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Excursion probability of certain non-centered smooth Gaussian random fields

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Abstract

Let $X = \{X(t), t \in T\}$ be a non-centered, unit-variance, smooth Gaussian random field indexed on some parameter space T, and let $A_u(X, T) = \{t \in T : X(t) \geq u\}$ be the excursion set. It is shown that, as $u \to \infty$, the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ can be approximated by the expected Euler characteristic of $A_u(X, T)$, denoted by $\mathbb{E}\{\chi(A_u(X, T))\}$, such that the error is super-exponentially small. The explicit formulae for $\mathbb{E}\{\chi(A_u(X, T))\}$ are also derived for two cases: (i) T is a rectangle and $X - \mathbb{E}X$ is stationary; (ii) T is an N-dimensional sphere and $X - \mathbb{E}X$ is isotropic.

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1. Introduction

Let $X = \{X(t),\ t \in T\}$ be a real-valued Gaussian random field living on some parameter space T. The excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ has been extensively studied in the literature due to its importance in both theory and applications in many areas. We refer to the survey Adler [1] and monographs Piterbarg [11], Adler and Taylor [2] and Azaïs and

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Wschebor [5] for the history, recent developments and related applications on this subject. To approximate the excursion probability for high exceeding level u, many authors have developed various powerful tools, including the double sum method [11], the tube method [15], the expected Euler characteristic approximation [1,16,17,2] and the Rice method [3–5].

In particular, the expected Euler characteristic approximation establishes a very general and profound result, building an interesting connection between the excursion probability and the geometry of the field. It was first rigorously proved by Taylor et al. [17] (see also Theorem 14.3.3 in [2]), showing that for a centered, unit-variance, smooth Gaussian random field, under certain conditions on the regularity of X and topology of T,

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} = \mathbb{E}\{\chi(A_u(X,T))\}\big(1+o\big(e^{-\alpha u^2}\big)\big) \quad \text{as } u\to\infty, \tag{1.1}$$

where $\chi(A_u(X,T))$ is the Euler characteristic of the excursion set $A_u(X,T)=\{t\in T : X(t)\geq u\}$ and $\alpha > 0$ is some constant. This verifies the "Expected Euler Characteristic Heuristic" for centered, unit-variance, smooth Gaussian random fields. Similar results can be found in [5] where the Rice method was applied. It had also been further developed by Cheng and Xiao [8] that (1.1) holds for certain Gaussian fields with stationary increments which have nonconstant variances. However, to the best of our knowledge, among the existing works on deriving the expected Euler characteristic approximation (1.1), the Gaussian field X is always assumed to be centered. In fact, the study of excursion probability for non-centered Gaussian fields is also very valuable since the varying mean function plays an important role in many models. Especially, when the Gaussian field is non-smooth, several results on the excursion probability have been obtained via the double sum method (see, for examples, [11,13,10]).

In this paper, we study the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t)\geq u\}$ for non-centered, unit variance, smooth (see condition (H1) below) Gaussian random fields. As the first contribution, we obtain in Theorem 3.5 that, in general, the expected Euler characteristic approximation 2.5 holds for such non-centered Gaussian fields when $T\subset\mathbb{R}^N$ is a compact rectangle. It shows that, comparing with the double sum method for non-smooth non-centered Gaussian fields (see [13] for example), we are able to obtain a much more accurate approximation for the excursion probability of smooth non-centered Gaussian fields such that the error is super-exponentially small. This is because the expected Euler characteristic approximation takes into account the effect of X over the boundary of T, which is ignored in the double sum method. By similar arguments in [3], such approximation can also be easily extended to the cases when $T\subset\mathbb{R}^N$ is a compact and convex set with smooth boundary or a compact and smooth manifold without boundary, see Theorem 2.6.

To apply the approximation in practice, one needs to find an explicit formula for the expected Euler characteristic $\mathbb{E}\{\chi(A_u(X,T))\}$. Under the assumption of centered Gaussian fields, Taylor and Adler [16] showed a very nice formula for $\mathbb{E}\{\chi(A_u(X,T))\}$ (see also [2]), involving the Lipschitz–Killing curvatures of the excursion set $A_u(X,T)$. However, there is lack of research to evaluate $\mathbb{E}\{\chi(A_u(X,T))\}$ for non-centered Gaussian fields. We provide here explicit formulae of $\mathbb{E}\{\chi(A_u(X,T))\}$ for two cases of non-centered Gaussian fields: (i) T is a rectangle and $X-\mathbb{E}X$ is stationary; (ii) T is an N-dimensional sphere and $X-\mathbb{E}X$ is isotropic; see respectively Theorems 3.5 and 3.11. The results show that, the mean function of the field does make the formula of $\mathbb{E}\{\chi(A_u(X,T))\}$ much more complicated than that of the centered field. In real applications, one usually needs to use the Laplace method to obtain explicit asymptotics for $\mathbb{E}\{\chi(A_u(X,T))\}$.

2. Excursion probability

2.1. Gaussian random fields on rectangles

We first consider the Gaussian field $X = \{X(t),\ t \in T\}$ with mean function $m(t) = \mathbb{E}\{X(t)\}$, where $T \subset \mathbb{R}^N$ is a compact rectangle. Throughout this paper, unless specified otherwise, X is assumed to be unit-variance, $m(\cdot)$ denotes the mean function of X and T denotes an N-dimensional compact rectangle. For a function $f(\cdot) \in C^2(T)$, we write $\frac{\partial f(t)}{\partial t_i} = f_i(t)$ and $\frac{\partial^2 f(t)}{\partial t_i \partial t_j} = f_{ij}(t)$. Denote by $\nabla f(t)$ and $\nabla^2 f(t)$ the column vector $(f_1(t), \dots, f_N(t))^T$ and the $N \times N$ matrix $(f_{ij}(t))_{i,j=1,\dots,N}$, respectively. We shall make use of the following smoothness condition (H1) and regularity condition (H2) for approximating the excursion probability, and also a weaker regularity condition (H2$'$) for evaluating the expected Euler characteristic $\mathbb{E}\{\chi(A_u(X,T))\}$ (note that (H2) implies (H2$'$)).

(H1) $X(\cdot) \in C^2(T)$ almost surely and its second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants $L > 0$ and $\eta \in (0, 1]$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(s))^2 \le L d(t,s)^{2\eta}, \quad \forall t, s \in T,\ i, j = 1, \dots, N, \tag{2.1}$$

where $d(t,s)$ is the distance of t and s.

(H2) For every pair $(t,s) \in T^2$ with $t \neq s$, the Gaussian random vector

$$(X(t), \nabla X(t), X_{ij}(t),\ X(s), \nabla X(s), X_{ij}(s), 1 \le i \le j \le N)$$

is non-degenerate.

(H2$'$) For every $t \in T$, $(X(t), \nabla X(t), X_{ij}(t), 1 \le i \le j \le N)$ is non-degenerate.

We may write $T = \prod_{i=1}^N [a_i, b_i]$, $-\infty < a_i < b_i < \infty$. Following the notation on page 134 in [2], we shall show that T can be decomposed into several faces of lower dimensions, based on which the Euler characteristic of the excursion set can be formulated.

A face J of dimension k is defined by fixing a subset $\sigma(J) \subset \{1, \dots, N\}$ of size k (if $k = 0$, we have $\sigma(J) = \emptyset$ by convention) and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0,1\}^{N-k}$ of size $N - k$, so that

$$\begin{aligned} J = \{t = (t_1, \dots, t_N) \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J), \\ t_j = (1 - \varepsilon_j) a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}. \end{aligned}$$

Denote by $\partial_k T$ the collection of all k-dimensional faces in T. Then the interior of T is given by $\overset{\circ}{T} = \partial_N T$ and the boundary of T is given by $\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J$. For $J \in \partial_k T$, denote by $\nabla X_{|J}(t)$ and $\nabla^2 X_{|J}(t)$ the column vector $(X_{i_1}(t), \dots, X_{i_k}(t))^T_{i_1,\dots,i_k \in \sigma(J)}$ and the $k \times k$ matrix $(X_{mn}(t))_{m,n \in \sigma(J)}$, respectively.

If $X(\cdot) \in C^2(T)$ and it is a Morse function a.s. (cf. Definition 9.3.1 in [2]), then according to Corollary 9.3.5 or pages 211–212 in [2], the Euler characteristic of the excursion set $A_u(X,T) = \{t \in T : X(t) \ge u\}$ is given by

$$\chi(A_u(X,T)) = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J) \tag{2.2}$$

with

$$\mu_i(J) := \#\big\{t \in J : X(t) \ge u, \nabla X_{|J}(t) = 0,\ \text{index } (\nabla^2 X_{|J}(t)) = i,$$

$$\varepsilon_j^* X_j(t) \ge 0 \text{ for all } j \notin \sigma(J)\big\},$$

where $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues.

