Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 125 (2015) 1009–1031

stochastic processes and their applications

www.elsevier.com/locate/spa

Asymptotic expansions for SDE's with small multiplicative noise

Sergio Albeverio [b,a,*], Boubaker Smii [a]

[a] *King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran 31261, Saudi Arabia*

[b] *Department of Applied Mathematics, University of Bonn, HCM, BiBoS, IZKS, Germany*

Received 4 April 2014; received in revised form 9 September 2014; accepted 10 September 2014
Available online 30 September 2014

Abstract

Asymptotic expansions are derived as power series in a small coefficient entering a nonlinear multiplicative noise and a deterministic driving term in a nonlinear evolution equation. Detailed estimates on remainders are provided.

Keywords: SDEs; Asymptotic expansions; Processes driven by multiplicative Lévy noise; Evolution equations

1. Introduction

A description of the evolution of dynamical systems of concern in disciplines like physics, biology, chemistry, ecology, geology, engineerings, economics in terms of differential equations is often appropriate. Sometimes it is natural to investigate to which extent an external small perturbation (forcing) can change the deterministic evolution. This can be discussed in the sense of asymptotic expansions in powers of a small parameter in front of the perturbation.

This problem has been studied in particular for the case where the perturbation is an additive noise of the Brownian or, more generally, Lévy type. In the case of evolution equations in a Hilbert space with global Lipschitz coefficient and Brownian additive noise, see [49]. For

* Corresponding author at: Department of Applied Mathematics, University of Bonn, HCM, BiBoS, IZKS, Germany. Tel.: +49 228733418.

E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), boubaker@kfupm.edu.sa (B. Smii).

http://dx.doi.org/10.1016/j.spa.2014.09.009

stochastic partial differential equations, related to evolutions on a Hilbert resp. Banach space, the problem has been discussed with non globally Lipschitz coefficients in a situation of dissipativity in [3] for the case where the additive noise is given by Brownian motion, and in [15] for the case of additive Lévy type noise. For related work determining the invariant measures in such cases, see, [4] resp. [3,5,6], (see also [50] for the special case of globally Lipschitz coefficients and additive Gaussian white noise).

In the present work we consider the finite dimensional case with multiplicative noise of Gaussian or Lévy type. Even for this relatively simple case to the best of our knowledge rigorous mathematical results seem quite scarce, despite the conceptual importance of the perturbation problem in relation, e.g., to classical mechanics. Rigorous "perturbation theory" is either limited to (general) linear systems and associated semi groups in Hilbert spaces, where a rich mathematical theory has been developed, particularly in connection with spectral problems in quantum theory (see, e.g., [35,45,51,56]), or else to particular nonlinear cases (see, e.g., [32,34,42,57]). For further motivations, mainly from physics,biology, engineerings and mathematical finance, see, e.g., [37,47,62].

A classical area where asymptotic perturbation methods originally arise is classical celestial mechanics (since the work by, e.g., S. Laplace, S. Poisson, C.F. Gauss, H. Poincaré). Here nonlinear and singularity effects are essential and particular methods have been developed, see, e.g., [25,27]. These are also related to perturbation theory around the solutions of the classical motion of harmonic oscillators, see, e.g., [28,29,57]. The stochastic case of Hamiltonian systems is studied in [11,10,63,64].

Perturbation theory in infinite dimensional systems has been studied in connection with hydrodynamics(small viscosity expansions, see, e.g. [7], small time expansions [32]), quantum field theory [8,9,12,35,42,40,41,60], neurobiological systems [2,3,15,53,61].

Let us also briefly mention connections with Laplace and stationary phase methods, see, e.g. [1,12,13,17,20,30,61].

The present paper considers deterministic, resp. stochastic finite dimensional differential equations, which are first order in time, and have smooth coefficients satisfying growth restrictions. The driving multiplicative forcing term resp. noise is of the general Lévy type. An asymptotic expansion in powers of a small parameter on which the diffusion coefficient depends is exhibited with good detailed control on remainders.

Some possible applications are mentioned at the end.

The structure of this paper is as follows:

In Section 2 we present the concrete small noise expansion (once it is assumed to exist) of the original stochastic differential equation, in terms of solutions of linear random differential equations, assuming that solutions exist and are unique.

In Section 3 we discuss existence and uniqueness of solutions of the original SDE. In Section 4 we discuss the solutions of the random equations for the expansion coefficients.

In Section 5 we prove the asymptotic character of the expansion. Section 6 is reserved to some comments on applications.

2. Small noise expansion of an evolution equation

We consider the evolution equation:

$$\begin{cases} u(t) = u(0) + \int_0^t \beta\,(u(s))\,ds + \int_0^t \sigma_\varepsilon\,(u(s))\,\eta(ds) \\ u(0) = u^0, \qquad u(t) \in \mathbb{R}^d, \quad t \in [0,\infty),\ \varepsilon > 0. \end{cases} \tag{1}$$

$\beta(.) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $d \in \mathbb{N}$, $\sigma_\varepsilon(.) : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are measurable functions resp. $d \times d$ matrix functions satisfying some additional assumptions (e.g., globally Lipschitz conditions).

η can be a signed bounded variation measure (in which case the integral is understood as a Stieltjes integral) or the heuristic derivative of a Lévy process in $\mathbb{R}^d$ (in which case the integral should be interpreted as a stochastic integral). For simplicity of notations we use the unified notation $\eta(ds)$, e.g., $\eta(ds) = dB(s)$, if B is a Brownian motion. Moreover if η has a jump component $u(s)$ in (1) should be understood as $u(s^-)$. We hope the meaning is always clear from the context in which we operate. See Section 3 for precise assumptions.

Our purpose is to show that under certain assumptions on $\beta, \sigma_\varepsilon, \eta$ the solution $u = u_\varepsilon$ of the evolution equation (1), assumed to exist, can be written as:

$$u_\varepsilon(t) = u_0(t) + \varepsilon u_1(t) + \cdots + \varepsilon^m u_m(t) + R_m(t, \varepsilon), \tag{2}$$

with $u_i : [0, \infty) \longrightarrow \mathbb{R}^d$ measurable and $\|R_m(t, \varepsilon)\| \leq C_m(t)\varepsilon^{m+1}$, for all $m \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small, for some $C_m(t) > 0$ independent of ε. Here $\|.\|$ denotes the norm in $\mathbb{R}^k$, for any $k \in \mathbb{N}$.

To obtain the desired expansion we shall assume that there are Taylor expansions of $\beta(x)$ and $\sigma_\varepsilon(x)$ in their variable $x \in \mathbb{R}^d$ and, moreover, $\varepsilon \longrightarrow \sigma_\varepsilon(x)$ is $\mathcal{C}^{M+1}$, $M \in \mathbb{N}$, for every fixed $x \in \mathbb{R}^d$.

Let us first introduce some useful notations:

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ (with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N} = \{1, 2, \ldots\}$), and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define:

- The length of α by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.
- $\alpha! := \alpha_1!\alpha_2! \cdots \alpha_d!$
- $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$.

The derivative of a function f of order $|\alpha| \in \mathbb{N}_0$ is defined by:

$$f^{(\alpha)} = D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}, \qquad D^0 f = f. \tag{3}$$

We have the following lemma:

Lemma 2.1. *Let f be a complex-valued function in $C^{p+1}(B(x_0, r))$, $r > 0$, $x_0 \in \mathbb{R}^d$, for some $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, where $B(x_0, r)$ is the open ball in $\mathbb{R}^d$ of center x_0 and radius r.*

Then for any $x \in B(x_0, r)$ we have Taylor's expansion formula:

$$f(x) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_p\left(f^{(p+1)}(x_0, x)\right), \tag{4}$$

with $D^\alpha f(x_0)$ the evaluation of $D^\alpha f$ at $x = x_0$ and $R_p\left(f^{(p+1)}(x_0, x)\right) = \sum_{|\alpha|=p+1} (x - x_0)^\alpha D^\alpha f(x_0 + \tau(x - x_0))$, with $\tau \in (0, 1)$. Alternatively

$$R_p\left(f^{(p+1)}(x_0, x)\right) = \sum_{|\alpha|=p+1} \frac{(x - x_0)^\alpha}{p!} \left(\int_0^1 (1 - s)^p D^\alpha f(x_0 + s(x - x_0))\, ds\right).$$

Moreover, setting for $|\alpha| = p+1$*:*
$C_p(x_0, x) = \frac{1}{p!}\left(\int_0^1 (1-s)^p \|D^\alpha f\,(x_0 + s(x-x_0))\,\| ds\right)$*, we have the bound*

$$\| R_p\left(f^{(p+1)}(x_0, x)\right)\| \le C_p(x_0, x)\|x - x_0\|^{p+1}, \tag{5}$$

with $\|C_p(x_0,\ x)\| \le \frac{1}{p+1}\ \sup_{s\in[0,1]} \|D^\alpha\ f(x_0 + s(x-x_0))\|$.

Proof. This is an easy consequence of, e.g. [38]. □

Using the previous lemma, we have the following:

Proposition 2.2. *Let* $u(\varepsilon)$ *be a* $\mathcal{C}^{N+1}$ *function of* $0 \le \varepsilon \le \varepsilon_0$ *for some* $N \in \mathbb{N}_0$*, with values in* $\mathbb{R}^d$*. Then*

$$u(\varepsilon) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots + \varepsilon^N u_N + R_N^u(\varepsilon), \tag{6}$$

with $u_i \in \mathbb{R}^d$*, independent of* ε*, and* $\|R_N^u(\varepsilon)\| \le C_N^u \varepsilon^{N+1}$*, with* $0 < C_N^u \le \tilde{C}_N^u := \frac{1}{N+1}\ \sup_{s\in[0,1]}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \|D^{N+1}\,u(s\,\varepsilon)\|$*, where* $\tilde{C}_N^u$ *is independent of* ε*. Moreover for any* $f \in C^{p+1}(\mathbb{R}^d)$*, with* $p \in \mathbb{N}_0$*, we have:*

$$\begin{aligned} f\,(u(\varepsilon)) &= \sum_{|\alpha|\le p} \frac{D^\alpha f\,(u_0)}{\alpha!}\,(u(\varepsilon) - u_0)^\alpha + R_p^{u(\varepsilon)}\left(f^{(p+1)}\,(u_0, u(\varepsilon))\right) \\ &= \sum_{|\alpha|\le p} \frac{D^\alpha f\,(u_0)}{\alpha!}\left(\sum_{l=1}^N \varepsilon^l u_l + R_N^u(\varepsilon)\right)^\alpha + R_p^{u(\varepsilon)}\left(f^{(p+1)}\,(u_0, u(\varepsilon))\right), \end{aligned} \tag{7}$$

with $R_p^{u(\varepsilon)}\left(f^{(p+1)}\,(u_0, u(\varepsilon))\right)$ *defined as* R_p *in* (4) *with* $f(x)$ *replaced by* $f(u(\varepsilon))$*.*
On $R_p^{u(\varepsilon)}\left(f^{(p+1)}\,(u_0, u(\varepsilon))\right)$ *we have the bound:*

$$\begin{aligned} R_p^{u(\varepsilon)}\left(f^{(p+1)}\,(u_0, u(\varepsilon))\right) &\le\ \varepsilon^{p+1}\,C_p(u_0,\ u(\varepsilon))\left\|\sum_{j=1}^p\ u_j\,\varepsilon^j\ +\ R_p^u(\varepsilon)\right\|^{p+1} \\ &\le \varepsilon^{p+1}\,C_p(u_0,\ u(\varepsilon))\left(\sum_{j=1}^p \|u_j\|\,\varepsilon_0^j + C_p^u\,\varepsilon_0^{p+1}\right)^{p+1}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &0 < C_p(u_0,\ u(\varepsilon)) \\ &\quad\le \frac{1}{p+1}\ \sup_{s\in[0,1]}\ \sup_{|\alpha|=p+1}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \left\|D^\alpha\ f\left(u_0 + s\left(\sum_{j=1}^p u_j\,\varepsilon^j\ + R_p^u(\varepsilon)\right)\right)\right\|, \end{aligned} \tag{9}$$

$C_p^u \le \tilde{C}_p^u$*, with* $\tilde{C}_p^u = \frac{1}{p+1}\ \sup_{s\in[0,1]}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \|D^{p+1}\ f(u(s\,\varepsilon))\|$.

