsymbol{K}} \oplus \boldsymbol{T}^{(j)})^{-1}(\boldsymbol{e}_j \otimes \boldsymbol{I})$, $\boldsymbol{I}$ having the same dimension as $\boldsymbol{T}^{(j)}$ and $\boldsymbol{T}$ is obtained by putting $\boldsymbol{T}^{(j)}$ on the main diagonal with all other entries zero. Also, the maximum $M = \sup S_t$ given $J_0 = i$ follows the distribution PH($E, \boldsymbol{\theta}^{(i)}, \boldsymbol{U}$) where the matrix $\boldsymbol{U}$ is formed by

$$u_{j\alpha,k\gamma} = \begin{cases} [\boldsymbol{T}^{(j)}]_{\alpha\gamma} + t^{(j)}_{\alpha}\theta^{(j)}_{j\gamma} & \text{for } j = k, \\ t^{(j)}_{\alpha}\theta^{(j)}_{k\gamma} & \text{for } j \neq k. \end{cases}$$

Here the symbols $\otimes$ and $\oplus$ stand for Kronecker product and Kronecker sum. For details about these two operations one can refer to Graham [7].

First we introduce some notations very similar to that in Asmussen and Perry [3]. For $u \geqslant 0$, $\alpha \in E^{(j)}$, define the first upcrossing and downcrossing probabilities

$$\pi^{+}_{i,j\alpha} = \mathbb{P}_i(S_t \text{ upcrosses 0 the first time in state } j, \text{ phase } \alpha \in E^{(j)}),$$
$$\pi^{-}_{i,j}(u) = \mathbb{P}_i(S_t \text{ downcrosses } -u \text{ the first time in state } j)$$

and barrier probabilities

$$p^{+}_{i,j\alpha}(u) = \mathbb{P}_i(S_t \text{ first upcrosses 0 in state } j, \text{ phase } \alpha \text{ before downcrossing } -u),$$
$$p^{-}_{i,j}(u) = \mathbb{P}_i(S_t \text{ first downcrosses } -u \text{ in state } j \text{ before upcrossing 0}).$$

It is obvious from the above and Section 3 that

$$\pi^{+}_{i,j\alpha} = (\boldsymbol{\theta}^{(i)}_{j\cdot})_{\alpha} \quad \text{and} \quad \pi^{-}_{i,j}(u) = \mathbb{P}_i(m_u = j) = \boldsymbol{e}_i' \mathrm{e}^{\boldsymbol{Q}u}\boldsymbol{e}_j.$$

Asmussen and Perry [3] derived the barrier probabilities in a more complicated situation in a queuing theory context, where they related $\{S_t\}$ to the virtual waiting time $\{V_t\}$ of a MAP/MMPH/1 queue (MAP = Markovian arrival process,

MMPH = Markov-modulated phase-type). Here we shall briefly go through their argument in order to give probabilistic interpretations to various auxiliary quantities needed to construct the barrier probabilities to be used in later sections.

First consider the event

$$\begin{aligned}
&\mathbb{P}_i(S_t \text{ upcrosses } 0 \text{ in state } j,\ \text{phase } \alpha \in E^{(j)})\\
&\quad = \mathbb{P}_i(S_t \text{ upcrosses } 0 \text{ in state } j,\ \text{phase } \alpha \in E^{(j)} \text{ before downcrossing } -u)\\
&\qquad + \sum_{k\in\mathscr{M}} \mathbb{P}_i(S_t \text{ upcrosses } 0 \text{ in state } j,\ \text{phase } \alpha \in E^{(j)} \text{ after}\\
&\qquad \text{downcrossing } -u \text{ in state } k)\\
&\quad = \mathbb{P}_i(S_t \text{ upcrosses } 0 \text{ in state } j,\ \text{phase } \alpha \in E^{(j)} \text{ before downcrossing } -u)\\
&\qquad + \sum_{k\in\mathscr{M}} \mathbb{P}_i(S_t \text{ downcrosses } -u \text{ in state } k \text{ before upcrossing } 0)\boldsymbol{\theta}^{(k)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}_{j\alpha} \qquad (9)
\end{aligned}$$

since $\mathbb{P}_k(S_t \text{ overshoots } u \text{ in } j, \alpha \in E^{(j)}) = \boldsymbol{\theta}^{(k)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}_{j\alpha}$. See Fig. 2 below for the decomposition above.