For $t \in J \in \partial_k T$, let

$$\begin{aligned}
&\Lambda_J(t) = (\lambda_{ij}(t))_{i,j\in\sigma(J)} := (\mathrm{Cov}(X_i(t), X_j(t)))_{i,j\in\sigma(J)} = \mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J}(t)),\\
&\{J_1, \dots, J_{N-k}\} = \{1, \dots, N\} \setminus \sigma(J),\\
&E(J) = \{(t_{J_1}, \dots, t_{J_{N-k}}) \in \mathbb{R}^{N-k} : t_j \varepsilon_j^* \ge 0,\ j = J_1, \dots, J_{N-k}\}.
\end{aligned} \tag{2.3}$$

Since X has unit variance, $\mathrm{Cov}(X(t), \nabla^2 X_{|J}(t)) = -\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J}(t)) = -\Lambda_J(t)$, which is negative definite. Define the number of *extended outward maxima* above level u as

$$\begin{aligned}
M_u^E(J) &:= \#\big\{t \in J : X(t) \ge u, \nabla X_{|J}(t) = 0,\ \text{index } (\nabla^2 X_{|J}(t)) = k,\\
&\qquad \varepsilon_j^* X_j(t) \ge 0 \text{ for all } j \notin \sigma(J)\big\}\\
&= \#\big\{t \in J : X(t) \ge u, \nabla X_{|J}(t) = 0,\ \text{index } (\nabla^2 X_{|J}(t)) = k,\\
&\qquad (X_{J_1}(t), \dots, X_{J_{N-k}}(t)) \in E(J)\big\}.
\end{aligned} \tag{2.4}$$

Notice that $M_u^E(J) = \mu_k(J)$ if J is a k-dimensional face. It can be seen from Lemma A.1 in the Appendix that $\mathbb{E}\{M_u^E(J)\}$, the expected number of extended outward maxima, has a very close relation to the excursion probability. In fact, Lemma A.1 is a crucial technique for showing our desired approximation, see the arguments below.

We call a function $h(u)$ *super-exponentially small* (when compared with $\mathbb{P}(\sup_{t\in T} X(t) \ge u)$), if there exists a constant $\alpha > 0$ such that $h(u) = o(e^{-\alpha u^2 - u^2/2})$ as $u \to \infty$. The sketch for proving the expected Euler characteristic approximation (1.1) consists of two steps. The first step, which is established in Lemma 2.1, is to show that the difference between the upper bound in (A.1) and the expected Euler characteristic $\mathbb{E}\{\chi(A_u(X, T))\}$ is super-exponentially small. Then we prove that the upper bound in (A.1) makes the major contribution since the last two terms in the lower bound in (A.1) are super-exponentially small, see Lemmas 2.2–2.4.

The approach described above is similar to that for proving the case of centered Gaussian fields. The main difference is that, for a non-centered Gaussian field, one also needs to deal with the mean function, especially its interaction with the covariance and the geometry of T. In particular, the proofs of Lemmas 2.1 and 2.4 contain certain new techniques for showing the error terms involving the mean function are still super-exponentially small. The important intuitive idea behind these rigorous proofs is that the mean function will make the error terms to be $o(e^{-\alpha'(u-b)^2 - u^2/2})$ for some positive constants α' and b as $u \to \infty$, where b is related to the mean function. This error is still $o(e^{-\alpha u^2 - u^2/2})$ for $\alpha \in (0, \alpha')$.

Lemma 2.1. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2′). *Then for each $J \in \partial_k T$ with $k \ge 1$, there exists some constant $\alpha > 0$ such that*

$$\mathbb{E}\left\{M_u^E(J)\right\} = \mathbb{E}\left\{(-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J)\right\} (1 + o(e^{-\alpha u^2})). \tag{2.5}$$

Proof. To simplify the notation, without loss of generality, we assume $\sigma(J) = \{1, \dots, k\}$ and that all elements in $\varepsilon(J)$ are 1, which implies $E(J) = \mathbb{R}_+^{N-k}$. Let $\mathcal{D}_i$ be the collection of all $k \times k$ matrices with index i. By the Kac–Rice metatheorem (cf. Theorem 11.2.1 or Corollary 11.2.2

in [2]), $\mathbb{E}\{M_u^E(J)\}$ equals

$$\int_J \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}} \mathbb{1}_{\{X(t)\ge u\}} \mathbb{1}_{\{(X_{k+1}(t),\dots,X_N(t))\in\mathbb{R}_+^{N-k}\}} |\nabla X_{|J}(t)=0\}$$
$$\times p_{\nabla X_{|J}(t)}(0)dt$$
$$= (-1)^k \int_J dt \int_u^\infty dx \int_0^\infty dy_{k+1}\cdots\int_0^\infty dy_N$$
$$\mathbb{E}\{\det \nabla^2 X_{|J}(t)\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}|X(t)=x,$$
$$X_{k+1}(t)=y_{k+1},\dots,X_N=y_N,\nabla X_{|J}(t)=0\}$$
$$\times\, p_{X(t),X_{k+1}(t),\dots,X_N(t)}(x,y_{k+1},\dots,y_N|\nabla X_{|J}(t)=0)p_{\nabla X_{|J}(t)}(0). \tag{2.6}$$

Since $\Lambda_J(t)$ is positive definite for every $t\in J$, there exists a $k\times k$ positive definite matrix Q_t such that $Q_t\Lambda_J(t)Q_t = I_k$, where I_k is the $k\times k$ identity matrix. We write $\nabla^2 X_{|J}(t) = Q_t^{-1}Q_t\nabla^2 X_{|J}(t)Q_tQ_t^{-1}$ and let $a_{ij}^l(t) = \mathrm{Cov}(X_l(t),(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij})$ for $l=1,\dots,N$. Recall that $m(\cdot)$ is the mean function of X, applying Lemma A.2 yields

$$\mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}|X(t)=x,\nabla X_{|J}(t)=0,$$
$$X_{k+1}(t)=y_{k+1},\dots,X_N(t)=y_N\}$$
$$=(Q_t\nabla^2 m_{|J}(t)Q_t)_{ij}+(-\delta_{ij},a_{ij}^1(t),\dots,a_{ij}^N(t))(\mathrm{Cov}(X(t),\nabla X(t)))^{-1}$$
$$\cdot(x,0,\dots,0,y_{k+1},\dots,y_N)^T. \tag{2.7}$$

Make change of variables $V(t)=(V_{ij}(t))_{1\le i,j\le k}$, where

$$V_{ij}(t)=(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}-(Q_t\nabla^2 m_{|J}(t)Q_t)_{ij}+x\delta_{ij},$$

i.e.,

$$Q_t\nabla^2 X_{|J}(t)Q_t = V(t)+Q_t\nabla^2 m_{|J}(t)Q_t - xI_k. \tag{2.8}$$

Denote the density of

$$((V_{ij}(t))_{1\le i\le j\le k}|X(t)=x,\nabla X_{|J}(t)=0,X_{k+1}(t)=y_{k+1},\dots,X_N(t)=y_N)$$

by $h_{t,y_{k+1},\dots,y_N}(v)$, $v=(v_{ij}:1\le i\le j\le k)\in\mathbb{R}^{k(k+1)/2}$. It follows from (2.7) and the independence of $X(t)$ and $\nabla X(t)$ that $h_{t,y_{k+1},\dots,y_N}(v)$ is independent of x. Let (v_{ij}) be the abbreviation of matrix $(v_{ij})_{1\le i,j\le k}$. Applying (2.8) yields

$$\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}|X(t)=x,\nabla X_{|J}(t)=0,$$
$$X_{k+1}(t)=y_{k+1},\dots,X_N(t)=y_N\}$$
$$=\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)\mathbb{1}_{\{Q_t\nabla^2 X_{|J}(t)Q_t\in\mathcal{D}_k\}}|X(t)=x,\nabla X_{|J}(t)=0,$$
$$X_{k+1}(t)=y_{k+1},\dots,X_N(t)=y_N\}$$
$$=\int_{\{v:(v_{ij})+Q_t\nabla^2 m_{|J}(t)Q_t-xI_k\in\mathcal{D}_k\}}\det\Big((v_{ij})+Q_t\nabla^2 m_{|J}(t)Q_t-xI_k\Big)$$
$$\times h_{t,y_{k+1},\dots,y_N}(v)\,dv. \tag{2.9}$$

Since $Q_t\nabla^2 m_{|J}(t)Q_t$ is continuous in t and T is compact, there exists some constant $c>0$ such that the following relation holds for all $t\in T$ and x large enough:

$$(v_{ij})+Q_t\nabla^2 m_{|J}(t)Q_t-xI_k\in\mathcal{D}_k,\quad \forall\ \|(v_{ij})\|<\frac{x}{c}. \tag{2.10}$$

Let

$$\begin{aligned} &W(t, x, y_{k+1}, \ldots, y_N) \\ &\quad = \int_{\{v:(v_{ij})+Q_t\nabla^2 m_{|J}(t)Q_t - xI_k \notin \mathcal{D}_k\}} \det\Big((v_{ij}) + Q_t\nabla^2 m_{|J}(t)Q_t - xI_k\Big) h_{t,y_{k+1},\ldots,y_N}(v)\,dv. \end{aligned}$$