Proof. This is immediate from Lemma 2.1, with $x \in \mathbb{R}^d$ replaced by $u(\varepsilon) \in \mathbb{R}^d$ and x_0 replaced by $u_0 \in \mathbb{R}^d$, and denoting the remainder R_p in (4) by $R_p^{u(\varepsilon)}$, to recall that it refers to the function $f(u(\varepsilon))$ instead of $f(x)$, and using (6). □

Remark 2.3. We point out that the remainder $R_N^u(\varepsilon)$ in (6), referring to the asymptotic expansion of $u(\varepsilon)$ as a function of ε, should not be confused with the remainder $R_p^{u(\varepsilon)}\left(f^{(p+1)}(u_0, u(\varepsilon))\right)$ in (7), for $p = N$ (which refers to the function $f(u(\varepsilon))$).

Now for any $N \in \mathbb{N}_0$ we have, by the definition of x^α given before and the binomial formula for the powers of the components y_i of the vector $y = \sum_{l=1}^N \varepsilon^l u_l + R_N^u(\varepsilon)$ in $\mathbb{R}^d$ on the left hand side of (10), to be taken to the multi power α, i.e. $y^\alpha = \prod_{i=1}^d y_i^{\alpha_i}$:

$$\begin{aligned}&\left(\sum_{l=1}^N \varepsilon^l u_l + R_N^u(\varepsilon)\right)^\alpha \\ &\quad= \prod_{i=1}^d \Big[\sum_{\substack{\alpha_{1,i},\dots,\alpha_{N+1,i}=0 \\ \alpha_{1,i}+\dots+\alpha_{N+1,i}=\alpha_i}}^{\alpha_i} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!} \varepsilon^{\alpha_{1,i}+2\alpha_{2,i}+\dots+N\alpha_{N,i}} u_{1,i}^{\alpha_{1,i}} u_{2,i}^{\alpha_{2,i}} \\ &\qquad \cdots u_{N,i}^{\alpha_{N,i}} \left(R_{N,i}^u(\varepsilon)\right)^{\alpha_{N+1,i}} \Big]. \end{aligned} \tag{10}$$

$u_{j,i}$, $i = 1, \dots, d$ is the ith component of the vector $u_j \in \mathbb{R}^d$, $j = 1, \dots, N$, and $R_{N,i}^u(\varepsilon)$ is the ith component of the vector $R_N^u(\varepsilon) \in \mathbb{R}^d$.

Note that $\left(R_{N,i}^u(\varepsilon)\right)^{\alpha_{N+1,i}}$ is bounded in norm by a positive constant $\left(C_N^u\right)^{\alpha_{N+1,i}}$ times $\varepsilon^{(N+1)\alpha_{N+1,i}}$ (since $\|R_N^u(\varepsilon)\| \le C_N^u\, \varepsilon^{N+1}$, $C_N^u \ge 0$ from (6)). We also point out that $\alpha_i \in \mathbb{N}_0$, $i = 1, \dots, d$ are the components of $\alpha \in \mathbb{N}_0^d$. We have $\alpha_{j,.} \in \mathbb{N}_0^d$ with $\alpha_{j,i} \in \mathbb{N}_0$, $j = 1, \dots, N+1$ restricted by the conditions appearing under the summation.

Thus we get, from Eqs. (7), (10) and by the assumption on $u(\varepsilon)$ in Proposition 2.2, for $N \in \mathbb{N}_0$, $p \in \mathbb{N}_0$:

$$\begin{aligned} f(u(\varepsilon)) &= \sum_{|\alpha|\le p} \frac{D^\alpha f(u_0)}{\alpha!} \prod_{i=1}^d \Big[\sum_{\substack{\alpha_{1,i},\dots,\alpha_{N+1,i}=0 \\ \alpha_{1,i}+\dots+\alpha_{N+1,i}=\alpha_i}}^{\alpha_i} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!} \varepsilon^{\alpha_{1,i}+2\alpha_{2,i}+\dots+N\alpha_{N,i}} \\ &\quad \times u_{1,i}^{\alpha_{1,i}} u_{2,i}^{\alpha_{2,i}} \cdots u_{N,i}^{\alpha_{N,i}} \left(R_{N,i}^u(\varepsilon)\right)^{\alpha_{N+1,i}} \Big] + R_p^{u(\varepsilon)}\left(f^{(p+1)}(u_0, u(\varepsilon))\right), \end{aligned} \tag{11}$$

with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$.

We can rewrite (11) by grouping the terms with the same power k of ε, $0 \le k \le N$. Denote the term of exact order k, $0 \le k \le N$, in ε appearing on the right hand side of (11) by $[f(u(\varepsilon))]_k$. To compute it we first observe that we have to take $\alpha_{N+1,i} = 0$, $i = 1, \dots, d$, otherwise, due to the bound on R_N, the effective presence of $\left(R_{N,i}^u(\varepsilon)\right)$ would give a term small of order at least $N+1$ (by the nature of $R_{N,i}^u(\varepsilon)$, and $N+1 > k$). Then we have to take the sum over the $\alpha_{j,i} \in \{0, 1, \dots, \alpha_i\}$, $j = 1, \dots, N$, $i = 1, \dots, d$ restricted by $\alpha_i \in \mathbb{N}_0$ and satisfying:

1. $\sum_{i=1}^d \sum_{j=1}^N j\,\alpha_{j,i} = k$,
2. $\alpha_i = \sum_{j=1}^N \alpha_{j,i}$.

For $k = 0$ we must then have $\alpha_{j,i} = 0$ for all j, i and thus:

$$[f(u(\varepsilon))]_0 = f(u_0). \tag{12}$$

For $k = 1$ we have from 1 that $\alpha_{1,i} = 1$ for some i, all other $\alpha_{j,l}$, $l \neq i$ being 0, for all $j = 1, \dots, N$. This implies $\alpha_{1,i} = 1, i = 1, \dots, d$, $\alpha_{j,i} = 0$, $j = 2, \dots, N$, $i = 1, \dots, d$.

Inserting this into (11) we get $[f(u(\varepsilon))]_1 = \sum_{i=1}^{d} \frac{\partial}{\partial y_i} f(y)_{|y=u_0} u_{1,i}$, (where we introduced the short notation $\partial_i f(u_0) := \frac{\partial}{\partial y_i} f(y)_{|y=u_0}$). We have thus:

$$[f(u(\varepsilon))]_1 = \sum_{i=1}^{d} \partial_i f(u_0) u_{1,i}. \tag{13}$$

For $k = 2$ we have from 1: $\sum_{i=1}^{d} \sum_{j=1}^{N} j\alpha_{j,i} = \sum_{i=1}^{d} \sum_{j=1}^{2} j\alpha_{j,i} = 2$. This gives the possibility $j = 2$ and $\alpha_{2,i} = 1$ for some i, $\alpha_{2,k} = 0$, $k \neq i$, $k \neq i$, $\alpha_{l,l'} = 0$, $l \neq 2$, $l' = 1, \ldots, d$. This provides the contribution $\sum_{i=1}^{d} \partial_i f(u_0) u_{2,i}$ to $[f(u(\varepsilon))]_2$. Another contribution is given by the case $j = 1$ and $\alpha_{1,i} = 1, \alpha_{1,i'} = 1$ for some $i, i' = 1, i \neq i'$, with $\alpha_{j,l} = 0, \forall j \neq 1$, $l = 1, \ldots, d$ and $\alpha_{1,m} = 0, \forall m \neq i'$, $m = 1, \ldots, d$ or $\alpha_{1,l} = 2$, $\alpha_{1,l'} = 0$, $l, l' = 1, \ldots, d$, $l \neq l'$. In this case we get the contributions:

$$\sum_{i=1}^{d} \sum_{i'=1}^{d} \frac{\partial^2}{\partial y_i \, \partial y_{i'}} f(y)_{|y=u_0} u_{1,i} u_{1,i'}, \quad \text{to } [f(u(\varepsilon))]_2. \tag{14}$$

Denoting $\partial_i \partial_{i'} f(u_0) := \frac{\partial^2}{\partial y_i \, \partial y_{i'}} f(y)_{|y=u_0}$, we have in total:

$$[f(u(\varepsilon))]_2 = \sum_{i=1}^{d} \partial_i f(u_0) u_{2,i} + \frac{1}{2!} \sum_{i=1}^{d} \sum_{i'=1}^{d} \partial_i \partial_{i'} f(u_0) u_{1,i} u_{1,i'}. \tag{15}$$

In a similar way we can get the contribution $[f(u(\varepsilon))]_k$ of order $k \geq 3$. It is easy to see that it contains the term $u_{k,i}$ only linearly and that it depends in a homogeneous way of total order k in components of the coefficients $u_{k-1}, u_{k-2}, \ldots, u_1, u_0$. Thus introducing the short notation:

$$\partial_1 \cdots \partial_k f(u_0) := \frac{\partial^k}{\partial y_1 \cdots \partial y_k} f(y)_{|y=u_0}, \tag{16}$$

we have

$$[f(u(\varepsilon))]_k = \sum_{i=1}^{d} \partial_i f(u_0) u_{k,i} + \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{d} \partial_{i_1} \cdots \partial_{i_k} f(u_0) u_{1,i_1} u_{1,i_2} \cdots u_{1,i_k}$$
$$+ B_k^f (u_0, \ldots, u_{k-1}), \tag{17}$$

for some function B_k^f which is a sum of monomials in components of the variables $u_0, \ldots, u_{k-1}$.

Let us summarize what we obtained from (6) to (17) in the following:

Proposition 2.4. *Let f, $u(\varepsilon)$ be as in Proposition 2.2. Then, for any $\varepsilon \in [0, \varepsilon_0]$, the asymptotic expansion of $f(u(\varepsilon))$ in powers of $\varepsilon \in [0, \varepsilon_0]$ up to order p (with $N \leq p$) is given by (11), with the estimates*

$$|\left(R_{N,i}^u(\varepsilon)\right)^{\alpha_{N+1,i}}| \leq C_{N,i}^u \, \varepsilon^{(N+1)\,\alpha_{N+1,i}}, \tag{18}$$

where

$$0 < C_{N,i}^u \leq \tilde{C}_{N,i}^{u_{(i)}} := \frac{1}{N+1} \sup_{s \in [0,1]} \sup_{\varepsilon \in [0,\varepsilon_0]} \|\left(D^{N+1} u_{(i)}\right)(s\,\varepsilon)\|, \tag{19}$$

with $\tilde{C}_{N,i}^{u_{(i)}}$ independent of ε, and $u_{(i)}(s\,\varepsilon)$ denoting the ith component of $u(s\varepsilon)$, $i = 1, \ldots, d$.