Denote the probability $\boldsymbol{\theta}^{(k)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}_{j\alpha}$ by $n^{-,+}_{k,j\alpha}(u)$, (9) becomes

$$\pi^{+}_{i,j\alpha} = p^{+}_{i,j\alpha}(u) + \sum_{k\in\mathscr{M}} p^{-}_{i,k}(u)n^{-,+}_{k,j\alpha}(u). \qquad (10)$$

Then consider Fig. 3 below for the decomposition of $\pi^{-}_{i,j}(u)$.

$$\begin{aligned}
&\mathbb{P}_i(S_t \text{ downcrosses } -u \text{ in state } j)\\
&\quad = \mathbb{P}_i(S_t \text{ downcrosses } -u \text{ in state } j \text{ before upcrossing } 0)\\
&\qquad + \sum_{k\in\mathscr{M}} \sum_{\alpha\in E^{(k)}} \mathbb{P}_i(S_t \text{ upcrosses } 0 \text{ in state } k,\ \text{phase } \alpha \in E^{(k)}\\
&\qquad \text{before downcrossing } -u \text{ in state } j). \qquad (11)
\end{aligned}$$

To calculate the probability in the double summation, first condition on the state of economy and phase of S_t to be $k \in \mathscr{M}, \alpha \in E^{(k)}$. Given this condition, the overshoot of S_t has distribution $\mathrm{PH}(E^{(k)}, \boldsymbol{e}'_{\alpha}, \boldsymbol{T}^{(k)})$ and conditional density function $\boldsymbol{e}'_{\alpha}\mathrm{e}^{\boldsymbol{T}^{(k)}x}\boldsymbol{t}^{(k)}$. Then further condition on the amount of the overshoot to be x. The event of interest becomes the probability that S_t downcrosses $-u$ the first time in state j given that downcrossing $-u$ did not happen before and now the state of economy is k and S is at a height of x above 0. This

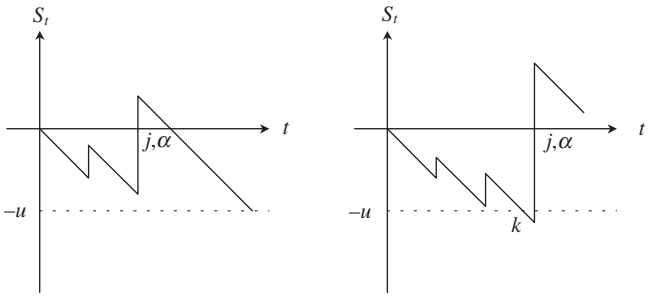

Fig. 2. The decomposition of $\pi^{+}_{i,j\alpha}$.

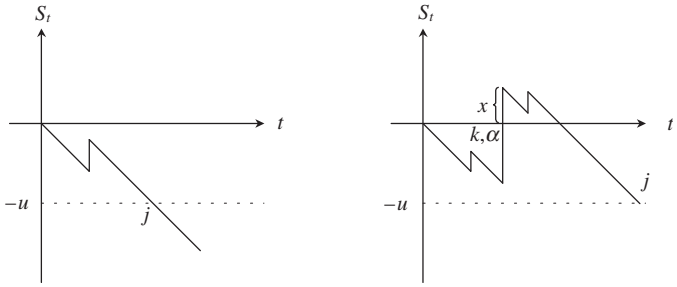

Fig. 3. The decomposition of $\pi^-_{i,j}(u)$.

probability is the same as the probability that starting with $J_0 = k, S_0 = 0$, S_t first descends level $-(x+u)$ in state j, which equals

$$\boldsymbol{e}'_k \mathrm{e}^{\boldsymbol{Q}(u+x)}\boldsymbol{e}_j.$$

Thus the sum in (11) equals

$$\sum_{k\in\mathscr{M}}\sum_{\alpha\in E^{(k)}} p^+_{i,k\alpha}(u)\int_0^\infty \boldsymbol{e}'_\alpha \mathrm{e}^{\boldsymbol{T}^{(k)}x}\boldsymbol{t}^{(k)}\boldsymbol{e}'_k \mathrm{e}^{\boldsymbol{Q}(u+x)}\boldsymbol{e}_j\,\mathrm{d}x.$$

The integral is the probability that starting at level u of an overshoot in state k, phase $\alpha\in E^{(k)}$, S_t first descends level 0 in state j. This can be evaluated as $(\boldsymbol{e}'_\alpha\otimes\boldsymbol{e}'_k)(-\boldsymbol{T}^{(k)}\oplus\boldsymbol{Q})^{-1}(\boldsymbol{t}^{(k)}\otimes \mathrm{e}^{\boldsymbol{Q}u})\boldsymbol{e}_j$, which is denoted by $n^+_{k\alpha,j}{}^-(u)$. Hence (11) becomes

$$\pi^-_{i,j}(u) = p^-_{i,j}(u) + \sum_{k\in\mathscr{M}}\sum_{\alpha\in E^{(k)}} p^+_{i,k\alpha}(u)n^+_{k\alpha,j}{}^-(u). \tag{12}$$

Rewriting in matrix notation and denoting l as the dimension of E, (10) and (12) become

$$\boldsymbol{\pi}^+ = \boldsymbol{p}^+(u) + \boldsymbol{p}^-(u)\boldsymbol{N}^{-+}(u),$$
$$\boldsymbol{\pi}^-(u) = \boldsymbol{p}^-(u) + \boldsymbol{p}^+(u)\boldsymbol{N}^{+-}(u)$$

and the unique solution is given by

$$\boldsymbol{p}^-(u) = [\boldsymbol{\pi}^-(u) - \boldsymbol{\pi}^+\boldsymbol{N}^{+-}(u)][\boldsymbol{I}_{d\times d} - \boldsymbol{N}^{-+}(u)\boldsymbol{N}^{+-}(u)]^{-1},$$
$$\boldsymbol{p}^+(u) = [\boldsymbol{\pi}^+ - \boldsymbol{\pi}^-(u)\boldsymbol{N}^{-+}(u)][\boldsymbol{I}_{l\times l} - \boldsymbol{N}^{+-}(u)\boldsymbol{N}^{-+}(u)]^{-1},$$

which is Theorem 6.1 of Asmussen and Perry [3].