Then (2.9) becomes

$$\begin{aligned} &\int_{\mathbb{R}^{k(k+1)/2}} \det\Big((v_{ij}) + Q_t\nabla^2 m_{|J}(t)Q_t - xI_k\Big) h_{t,y_{k+1},\ldots,y_N}(v)\,dv \\ &\quad - W(t, x, y_{k+1}, \ldots, y_N). \end{aligned} \tag{2.11}$$

It follows from (2.10) that

$$\begin{aligned} I(t,x) &:= \int_0^\infty dy_{k+1} \cdots \int_0^\infty dy_N\, p_{X(t),X_{k+1}(t),\ldots,X_N(t)}(x, y_{k+1}, \ldots, y_N|\nabla X_{|J}(t) = 0) \\ &\quad \times |W(t, x, y_{k+1}, \ldots, y_N)| \\ &\le \int_0^\infty dy_{k+1} \cdots \int_0^\infty dy_N\, p_{X(t),X_{k+1}(t),\ldots,X_N(t)}(x, y_{k+1}, \ldots, y_N|\nabla X_{|J}(t) = 0) \\ &\quad \times \int_{\|(v_{ij})\|\ge \frac{x}{c}} \Big|\det\Big((v_{ij}) + Q_t\nabla^2 m_{|J}(t)Q_t - xI_k\Big)\Big| h_{t,y_{k+1},\ldots,y_N}(v)dv \\ &\le p_{X(t)}(x|\nabla X_{|J}(t) = 0) \int_{\|(v_{ij})\|\ge \frac{x}{c}} \Big|\det\Big((v_{ij})+Q_t\nabla^2 m_{|J}(t)Q_t - xI_k\Big)\Big| f_t(v)dv, \end{aligned} \tag{2.12}$$

where $f_t(v)$ is the density of $((V_{ij}(t))_{1\le i\le j\le k}|X(t) = x, \nabla X_{|J}(t) = 0)$ and the last inequality comes from replacing the integral domain $\mathbb{R}_+^{N-k}$ by $\mathbb{R}^{N-k}$. Notice that the last integral in (2.12) is $o(e^{-\alpha x^2})$ for some $\alpha > 0$ as $x \to \infty$, implying $\int_J \int_u^\infty I(t,x)dxdt = o(e^{-\alpha u^2 - u^2/2})$ as $u \to \infty$. Plugging this, together with (2.9) and (2.11), into (2.6), we see that $\mathbb{E}\{M_u^E(J)\}$ becomes

$$\begin{aligned} &(-1)^k \int_J \det(\Lambda_J(t))dt \int_u^\infty dx \int_0^\infty dy_{k+1} \cdots \int_0^\infty dy_N \\ &\quad \times \mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}}|X(t) = x, \\ &\qquad\qquad X_{k+1}(t) = y_{k+1}, \ldots, X_N = y_N, \nabla X_{|J}(t) = 0\} \\ &\quad \times p_{X(t),X_{k+1}(t),\ldots,X_N(t)}(x, y_{k+1}, \ldots, y_N|\nabla X_{|J}(t) = 0)p_{\nabla X_{|J}(t)}(0)dt. \\ &= (-1)^k\Big[\int_J dt \int_u^\infty dx \int_0^\infty dy_{k+1} \cdots \int_0^\infty dy_N \\ &\quad \times \mathbb{E}\{\det \nabla^2 X_{|J}(t)|X(t) = x, X_{k+1}(t) = y_{k+1}, \ldots, X_N = y_N, \nabla X_{|J}(t) = 0\} \\ &\quad \times p_{X(t),X_{k+1}(t),\ldots,X_N(t)}(x, y_{k+1}, \ldots, y_N|\nabla X_{|J}(t) = 0)p_{\nabla X_{|J}(t)}(0)\Big] + o(e^{-\alpha u^2 - u^2/2}) \\ &= \mathbb{E}\left\{(-1)^k \sum_{i=0}^k (-1)^i \mu_i(J)\right\} (1 + o(e^{-\alpha u^2})), \end{aligned}$$

where the last line is due to the Kac–Rice metatheorem and the fact that

$$\sum_{i=0}^k (-1)^i |\det \nabla^2 X_{|J}(t)|\mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\}} = \det \nabla^2 X_{|J}(t), \quad \text{a.s.} \quad \square$$

Let $\mathbb{S}^{k-1}$ be the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$. The following result shows that the factorial moments of $M_u^E(J)$ are super-exponentially small.

Lemma 2.2. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2). *Then for all $J \in \partial_k T$, $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}$ are super-exponentially small.*

Proof. If $k = 0$, then $M_u^E(J)$ is either 0 or 1 and hence $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\} = 0$. If $k \geq 1$, then, thanks to Lemma A.3, it suffices to show $\beta_J^2 < 1$. By Lemma A.2, for every $e \in \mathbb{S}^{k-1}$ and $t \in T$, $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) \leq 1$. On the other hand,

$$\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) = 1 \Longrightarrow \mathrm{Cov}(X(t), \nabla^2 X_{|J}(t)e) = 0. \tag{2.13}$$

Note that the right hand side of (2.13) is equivalent to $\Lambda_J(t)e = 0$. However, by (H2), $\Lambda_J(t)$ is positive definite, which implies $\Lambda_J(t)e \neq 0$ for all $e \in \mathbb{S}^{k-1}$. Thus for every $e \in \mathbb{S}^{k-1}$ and $t \in T$, $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) < 1$. Combining this with the continuity of $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e)$ in (e,t), we conclude $\beta_J^2 < 1$. □

By similar arguments for showing Lemma 4.5 in [8], one can easily obtain that the cross terms $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ in (A.1) are super-exponentially small if J and J' are not adjacent. In particular, as the main step therein, Eq. (4.13) is essentially not affected by the mean function of the field. We thus have the following result.

Lemma 2.3. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2). *Let J and J' be two faces of T such that their distance is positive, i.e., $\inf_{t\in J, s\in J'} \|s-t\| > \delta_0$ for some $\delta_0 > 0$. Then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.*

Next we turn to the alternative case when J and J' are adjacent. In such case, it is more technical to prove that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small. To shorten the arguments for deriving Lemma 2.4, we will quote certain results in the proof of Theorem 4.8 in [8] (or Theorem 4 in [3]).

Lemma 2.4. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2). *Let J and J' be two faces of T such that they are adjacent, i.e., $\inf_{t\in J, s\in J'} \|s-t\| = 0$. Then $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small.*

Proof. Let $I := \bar{J} \cap \bar{J'} \neq \emptyset$. Without loss of generality, we assume

$$\sigma(J) = \{1,\ldots,l,l+1,\ldots,k\}, \qquad \sigma(J') = \{1,\ldots,l,k+1,\ldots,k+k'-l\}, \tag{2.14}$$

where $0 \leq l \leq k \leq k' \leq N$ and $k' \geq 1$. Recall that, if $k = 0$, then $\sigma(J) = \emptyset$. Under assumption (2.14), we have $J \in \partial_k T$, $J' \in \partial_{k'} T$ and $\dim(I) = l$. Assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, which implies $E(J) = \mathbb{R}_+^{N-k}$ and $E(J') = \mathbb{R}_+^{N-k'}$.

We first consider the case $k \geq 1$. Applying the Kac–Rice metatheorem and removing the indicator functions for negative definiteness of the Hessian of the field, we obtain that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is bounded from above by

$$\begin{aligned}&\int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_{k+1} \cdots \int_0^\infty dz_{k+k'-l} \int_0^\infty dw_{l+1} \cdots \int_0^\infty dw_k \\ &\mathbb{E}\Big\{|\det \nabla^2 X_{|J}(t)|\,|\det \nabla^2 X_{|J'}(s)|\Big| X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, X_{k+1}(t) = z_{k+1}, \\ &\qquad \ldots, X_{k+k'-l}(t) = z_{k+k'-l}, \nabla X_{|J'}(s) = 0, X_{l+1}(s) = w_{l+1}, \ldots, X_k(s) = w_k\Big\}\end{aligned}$$

$$\times\, p_{t,s}(x, y, 0, z_{k+1}, \ldots, z_{k+k'-l}, 0, w_{l+1}, \ldots, w_k)$$
$$:= \iint_{J\times J'} A(t,s)\,dtds, \tag{2.15}$$

where $p_{t,s}(x, y, 0, z_{k+1}, \ldots, z_{k+k'-l}, 0, w_{l+1}, \ldots, w_k)$ is the density of

$$(X(t), X(s), \nabla X_{|J}(t), X_{k+1}(t), \ldots, X_{k+k'-l}(t), \nabla X_{|J'}(s), X_{l+1}(s), \ldots, X_k(s))$$

evaluated at $(x, y, 0, z_{k+1}, \ldots, z_{k+k'-l}, 0, w_{l+1}, \ldots, w_k)$.