The term $R_p^{u(\varepsilon)}\left(f^{(p+1)}(u_0, u(\varepsilon))\right)$ *in* (11) *satisfies the estimate* (8) *with* (9) *and* $R_p^u(\varepsilon)$ *satisfying* $\|R_p^u(\varepsilon)\| \le C_p^u\, \varepsilon^{p+1}$, $0 < C_p^u \le \tilde{C}_p^u =: \frac{1}{p+1} \sup_{s\in[0,1]} \sup_{\varepsilon\in[0,\varepsilon_0]} \|\Big(D^{p+1} u\Big)(s\,\varepsilon)\|$.

We can also write the asymptotic expansion in powers of ε up to order p in the following form

$$f(u(\varepsilon)) = \sum_{k=0}^{p} [f(u(\varepsilon))]_k\, \varepsilon^k + \tilde{R}_p^{u(\varepsilon)}, \tag{20}$$

with $[f(u(\varepsilon))]_k$ defined by (12)–(17), $\tilde{R}_p^{u(\varepsilon)}$ being defined as the sum of all terms in (11) having absolute bound of order at least $p+1$ in ε. More precisely

$$\begin{aligned}\tilde{R}_p^{u(\varepsilon)} &= \sum_{|\alpha|\le p} \frac{D^\alpha f(u_0)}{\alpha!} \\ &\quad\times \prod_{i=1}^{d} \Big[\sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!}\, \varepsilon^{\sum_{j=1}^{N} j\,\alpha_{j,i}}\, u_{1,i}^{\alpha_{1,i}} u_{2,i}^{\alpha_{2,i}} \cdots u_{N,i}^{\alpha_{N,i}} \left(R_{N,i}^u(\varepsilon)\right)^{\alpha_{N+1,i}}\Big] \\ &\quad + R_p^{u(\varepsilon)}\Big(f^{(p+1)}(u_0, u(\varepsilon))\Big),\end{aligned} \tag{21}$$

$\sum_{\alpha_{j,i}}^{\star}$ meaning sum over $\alpha_{j,i} \in \{0,\ldots,\alpha_i\}$, $j = 1,\ldots,N+1$, $\alpha_{N+1,i} \in \{1,\ldots,\alpha_i\}$, $\sum_{j=1}^{N+1} \alpha_{j,i} = \alpha_i$, $\sum_{i=1}^{d} \alpha_i = |\alpha|$.

The following bound holds:

$$\begin{aligned}\|\tilde{R}_p^{u(\varepsilon)}\| \le\; & \varepsilon^{p+1} \sum_{|\alpha|\le p} \frac{\|D^\alpha f(u_0)\|}{\alpha!} \prod_{i=1}^{d} \Big[\sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!} \\ & \varepsilon_0^{\sum_{j=1}^{N} j\,\alpha_{j,i}}\, u_{1,i}^{\alpha_{1,i}} u_{2,i}^{\alpha_{2,i}} \cdots u_{N,i}^{\alpha_{N,i}}\, \|\left(R_{N,i}^u(\varepsilon)\right)\|^{\alpha_{N+1,i}}\Big] \\ & + \left\|R_p^{u(\varepsilon)}\Big(f^{(p+1)}(u_0, u(\varepsilon))\Big)\right\|,\end{aligned} \tag{22}$$

with the estimates on $R_{N,i}^u(\varepsilon)$ and $R_p^{u(\varepsilon)}\left(f^{(p+1)}(u_0, u(\varepsilon))\right)$ given by (18), (19) resp. (8), (9).

$$\begin{aligned}\|\tilde{R}_p^{u(\varepsilon)}\| \le\; & \varepsilon^{p+1} \sup_{|\alpha|\le p} \sup_{\varepsilon\in[0,\varepsilon_0]} \left\|\frac{D^\alpha f(u_0)}{\alpha!}\right\| \prod_{i=1}^{d} \Big[\sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!}\, \varepsilon_0^{\sum_{j=1}^{N} j\,\alpha_{j,i}} \\ & \times \prod_{j=1}^{N} \|u_{j,i}\|^{\alpha_{j,i}} \frac{1}{N+1} \sup_{s\in[0,1]} \sup_{\varepsilon\in[0,\varepsilon_0]} \left\|D^{N+1} f\Big(u_{(i)}(s\,\varepsilon)\Big)\right\|\Big], \quad i = 1,\ldots,d.\end{aligned} \tag{23}$$

In an entirely similar way we prove the following:

Proposition 2.5. *Let* $f \in \mathcal{C}^{p+1}(\mathbb{R}^d)$ *and consider for* $N \in \mathbb{N}_0$, $\varepsilon \in [0,\varepsilon_0]$, $y \in \mathbb{R}^d$, $y_j \in \mathbb{R}^d$, $j = 0,\ldots,N$; $f\Big(\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y\Big)$. *This function has an asymptotic expansion in powers of* ε. *Calling* $f_j(y_0,\ldots,y_j)$ *the coefficient in the term of exact order* j *in the asymptotic* *expansion of* $f\left(\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y\right)$ *in powers of* ε, *we have*

$$f\Big(\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y\Big) = \sum_{j=0}^{p} \varepsilon^j\, f_j(y_0, \ldots, y_j) + A_p^f(\underline{y}; \varepsilon), \tag{24}$$

with $\underline{y} := (y_0, \ldots, y_N, y) \in \mathbb{R}^{(N+2)d}$ *and* $|A_p^f(\underline{y}; \varepsilon)| \le \varepsilon^{p+1}\, K_{p,N,\varepsilon_0}$, *with*

$$K_{p,N,\varepsilon_0} := \sup_{|\alpha|\le p}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \left\| \frac{D^\alpha\, f(u_0)}{\alpha!} \right\| \prod_{i=1}^{d} \Big[\sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!}\, \varepsilon_0^{\sum_{j=1}^{N} j\,\alpha_{j,i}} \prod_{j=1}^{N} |u_{j,i}|^{\alpha_{j,i}}$$
$$\times \frac{1}{N+1} \sup_{s\in[0,1]}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \Big\| D^{N+1} f\Big(\sum_{j=0}^{N} s^j\, \varepsilon^j\, y_{j,(i)} + s^{N+1}\, \varepsilon^{N+1}\, y_{(i)}\Big) \Big\| \Big], \tag{25}$$

$y_{j,(i)}$ *resp.* $y_{(i)}$ *standing for the* i*th components of the vectors* y_j *resp.* $y \in \mathbb{R}^d$, $i = 1, \ldots, d$. *Moreover,* $\lim_{\varepsilon\downarrow 0} \varepsilon^{-(p+1)}\, A_p^f(\underline{y}; \varepsilon)$ *exists and is equal to*

$$\frac{D^\alpha\, f(u_0)}{\alpha!} \prod_{i=1}^{d} \sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!} \prod_{j=1}^{N} (y_{j,(i)})^{\alpha_{j,i}}\, D^{N+1} f(y_{(i)}(s\varepsilon))_{|\varepsilon=0}. \tag{26}$$

Proof. It suffices to replace $u(\varepsilon)$ resp. u_j resp. $R_N^u(\varepsilon)$ in Proposition 2.4 by $\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y$ resp. y_j, resp. $\varepsilon^{N+1} y$. Then $f_j(y_0, \ldots, y_j) = [f\left(\sum_{j=0}^{N} \varepsilon^j y_j + \varepsilon^{N+1} y\right)]_j$ are given by (12)–(17). We also have $A_p^f(\underline{y}; \varepsilon) = \tilde{R}_p\left(\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y\right)$, with $\tilde{R}_p\left(\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y\right)$ denoting $\tilde{R}_p^{u(\varepsilon)}$ as defined by (21) with $u(\varepsilon)$ replaced by $\sum_{j=0}^{N} \varepsilon^j\, y_j + \varepsilon^{N+1}\, y$.

The final estimates follow from (23) with the estimate on the right hand side as in the estimate of (22) in Proposition 2.4. □

Assume now the components β_l, $l = 1, \ldots, d$ of β in (1) are in $\mathcal{C}^{N+1}(\mathbb{R}^d)$. By replacing simply f by β_l formula (17) yields the term of exact order $0 \le k \le N$, in the asymptotic expansion (20) in powers of ε of the lth components of the coefficient $\beta(u(\varepsilon))$ in (1), with remainders estimates given by (22).

In order to get a corresponding expansion in powers of ε for the matrix elements $(\sigma_\varepsilon)_{l,l'}(u(\varepsilon))$, $l, l' = 1, \ldots, d$ of the matrix σ_ε in powers of ε we have to take care of the fact that, as opposite to f and β_l, $(\sigma_\varepsilon)_{l,l'}$ also depends on ε, not only on its argument.

Let us assume that, for $\varepsilon \in [0,\ \varepsilon_0)$:

$$\sigma_\varepsilon(x) = \sum_{j=0}^{M} \sigma_j(x)\varepsilon^j + R_M^\sigma(\varepsilon)(x), \qquad \text{for any } x \in \mathbb{R}^d \tag{27}$$

with $\sup_{x\in\mathbb{R}^d} \|R_M^\sigma(x, \varepsilon)\| \le C_{M,\sigma}\, \varepsilon^{M+1}$, ($\|.\|$ denoting here the norm of the matrix $R_M^\sigma(x, \varepsilon)$), and $C_{M,\sigma} \ge 0$. Note that the σ_j are coefficients of σ_ε, the expansion in powers of ε, and should not be confused with the values of σ_ε for $\varepsilon = j$. Let us also assume that the elements $(\sigma_j(x))_{l,l'}$, $l, l' = 1, \ldots, d$ of the matrix $\sigma_j(x)$, $j = 1, \ldots, N$, belong to $\mathcal{C}^{s+1}(\mathbb{R}^d)$ as functions of $x \in \mathbb{R}^d$.

For any $M,\ N \in \mathbb{N}_0$, $s \in \mathbb{N}_0$ we have, from (2), (27), using the right hand side of (11):

$$\begin{aligned}
\sigma_\varepsilon\left(u(\varepsilon)\right) &= \sum_{j=0}^{M} \sigma_j\left(u(\varepsilon)\right)\varepsilon^j + R_M^\sigma(\varepsilon)\\
&= \sum_{j=0}^{M} \sigma_j\left(\sum_{k=0}^{N} \varepsilon^k u_k + R_N^u(\varepsilon)\right)\varepsilon^j + R_M^\sigma(\varepsilon)\\
&= \sum_{j=0}^{M} \varepsilon^j\left[\sum_{|\gamma|\le s} \frac{D^\gamma \sigma_j(u_0)}{\gamma!}\left(\sum_{k=0}^{N} \varepsilon^k u_k + R_N^u(\varepsilon) - u_0\right)^\gamma + R_s^{\sigma_j}\right] + R_M^\sigma(\varepsilon)\\
&= \sum_{j=0}^{M} \varepsilon^j\left[\sum_{|\gamma|\le s} \frac{D^\gamma \sigma_j(u_0)}{\gamma!}\prod_{i=1}^{d} \sum_{\substack{\gamma_{1,i},\dots,\gamma_{N+1,i}=0\\ \gamma_{1,i}+\dots+\gamma_{N+1,i}=\gamma_i}}^{\gamma_i} \frac{\gamma_i!}{\gamma_{1,i}!\gamma_{2,i}!\cdots\gamma_{N+1,i}!}\varepsilon^{\gamma_{1,i}+2\gamma_{2,i}+\dots+N\gamma_{N,i}}\right.\\
&\qquad\left.\times u_{1,i}^{\gamma_{1,i}} u_{2,i}^{\gamma_{2,i}}\cdots u_{N,i}^{\gamma_{N,i}}\left(R_N^{u(\varepsilon)}(\varepsilon)\right)^{\gamma_{N+1,i}} + R_s^{\sigma_j}\right] + R_M^\sigma(\varepsilon). \qquad (28)
\end{aligned}$$

Here $R_s^{\sigma_j}$ is a short notation for $R_s^{\sigma_j}\left(\sum_{k=0}^{N} \varepsilon^k u_k + R_N^u(\varepsilon)\right)$.