Finally, let $x, y\geqslant 0$ and define

$$p^+_{i,j\alpha}(x,y) = \mathbb{P}_i(S_t \text{ upcrosses } x \text{ in state } j,\ \text{phase } \alpha\in E^{(j)}$$
$$\text{before downcrossing } -y).$$

For $0<z<u$, consider Fig. 4 for the decomposition of $p^+_{i,j\alpha}(u)$:

$$p^+_{i,j\alpha}(u) = p^+_{i,j\alpha}(z) + \sum_{k\in\mathscr{M}} \mathbb{P}_i(S_t \text{ downcrosses } -z \text{ in state } k \text{ before}$$
$$\text{upcrossing } 0)\, p^+_{k,j\alpha}(z,u-z). \tag{13}$$

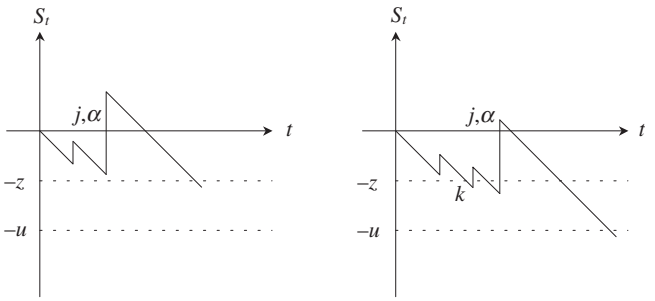

Fig. 4. The decomposition of $p^+_{i,j\alpha}(u)$.

It follows from the definition of $p^+_{i,j\alpha}(x,y)$ and $p^-_{i,j}(u)$ that (13) can be rewritten as

$$p^+_{i,j\alpha}(u) = p^+_{i,j\alpha}(z) + \sum_{k\in\mathcal{M}} p^-_{i,k}(z)p^+_{k,j\alpha}(z,u-z). \tag{14}$$

Rewriting (14) in matrix notation and letting $u = x + y$ and $z = x$,

$$\boldsymbol{p}^+(x+y) = \boldsymbol{p}^+(x) + \boldsymbol{p}^-(x)\boldsymbol{p}^+(x,y).$$

Assuming the invertibility of $\boldsymbol{p}^-(x)$,

$$\boldsymbol{p}^+(x,y) = [\boldsymbol{p}^-(x)]^{-1}[\boldsymbol{p}^+(x+y) - \boldsymbol{p}^+(x)].$$

6. The joint distribution of surplus before and after ruin in the case of phase-type claims

In this section, we shall derive explicit formulae for $f_{ij}(u,x,y)$, $F_{ij}(u,\infty,y)$ and $F_i(u,\infty,y)$ in the case of phase-type claims. The barrier probabilities and the distribution of the first overshoot of $\{S_t\}$ above level zero obtained in the Section 5 will be the basic building blocks. The dual process $\{\check{S}_t\}$ defined similar to $\{S_t\}$ but with the state of economy $\{J_t\}$ replaced by the time-reversed version $\{\check{J}_t\}$ will also be used. All barrier probabilities and other symbols related to the time-reversed version $\{\check{S}_t\}$ will be labelled with the notation $\check{}$. Recall the notation $\boldsymbol{s}(x) = \mathrm{diag}(\beta_i b_i(x))$ defined in Theorem 2.

Theorem 4. *Let $\boldsymbol{f}(u,x,y)$ be a matrix with ijth element $f_{ij}(u,x,y)$. Assuming the invertibility of the appropriate matrices, for $u \leqslant x$,*

$$\boldsymbol{f}(u,x,y) = [\boldsymbol{p}^-(u)]^{-1}\mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{s}(x+y)$$

for $u > x$,

$$\boldsymbol{f}(u,x,y) = [\Delta\boldsymbol{p}^-(u)]^{-1}[\check{\boldsymbol{p}}^-(y)\boldsymbol{s}(x+y)\check{\boldsymbol{p}}^+(u-x,x)\check{\boldsymbol{N}}^{+-}(u)]'\Delta[\boldsymbol{p}^-(y)]^{-1}.$$

Proof. If $x \geqslant u$, let $\boldsymbol{f}(u,x)$ be a matrix with ijth element

$$f_{ij}(u,x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_i(\tau(u) < \infty, J_{\tau(u)} = j, R_{\tau(u)^-} \leqslant x | R_0 = u).$$

Consider Fig. 5 for the decomposition of $f_{ij}(0,x)$:

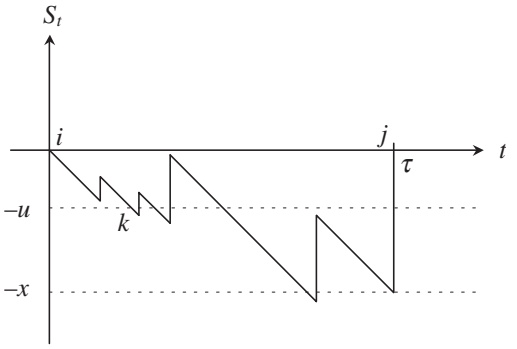

Fig. 5. The decomposition of $f_{ij}(0,x)$.