Let $\{e_1, e_2, \ldots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For each $t \in T$, let $\Lambda(t) = (\lambda_{ij}(t))_{1\le i,j\le N}$, where $\lambda_{ij}(t) = \mathrm{Cov}(X_i(t)X_j(t))$ as defined in (2.3). For $t \in J$ and $s \in J'$, let $e_{t,s} = (s-t)^T/\|s-t\|$ and let $\alpha_i(t,s) = \langle e_i, \Lambda(t)e_{t,s}\rangle$. Then

$$\Lambda(t)e_{t,s} = \sum_{i=1}^N \langle e_i, \Lambda(t)e_{t,s}\rangle e_i = \sum_{i=1}^N \alpha_i(t,s)e_i. \tag{2.16}$$

Notice that $\Lambda(t) = \mathrm{Cov}(\nabla X(t), \nabla X(t))$ is positive definite for each $t \in T$. On the other hand, $\Lambda(\cdot)$ is continuous as a function in t and T is compact, thus $\Lambda(t)$ are uniformly positive definite for all $t \in T$. Therefore, there exists some $\alpha_0 > 0$ such that

$$\langle e_{t,s}, \Lambda(t)e_{t,s}\rangle \ge \alpha_0 \tag{2.17}$$

for all t and s. Let

$$\begin{aligned}
D_i &= \{(t,s) \in J \times J' : \alpha_i(t,s) \ge \beta_i\}, \quad \text{if } l+1 \le i \le k,\\
D_i &= \{(t,s) \in J \times J' : \alpha_i(t,s) \le -\beta_i\}, \quad \text{if } k+1 \le i \le k+k'-l,\\
D_0 &= \left\{(t,s) \in J \times J' : \sum_{i=1}^l \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \ge \beta_0\right\},
\end{aligned} \tag{2.18}$$

where $\beta_0, \beta_1, \ldots, \beta_{k+k'-l}$ are positive constants such that $\beta_0 + \sum_{i=l+1}^{k+k'-l} \beta_i < \alpha_0$. It follows from (2.18) that, if (t,s) does not belong to any of $D_0, D_m, \ldots, D_{k+k'-m}$, then by (2.16) and (2.14),

$$\langle \Lambda(t)e_{t,s}, e_{t,s}\rangle = \sum_{i=1}^N \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \le \beta_0 + \sum_{i=l+1}^{k+k'-l} \beta_i < \alpha_0,$$

contradicting (2.17). Therefore, $D_0 \cup \bigcup_{i=l+1}^{k+k'-l} D_i$ is a covering of $J \times J'$. By (2.15),

$$\mathbb{E}\{M_u^E(J)M_u^E(J')\} \le \iint_{D_0} A(t,s)\,dtds + \sum_{i=l+1}^{k+k'-l} \iint_{D_i} A(t,s)\,dtds.$$

It follows from the same arguments in the proof of Theorem 4.8 in [8] that $\iint_{D_0} A(t,s)\,dtds$ is super-exponentially small. Next we show that $\iint_{D_i} A(t,s)\,dtds$ is super-exponentially small for $i = l+1, \ldots, k$.

It follows from (2.15) that $\iint_{D_i} A(t,s)\,dtds$ is bounded above by

$$\begin{aligned}
&\iint_{D_i} dtds \int_u^\infty dx \int_0^\infty dw_i\, p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x, 0, w_i, 0)\\
&\quad\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)|\,|\det \nabla^2 X_{|J'}(s)|\,|X(t) = x,\\
&\qquad\qquad \nabla X_{|J}(t) = 0, X_i(s) = w_i, \nabla X_{|J'}(s) = 0\}.
\end{aligned} \tag{2.19}$$

Notice that if a subset $B \subset D_i$ satisfies $\inf_{t\in B\cap J,\, s\in B\cap J'} \|s-t\| > \eta_0$ for some $\eta_0 > 0$, then similarly to Lemma 2.3, $\iint_B A(t,s)\,dtds$ is super-exponentially small. Therefore, in the arguments below, we only treat the case when t and s are close enough or $\|t-s\| \to 0$.

There exists some positive constant C_1 such that

$$
\begin{aligned}
&p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)\\
&\quad = p_{\nabla X_{|J'}(s),X_1(t),\ldots,X_{i-1}(t),X_{i+1}(t),\ldots,X_k(t)}(0|X(t)=x, X_i(s)=w_i, X_i(t)=0)\\
&\qquad \times p_{X(t)}(x|X_i(s)=w_i, X_i(t)=0)\,p_{X_i(s)}(w_i|X_i(t)=0)\,p_{X_i(t)}(0)\\
&\quad \le C_1(\mathrm{detCov}(X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)))^{-1/2}\\
&\qquad \times \exp\left\{-\frac{(x-\xi_2(t,s))^2}{2\sigma_2^2(t,s)}\right\}\exp\left\{-\frac{(w_i-\xi_1(t,s))^2}{2\sigma_1^2(t,s)}\right\},
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{aligned}
&\xi_1(t,s) = \mathbb{E}\{X_i(s)|X_i(t)=0\} = m_i(s),\\
&\sigma_1^2(t,s) = \mathrm{Var}(X_i(s)|X_i(t)=0) = \frac{\mathrm{detCov}(X_i(s),X_i(t))}{\lambda_{ii}(t)},\\
&\xi_2(t,s) = \mathbb{E}\{X(t)|X_i(s)=w_i, X_i(t)=0\},\\
&\sigma_2^2(t,s) = \mathrm{Var}(X(t)|X_i(s)=w_i, X_i(t)=0).
\end{aligned}
$$

In particular, applying Taylor's formula to $X_i(s)$ (see Eq. (4.23) in [8] or [12]), one has

$$
\begin{aligned}
\xi_2(t,s) &= \mathbb{E}\{X(t)|\langle\nabla X_i(t),e_{t,s}\rangle = w_i/\|s-t\| + o(1), X_i(t)=0\},\\
&= m(t) + (\mathrm{Cov}(X(t),\langle\nabla X_i(t),e_{t,s}\rangle),0)\begin{pmatrix}\frac{1}{\mathrm{Var}(\langle\nabla X_i(t),e_{t,s}\rangle)} & 0\\ 0 & \frac{1}{\lambda_{ii}(t)}\end{pmatrix}\\
&\qquad \cdot\begin{pmatrix} w_i/\|s-t\| + o(1) - \langle\nabla m_i(t),e_{t,s}\rangle\\ -m_i(t)\end{pmatrix}\\
&= m(t) - \frac{\alpha_i(t,s)[w_i/\|s-t\| - \langle\nabla m_i(t),e_{t,s}\rangle + o(1)]}{\mathrm{Var}(\langle\nabla X_i(t),e_{t,s}\rangle)}
\end{aligned}
\tag{2.21}
$$

and

$$
\sigma_2^2(t,s) = \mathrm{Var}(X(t)|\langle\nabla X_i(t),e_{t,s}\rangle, X_i(t)) + o(1) \le 1-\delta_0 \tag{2.22}
$$

for some $\delta_0 > 0$.

Also, by the same arguments in the proof of Theorem 4.8 in [8], there exist positive constants C_2, C_3, N_1 and N_2 such that

$$
\mathrm{detCov}(\nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)) \ge C_2\|s-t\|^{2(l+1)} \tag{2.23}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)|\,\big|X(t)=x, \nabla X_{|J}(t)=0, X_i(s)=w_i, \nabla X_{|J'}(s)=0\right\}\\
&\quad = \mathbb{E}\big\{|\det\nabla^2 X_{|J}(t)||\det\nabla^2 X_{|J'}(s)|\,\big|X(t)=x, \nabla X_{|J}(t)=0,\\
&\qquad \langle\nabla X_i(t),e_{t,s}\rangle = w_i/\|s-t\| + o(1), \nabla X_{|J'}(s)=0\big\}\\
&\quad \le C_3(x^{N_1} + |w_i/\|s-t\||^{N_2} + 1).
\end{aligned}
\tag{2.24}
$$