Let us note that (28) is a relation between matrices, to be understood element by element. $D^\gamma\sigma_j(u_0)$ has to be interpreted as D^γ applied to the elements $(\sigma_j)_{l,l'},\ l,l' = 1,\dots,d$ of the matrix σ_j, evaluated then at u_0.

Proceeding as in the case of the expansions of f and β_l we exhibit the coefficient $[\sigma_\varepsilon\left(u(\varepsilon)\right)]_k$ of the power k, $0 \le k \le \min(M,N)$ in the development of $\sigma_{\varepsilon,l,l'}\left(u(\varepsilon)\right)$ on the right hand side of (28). We shall write $[\sigma_\varepsilon\left(u(\varepsilon)\right)]_k$ in matrix form, but it should be understood element by element. As we did for f, for this we have to set $\gamma_{N+1,i} = 0$, $i = 1,\dots,d$. Moreover we observe that (28) contains a sum of products of ε^j times a sum of terms with power $\sum_{i=1}^{d}\left(\gamma_{1,i} + 2\gamma_{2,i} + \dots + N\gamma_{N,i}\right)$ in ε, hence the analogues of (1), (6) we had for $[f\left(u(\varepsilon)\right)]_k$ are:

1. $j + \sum_{i=1}^{d}\sum_{l=1}^{N} l\,\gamma_{l,i} = k,\ j = 0,\dots,M.$
2. $\gamma_i = \sum_{l=1}^{N} \gamma_{l,i}$, with $\gamma_{l,i} = \{0,1,\dots,\gamma_i\},\ l = 1,\dots,N, i = 1,\dots,d, \gamma_i \in \mathbb{N}_0.$

We see from 1. that we must have $j \le k$. Let us first compute the terms for $k = 0,1,2$. We have:

$$[\sigma_\varepsilon\left(u(\varepsilon)\right)]_0 = \sigma_0\left(u_0\right), \qquad (29)$$

since $k = 0$ implies $j = 0$, $\gamma_{j,i} = 0$ for all j, i. To obtain $[\sigma_\varepsilon\left(u(\varepsilon)\right)]_1$ we observe that from 1. we have the possibilities (a) $j = 0$ and $\gamma_{1,i} = 1$ for some i, $\gamma_{1,l} = 0\ \forall l \ne i$, $\gamma_{2,i} = \dots = \gamma_{N,i} = 0$, for all $i = 1,\dots,d$, or (b) $j = 1$, $\gamma_{l,i} = 0$, $\forall l, i$. Thus we have:

$$[\sigma_\varepsilon\left(u(\varepsilon)\right)]_1 = \sum_{i=1}^{d} \partial_i\,\sigma_0\left(u_0\right)\,u_{1,i} + \sigma_1\left(u_0\right). \qquad (30)$$

For $k = 2$, we have for $j = 0$ only the possibilities we already discussed for $[f(u(\varepsilon))]_2$, so we get a contribution

$$\sum_{i=1}^{d} \partial_i \sigma_0(u_0) u_{2,i} + \frac{1}{2!} \sum_{i,i'=1}^{d} \partial_i \partial_{i'} \sigma_0(u_0) u_{1,i} u_{1,i'}. \tag{31}$$

For $j = 1$ we have the possibilities given by $\sum_{i=1}^{d} \left(\gamma_{1,i} + 2\gamma_{2,i} + \cdots + N\gamma_{N,i}\right) = 1$, which are those discussed for $[f(u(\varepsilon))]_1$, and the possibility $\gamma_{j,i} = 0, \forall j, i$. Hence we get the contribution $\sum_{i=1}^{d} \partial_i \sigma_1(u_0) u_{1,i} + \sigma_2(u_0)$. In total then we get for any $l, l' = 1, \ldots, d$:

$$[\sigma_\varepsilon(u(\varepsilon))]_2 = \sum_{i=1}^{d} \partial_i \sigma_0(u_0) u_{2,i} + \frac{1}{2!} \sum_{i,i'=1}^{d} \partial_i \partial_{i'} \sigma_0(u_0) u_{1,i} u_{1,i'} + \sum_{i=1}^{d} \partial_i \sigma_1(u_0) u_{1,i} + \sigma_2(u_0). \tag{32}$$

In the general case $k \geq 2$ we see that

$$[\sigma_\varepsilon(u(\varepsilon))]_k = \sum_{i=1}^{d} \partial_i \sigma_0(u_0) u_{k,i} + \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{d} \partial_{i_1} \cdots \partial_{i_k} \sigma_0(u_0) u_{1,i_1} u_{1,i_2} \cdots u_{1,i_k} + \sigma_k(u_0) + A_k^\sigma(u_0, \ldots, u_{k-1}), \tag{33}$$

where $A_k^\sigma(u_0, \ldots, u_{k-1})$ is a $d \times d$ matrix which depends only on the indicated variables. Note that (33) is also valid for $k = 2$, with $A_2^\sigma(u_0, u_1) = \sum_{i=1}^{d} \partial_i \sigma_1(u_0) u_{1,i}$.

We shall now apply the formulae we have obtained for $[\beta(u(\varepsilon))]_k$ and $[\sigma_\varepsilon(u(\varepsilon))]_k$, $k \in \mathbb{N}_0$, to the case where $u(\varepsilon)$ is replaced by the pathwise solution $u_\varepsilon(s)$ of (1), assumed first to exist and to have an asymptotic expansion in ε of the form (6), (see Section 5 for the justification of this assumption). By matching coefficients of the same order k on both sides of (1), i.e. $u_k(t)$ resp. $[u(0)]_k + \int_0^t [\beta(u_\varepsilon(s))]_k \, ds$ and $\int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_k \, \eta(ds)$, we get the following proposition:

Proposition 2.6. *Let us assume that the coefficient σ_ε is $\mathcal{C}^{M+1}$ in $\varepsilon \in [0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, in the sense that (27) holds. Moreover assume that $\beta \in \mathcal{C}^{p+1}(\mathbb{R}^d)$, $\sigma_\varepsilon \in \mathcal{C}^{s+1}(\mathbb{R}^d)$, for any $\varepsilon \in [0, \varepsilon_0)$, for some $p \in \mathbb{N}_0$, $s \in \mathbb{N}_0$. Furthermore assume that the stochastic equation (1) has a pathwise solution u_ε for all $t \in [0, T]$, $T > 0$, and the solution $u_\varepsilon(t)$ is $\mathcal{C}^{m+1}$, for some $m \in \mathbb{N}$, in $\varepsilon \in [0, \varepsilon_0)$, i.e. (6) holds for $u(\varepsilon) = u_\varepsilon(t)$. Then the expansion coefficients $u_k(t)$ of the solution $u_\varepsilon(t)$ of (1) satisfy the following equations:*

$$u_0(t) = u^0 + \int_0^t \beta(u_0) \, ds + \int_0^t \sigma_0(u_0) \eta(ds); \tag{34}$$

$$\begin{aligned} u_1(t) &= \sum_{i=1}^{d} \Big[\int_0^t \partial_i \beta(u_0) u_{1,i} ds + \int_0^t \partial_i \sigma_0(u_0) \, u_{1,i} \, \eta(ds) \Big] + \int_0^t \sigma_1(u_0) \eta(ds) \\ &= \int_0^t [\beta(u_\varepsilon(s))]_1 \, ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_1 \, \eta(ds), \end{aligned} \tag{35}$$

with

$$[\beta\,(u_\varepsilon(s))]_1 = \sum_{i=1}^{d} \partial_i\, \beta(u_0)\, u_{1,i}, \qquad [\sigma_\varepsilon\,(u_\varepsilon(s))]_1 = \partial_i\, \sigma_0(u_0)\, u_{1,i} + \sigma_1(u_0);$$

$$\begin{aligned} u_2(t) \;&=\; \sum_{i=1}^{d} \Big[\int_0^t \partial_i\, \beta(u_0)\, u_{2,i}\, ds + \frac{1}{2!}\int_0^t \sum_{i,i'=1}^{d} \partial_i \partial_{i'}\, \beta(u_0)\, u_{1,i}\, u_{1,i'} ds\Big] \\ &\quad + \sum_{i=1}^{d} \int_0^t \Big[\partial_i\, \sigma_0(u_0)\, u_{2,i} + \frac{1}{2!} \sum_{i,i'=1}^{d} \partial_i \partial_{i'}\, \sigma_0(u_0)\, u_{1,i}\, u_{1,i'} \\ &\quad + \partial_i\, \sigma_1(u_0)\, u_{1,i}\,\Big]\eta(ds) + \int_0^t \sigma_2(u_0)\, \eta(ds) \\ &= \int_0^t [\beta\,(u_\varepsilon(s))]_2\, ds + \int_0^t [\sigma_\varepsilon\,(u_\varepsilon(s))]_2\, \eta(ds), \end{aligned} \tag{36}$$

with

$$[\beta\,(u_\varepsilon(s))]_2 = \partial_i\, \beta(u_0)\, u_{2,i} + \frac{1}{2!}\, \partial_i \partial_{i'}\, \beta(u_0)\, u_{1,i}\, u_{1,i'} \tag{37}$$

and

$$\begin{aligned} [\sigma_\varepsilon\,(u_\varepsilon(s))]_2 \;&=\; \partial_i\, \sigma_0(u_0)\, u_{2,i} + \frac{1}{2!} \sum_{i,i'=1}^{d} \partial_i \partial_{i'}\, \sigma_0(u_0)\, u_{1,i}\, u_{1,i'} \\ &\qquad + \partial_i\, \sigma_1(u_0)\, u_{1,i} + \sigma_2(u_0); \end{aligned} \tag{38}$$

and for all $1 \le k \le \min(Mp,\ Ns)$*:*

$$u_k(t) = \int_0^t [\beta\,(u_\varepsilon(s))]_k\, ds + \int_0^t [\sigma_\varepsilon\,(u_\varepsilon(s))]_k\, \eta(ds), \tag{39}$$

where $[\beta\,(u_\varepsilon(s))]_k$ *and* $[\sigma_\varepsilon\,(u_\varepsilon(s))]_k$ *are given in* (17)*, (by replacing* f *by* β *and* $u(\varepsilon)$ *by* $u_\varepsilon(s)$*), resp. in* (33) *(*$u(\varepsilon)$ *being replaced by* $u_\varepsilon(s)$*).*

Proof. The proof was already carried through before the proposition. □

Remark 2.7. 1. For the existence and uniqueness of solutions of (1) see Section 3.

2. If $\beta(\,x\,) = Ax + b$, (with $b \in \mathbb{R}^d$ independent of x and A a $d \times d$-matrix independent of x) and $\sigma_\varepsilon(x) = \sigma_0 + \varepsilon\sigma_1(x)$, with $\sigma_0 = c$ and $(\sigma_1(x))_{l,l'} = \lambda_{l,l'}\, x_{l'},\ x \in \mathbb{R}^d$, with $c,\ \lambda$ constant $d \times d$-matrices, then $\sigma_i = 0,\ i = 1, \dots, d$ and $\partial_{i_1} \cdots \partial_{i_n} \beta = 0,\ n = 2, \dots, d$. Moreover, $(\partial_i \beta)_l(x) = \frac{\partial}{\partial x_i} \sum_{k=1}^{d} A_{lk}\, x_k = A_{li},\ i, l = 1, \dots, d$ and thus

$$\sum_{i=1}^{d} \partial_i \beta(u_0)\, u_{k,i} = A\, u_k, \quad k \in \mathbb{N}. \tag{40}$$

Moreover,

$$\partial_{i_1} \cdots \partial_{i_k}\, \sigma_0(x) = 0, \quad \forall\, k \in \mathbb{N},\ x \in \mathbb{R}^d, \qquad (\partial_i \sigma_1(x))_{l,l'} = \lambda_{l,i}\, \delta_{i,l'}, \tag{41}$$

with $\delta_{i,l'}$ the Kronecker symbol, $l, l', i = 1, \dots, d$.