Conditioning on J_t at the first time S_t downcrosses $-u$ before overshooting 0,

$$f_{ij}(0,x)=\sum_{k\in\mathcal{M}} p^{-}_{i,k}(u)f_{kj}(u,x).$$

Rewriting in matrix notation and assuming the invertibility of $\boldsymbol{p}^{-}(u)$,

$$\boldsymbol{f}(u,x)=[\boldsymbol{p}^{-}(u)]^{-1}\boldsymbol{f}(0,x),$$

where by (6),

$$\begin{aligned}\boldsymbol{f}(0,x)&=-\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{G}(0,x,0)\\&=\mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{S}((x,\infty)).\end{aligned}$$

Hence $f_{ij}(u,x,y)$ can be obtained from

$$f_{ij}(u,x,y)=f_{ij}(u,x)\,\frac{b_j(x+y)}{\bar{B}_j(x)}.$$

In matrix notation,

$$\begin{aligned}\boldsymbol{f}(u,x,y)&=\boldsymbol{f}(u,x)\,\mathrm{diag}\left(\frac{b_j(x+y)}{\bar{B}_j(x)}\right)\\&=[\boldsymbol{p}^{-}(u)]^{-1}\mathrm{e}^{\check{\boldsymbol{K}}x}\,\mathrm{diag}(\beta_j\bar{B}_j(x))\,\mathrm{diag}\left(\frac{b_j(x+y)}{\bar{B}_j(x)}\right)=[\boldsymbol{p}^{-}(u)]^{-1}\mathrm{e}^{\check{\boldsymbol{K}}x}\boldsymbol{s}(x+y).\end{aligned}$$

If $x<u$, let $\tau=\tau(0)$ be the time of ruin with initial zero reserve as usual and T be the time of recovery, that is, the time when S_t first downcrosses 0 after upcrossing 0. In the classical compound Poisson risk model, Dickson [4] obtained $f(u,x,y)$ by time-reversion. Consider stationary version of J_t and $\check{J}_t$, we may then assume $\check{J}_v=J_{T-v}$ and $\check{S}_v=-S_{T-v}$ where $v\in[0,T]$. Let x,y,i and j be fixed. Consider the event A that $J_0=i$ and S_t downcrosses $-u$ before overshooting 0, overshoots 0 with $-S_{\tau-}\in[x,x+\mathrm{d}x]$ and $S_\tau\in[y,y+\mathrm{d}y]$ at τ, and S_t does not overshoot y before downcrossing 0 at state j for any $\tau<t<T$. The dual event of A is the event that $\check{J}_0=j$, S_t cannot descend below level $-u$ before $\check{\tau}$, $-\check{S}_{\check{\tau}-}\in[y,y+\mathrm{d}y]$, $\check{S}_{\check{\tau}}\in[x,x+\mathrm{d}x]$, and $\check{S}_t$ has to overshoot u before recovering at state of economy i. One sample path realization of event A and its dual sample path is shown in Fig. 6.

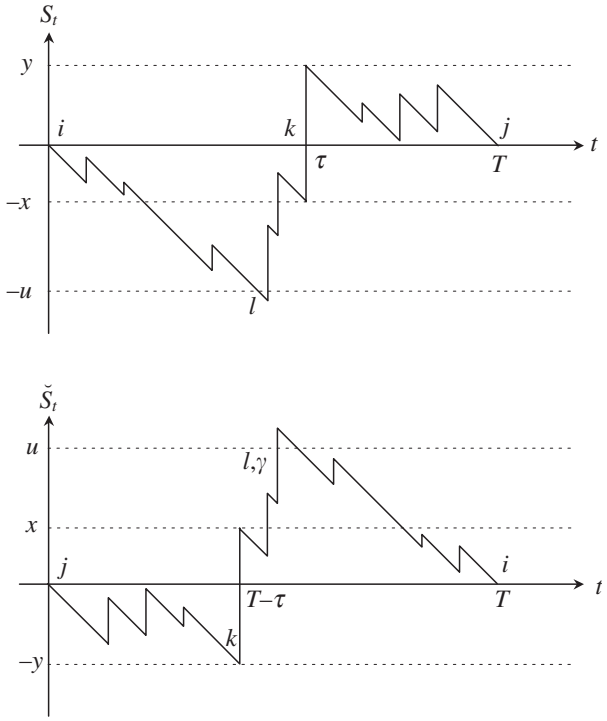

Fig. 6. The dual sample path.