Combining (2.19)–(2.24), and making change of variable $w = w_i/\|s-t\|$, we obtain that there exist positive constants C_4, C_5, C_6 and C_7 such that $\iint_{D_i} A(t,s)\,dtds$ is bounded above by

$$\begin{aligned}
&C_4 \iint_{D_i} dtds \|s-t\|^{-l-1} \int_u^\infty dx \int_0^\infty dw_i (x^{N_1} + |w_i/\|s-t\||^{N_2} + 1)\\
&\quad\times \exp\left\{-\frac{(x-\xi_2(t,s))^2}{2\sigma_2^2(t,s)}\right\} \exp\left\{-\frac{(w_i-m_i(s))^2}{2\sigma_1^2(t,s)}\right\}\\
&= C_4 \iint_{D_i} dtds \|s-t\|^{-l} \int_u^\infty dx \int_0^\infty dw (x^{N_1} + |w|^{N_2} + 1)\\
&\quad\times \exp\left\{-\frac{\left(x - m(t) + \frac{\alpha_i(t,s)[w - \langle \nabla m_i(t), e_{t,s}\rangle + o(1)]}{\mathrm{Var}(\langle \nabla X_i(t), e_{t,s}\rangle)}\right)^2}{2\sigma_2^2(t,s)}\right\} \exp\left\{-\frac{(w-\widetilde{m}_i(t,s))^2}{2\widetilde{\sigma}_1^2(t,s)}\right\}\\
&\le C_4 \iint_{D_i} dtds \|s-t\|^{-l} \int_u^\infty \exp\left\{-\frac{[x - C_5 + \beta_i(C_6 w - C_7)]^2}{2(1-\delta_0)}\right\}dx\\
&\quad\times \int_0^\infty (x^{N_1} + |w|^{N_2} + 1) \exp\left\{-\frac{(w-\widetilde{m}_i(t,s))^2}{2\widetilde{\sigma}_1^2(t,s)}\right\}dw,
\end{aligned}$$

where $\widetilde{\sigma}_1(t,s) = \sigma_1(t,s)/\|s-t\|$, $\widetilde{m}_i(t,s) = m_i(s)/\|s-t\|$ and we have used the fact $\alpha_i(t,s) \ge \beta_i > 0$ for the last inequality. This, in turn, ensures that there exists some $\delta_1 \in (0, \delta_0)$ such that for sufficiently large u,

$$\begin{aligned}
&\iint_{D_i} A(t,s)\,dtds\\
&\le C_4 \exp\left\{-\frac{u^2}{2(1-\delta_1)}\right\} \iint_{D_i} \|s-t\|^{-l} dtds\\
&\quad\times \int_0^\infty \exp\left\{-\frac{\beta_i^2 C_6^2 w^2}{2(1-\delta_1)}\right\}(u^{N_1} + |w|^{N_2} + 1) \exp\left\{-\frac{(w-\widetilde{m}_i(t,s))^2}{2\widetilde{\sigma}_1^2(t,s)}\right\}dw\\
&\le C_4 \exp\left\{-\frac{u^2}{2(1-\delta_1)}\right\} \iint_{D_i} \|s-t\|^{-l} dtds \int_0^\infty\\
&\quad\times \exp\left\{-\frac{\beta_i^2 C_6^2 w^2}{2(1-\delta_1)}\right\}(u^{N_1} + |w|^{N_2} + 1)dw.
\end{aligned}$$

Since $\|s-t\|^{-l}$ is integrable on $J \times J'$, we conclude that $\iint_{D_i} A(t,s)\,dtds$ is finite and super-exponentially small.

It is similar to show that $\iint_{D_i} A(t,s)\,dtds$ is super-exponentially small for $i = k+1, \ldots, k+k'-l$. The case when $k = 0$ can also be proved similarly. □

Now we can derive our main result of this section.

Theorem 2.5. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2). *Then there exists some $\alpha > 0$ such that the expected Euler characteristic approximation* (1.1) *holds.*

Proof. The result follows immediately from combining (A.1), Lemmas 2.1–2.4. □

2.2. Gaussian random fields on other sets

Adler and Taylor [2] obtained the expected Euler characteristic approximation (1.1) for centered Gaussian fields living on quite general manifolds. Since the method used in this paper is different, and it may require more powerful techniques and careful arguments to extend the parameter sets to general manifolds, we will not attempt to achieve such extension here. However, similarly to Azaïs and Delmas [3], we can easily extend the approximation to the cases of smooth and compact manifolds without boundary or convex and compact sets with smooth boundary.

We first introduce some notation. Let (T, g) be an N-dimensional Riemannian manifold, where g is the Riemannian metric, and let f be a real-valued smooth function on T. Then the *gradient* of f, denoted by ∇f, is the unique continuous vector field on T such that $g(\nabla f, \xi) = \xi f$ for every vector field ξ. The *Hessian* of f, denoted by $\nabla^2 f$, is the double differential form defined by $\nabla^2 f(\xi, \zeta) = \xi\zeta f - \nabla_\xi \zeta f$, where ξ and ζ are vector fields and ∇_ξ is the Levi-Civitá connection of (T, g). To make the notation consistent with the Euclidean case, we fix an orthonormal frame $\{E_i\}_{1\le i\le N}$, and let

$$\nabla f = (f_1, \dots, f_N) = (E_1 f, \dots, E_N f),$$
$$\nabla^2 f = (f_{ij})_{1\le i,j\le N} = (\nabla^2 f(E_i, E_j))_{1\le i,j\le N}.$$

Note that if t is a critical point, i.e. $\nabla f(t) = 0$, then $\nabla^2 f(E_i, E_j)(t) = E_i E_j f(t)$, which is similar to the Euclidean case. As in the Euclidean space, we denote by d the distance function induced by Riemannian metric g, which is also called the geodesic distance on (T, g).

If $X(\cdot) \in C^2(T)$, where T is a smooth and compact manifold without boundary, and it is a Morse function a.s., then according to Corollary 9.3.5 in [2], the Euler characteristic of the excursion set $A_u(X, T) = \{t \in T : X(t) \ge u\}$ is given by

$$\chi(A_u(X, T)) = (-1)^N \sum_{i=0}^{N} (-1)^i \mu_i(T) \tag{2.25}$$

with

$$\mu_i(T) := \#\big\{t \in T : X(t) \ge u, \nabla X(t) = 0, \ \text{index}\ (\nabla^2 X(t)) = i\big\}.$$

If T is a convex and compact sets with smooth boundary, then we have

$$\chi(A_u(X, T)) = (-1)^N \sum_{i=0}^{N} (-1)^i \mu_i(\overset{\circ}{T}) + (-1)^{N-1} \sum_{i=0}^{N-1} (-1)^i \mu_i(\partial T)$$

with

$$\mu_i(\overset{\circ}{T}) := \#\big\{t \in \overset{\circ}{T} : X(t) \ge u, \nabla X(t) = 0, \ \text{index}\ (\nabla^2 X(t)) = i\big\},$$
$$\mu_i(\partial T) := \#\big\{t \in \partial T : X(t) \ge u, \nabla X_{|\partial T}(t) = 0, \ \text{index}\ (\nabla^2 X_{|\partial T}(t)) = i\big\}.$$

By similar arguments for Gaussian fields on rectangles in the previous section, together with the projection technique in [3] or the argument by charts in Theorem 12.1.1 in [2], we can obtain the following extension, whose proof is omitted here.

Theorem 2.6. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2)*, where T is a smooth and compact manifold without boundary or a convex and compact set with* *smooth boundary. Then there exists some $\alpha > 0$ such that the expected Euler characteristic approximation (1.1) holds.*

3. The expected Euler characteristic

We now turn to computing the expected Euler characteristic $\mathbb{E}\{\chi(A_u(X,T))\}$. To do this, we need some preliminary results on calculations of certain Gaussian matrices.

3.1. Preliminary computations on Gaussian matrices

The following lemma can be obtained by elementary calculations. See also Lemma 11.6.1 in [2] for reference.

Lemma 3.1 (*Wick Formula*)**.** *Let $(Z_1, Z_2, \dots, Z_N)$ be a centered Gaussian random vector. Then for any integer k,*

$$\mathbb{E}\{Z_1 Z_2 \cdots Z_{2k+1}\} = 0,$$
$$\mathbb{E}\{Z_1 Z_2 \cdots Z_{2k}\} = \sum \mathbb{E}\{Z_{i_1} Z_{i_2}\} \cdots \mathbb{E}\{Z_{i_{2k-1}} Z_{i_{2k}}\},$$

where the sum is taken over the $(2k)!/(k!2^k)$ different ways of grouping $Z_1, \dots, Z_{2k}$ into k pairs.

Let $\Delta_N = (\Delta_{i,j})_{1\le i,j\le N}$ and $\Xi_N = (\Xi_{i,j})_{1\le i,j\le N}$ be two $N \times N$ symmetric centered Gaussian matrices satisfying the following properties:

$$\begin{aligned} &\mathbb{E}\{\Delta_{i,j}\Delta_{k,l}\} = \mathcal{E}(i,j,k,l) - \delta_{ij}\delta_{kl},\\ &\mathbb{E}\{\Xi_{i,j}\Xi_{k,l}\} = \mathcal{F}(i,j,k,l), \end{aligned} \tag{3.1}$$

where $\mathcal{E}$ and $\mathcal{F}$ are both symmetric functions of i, j, k, l, and δ_{ij} is the Kronecker delta function.

The following result is an extension of Lemma 11.6.2 in [2]. It will be used for computing the expected Euler characteristic of stationary or isotropic Gaussian fields.

Lemma 3.2. *Let $B_N = (B_{i,j})_{1\le i,j\le N}$ be an $N \times N$ real symmetric matrix. Then, under (3.1),*

$$\begin{aligned} &\mathbb{E}\{\det(\Delta_N + B_N)\} = \sum_{k=0}^{\lfloor N/2\rfloor} \frac{(-1)^k(2k)!}{k!2^k} S_{N-2k}(B_N),\\ &\mathbb{E}\{\det(\Xi_N + B_N)\} = \det(B_N), \end{aligned} \tag{3.2}$$

where $S_j(B_l)$ denotes the sum of the $\binom{l}{j}$ principle minors of order j in B_l, and $S_0(B_l) = 1$ by convention.