Furthermore $\partial_{i_1} \cdots \partial_{i_k} \sigma_1(x) = 0,\ \forall k \geq 2,\ \partial_{i_1} \cdots \partial_{i_k} \sigma_j(x) = 0,\ \forall j \geq 2,\ k \in \mathbb{N}_0$. Hence from Proposition 2.6 we get:

$$u_0(t) = u^0 + \int_0^t Au_0(s)\,ds + b\int_0^t u_0(s)\,ds + c\eta(t), \tag{42}$$

and

$$(u_k(t))_l = \int_0^t (A\,u_k(s))_l\,ds + \sum_{i=1}^d \lambda_{l,i} \int_0^t (u_{k-1}(s))_i \eta_i(ds),$$
$$l = 1, \ldots, d,\ k \in \mathbb{N}. \tag{43}$$

3. β and σ_ε are as in 2., however with $\sigma_0 = c$ replaced by $\sigma_0(x) = \Pi\,x$, Π a constant $d \times d$-matrix, $x \in \mathbb{R}^d$, then (40) holds, the first equation in (41) is for $k = 1$ replaced by $(\partial_{i_1} \Pi\,x)_l = \Pi_{l,i_1}$, thus (42) is replaced by

$$u_0(t) = u^0 + \int_0^t Au_0(s)\,ds + b\int_0^t u_0(s)\,ds + \Pi\,\eta(t), \tag{44}$$

and

$$(u_1(t))_l = A\int_0^t (u_1(s))_l\,ds + \sum_{i=1}^d \lambda_{l,l'} \int_0^t (u_0(s))_i \eta_i(ds)$$
$$+ \sum_{i=1}^d \Pi_{l,i} \int_0^t u_{1,\,i}\,\eta_i(ds), \tag{45}$$

for $k \geq 2,\ l = 1, \ldots, d$:

$$(u_k(t))_l = \int_0^t (A\,u_k(s))_l\,ds + \sum_{l'=1}^d \lambda_{l,l'} \int_0^t (u_{k-1}(s))_{l'} \eta_{l'}(ds)$$
$$+ \sum_{i=1}^d \Pi_{l,i} \int_0^t u_{k,\,i}\,\eta_i(ds). \tag{46}$$

4. If $\beta(x) = Ax + F(x)$, A as in 2., $F \in \mathcal{C}^{p+1}(\mathbb{R}^d)$ and $\sigma_0 = 0$, $\sigma_1(x) = \Lambda$, with Λ a constant $d \times d$ matrix, so that (1) has additive noise, then

$$u_0(t) = u^0 + \int_0^t Au_0(s)\,ds + \int_0^t F(u_0)\,ds, \tag{47}$$

and

$$u_1(t) = \int_0^t Au_1\,ds + \sum_{i=1}^d \int_0^t \partial_i\,F(u_0)\,u_{1,i}ds + \Lambda\,\eta(t). \tag{48}$$

The $u_k(t)$, $k \geq 2$ are in this case given by linear nonhomogeneous stochastic equations with random coefficients, depending only on the $u_0, \ldots, u_{k-1}$, without any external noise term. The expansion is then a particular case of the one explicitly given in [6] (specialized to our present case where the Hilbert space is $\mathbb{R}^d$).

5. Eq. (34) is of the same type as Eq. (1) with $\varepsilon = 0$. Only for $\sigma_0 \equiv 0$ we have a purely deterministic equation. For $\sigma_0 \neq 0$ the expansion in powers of ε of the solution of (1) is really useful whenever (34) can be better handled than the original equation (1), which

happens whenever σ_0 has a simpler dependence on x than σ_ε itself. See Section 6, for some examples.

Let us also underline that Eqs. (35)–(39) for the $u_k(t)$ are linear nonhomogeneous, with random coefficients involving only $u_0, \ldots, u_{k-1}$, hence to be solved recursively.

6. If the coefficient β in (1) depends itself on $\varepsilon \in [0, \varepsilon_0)$ and is in $\mathcal{C}^{\tilde{M}+1}$ as a function of ε, thus has an expansion $\beta(x) = \sum_{i=1}^{\tilde{M}} \beta_i(x)\,\varepsilon^i + R^{\beta}_{\tilde{M}}(\varepsilon, x)$, $\forall x \in \mathbb{R}^d$, then the expansion (12)–(17) with f replaced by any component of β has to be replaced by (29)–(33), with the matrix elements of σ_ε replaced by the components of β_ε, i.e. $[\beta_\varepsilon(u(\varepsilon))]_0 = \beta_0(u_0)$, $[\beta_\varepsilon(u(\varepsilon))]_1 = \sum_{i=1}^{d} \partial_i\, \beta_0\,(u_0)\; u_{1,i} + \beta_1\,(u_0)$ and correspondingly for (32). (33) holds for β replaced by β_0, whereas in the equations for the $u_k(t)$, $k \in \mathbb{N}$ we have to replace $[\beta\,(u(s, \varepsilon))]_k$ by $[\beta_\varepsilon\,(u(s, \varepsilon))]_k$, with

$$[\beta_\varepsilon\,(u(s, \varepsilon))]_k = \sum_{i=1}^{d} \partial_i\, \beta_0(u_0)\, u_{k,i} + \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{d} \partial_{i_1}\cdots\partial_{i_k}\, \beta_0(u_0)\, u_{1,i_1}u_{1,i_2}\cdots u_{1,i_k} + \beta_k(u_0) + A^{\beta}_k(u_0, \ldots, u_{k-1}). \tag{49}$$

3. Existence and uniqueness results for the original SDE

Let $\beta : \mathbb{R}^d \to \mathbb{R}^d,\ \beta = (\beta^1, \ldots, \beta^d),\ \beta_i : \mathbb{R}^d \to \mathbb{R},\ i = 1, \ldots, d$, and let $\sigma = \left(\sigma^i_j\right)$, with $\sigma^i_j : \mathbb{R}^d \to \mathbb{R}, i, j = 1, \ldots, d$.

We assume that β is globally Lipschitz, i.e. $\|\beta(x) - \beta(y)\| \le k_\beta \|x - y\|$, for all $x, y \in \mathbb{R}^d$, for some constant $k_\beta > 0$.

We also assume σ^i_j are globally Lipschitz, i.e. $\|\sigma^i_j(x) - \sigma^i_j(y)\| \le k_{\sigma^i_j}\|x - y\|$, for some constant $k_{\sigma^i_j} > 0$ (independent of x, y) and all $x, y \in \mathbb{R}^d,\ i, j = 1, \ldots, d$.

Let $\tilde{L}(t)$ be a Lévy process on $\mathbb{R}^d$, without Gaussian and deterministic component, i.e. with characteristic function:

$$E\left(e^{i\langle u, \tilde{L}(t)\rangle}\right) = e^{\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle\, \chi_B(y)\right)\nu(dy)}, \tag{50}$$

$u \in \mathbb{R}^d$, B the unit ball in $\mathbb{R}^d$. ν is the intensity measure, also called Lévy measure, satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(|y|^2 \wedge 1)\,\nu(dy) < \infty$. For information on Lévy processes and related equations, see, e.g., [23,31,48,52,58].

The following Lévy–Itô decomposition holds (see, e.g., [18,19], ([23, p. 108–109]), [48]):

$$\tilde{L}_t = \int_B x\tilde{N}(t, dx) + \int_{\mathbb{R}^d\setminus B} xN(t, dx), \quad t \ge 0, \tag{51}$$

with N a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ (the Poisson random measure associated with the jumps $\Delta Z_t := \tilde{L}_t - \tilde{L}_{t\_}$, i.e. $N([0, t) \times A) = \{0 \le s < t | \Delta Z_s \in A\}$, for each $t \ge 0$, $A \in \mathcal{B}(\mathbb{R}^d\setminus\{0\})$, $\tilde{N}(t, A) := N(t, A) - t\nu(A)$, for all $A \in \mathcal{B}(\mathbb{R}^d, \{0\})$, $0 \in \overline{A}$, $\overline{A}$ the closure of A). We have $\nu(A) = E(N(1, A))$; for each $t > 0$, $\omega \in \Omega$, $\tilde{N}(t, \cdot)(\omega)$ is the compensated Poisson random measure (to $N(t, \cdot)(\omega)$) on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d\setminus\{0\})$; $\tilde{N}(t, A)$, $t \ge 0$ is, in particular, a martingale-valued measure.

It is known that the solution $u(t)$ of (1) with $\eta(ds) = d\tilde{L}_s + b\,ds + dB_A(s)$, $b \in \mathbb{R}^d$, B_A a Brownian motion in $\mathbb{R}^d$ with covariance matrix A, can be identified with the solution X_t of the following equation, see, e.g. [23]:

$$X_t(x) = x + \int_0^t \beta(X_{s\_})\,ds + b\,t + \sigma_\varepsilon\,(X_{t\_})\,dB_A(t) + \int_{0<|x|\leq 1} \sigma_\varepsilon(X_{t\_})\tilde{N}(dt,dx)$$
$$+ \int_{|x|>1} \sigma_\varepsilon(X_{t\_})N(dt,dx). \tag{52}$$

The following theorem holds:

Theorem 3.1. *If the coefficients β, σ satisfy the above global Lipschitz conditions and η is as above then there exists a strong, càdlàg, adapted solution of the SDE (1) or (52) and the solution is unique, for any initial condition u^0 resp. $x \in \mathbb{R}^d$.*

Proof. This is a particular case of results given, e.g., in ([39, pp. 237]), [55], ([43, p. 231]) and [14]. □

Remark 3.2. Other existence and uniqueness conditions are known. Particularly the Lipschitz conditions can be relaxed to local ones, with a condition of at most linear growth at infinity see, e.g., [39,59]. This (and the previous result) also holds for the non autonomous case where β, σ have an additional explicit measurable dependence on t and all constants entering the Lipschitz and additional growth conditions are uniform in t.