Mathematically,

$$\begin{aligned}
\pi_i \mathbb{P}_i(A) &= \mathbb{P}_{\boldsymbol{\pi}}(J_0 = i, A) \\
&= \mathbb{P}_{\boldsymbol{\pi}}(J_0 = i, S_v < 0 \ \forall v \in (0,\tau), \exists \zeta \in (0,\tau) \text{ such that } S_\zeta = -u, \\
&\quad - S_{\tau^-} \in [x, x + \mathrm{d}x], S_\tau \in [y, y + \mathrm{d}y], S_v < y \ \forall v \in (\tau, T), J_T = j) \\
&= \mathbb{P}_{\boldsymbol{\pi}}(\check{J}_0 = j, \check{S}_v > -y \ \forall v \in (0, T-\tau), -\check{S}_{T-\tau^-} \in [y, y + \mathrm{d}y], \\
&\quad \check{S}_{t-\tau} \in [x, x + \mathrm{d}x], \check{S}_v > 0 \ \forall v \in (T-\tau, T), \\
&\quad \exists \zeta \in (T-\tau, T) \text{ such that } \check{S}_\zeta = u, \check{J}_T = i) \\
&= \pi_j \mathbb{P}_j(\check{S}_v > -y \ \forall v \in (0, T-\tau), -\check{S}_{T-\tau^-} \in [y, y + \mathrm{d}y], \\
&\quad \check{S}_{t-\tau} \in [x, x + \mathrm{d}x], \check{S}_v > 0 \ \forall v \in (T-\tau, T), \\
&\quad \exists \zeta \in (T-\tau, T) \text{ such that } \check{S}_\zeta = u, \check{J}_T = i). \qquad (15)
\end{aligned}$$

Given that $J_0 = i$, the probability that S_t downcrosses $-u$ the first time in state of economy l before overshooting 0 is $p^-_{i,l}(u)$. Given that $R_0 = u$ and $J_0 = l$, the probability that $R_{\tau(u)^-} \in [x, x + \mathrm{d}x]$, $-R_{\tau(u)} \in [y, y + \mathrm{d}y]$ and $J_\tau(u) = k$ is $f_{lk}(u, x, y)\,\mathrm{d}x\,\mathrm{d}y$. Given that now the state of economy is k and $S_t = y$, the probability that S_t will downcross 0 in state of economy j before overshooting y is the same as the probability that given the state of economy is k and $S_0 = 0$, the probability that S_t will downcross $-y$ in state of economy j before overshooting 0, which is $p^-_{k,j}(y)$. Thus conditioning on $J_t = l$ when S_t first downcrosses $-u$ and $J_t = k$ when S_t overshoots 0 and then summing over all l and $k \in \mathscr{M}$,

the probability on the left-hand side of (15) can be written as

$$\pi_i \sum_{l,k\in\mathscr{M}} p^-_{i,l}(u) f_{lk}(u,x,y)\,\mathrm{d}x\,\mathrm{d}y\,p^-_{k,j}(y).$$

Similarly, conditioning on $\check{J}_t = k$ when $\check{S}_t$ attains $-y$ and $\check{J}_t = l$ and the phase of the claim to be $\gamma \in E^{(l)}$ when $\check{S}_t$ upcrosses u for the first time after attaining x and then summing over all $k, l \in \mathscr{M}$ and $\gamma \in E^{(l)}$, the probability on the right-hand side of (15) can be written as

$$\pi_j \sum_{k\in\mathscr{M}} \left[\check{p}^-_{j,k}(y)\beta_k \boldsymbol{\alpha}^{(k)} \mathrm{e}^{\boldsymbol{T}^{(k)}(x+y)} \boldsymbol{t}^{(k)}\,\mathrm{d}x\,\mathrm{d}y \sum_{l,\gamma\in E^{(l)}} \check{p}^+_{k,l_\gamma}(u-x,x)\check{n}^{+-}_{l_\gamma,i}(u) \right].$$

Rewrite (15) in matrix notation,

$$\Delta \boldsymbol{p}^-(u)\boldsymbol{f}(u,x,y)\boldsymbol{p}^-(y) = [\check{\boldsymbol{p}}^-(y)\boldsymbol{s}(x+y)\check{\boldsymbol{p}}^+(u-x,x)\check{\boldsymbol{N}}^{+-}(u)]'\Delta.$$

Assuming the invertibility of $\boldsymbol{p}^-(u)$, $\boldsymbol{p}^-(y)$ and $\check{\boldsymbol{p}}^-(u-x)$ (due to the presence of $\check{\boldsymbol{p}}^+(u-x,x)$), the joint density can be obtained by

$$\boldsymbol{f}(u,x,y) = [\Delta \boldsymbol{p}^-(u)]^{-1}[\check{\boldsymbol{p}}^-(y)\boldsymbol{s}(x+y)\check{\boldsymbol{p}}^+(u-x,x)\check{\boldsymbol{N}}^{+-}(u)]'\Delta[\boldsymbol{p}^-(y)]^{-1}.$$

The joint density of surplus before and after ruin can be obtained by summing over all j. □

The necessary condition for the invertibility of the matrices is hard to establish, but we shall consider a numerical example to illustrate that the assumption is not fictitious. Although the formula for $x<u$ is not as explicit as that for $x\geqslant u$, it is easy to program using mathematical languages like MATLAB.