Proof. We first consider the case when N is even, say $N = 2l$. Then

$$\mathbb{E}\{\det(\Delta_{2l} + B_{2l})\} = \sum_P \eta(p)\mathbb{E}\{(\Delta_{1,i_1} + B_{1,i_1}) \cdots (\Delta_{2l,i_{2l}} + B_{2l,i_{2l}})\},$$

where $p = (i_1, i_2 \cdots, i_{2l})$ is a permutation of $(1, 2, \dots, 2l)$, P is the set of the $(2l)!$ such permutations, and $\eta(p)$ equals $+1$ or -1 depending on the order of the permutation p. It follows from Lemma 3.1 that for $k \le l$, $\mathbb{E}\{\Delta_{1,i_1} \cdots \Delta_{2k-1,i_{2k-1}}\} = 0$ and

$$\begin{aligned} \mathbb{E}\{\Delta_{1,i_1} \cdots \Delta_{2k,i_{2k}}\} = &\sum_{Q_{2k}} \{\mathcal{E}(1,i_1,2,i_2) - \delta_{1i_1}\delta_{2i_2}\} \times \cdots \\ &\times \{\mathcal{E}(2k-1,i_{2k-1},2k,i_{2k}) - \delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}}\}, \end{aligned}$$

where Q_{2k} is the set of the $(2k)!/(k!2^k)$ ways of grouping $(i_1, i_2, \ldots, i_{2k})$ into pairs without regard to order, keeping them paired with the first index. Hence

$$\begin{aligned}
&\sum_P \eta(p)\mathbb{E}\{\Delta_{1,i_1}\cdots\Delta_{2k,i_{2k}}\}B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&\quad= \sum_P \eta(p)\Bigg(\sum_{Q_{2k}}\{\mathcal{E}(1,i_1,2,i_2)-\delta_{1i_1}\delta_{2i_2}\}\cdots\{\mathcal{E}(2k-1,i_{2k-1},2k,i_{2k})\\
&\qquad -\delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}}\}\Bigg)B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&\quad= \sum_P \eta(p)\left(\sum_{Q_{2k}}(-1)^k(\delta_{1i_1}\delta_{2i_2})\cdots(\delta_{2k-1,i_{2k-1}}\delta_{2k,i_{2k}})\right)B_{2k+1,i_{2k+1}}\cdots B_{2l,i_{2l}}\\
&\quad= \frac{(-1)^k(2k)!}{k!2^k}\det((B_{i,j})_{2k+1\le i,j\le 2l}),
\end{aligned}$$

where the second equality is due to the fact that all products involving at least one $\mathcal{E}$ term will cancel out because of their symmetry property, and the last equality comes from changing the order of summation and then noting that the delta functions are nonzero only in those permutations in P with $(i_1, i_2, \ldots, i_{2k-1}, i_{2k}) = (1, 2, \ldots, 2k-1, 2k)$. Thus

$$\mathbb{E}\{\det(\Delta_{2l}+B_{2l})\} = \sum_{k=0}^{l}\frac{(-1)^k(2k)!}{k!2^k}S_{2l-2k}(B_{2l}).$$

Similarly, we obtain that when $N = 2l + 1$,

$$\mathbb{E}\{\det(\Delta_{2l+1}+B_{2l+1})\} = \sum_{k=0}^{l}\frac{(-1)^k(2k)!}{k!2^k}S_{2l+1-2k}(B_{2l+1}).$$

The proof for the first line in (3.2) is completed. The second line in (3.2) follows similarly. □

Let $B_N(i_1, \ldots, i_n; i_1, \ldots, i_n) = (B_{i_j,i_k})_{1\le j,k\le n}$ be the $n \times n$ principle submatrix of B_N extracted from the $i_1, \ldots, i_n$ rows and $i_1, \ldots, i_n$ columns in B_N, where $1 \le i_1 < \cdots < i_n \le N$.

Proposition 3.3. *Let Δ_N and Ξ_N be two $N \times N$ symmetric centered Gaussian matrices satisfying (3.1), and let B_N be an $N \times N$ real symmetric matrix. Then for $x \in \mathbb{R}$,*

$$\begin{aligned}
\mathbb{E}\{\det(\Delta_N+B_N-xI_N)\} &= \sum_{n=0}^{N}\frac{(-1)^{N-n}}{(N-n)!}\\
&\quad\times\left(\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k(N-n+2k)!}{k!2^k}S_{n-2k}(B_N)\right)x^{N-n},\\
\mathbb{E}\{\det(\Xi_N+B_N-xI_N)\} &= \sum_{n=0}^{N}(-1)^{N-n}S_n(B_N)x^{N-n},
\end{aligned} \tag{3.3}$$

where $S_j(\cdot)$ is defined in Lemma 3.2.

Proof. Applying the Laplace expansion of the determinant yields

$$\mathbb{E}\{\det(\Delta_N+B_N-xI_N)\} = \sum_{n=0}^{N}(-1)^{N-n}\mathbb{E}\{S_n(\Delta_N+B_N)\}x^{N-n}. \tag{3.4}$$

By Lemma 3.2,

$$\begin{aligned}
\mathbb{E}\{S_n(\Delta_N + B_N)\} &= \sum_{1\le i_1<\cdots<i_n\le N} \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k(2k)!}{k!2^k} S_{n-2k}(B_N(i_1,\ldots,i_n;i_1,\ldots,i_n)) \\
&= \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k(2k)!}{k!2^k} \frac{\binom{N}{n}\binom{n}{n-2k}}{\binom{N}{n-2k}} S_{n-2k}(B_N) \\
&= \frac{1}{(N-n)!} \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k(N-n+2k)!}{k!2^k} S_{n-2k}(B_N),
\end{aligned}$$

where the second equality is due to the observation that the sum on all principle submatrices of order n in the first line makes every principal minor of order $n-2k$ appear $\binom{N}{n}\binom{n}{n-2k}/\binom{N}{n-2k}$ many times. Plugging this into (3.4) yields the first line in (3.3).

By Lemma 3.2 again,

$$\mathbb{E}\{S_n(\Xi_N + B_N)\} = \sum_{1\le i_1<\cdots<i_n\le N} \det(B_N(i_1,\ldots,i_n;i_1,\ldots,i_n)) = S_n(B_N).$$

Plugging this into (3.4), with Δ_N being replaced by Ξ_N, yields the second line in (3.3). $\square$

Remark 3.4. Let $B_N \equiv 0$. Then it can be derived from Proposition 3.3 that

$$\mathbb{E}\{\det(\Delta_N + B_N - xI_N)\} = (-1)^N H_N(x),$$

coinciding with the result in Corollary 11.6.3 in [2], where $H_N(x)$ is the Hermite polynomial of order N. Meanwhile,

$$\mathbb{E}\{\det(\Xi_N + B_N - xI_N)\} = (-1)^N x^N. \quad \square$$

3.2. *Non-centered stationary Gaussian fields on rectangles*

Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field such $X(t) = Z(t) + m(t)$, where Z is a centered unit-variance stationary Gaussian random field, $m(\cdot)$ is the mean function of X, and as usual, T is a compact rectangle. By classical spectral representation for stationary Gaussian fields (cf. Chapter 5 in [2]), the field Z has representation

$$Z(t) = \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle} W(d\lambda)$$

and covariance

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle} \nu(d\lambda),$$

where W is a complex-valued Gaussian random measure and ν is the spectral measure satisfying $\nu(\mathbb{R}^N) = C(0) = \sigma^2$. We introduce the second-order spectral moments

$$\lambda_{ij} = \int_{\mathbb{R}^N} \lambda_i\lambda_j \nu(d\lambda),$$

and for any face $J \in \partial_k T$ with $k \ge 1$, denote $\Lambda_J = (\lambda_{ij})_{i,j\in\sigma(J)}$. Notice that λ_{ij} and Λ_J do not depend on t, and they are respectively the same as $\lambda_{ij}(t)$ and $\Lambda_J(t)$ defined in (2.3). In particular, we have

$$\mathrm{Cov}(\nabla Z_{|J}(t), \nabla Z_{|J}(t)) = -\mathrm{Cov}(Z(t), \nabla^2 Z_{|J}(t)) = \Lambda_J$$

and that

$$\mathcal{E}_0(i,j,k,l) := \mathbb{E}\{Z_{ij}(t)Z_{kl}(t)\} = \int_{\mathbb{R}^N} \lambda_i\lambda_j\lambda_k\lambda_l \nu(d\lambda)$$

is a symmetric function of i, j, k, l.

Recall that for a $k \times k$ positive definite matrix B, the *principal square root* of B^{-1}, which is usually denoted by $B^{-1/2}$, is the unique $k \times k$ positive definite matrix Q such that $QBQ = I_k$. Denote by $\Psi(x)$ the tail probability of a standard Gaussian distribution, that is $\Psi(x) = (2\pi)^{-1/2}\int_x^\infty e^{-y^2/2}dy$. Notice that in (3.5), for every $\{t\} \in \partial_0 T$, $\nabla X(t) \in E(\{t\})$ specifies the signs of the partial derivatives $X_j(t)$ ($j = 1, \dots, N$) and, for $J \in \partial_k T$ with $k \geq 1$, the set $\{J_1, \dots, J_{N-k}\}$ and $E(J)$ are defined in (2.3).