4. Discussion of the equations for the expansion coefficients

In this section we shall provide solutions as explicit as possible to Eqs. (39), for the expansion coefficients of the solution of (1) in powers of the small parameter ε. We first observe that (39) is a nonhomogeneous linear equation in u_k of the form:

$$du_{k,l}(t) = \left[\tilde{F}_{k,l}(t) + \sum_{l'=1}^{d} \tilde{\gamma}_{k,l,l'}(t,u_0)(t)\,u_{k,l'}(t)\right]dt$$
$$+ \sum_{j=1}^{d} \tilde{G}_{k,l,j}(t,u_0(t),u_k(t))\,d\eta_j(t)$$
$$+ \sum_{l'=1}^{d} \tilde{g}_{k,l,l'}(t,u_0)\,d\eta_{l'}(t), \quad k \in \mathbb{N},\ k \leq K, \text{ for some } K > 0,\ l = 1,\ldots,d, \tag{53}$$

with

$$\begin{cases} \tilde{F}_{k,l}(t) := [\beta_l(u(t,\varepsilon))]_k \quad (\text{with, } [.]_k \text{ given by (12)–(17), for } k \geq 2), \qquad \tilde{F}_{1,l}(t) := 0; \\ \tilde{\gamma}_{k,l,l'}(t,u_0) = \partial_{l'}\beta_l(u_0), \quad l = l' = 1,\ldots,d,\ k \in \mathbb{N}; \\ \tilde{G}_{k,l,j}(t,u_0(t),u_k(t)) = \sum_{i=1}^{d} \partial_i\sigma_{0,l,j}(u_0(t))\,u_{k,i}, \quad \text{and, for } k \geq 2, \\ \tilde{g}_{k,l,l'}(t) = \frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{d} \partial_{i_1}\cdots\partial_{i_k}\sigma_0(u_0)\,u_{1,i_1}u_{1,i_2}\cdots u_{1,i_k} + \sigma_k(u_0) + A_k^\sigma(u_0,\ldots,u_{k-1}). \end{cases} \tag{54}$$

We observe that (54) constitutes a set of recursive equations, where the kth order equation, for $X_{k,l}$, $l \in \{1, \ldots, d\}$, only involves the components of X_k, in a linear way, with random coefficients f_k, g_k depending on the vectors $X_0, \ldots, X_{k-1}$, and with a random inhomogeneity depending on $X_{k,l'}$, $l' \neq l$. It is thus of the form

$$dX_{k,l}(t) = f_k(X_0, \ldots, X_{k-1})\, X_{k,l}\, dt + g_k(X_0, \ldots, X_{k-1})\, X_{k,l}\, \eta(dt) + h_k(X_0)\, \eta(dt) + h_k(X_{0,l'}, \ldots, X_{k-1,l'})\, dt, \quad l' = 1, \ldots, d,\ l' \neq l,\ l = 1, \ldots, d,\ k = 1, \ldots, K. \tag{55}$$

Under Lipschitz assumptions and at most polynomial growth at infinity in the space variable for β, σ_ε and their derivatives up to order K, we can apply methods similar to the one used in [6,15] (in the infinite dimensional case, however with additive noise) to show that existence and uniqueness of solutions hold. Also proofs can be adopted to cover our case starting from literature on the martingale method, see, e.g., [54].

Yet still in the additive noise case and even for η a Brownian motion no "explicit" solutions are known.

In general, even in the 1-dimensional case, it is difficult to find explicit solutions. In fact already the equation for u_1 is a nonhomogeneous linear stochastic differential equation involving random coefficients and an inhomogeneity depending on the solution u_0, and the coefficients are in general nonlinear in u_0.

In the special d-dimensional case where $\sigma_1(y) = a\,y$ and $\sigma_0(y) = b\,y$ for some constant $d \times d$ matrices a, b, and $\beta(x) = c\,x + d$, for some constant coefficient matrix c and $d \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, then the linear equations for u_0, u_1 have constant coefficients and it is easy from (39), (17) and (33), to see that also the equations for the u_k, $k \geq 2$ are of this type. In this case, at least for $\eta = B$ a Brownian motion, we can apply results on systems of linear equations with terms of at most first order in the state variables, which are to be found, e.g., in [24,36], to find an explicit expansion for u_k.

In this special case we can thus apply to the discussion of (39) results on the solution of linear deterministic resp. stochastic evolution equations, according to the following proposition:

Proposition 4.1. *Consider a system of K coupled linear stochastic evolution equation with random coefficients, the coefficients of the equation for the kth component $k = 1, \ldots, K$ being only dependent of the components of index* $0, 1, 2, \ldots, k-1$. *The equation for the lth component of the kth vector, $X_{k,l}$ is of the form:*

$$dX_{k,l}(t) = \left[F_{k,l}(t) + \sum_{l'=1}^{d} \gamma_{k,l,l'}(t)\, X_{k,l'}(t) \right] dt + \sum_{j=1}^{m} G_{k,l,j}(t,\ X_k(t))\, dB_j(t) + \sum_{l''=1}^{d} g_{k,l,l''}(t)\, dB_{l''}(t), \tag{56}$$

with all components of γ_k, g_k independent of X_k, and $F_{k,l}$ as well as $G_{k,l,j}$ linear in the components $X_{k,l}$ of X_k and independent of other state variables.

All coefficients F, γ, G, g are supposed to be locally Lipschitz and satisfy the linear growth conditions, with constant uniform in t. The explicit dependence of all coefficients on t is supposed to be measurable.

The solution of (56) is given by:

$$X_{k,l}(t) = \sum_{k',l'} \Phi_{k,l,k',l'}(t) \left\{ \sum_{k'',l''} \int_0^t \Phi^{-1}_{k',l',k'',l''}(s)[F_{k'',l''}(s) - G_{k'',l',l''}(s)\, g_{k'',l',l''}(s)]ds \right.$$
$$\left. + \sum_{k'',l''} \int_0^t \Phi^{-1}_{k',l',k'',l''}(s)\, g_{k'',l',l''}(s)\, dB_{l''}(s) \right\}, \tag{57}$$

the summation being over $k', k'' = 1, \ldots, K$ and $l', l'' = 1, \ldots, d$, for all $k = 1, \ldots, K$, $l = 1, \ldots, d$. For $k = 0$, $X_{0,l}(t)$ is the solution of (34).

Φ is the fundamental $Kd \times Kd$ matrix of the corresponding homogeneous equation, i.e. Eq. (56) with $F = g = 0$, normalized so that $\Phi(0)$ is the unit matrix, and the integrals being understood in Itô's sense.

Proof. The proof uses Itô′s formula to identify dX_t as given by the derivative of the right hand side of (57) with the right hand side of (56). The presence of $\Phi(t)$ is for similar reasons as in Lagrange's method for systems of ODEs, see [24,36], to which we refer for details. □

Remark 4.2. For $K = 1$, $d = 1$, the fundamental $Kd \times Kd$ matrix reduces to a scalar Φ. In this case we have (see e.g., [36, p. 113]):

$$\Phi(t) = \exp\Big(\int_0^t \Big[\gamma(s) - \frac{1}{2}G^2(s)\Big]ds + \int_0^t G(s)\, dB(s)\Big). \tag{58}$$

In the case where η is the sum of B and a jump component η_J we have instead:

$$\Phi(t) = \exp\Big(\int_0^t \Big[\gamma(s) - \frac{1}{2}G^2(s)\Big]ds + \int_0^t G(s)\, dB(s)\Big) \prod_{0<s\le t} (1 + \Delta\eta_J(s))e^{-\Delta\eta_J(s)}, \tag{59}$$

where $\Delta\eta_J(s) := \eta_J(s) - \eta_J(s^-)$, is the jump of η_J between s^- and s. The product term is the Doléans-Dade exponential of a Lévy jump process, see, e.g. [23, p. 247], and [59].

Remark 4.3. 1. The corresponding results hold also in the deterministic case where B is replaced by a function η of bounded variation.

The concrete expression for Φ changes, due to the fact that there is no correction term in the exponents as in the Brownian motion exponential. For example instead of (58) we simply have then

$$\Phi(t) = \exp\Big(\int_0^t \gamma(s)\, ds + \int_0^t G(s)\, \eta(ds)\Big), \tag{60}$$

the second integral being a Stieltjes one.

2. In the case where η contains a nontrivial jump component, we were not able to find in the literature a general result of the type of (57).

For particular cases, see however [46,33]. As in (59) in the expression for Φ an additional Doléans-Dade term (stochastic exponential of a Lévy jump process) appears.

As we stated before Proposition 4.1, that proposition can be applied to the case where $\eta = B$ and, as in Remarks 2.7.2 and 2.7.3, $\beta(x) = Ax + b$, $\sigma_\varepsilon(x) = \sigma_0(x) + \varepsilon\,\sigma_1(x)$, with $\sigma_1(x) = \lambda\, x$, $\sigma_0(x) = \Pi\, x$ for all $x \in \mathbb{R}^d$, $b \in \mathbb{R}^d$.

From Proposition 4.1 and (54) we get the following:

Proposition 4.4. *Let $\beta(x) = Ax + b, \sigma_\varepsilon(x) = \sigma_0(x) + \varepsilon\,\sigma_1(x)$, $\sigma_1(x) = \lambda\, x$, $\sigma_0(x) = \Pi\, x$ for all $x \in \mathbb{R}^d$, $b \in \mathbb{R}^d$. A, λ, Π are constant $d \times d$ matrices. Consider the solution of the equation $du = \beta(u)\, dt + \sigma_\varepsilon(u)\,\eta(ds)$.*

Let u_k, $k \in \mathbb{N}_0$ be the expansion coefficients which satisfy the equations in Proposition 2.6. Then the lth component $u_{k,l}$ of u_k is given, for $k = 1, \ldots, N$, by

$$u_{k,l}(t) = \sum_{k',l'} \Phi_{k,l,k',l'}(t)\Big\{-\sum_{k'',i}\int_0^t \Phi^{-1}_{k',l',k'',i}(s)\,u_{k-1,i}(s) \times \Big(\lambda_{l,i}\,u_{k'',i}(s)\,ds + \lambda_{l',i}\,dB_i(s)\Big)\Big\}. \tag{61}$$

Φ is the fundamental matrix of the system

$$du_{k,l}(s) = (Au_k)_l(s)\,ds + \sum_{i=1}^{d} \Pi_{l,i}\,u_{k,i}\,dB_i(s), \quad k = 1, \ldots, N,\ l = 1, \ldots, d. \tag{62}$$

Moreover $u_0(t)$ (with components $u_{0,i}$, $i = 1, \ldots, d$) solves (34).

5. The asymptotic character of the expansion

In this section we shall prove the asymptotic character of the expansion of the solution u_ε of (1) in powers of ε.

Let $\beta \in \mathcal{C}^{p+1}(\mathbb{R}^d; \mathbb{R}^d)$, for some $p \in \mathbb{N}$. Consider as in Proposition 2.5, $\underline{y} = (y_0, \ldots, y_N, y) \in \mathbb{R}^{(N+2)d}$. Let $A^\beta_p(\underline{y}; \varepsilon) = \beta(\sum_{j=0}^N \varepsilon^j\,y_j + \varepsilon^{N+1}\,y) - \sum_{j=0}^p \varepsilon^j\,\beta_j(y_0, \ldots, y_j)$, β_j being the coefficients in the term of exact order j in the asymptotic expansion of $\beta(\sum_{j=0}^N \varepsilon^i\,y_i + \varepsilon^{N+1}\,y)$ in powers of $\varepsilon \in [0, \varepsilon_0]$. By Proposition 2.5 applied to the components $\beta_{(i)}$, $i = 1, \ldots, d$ of β we get $\|\,A^\beta_p(\underline{y}; \varepsilon)\| \le \varepsilon^{p+1}\,K^\beta_{p,N,\varepsilon_0}$, with $K^\beta_{p,N,\varepsilon_0}$ defined according to Proposition 2.5, with f replaced by $\beta_{(i)}$, by:

$$K^\beta_{p,N,\varepsilon_0} := \max_{i=1,\ldots,d} K^{\beta_{(i)}}_{p,N,\varepsilon_0}, \tag{63}$$

with

$$K^{\beta_{(i)}}_{p,N,\varepsilon_0} := \sup_{|\alpha|\le p}\ \sup_{\varepsilon\in[0,\varepsilon_0]} \left\|\frac{D^\alpha\,\beta_{(i)}(u_0)}{\alpha!}\right\| \prod_{i=1}^{d}\Big[\sum_{\alpha_{j,i}}^{\star} \frac{\alpha_i!}{\alpha_{1,i}!\cdots\alpha_{N+1,i}!}\,\varepsilon_0^{\sum_{j=1}^N j\,\alpha_{j,i}} \times \prod_{j=1}^{N} \|u_{j,i}\|^{\alpha_{j,i}}\,\frac{1}{N+1}\,\sup_{s\in[0,1]}\ \sup_{\varepsilon\in[0,\varepsilon_0]}\Big\|D^{N+1}\beta_{(i)}\Big(\sum_{j=0}^N s^j\,\varepsilon^j\,y_{j,(i)} + \varepsilon^{N+1}\,s^{N+1}\,y_{(i)}\Big)\Big\|\Big]. \tag{64}$$

From this we have $\varepsilon^{-(p+1)}\|A^\beta_p(\underline{y}; \varepsilon)\| \le K^\beta_{p,N,\varepsilon_0}$, $0 < \varepsilon \le \varepsilon_0$.