From the above results on the joint distribution of the surplus before and after ruin, by taking proper limits, we can obtain the closed form solutions for the distribution of the surplus before ruin and the distribution of the deficit at ruin. But for the distribution of the deficit at ruin, we can obtain a very explicit result using a simpler argument. Recall the $j\alpha$th unit column vector $\boldsymbol{e}_{j\alpha}$ in (9). Let $\boldsymbol{e}^j = \sum_{\alpha\in E^{(j)}} \boldsymbol{e}_{j\alpha}$ be a column vector with entries equal to 1 at positions $j\alpha$ for all $\alpha \in E^{(j)}$ and $\boldsymbol{I}^j$ be the matrix formed by placing the elements of $\boldsymbol{e}^j$ on the main diagonal and letting all other entries equal to 0.

Theorem 5. *The distribution of the deficit at ruin starting with initial surplus u and state of economy i and ruins at state of economy j is given by*

$$F_{ij}(u,\infty,y) = \boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}^j - \boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{I}^j\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}$$

and the distribution of the deficit at ruin starting with initial surplus u and state of economy i is given by

$$F_i(u,\infty,y) = \boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e} - \boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}.$$

Proof. Consider the event that ruin occurs and the deficit is greater than y, that is, the overshoot of S_t above level u is greater than y. This is the same as the event $\tau(u)<\infty$ and $S_{\tau(u)}>u+y$. To calculate this event, we further partition this into disjoint events by

considering the phase of the claim when S_t first upcrosses level u:

$$\begin{aligned}&\mathbb{P}_i(\tau(u)<\infty, J_{\tau(u)}=j, R_{\tau(u)}\leqslant -y|R_0=u)\\&\quad=\mathbb{P}_i(\tau(u)<\infty, J_{\tau(u)}=j, S_{\tau(u)}>u+y)\\&\quad=\sum_{\alpha\in E^{(j)}}\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}_{j\alpha}\times\boldsymbol{e}'_{j\alpha}\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}=\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{I}^j\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}.\end{aligned}$$

Hence

$$\begin{aligned}F_{ij}(u,\infty,y)&=\mathbb{P}_i(\tau(u)<\infty, J_{\tau(u)}=j, R_{\tau(u)}>-y|R_0=u)\\&=\psi_{ij}(u)-\mathbb{P}_i(\tau(u)<\infty, J_{\tau(u)}=j, R_{\tau(u)}\leqslant -y|R_0=u)\\&=\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}^j-\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{I}^j\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}\end{aligned}$$

and the distribution of the deficit at ruin starting with initial surplus u and state of economy i is given by

$$F_i(u,\infty,y)=\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\boldsymbol{e}-\boldsymbol{\theta}^{(i)}\mathrm{e}^{\boldsymbol{U}u}\mathrm{e}^{\boldsymbol{T}y}\boldsymbol{e}. \qquad \square$$

7. Numerical illustration

In this section, we shall consider one numerical example for the calculation of the joint density function of surplus before and immediately after ruin for a Markov-modulated risk model with three states of economy. Suppose that

$$\boldsymbol{\Lambda}=\begin{bmatrix}-\frac{1}{3} & \frac{1}{9} & \frac{2}{9}\\ \frac{1}{9} & -\frac{1}{3} & \frac{2}{9}\\ \frac{1}{6} & 0 & -\frac{1}{6}\end{bmatrix} \quad\text{and}\quad \boldsymbol{\beta}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ 1\end{bmatrix}.$$

In state of economy 1, the claims sizes are exponentially distributed with mean 1. In state of economy 2, the claim sizes are exponentially distributed with mean 6. In state of economy 3, the claim sizes are hyperexponentially distributed with two channels and the density is

$$\tfrac{3}{4}\mathrm{e}^{-x}+\tfrac{1}{2}\mathrm{e}^{-2x}.$$

The stationary distribution of the continuous-time Markov chain is

$$\boldsymbol{\pi}=\begin{bmatrix}\frac{9}{28} & \frac{3}{28} & \frac{4}{7}\end{bmatrix}$$

and $\sum_{i=1}^{3}\pi_i\beta_i\mu_{B_i}=\frac{7}{8}$ which means that the relative security loading is $\frac{1}{7}$.