Theorem 3.5. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field such that $X(t) = Z(t)+m(t)$, where Z is a centered unit-variance stationary Gaussian random field and $m(\cdot)$ is the mean function of X. If X satisfies conditions* (H1) *and* (H2′)*, then*

$$\begin{aligned}
&\mathbb{E}\{\chi(A_u(X,T))\} \\
&= \sum_{\{t\}\in\partial_0 T} \mathbb{P}\{\nabla X(t) \in E(\{t\})\}\Psi(u-m(t)) + \sum_{k=1}^N \sum_{J\in\partial_k T} \frac{(\det(\Lambda_J))^{1/2}}{(2\pi)^{(k+1)/2}} \\
&\quad \times \int_J dt \int_u^\infty dx\, \exp\left\{-\frac{1}{2}\left[(x-m(t))^2 + (\nabla m_{|J}(t))^T \Lambda_J^{-1} \nabla m_{|J}(t)\right]\right\} \\
&\quad \times \mathbb{P}\{(X_{J_1}(t), \dots, X_{J_{N-k}}(t)) \in E(J) | \nabla X_{|J}(t) = 0\} \\
&\quad \times \left[\sum_{j=0}^k \frac{(-1)^j}{(k-j)!}\left(\sum_{i=0}^{\lfloor j/2\rfloor} \frac{(-1)^i (k-j+2i)!}{i!2^i} S_{j-2i}\left(\Lambda_J^{-1/2}\nabla^2 m_J(t)\Lambda_J^{-1/2}\right)\right) x^{k-j}\right],
\end{aligned} \tag{3.5}$$

where $S_{j-2i}(\cdot)$ is defined in Lemma 3.2 and $\Lambda_J^{-1/2}$ is principal square root of Λ_J^{-1}.

Proof. If $J = \{t\} \in \partial_0 T$, then

$$\begin{aligned}
\mathbb{E}\{\mu_0(J)\} &= \mathbb{P}\{X(t) \geq u, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } 1 \leq j \leq N\} \\
&= \mathbb{P}\{\nabla X(t) \in E(\{t\})\}\Psi(u - m(t)),
\end{aligned} \tag{3.6}$$

where the last equality is due to the independence of $X(t)$ and $\nabla X(t)$ for each fixed t.

Let $J \in \partial_k T$ with $k \geq 1$ and let $\mathcal{D}_i$ be the collection of all $k \times k$ matrices with index i. Applying the Kac–Rice metatheorem, similarly to the proof of Lemma 2.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{i=0}^k (-1)^i \mu_i(J)\right\} \\
&= \int_J p_{\nabla X_{|J}(t)}(0)dt \sum_{i=0}^k (-1)^i \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\}} \\
&\quad \times \mathbb{1}_{\{X(t)\geq u\}} \mathbb{1}_{\{(X_{J_1}(t),\dots,X_{J_{N-k}}(t))\in E(J)\}} | \nabla X_{|J}(t) = 0\}
\end{aligned}$$

$$
\begin{aligned}
&= \int_J p_{\nabla X_{|J}(t)}(0)dt\, \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{X(t)\geq u\}} \mathbb{1}_{\{(X_{J_1}(t),\ldots,X_{J_{N-k}}(t))\in E(J)\}} | \nabla X_{|J}(t) = 0\} \\
&= \frac{1}{(2\pi)^{(k+1)/2}(\det(\Lambda_J))^{1/2}} \int_J dt \\
&\quad \times \int_u^\infty dx \exp\left\{-\frac{1}{2}\left[(x-m(t))^2 + (\nabla m_{|J}(t))^T \Lambda_J^{-1} \nabla m_{|J}(t)\right]\right\} \\
&\quad \times \mathbb{P}\{(X_{J_1}(t),\ldots,X_{J_{N-k}}(t)) \in E(J) | \nabla X_{|J}(t) = 0\} \mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x\}, \quad (3.7)
\end{aligned}
$$

where the last equality is due to the fact that $\nabla X(t)$ is independent of both $X(t)$ and $\nabla^2 X(t)$ for each fixed t.

Now we turn to computing $\mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x\}$. To simplify the notation, let $Q = \Lambda_J^{-1/2}$. Then

$$
\mathbb{E}\{Z(t)(Q\nabla^2 Z_{|J}(t)Q)_{ij}\} = -(Q\Lambda_J Q)_{ij} = -\delta_{ij} \tag{3.8}
$$

and we can write

$$
\begin{aligned}
\mathbb{E}\{\det(Q\nabla^2 X_{|J}(t)Q) | X(t) = x\} &= \mathbb{E}\{\det(Q\nabla^2 Z_{|J}(t)Q + Q\nabla^2 m_J(t)Q) | X(t) = x\} \\
&= \mathbb{E}\{\det(\Delta(x) + Q\nabla^2 m_J(t)Q)\},
\end{aligned}
$$

where $\Delta(x) = (\Delta_{ij}(x))_{i,j\in\sigma(J)}$ is a Gaussian matrix. Applying Lemma A.2 and (3.8), we obtain

$$
\mathbb{E}\{\Delta_{ij}(x)\} = \mathbb{E}\{(Q\nabla^2 Z_{|J}(t)Q)_{ij} | X(t) = x\} = -x\delta_{ij}
$$

and

$$
\begin{aligned}
&\mathbb{E}\{[\Delta_{ij}(x) - \mathbb{E}\{\Delta_{ij}(x)\}][\Delta_{kl}(x) - \mathbb{E}\{\Delta_{kl}(x)\}]\} \\
&\quad = \mathbb{E}\{(Q\nabla^2 Z_{|J}(t)Q)_{ij}(Q\nabla^2 Z_{|J}(t)Q)_{kl}\} - \delta_{ij}\delta_{kl} = \mathcal{E}(i,j,k,l) - \delta_{ij}\delta_{kl},
\end{aligned}
$$

where $\mathcal{E}$ is a symmetric function of i, j, k, l. Therefore,

$$
\mathbb{E}\{\det(Q\nabla^2 X_{|J}(t)Q) | X(t) = x\} = \mathbb{E}\{\det(\Delta + Q\nabla^2 m_J(t)Q - xI_k)\},
$$

where $\Delta = (\Delta_{ij})_{i,j\in\sigma(J)}$ and Δ_{ij} are Gaussian variables satisfying

$$
\mathbb{E}\{\Delta_{ij}\} = 0, \qquad \mathbb{E}\{\Delta_{ij}\Delta_{kl}\} = \mathcal{E}(i,j,k,l) - \delta_{ij}\delta_{kl}.
$$

It follows from Proposition 3.3 that

$$
\begin{aligned}
&\mathbb{E}\{\det(Q\nabla^2 X_{|J}(t)Q) | X(t) = x\} \\
&\quad = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} \left(\sum_{i=0}^{\lfloor j/2\rfloor} \frac{(-1)^i (k-j+2i)!}{i!2^i} S_{j-2i}(Q\nabla^2 m_J(t)Q)\right) x^{k-j}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x\} = \det(\Lambda_J)\mathbb{E}\{\det(Q\nabla^2 X_{|J}(t)Q) | X(t) = x\} \\
&\quad = \det(\Lambda_J) \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} \left(\sum_{i=0}^{\lfloor j/2\rfloor} \frac{(-1)^i (k-j+2i)!}{i!2^i} S_{j-2i}(Q\nabla^2 m_J(t)Q)\right) x^{k-j}.
\end{aligned}
$$

Plugging this into (3.7), together with (3.6) and (2.2), yields the desired result. $\square$

Corollary 3.6. *Let the conditions in Theorem 3.5 hold. Assume additionally that t_0, an interior point in T, is the unique maximum point of $m(t)$ and that $\nabla^2 m(t_0)$ is nondegenerate. Then as $u \to \infty$,*

$$\mathbb{E}\{\chi(A_u(X,T))\} = \frac{\sqrt{\det(\Lambda_J)}u^{N/2}}{\sqrt{\det(-\nabla^2 m(t_0))}}\Psi(u - m(t_0))(1 + o(1)). \tag{3.9}$$

Proof. By Theorem 3.5,

$$\mathbb{E}\{\chi(A_u(X,T))\} = \frac{\sqrt{\det(\Lambda_J)}}{(2\pi)^{(N+1)/2}}\int_u^\infty x^N dx \times \int_J \exp\left\{-\frac{1}{2}\left[(x - m(t))^2 + (\nabla m(t))^T \Lambda^{-1}\nabla m(t)\right]\right\} dt(1 + o(1)).$$

Applying the Laplace method (see, e.g., [18]), we obtain that as $x \to \infty$,

$$\int_J \exp\left\{-\frac{1}{2}\left[(x - m(t))^2 + (\nabla m(t))^T \Lambda^{-1}\nabla m(t)\right]\right\} dt = \frac{(2\pi)^{N/2}}{x^{N/2}\sqrt{\det(-\nabla^2 m(t_0))}}\exp\left\{-\frac{1}{2}(x - m(t_0))^2\right\}(1 + o(1)).$$

Thus as $u \to \infty$,

$$\begin{aligned}\mathbb{E}\{\chi(A_u(X,T))\} &= \frac{\sqrt{\det(\Lambda_J)}}{\sqrt{2\pi}\sqrt{\det(-\nabla^2 m(t_0))}}\int_u^\infty x^{N/2} \\ &\quad\times \exp\left\{-\frac{1}{2}(x - m(t_0))^2\right\} dx(1 + o(1)) \\ &= \frac{\sqrt{\det(\Lambda_J)}u^{N/2}}{\sqrt{\det(-\nabla^2 m(t_0))}}\Psi(u - m(t_0))(1 + o(1)). \qquad \square\end{aligned}$$

Remark 3.7. The asymptotic approximation in (3.9) is a special case of Theorem 5 in [13] when the index α therein equals 2, which implies the Gaussian field is smooth. However, in our result, a higher-order approximation, such as letting the error term in (3.9) be $o(u^{-1})$ instead of $o(1)$, is also available by applying a higher-order Laplace approximation to $\mathbb{E}\{\chi(A_u(X,T))\}$ (see, e.g., [18]). Since the calculation is tedious, it is omitted here.