Similarly we show that $\varepsilon^{-(p+1)}\|A^\sigma_p(\underline{y}; \varepsilon)\| \le K^\sigma_{p,N,\varepsilon_0}$, $0 < \varepsilon \le \varepsilon_0$, with $A^\sigma_p(\underline{y}; \varepsilon)$ resp. $K^\sigma_{p,N,\varepsilon_0}$ defined in a corresponding way with β replaced by $\sigma = \sigma_\varepsilon$.

The quantities $K^{\beta}_{p,N,\varepsilon}$ and $K^{\sigma}_{p,N,\varepsilon}$ are by construction independent of $\varepsilon \in [0, \varepsilon_0]$. Hence we have

$$\lim_{\varepsilon\downarrow 0}\sup\, \varepsilon^{-(p+1)}\|A^{\gamma}_{p}(\underline{y};\varepsilon)\| \le K^{\gamma}_{p,N,\varepsilon_0}, \tag{65}$$

with γ standing for β resp. σ.

On the other hand, since by assumptions β and σ_ε are smooth, we also have (from the expression for A^{f}_{p} in the proof of Proposition 2.5 with f replaced by γ) that $\lim_{\varepsilon\downarrow 0} \varepsilon^{-(p+1)}A^{\gamma}_{p}(\underline{y};\varepsilon)$ exists and from (22) it is equal to the expression given by (26) (for f replaced by γ).

Let us now consider the case where $y_0 = u_0$, $y_j = u_j(t)$, $j \in \mathbb{N}$, with $u_j(t)$ solutions of the recursive system of equations in Section 3, which we assume to exist. Let us consider

$$y_N(t,\varepsilon) := \varepsilon^{-(N+1)}\Big[u_\varepsilon(t) - \sum_{j=0}^{N} \varepsilon^j\, u_j(t)\Big]. \tag{66}$$

Since u_ε satisfies by assumption Eq. (1) we have

$$\begin{aligned} y_N(t,\varepsilon) = \varepsilon^{-(N+1)}\Big[&u(0) + \int_0^t \beta\,(u_\varepsilon(s))\,ds \\ &+ \int_0^t \sigma_\varepsilon\,(u_\varepsilon(s))\,\eta(ds) - \sum_{j=0}^{N}\varepsilon^j\, u_j(t)\Big]. \end{aligned} \tag{67}$$

By the definition (66) of $y_N(t,\varepsilon)$ we have

$$\int_0^t \beta\,(u_\varepsilon(s))\,ds = \int_0^t \beta\Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N}\varepsilon^j\, u_j(s)\Big]ds, \tag{68}$$

and correspondingly

$$\int_0^t \sigma_\varepsilon\,(u_\varepsilon(s))\,\eta(ds) = \int_0^t \sigma_\varepsilon\Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N}\varepsilon^j\, u_j(s)\Big]\eta(ds). \tag{69}$$

Inserting this into (67) we get (minding $u(0) = u^0$):

$$\begin{aligned} y_N(t,\varepsilon) = \varepsilon^{-(N+1)}\Big[&u^0 + \int_0^t \beta\Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N}\varepsilon^j\, u_j(s)\Big]ds \\ &+ \int_0^t \sigma_\varepsilon\Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N}\varepsilon^j\, u_j(s)\Big]\eta(ds) - \sum_{j=0}^{N}\varepsilon^j\, u_j(t)\Big]. \end{aligned} \tag{70}$$

From Proposition 2.4 we have, on the other hand, for $j \in \mathbb{N}$:

$$u_j(t) = \int_0^t [\beta(u_\varepsilon(s))]_j\, ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_j\, \eta(ds). \tag{71}$$

Inserting into (70) we get:

$$
\begin{aligned}
y_N(t,\varepsilon) = \varepsilon^{-(N+1)} \Big[u^0 &+ \int_0^t \beta\Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N} \varepsilon^j\, u_j(s)\Big] ds \\
&+ \int_0^t \sigma_\varepsilon \Big[\varepsilon^{N+1} y_N(s,\varepsilon) + \sum_{j=0}^{N} \varepsilon^j\, u_j(s)\Big] \eta(ds) \\
&- \sum_{j=0}^{N} \varepsilon^j \Big[\int_0^t [\beta(u_\varepsilon(s))]_j \; ds + \int_0^t [\sigma_\varepsilon(u_\varepsilon(s))]_j \; \eta(ds) \Big]\Big].
\end{aligned} \tag{72}
$$

Recalling the definitions of $A_N^\beta(\underline{y};\varepsilon)$ and $A_N^\sigma(\underline{y};\varepsilon)$, as A_N^f in Proposition 2.5, with f replaced by β resp. σ, we can write this as

$$
y_N(t,\varepsilon) = \varepsilon^{-(N+1)} \Big[u^0 + \int_0^t A_N^\beta(\underline{y};\varepsilon)\, ds + \int_0^t A_N^\sigma(\underline{y};\varepsilon)\, \eta(ds) \Big], \tag{73}
$$

with $\underline{y} = \underline{y}(s) = \Big\{ u_0(s), u_1(s), \ldots, u_k(s), \varepsilon^{N+1}\, y_N(s,\varepsilon) \Big\}$.

From this, noting that from (66) we have $y_N(0,\varepsilon) = 0$ for all $\varepsilon \in [o, \varepsilon_0]$, we see that y_N satisfies the stochastic differential equation

$$
\begin{cases} dy_N(t,\varepsilon) = a_N^\beta(\underline{y}(t);\varepsilon)dt + a_N^\sigma(\underline{y}(t);\varepsilon)\eta(dt), \quad t \geq 0. \\ y_N(0,\varepsilon) = 0, \end{cases} \tag{74}
$$

with $a_N^\gamma(\underline{y}(t);\varepsilon) = \varepsilon^{-(N+1)}\, A_N^\gamma(\underline{y};\varepsilon)$, where γ stands for β resp. σ.

We saw before that the coefficients $a_N^\gamma(\underline{y}(t);\varepsilon)$ satisfy:

$$
\sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{[0,T]} |a_N^\gamma(\underline{y};\varepsilon)| \leq K_{N,\varepsilon_0}^\gamma, \tag{75}
$$

with $K_{N,\varepsilon_0}^\gamma$ independent of ε, and converge P-almost surely as $\varepsilon \downarrow 0$ (P-being the underlying probability measure).

Remark 5.1. We only obtain P-a.s convergence since our solutions u_j in Section 3, are, in general, only P-a.s.

Assuming the noise η in (1) is either a Brownian motion or is deterministic, or has also a jump component given by a Poisson process as in [39, p. 279], then because of our smoothness assumptions on $\beta, \sigma_\varepsilon$ and the results on the u_j in Section 3, we can apply Theorem 4 in chapter 2, Section 8, p. 279 in [39] (cfs. also Theorem 2 and Corollary 1, p. 52–53, for the case where η is a Brownian motion,) and obtain that there exists $y_N(t,0) \in L^2(P)$ such that

$$
\lim_{\varepsilon \downarrow 0} \sup_{0 \leq t \leq T} \mathbb{E}\Big\{ \|y_N(t,\varepsilon) - y_N(t,0)\|^2 \Big\} = 0. \tag{76}
$$

Let us summarize these results in the following

Theorem 5.2. *Let us consider the stochastic differential equation* (1) *with* $\beta, \sigma_\varepsilon$ *which are* $\mathcal{C}^{k+1}$, $k \in \mathbb{N}_0$ *in the space variables and such that* σ_ε *is* $\mathcal{C}^M$ *in* ε, $\varepsilon \in [0, \varepsilon_0]$, *for some* $M \geq k+1$, *with uniformly bounded derivatives.*

Assume that η is such that solutions $u_\varepsilon(t)$ of (1) and the $u_j(t)$ of the equations in Section 3 exist in $L^2(P)$, with P the underlying probability measure. Moreover assume that the solution of (1) depends $L^2(P)$-continuously on ε, $\varepsilon \in [0, \varepsilon_0]$. Then the solution $u_\varepsilon(t)$ of (1) has the following asymptotic expansion in powers of $\varepsilon \in [0, \varepsilon_0]$:

$$u_\varepsilon(t) = \sum_{j=0}^{N} \varepsilon^j\, u_j(t) + \varepsilon^{N+1}\, y_N(t, \varepsilon), \quad t > 0, \tag{77}$$

with $y_N(t, \varepsilon) \in L^2(P)$ for all $t \geq 0$, with

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\Big\{ \sup_{0 \leq t \leq T} \|y_N(t, \varepsilon) - y_N(t, 0)\|^2 \Big\} = 0. \tag{78}$$

The remainder $y_N(t, \varepsilon)$ satisfies the estimate

$$\sup_{0 \leq t \leq T} \Big[\mathbb{E}\|y_N(t, \varepsilon)\|^2 \Big]^{\frac{1}{2}} \leq \tilde{K}_{N,T,\varepsilon_0}, \tag{79}$$

with $\tilde{K}_{N,T,\varepsilon_0} > 0$ independent of ε for all $\varepsilon \in [0, \varepsilon_0]$.

Remark 5.3. The idea of the proof comes from a sketch given in [37]. The same basic conclusions can be drawn whenever one has an $L^2(P)$-result on continuity of solutions of stochastic differential equations with respect to parameters appearing in the coefficients. Gihman–Skorohod's result is only one among other possible results. Let us note that it is formulated with intrinsically random coefficients only for η being a one-dimensional Brownian motion (cf. Theorem. 2, in [39, pp. 52–53]). In the case of $\mathbb{R}^d$ and with non intrinsically random coefficients this problem is discussed in [39, p. 279]. An adaptation to our case is possible, but we are not aware of any specific reference.

The case of dissipative semi-linear stochastic equations with additive Lévy-type noise has been treated by us in details in finite or infinite dimensions, [15,6].

Similar results should be obtainable replacing the $L^2(P)$-continuity by other types of continuity in the probabilistic sense. However we were not able to locate specific references in this sense.

6. A remark on some applications

Heuristic asymptotic expansions in small parameters, to a certain order and more often without any proof of their asymptotic character (because of lack of suitable estimates on the remainders) appear often in the literature. E.g, in neurobiology, stochastic models of the Fitz Hugh Nagumo type without space dependence have been discussed extensively, at least with additive Gaussian noise. Our method can be applied to them. Examples are discussed basically with additive noise, e.g., in [2,61].

Another area where we find examples is mathematical finance. If we take $\sigma_0 = 0$, $\sigma_1(x) = \tilde{\sigma}\, x$, $\tilde{\sigma} > 0$, $\beta(x) = r\, x$, $x \in \mathbb{R}$, i.e., we take the model of example 2 in Remark 2.7, then u_ε satisfies the equation of a Black–Scholes model with volatility parameter $\varepsilon\, \tilde{\sigma}$ and our expansion is then a small volatility expansion, see also [47]. If we take instead $\sigma_0(x) = \tilde{\sigma}\, x$, $\sigma_i(x) \neq 0$ for some $i \in \mathbb{N}$, $\beta(x) = rx$, $x \in \mathbb{R}$, then we have a stochastic volatility model with leading order given by the Black–Scholes solution and the expansions yields corrections around the Black–Scholes model.