By using the iteration scheme, it is found that, up to five decimal places of accuracy,

$$\boldsymbol{Q}=\begin{bmatrix}-0.46500 & 0.14747 & 0.31753\\ 0.21378 & -0.56527 & 0.35149\\ 0.33403 & 0.02722 & -0.36125\end{bmatrix},$$

$$\breve{\boldsymbol{Q}}=\begin{bmatrix}-0.46524 & 0.05651 & 0.40874\\ 0.45329 & -0.56831 & 0.11502\\ 0.27141 & 0.08656 & -0.35797\end{bmatrix},$$

$$\boldsymbol{\pi}^{+} = \begin{bmatrix} 0.36809 & 0.23991 & 0.21527 & 0.02250 \\ 0.05840 & 0.59014 & 0.19750 & 0.02097 \\ 0.06325 & 0.12940 & 0.59188 & 0.10841 \end{bmatrix},$$

$$\check{\boldsymbol{\pi}}^{+} = \begin{bmatrix} 0.36833 & 0.20535 & 0.24481 & 0.02637 \\ 0.10908 & 0.60837 & 0.12382 & 0.01067 \\ 0.05361 & 0.14542 & 0.58908 & 0.10817 \end{bmatrix},$$

$$\boldsymbol{U} = \begin{bmatrix} -0.63191 & 0.23991 & 0.21527 & 0.02550 \\ 0.00973 & -0.06831 & 0.03292 & 0.00350 \\ 0.06325 & 0.12940 & -0.40812 & 0.10841 \\ 0.12650 & 0.25880 & 1.18376 & -1.78318 \end{bmatrix},$$

$$\check{\boldsymbol{U}} = \begin{bmatrix} -0.63167 & 0.20535 & 0.24481 & 0.02637 \\ 0.01818 & -0.06527 & 0.02064 & 0.00178 \\ 0.05361 & 0.14542 & -0.41092 & 0.10817 \\ 0.10722 & 0.29085 & 1.17816 & -1.78366 \end{bmatrix}.$$

The matrices $\boldsymbol{N}^{+-}(u)$, etc. can be found by the formulae in Section 5, say, $\check{\boldsymbol{\pi}}^{-}(u) = \mathrm{e}^{\check{\boldsymbol{Q}}u}$ and $\boldsymbol{N}^{-+}(u) = \boldsymbol{\pi}^{+}\mathrm{e}^{\boldsymbol{U}u}$.

By summing up each row of $\boldsymbol{\pi}^{+}$, the ruin probabilities with zero initial reserve are found to be $\psi_1(0) = 0.8458$, $\psi_2(0) = 0.8670$ and $\psi_3(0) = 0.8929$.

Figs. 7 and 8 are some graphs of the joint density function $f_{ij}(u, x, y)$ for various combinations of u, i and j. Notice that for each graph there is a ridge at the line $x = u$ because of the structural change of the joint density function. From Fig. 7, we note that

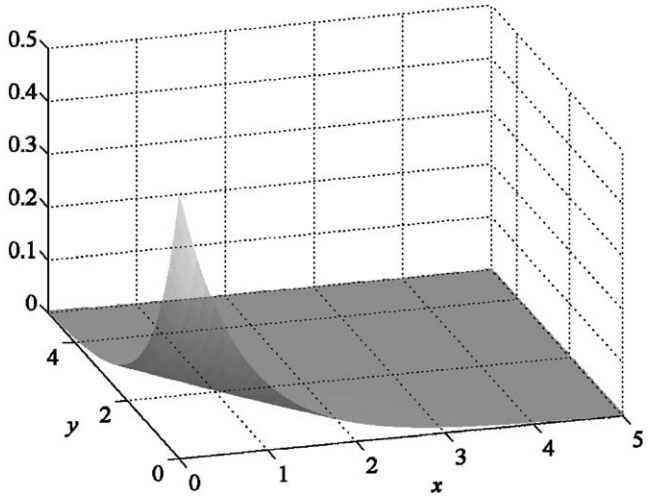

Fig. 7. $f_{11}(0, x, y)$.

$f_{11}(0,x,y)$ has a unique mode and is a decreasing function in both x and y. We observe, from Fig. 8, that $f_{23}(2,x,y)$ has two modes, although one of them is much smaller. This bimodal feature becomes more obvious in Figs. 9 and 10.

Figs. 9 and 10 are the graphs of the joint density function $f_i(u,x,y)$ for $u=1, i=1$ and $u=2, i=2$.

Fig. 11 is the graph of the distribution of the deficit at ruin $F_i(u,\infty,y)$ for $u=1$, $i=1$ and $u=2, i=2$. From the figures we see that $F_2(2,\infty,y)$ has heavier tail than $F_1(1,\infty,y)$.

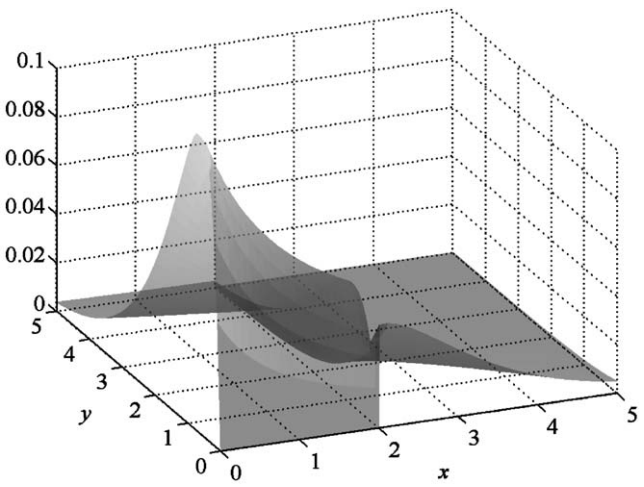

Fig. 8. $f_{23}(2,x,y)$.