Similar applications can be given to the multidimensional Black–Scholes model, see, e.g., [47,62] and the AS-model for interacting assets discussed in [16,21,22,47].

For other applications in this area see also [26,44].

Acknowledgments

This work was supported by King Fahd University of Petroleum and Minerals under the project ♯ IN111009. The authors gratefully acknowledge this support.

We are also grateful to Luca Di Persio and Elisa Mastrogiacomo for stimulating discussions and collaborations on related problems.

References

[1] S. Albeverio, Wiener and Feynman-path integrals and their applications, in: Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), in: Proc. Sympos. Appl. Math., vol. 52, Amer. Math. Soc., Providence, RI, 1997, pp. 153–194.

[2] S. Albeverio, L. Di Persio, Some stochastic dynamical models in neurobiology: recent developments, Eur. Commun. Math. Theor. Biol. 14 (2011) 44–53.

[3] S. Albeverio, L. Di Persio, E. Mastrogiacomo, Small noise asymptotic expansions for stochastic PDE's I. The case of a dissipative polynomially bounded nonlinearity, Tohoku Math. J. 63 (2011) 877–898.

[4] S. Albeverio, L. Di Persio, E. Mastrogiacomo, Invariant measures for stochastic differential equations on networks, Proc. Sympos. Pure Math. 87 (2013) 1–33.

[5] S. Albeverio, L. Di Persio, E. Mastrogiacomo, B. Smii, A class of Lévy driven SDEs and their explicit invariant measures, Preprint, 2014. http://arxiv.org/abs/1407.3943.

[6] S. Albeverio, L. Di Persio, E. Mastrogiacomo, B. Smii, Explicit invariant measures for infinite dimensional SDE driven by Lévy noise with dissipative nonlinear drift I, Preprint, 2014. http://arxiv.org/abs/1312.2398.

[7] S. Albeverio, B. Ferrario, Some Methods of Infinite Dimensional Analysis in Hydrodynamic: Recent Progress and Prospects, in: Lecture Notes in Math., vol. 1942, Springer, Berlin, 2008, pp. 1–50.

[8] S. Albeverio, H. Gottschalk, J.-L. Wu, Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions, Rev. Math. Phys. 8 (6) (1996) 763–817.

[9] S. Albeverio, H. Gottschalk, M.W. Yoshida, System of classical particles in the grand canonical ensemble, scaling limits and quantum field theory, Rev. Math. Phys. 17 (02) (2005) 175–226.

[10] S. Albeverio, A. Hilbert, Kolokoltsov, Estimates uniform in time for the transition probability of diffusions with small drift and for stochastically perturbed Newton equations, J. Theoret. Probab. 12 (1999) 293–300.

[11] S. Albeverio, A. Hilbert, Kolokoltsov, Uniform asymptotic bounds for the heat kernel and the trace of a stochastic geodesic flow, Stochastics 84 (2–3) (2012) 315–333.

[12] S. Albeverio, R. Höegh-Krohn, S. Mazzuchi, Mathematical Theory of Feynamn Path Integrals, second ed., Springer, Berlin, 2008.

[13] S. Albeverio, S. Liang, Asymptotic expansions for the Laplace approximations of sums of Banach space-valued random variables, Ann. Probab. 33 (2005) 300–336.

[14] S. Albeverio, V. Mandrekar, B. Rüdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise, Stochastic Process. Appl. 119 (3) (2009) 835–863.

[15] S. Albeverio, E. Mastrogiacomo, B. Smii, Small noise asymptotic expansions for stochastic PDE's driven by dissipative nonlinearity and Lévy noise, Stochastic Process. Appl. 123 (2013) 2084–2109.

[16] S. Albeverio, A. Popovici, V. Steblovskaya, A numerical analysis of the extended Black–Scholes model, Int. J. Theor. Appl. Finance 9 (1) (2006) 69–89.

[17] S. Albeverio, H. Röckle, V. Steblovskaya, Asymptotic expansions for Ornstein–Uhlenbeck semigroups perturbed by potentials over Banach spaces, Stoch. Stoch. Rep. 69 (2000) 195–238.

[18] S. Albeverio, B. Rüdiger, Stochastic integrals and the Lévy–Itô decomposition theorem on separable Banach spaces, Stoch. Anal. Appl. 23 (2) (2005) 217–253.

[19] S. Albeverio, B. Rüdiger, J.-L. Wu, Invariant measures and symmetry property of Lévy type operators, Potential Anal. 13 (2000) 147–168.

[20] S. Albeverio, V. Steblovskaya, Asymptotics of infinite-dimensional integrals with respect to smooth measures. I, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2 (1999) 529–556.

[21] S. Albeverio, V. Steblovskaya, Financial market with interacting assets. Pricing barrier options, Tr. Mat. Inst. Steklova (2002) 173–184.
[22] S. Albeverio, V. Steblovskaya, K. Wallbaum, Valuation of equity-linked life insurance contracts using a model with interacting assets, Stoch. Anal. Appl. 27 (2009) 1077–1095.
[23] D. Applebaum, Lévy Processes and Stochastic Calculus, second ed., Cambridge University Press, 2009.
[24] L. Arnold, Stochastic Differential Equations. Theory and Applications, Wiley, 1974.
[25] V. Arnold, Mathematical Methods of Classical Mechanics, Springer, 1978.
[26] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics, J. R. Stat. Soc. Ser. B Stat. Methodol. 63 (2) (2001) 167–241.
[27] G.D. Birkhoff, Dynamical Systems, AMS, Providence, 1927.
[28] K. Bogoliubov, Introduction to Nonlinear Mechanics, Kraus, 1970.
[29] N. Bogoliubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Non-Linear Oscillations, Gordon and Breach, New York, 1961.
[30] K.W. Breitung, Asymptotic Approximations for Probability Integrals, in: Lecture Notes in Mathematics, Springer, 1994.
[31] Z. Brzeźniak, E. Hausenblas, Uniqueness in law of the Itô integral with respect to Lévy noise, in: Seminar Stoch. Anal., Random Fields and Appl., VI, Birkhäuser, Basel, 2011, pp. 37–57.
[32] O. Costin, S. Tanveer, Nonlinear evolution PDEs in $\mathbb{R}_+ \times C^d$: existence and uniqueness of solutions, asymptotics and Borel summability, Ann. Inst. H. Poincaré Anal. Nonlin. 24 (5) (2007) 795–823.
[33] J. Duan, J. Yan, General matrix valued inhomogeneous linear stochastic differential equations and applications, Statist. Probab. Lett. (2008) 2361–2365.
[34] V. Eckhaus, Asymptotic Analysis of Singular Perturbations, Elsevier Science Ltd., 1979.
[35] J.-P. Eckmann, H. Epstein, J. Fröhlich, Asymptotic perturbation expansion for the S-matrix and the definition of time ordered functions in relativistic quantum field models, Ann. Inst. H. Poincaré Phys. Theor. 25 (1976) 1–34.
[36] T. Gard, Introduction to Stochastic Differential Equations, Marcel Dekker Inc., 1988.
[37] C. Gardiner, Stochastic Methods: A Handbook for the Natural and Social Sciences, Springer-Verlag, Berlin, Heidelberg, 2009.
[38] M. Giaquinta, G. Modica, An Introduction to Functions of Several Variables, Birkhäuser, 2000.
[39] I.I. Gikhman, A.V. Skorohod, Stochastic Differential Equations, Springer, Berlin, 1972.
[40] H. Gottschalk, B. Smii, How to determine the law of the solution to a SPDE driven by a Lévy space–time noise, J. Math. Phys. 43 (2007) 1–22.
[41] H. Gottschalk, B. Smii, H. Ouerdiane, Convolution calculus on white noise spaces and Feynman graph representation of generalized renormalization flows, Math. Anal. Random Phenom. 101–110 (2007).
[42] H. Gottschalk, B. Smii, H. Thaler, The Feynman graph representation of general convolution semigroups and its applications to Lévy statistics, Bernoulli 14 (2008), no. 2, pp. 322–351.
[43] M. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodamsha, Amesterdam, Tokyo, 1981.
[44] P. Imkeller, I. Pavlymkevich, T. Wetzel, Lévy driven diffusions with exponentially light jumps, Ann. Probab. (2009) 530–564.
[45] T. Kato, Perturbation Theory for Linear Operators, second ed., Springer, Berlin, Heidelberg, 1980.
[46] R. Léandre, Flot d'une équation différentielle stochastique, in: Série de Prob. XIX, in: LN Math., Springer, 1985, pp. 271–278.
[47] E. Lütkebohmert, An asymptotic expansion for a Black–Scholes type model, Bull. Sci. Math. (2004) 661–685.
[48] V. Mandrekar, B. Rüdiger, Stochastic Integration in Banach Spaces, VII, Springer, Berlin, 2015.
[49] R. Marcus, Parabolic Itô equations, Trans. Amer. Math. Soc. 198 (1974) 177–190.
[50] R. Marcus, Parabolic Itô equations with monotone nonlinearities, J. Funct. Anal. 29 (3) (1978) 275–286.
[51] V. Maslov, Perturbation theory for the multidimensional Schrödinger equation, Uspekhi Mat. Nauk 16, 3 (99) (1961) 217.
[52] T. Meyer-Brandis, F. Proske, Explicit representation of strong solutions of SDEs driven by infinite dimensional Lévy processes, J. Theoret. Probab. 23 (2010) 301–314.
[53] P. Milan, K. Miroslav, Stochastic equations for simple discrete models of epitaxial growth, Phys. Rev. E 54 (4) (1996).
[54] I. Negoro, M. Tsuchiya, Convergence and uniqueness theorems for Markov processes associated with Lévy operators, in: Prob. Theo. and Math. Stat., Lecture Notes in Math., Springer, Berlin, 1988, pp. 348–356.
[55] P. Protter, Stochastic Integrations and Differential Equations, Springer, 2005.
[56] M. Reed, B. Simon, Methods of Modern Mathematical Physics. Vols. 1–4, Academic Pr. Inc., 1981.
[57] J. Sanders, F. Verhulst, Averaging Methods in Nonlinear Dynamical Systems, Springer-Verlag, 1985.

[58] K.I. Sato, Lévy Processes and Infinite Divisible Distributions, Cambridge University Press, 1999.
[59] R. Situ, Theory of Stochastic Differential Equations with Jumps and Applications, Springer, New York, 2005.
[60] B. Smii, A linked cluster theorem of the solution of the generalized Burger equation, Appl. Math. Sci. (Ruse) 6 (2012), no. 1–4, pp. 21–38.
[61] H.C. Tuckwell, Introduction to Theoretical Neurobiology. Vol. 2, Nonlinear and Stochastic Theories, in: Cambridge Studies in Mathematical Biology, vol. 8, Cambridge University Press, Cambridge, 1988, p. xii+265.
[62] M. Uchida, N. Yoshida, Asymptotic expansion for small diffusions applied to option pricing, Stat. Inference Stoch. Process. 7 (2004) 189–223.
[63] A. Vasil'eva, N. Uspekhi, Asymptotic behaviour of solutions to certain problems involving non-linear differential equations containing a small parameter multiplying the highest derivatives, Uspekhi Mat. Nauk 18, 3 (111) (1963) 15–86.
[64] F. Wentzell, Random Perturbations of Dynamical Systems, Springer Verlag, 2012.