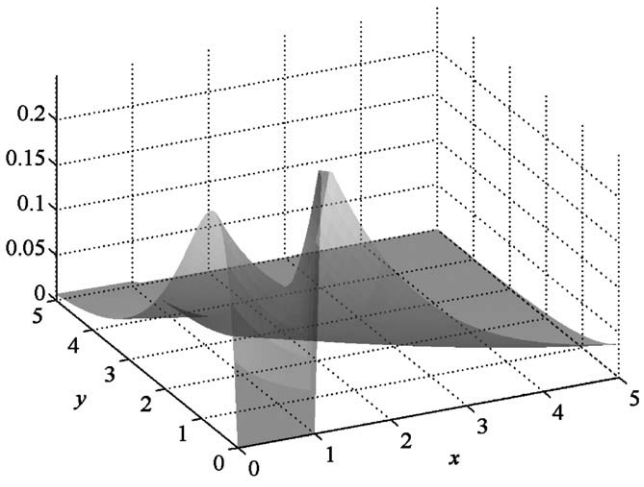

Fig. 9. $f_1(1,x,y)$.

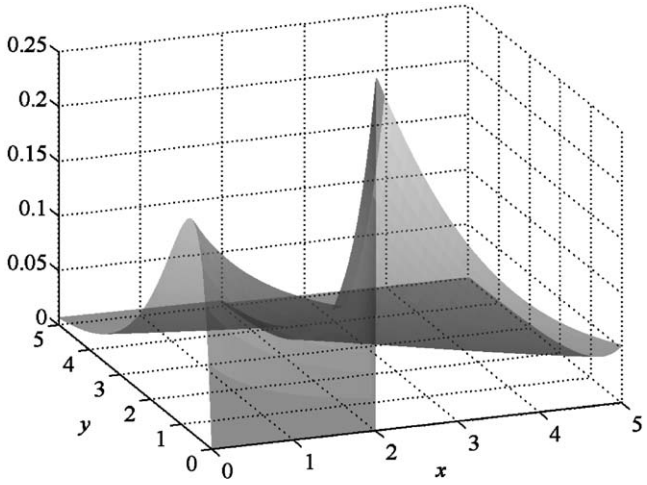

Fig. 10. $f_2(2, x, y)$.

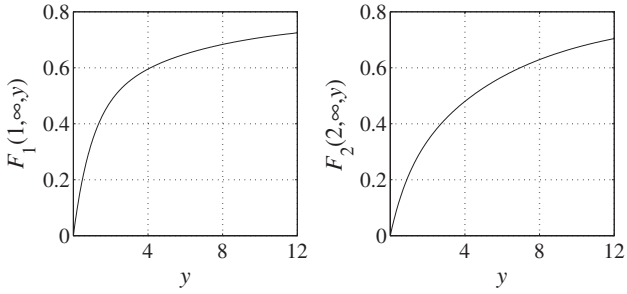

Fig. 11. $F_1(1, \infty, y)$ and $F_2(2, \infty, y)$.

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References

[1] S. Asmussen, Risk theory in a Markovian environment, Scand. Actuar. J. 1989 (1989) 69–100.
[2] S. Asmussen, Ruin Probabilities, World Scientific, Singapore, 2000.
[3] S. Asmussen, D. Perry, On cycle maxima, first passage problems and extreme value theory for queues, Stochastic Models 8 (1992) 421–458.
[4] D.C.M. Dickson, On the distribution of the surplus prior to ruin, Insurance: Math. Econom. 11 (1992) 191–207.
[5] H.U. Gerber, An Introduction to Mathematical Risk Theory, S.S. Huebner Foundation Monograph Series No. 8, R. Irwin, Homewood, IL, 1979.
[6] H.U. Gerber, E.S.W. Shiu, On the time value of ruin, North American Actuar. J. 2 (1) (1998) 48–78.

[7] A. Graham, Kronecker Products and Matrix Calculus with Applications, Ellis Horwood Series in Mathematics and its Applications, Chichester, Horwood, 1981.
[8] J. Grandell, Aspects of Risk Theory, Springer, New York, 1990.
[9] M.R. Hardy, A regime-switching model of long-term stock returns, North American Actuar. J. 5 (2) (2001) 41–53.
[10] S. Li, J. Garrido, On ruin for the Erlang(n) risk process, Insurance: Math. Econom. 34 (2004) 391–408.
[11] S. Li, J. Garrido, On a general class of renewal risk process: analysis of the Gerber–Shiu function, Adv. Appl. Probab. 37 (2005) 836–856.
[12] M.F. Neuts, Matrix-geometric Solutions in Stochastic Models, Johns Hopkins University Press, Baltimore, London, 1981.
[13] T. Rolski, H. Schmidli, V. Schmidt, J. Teugels, Stochastic Processes for Insurance and Finance, Wiley, New York, 1999.