By Theorem 2.5, the expected Euler characteristic (3.5) approximates the excursion probability with a super-exponentially small error. Comparing with the $o(1)$ error, we see that the expected Euler characteristic is much more accurate than the asymptotic approximation in (3.9). This accuracy will be very useful in statistical applications, such as detecting significant regions by computing p-values via excursion probability, especially when the threshold u is not very high.

Meanwhile, the mean function in Theorem 3.5 can be very general. In contrast, Corollary 3.6 only deals with a special case when the maximum of the mean function is achieved at a unique point. In fact, the classical double sum method used in [13] usually requires the mean function to be nice enough, and to the best of our knowledge, it cannot completely solve the case when the maximum of the mean function is achieved on a general subset of T so far. Since (3.5) provides a general (although complicated) formula for the expected Euler characteristic, one can always approximate the integral therein to find suitable asymptotics for the excursion probability.

Besides the super-exponentially small error and the generality of the mean function, the expected Euler characteristic approximation has another important advantage on the geometric interpretation. Especially in real applications, such as image analysis, instead of using the complicated formula in (3.5), one may try to simulate the mean value of observed Euler characteristic of the excursion sets to estimate the excursion probability. □

Corollary 3.8. *Let the conditions in Theorem 3.5 hold. If Z is an isotropic Gaussian random field with* $\mathrm{Var}(Z_1(t)) = \gamma^2$, *then*

$$\begin{aligned}\mathbb{E}\{\chi(A_u(X,T))\} &= \sum_{\{t\}\in\partial_0 T} \mathbb{P}\{\nabla X(t) \in E(\{t\})\}\Psi(u-m(t)) + \sum_{k=1}^{N}\sum_{J\in\partial_k T}\frac{\gamma^k}{(2\pi)^{(k+1)/2}}\\ &\quad\times \int_J dt \int_u^\infty dx\, \exp\left\{-\frac{1}{2}\left[(x-m(t))^2 + \gamma^{-2}\|\nabla m_{|J}(t)\|^2\right]\right\}\\ &\quad\times \mathbb{P}\{(X_{J_1}(t),\ldots,X_{J_{N-k}}(t)) \in E(J)\}\\ &\quad\times \left[\sum_{j=0}^{k}\frac{(-1)^j}{(k-j)!}\left(\sum_{i=0}^{\lfloor j/2\rfloor}\frac{(-1)^i(k-j+2i)!}{i!2^i}\gamma^{-2(j-2i)}S_{j-2i}\left(\nabla^2 m_J(t)\right)\right)x^{k-j}\right].\end{aligned}$$

Proof. The result follows immediately from Theorem 3.5, the independence of $X_i(t)$ and $X_j(t)$ when $i \neq j$ and that, for $J \in \partial_k T$ with $k \geq 1$, $\Lambda_J = \gamma^2 I_k$, which implies $\Lambda_J^{-1/2} = \gamma^{-1} I_k$. □

3.3. Non-centered isotropic Gaussian fields on spheres

Let $\mathbb{S}^N$ denote the N-dimensional unit sphere and let $X = \{X(t),\ t \in \mathbb{S}^N\}$ be a Gaussian random field such $X(t) = Z(t) + m(t)$, where Z is a centered unit-variance isotropic Gaussian random field on $\mathbb{S}^N$ and $m(\cdot)$ is the mean function of X.

The following theorem by Schoenberg [14] characterizes the covariances of isotropic Gaussian fields on $\mathbb{S}^N$ (see also [9]).

Theorem 3.9. *A continuous function $C(\cdot,\cdot) : \mathbb{S}^N \times \mathbb{S}^N \to \mathbb{R}$ is the covariance of an isotropic Gaussian field on $\mathbb{S}^N$ if and only if it has the form*

$$C(t,s) = \sum_{n=0}^{\infty} a_n P_n^\lambda(\langle t,s\rangle), \quad t,s \in \mathbb{S}^N, \tag{3.10}$$

where $\lambda = (N-1)/2$, $a_n \geq 0$, $\sum_{n=0}^\infty a_n P_n^\lambda(1) < \infty$, and P_n^λ is the ultraspherical polynomials defined by the expansion

$$(1-2rx+r^2)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(x), \quad x \in [-1,1].$$

If X is centered, then the Gaussian field only depends on the covariance function which behaves isotropically over $\mathbb{S}^N$. Therefore, as discussed in [7,6], we do not need to introduce special coordinate system on the sphere. However, if X is non-centered, due to arbitrary behaviors of the mean function, it is much more convenient to adopt the usual spherical coordinates, especially for obtaining exact asymptotics. To achieve this, for $t = (t_1,\ldots,t_{N+1}) \in \mathbb{S}^N$, we define the corresponding spherical coordinate $\theta = (\theta_1, \ldots, \theta_N)$ as follows.

$$\begin{aligned}
t_1 &= \cos\theta_1, \\
t_2 &= \sin\theta_1 \cos\theta_2, \\
t_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\ \vdots \\
t_N &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \cos\theta_N, \\
t_{N+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \sin\theta_N,
\end{aligned}$$

where $\theta \in \Theta := [0, \pi]^{N-1} \times [0, 2\pi)$. We also define the Gaussian random field $\widetilde{X} = \{\widetilde{X}(\theta), \theta \in \Theta\}$ by $\widetilde{X}(\theta) = X(t)$ and denote by $\widetilde{C}$ the covariance function of $\widetilde{X}$ accordingly. Similarly, let $\widetilde{Z}(\theta) = Z(t)$ and $\widetilde{m}(\theta) = m(t)$. We introduce the following orthonormal basis on the sphere,

$$\frac{\partial}{\partial\tilde{\theta}_1} = \frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\tilde{\theta}_2} = \frac{1}{\sin\theta_1}\frac{\partial}{\partial\theta_2}, \ldots, \frac{\partial}{\partial\tilde{\theta}_N} = \frac{1}{\prod_{i=1}^{N-1}\sin\theta_i}\frac{\partial}{\partial\theta_N}.$$

For a real-valued function on the sphere, say $f(t)$ $(t \in \mathbb{S}^N)$, define respectively the gradient and Hessian of the function under the spherical coordinate, $\tilde{f}(\theta)$ $(\theta \in \Theta)$, as

$$\begin{aligned}
\nabla\tilde{f}(\theta) &= \left(\tilde{f}_i(\theta), \ldots, \tilde{f}_N(\theta)\right)^T := \left(\frac{\partial}{\partial\tilde{\theta}_1}\tilde{f}(\theta), \ldots, \frac{\partial}{\partial\tilde{\theta}_N}\tilde{f}(\theta)\right)^T, \\
\nabla^2\tilde{f}(\theta) &= \left(\tilde{f}_{ij}(\theta)\right)_{1\le i,j\le N} := \left(\frac{\partial^2}{\partial\tilde{\theta}_i\partial\tilde{\theta}_j}\tilde{f}(\theta)\right)_{1\le i,j\le N}.
\end{aligned}$$

Lemma 3.10, characterizing the covariance of $(\widetilde{X}(\theta), \nabla\widetilde{X}(\theta), \nabla^2\widetilde{X}(\theta))$, can be obtained easily by elementary calculations. The proof is omitted here.

Lemma 3.10. *Let $X = \{X(t),\ t \in \mathbb{S}^N\}$ be a non-centered isotropic Gaussian random field with covariance (3.10) and satisfying* (H1) *and* (H2′). *Then*

$$\begin{aligned}
&\left.\frac{\partial\widetilde{C}(\theta,\varphi)}{\partial\tilde{\theta}_i}\right|_{\theta=\varphi} = \left.\frac{\partial^3\widetilde{C}(\theta,\varphi)}{\partial\tilde{\theta}_i\partial\tilde{\varphi}_j\partial\tilde{\varphi}_k}\right|_{\theta=\varphi} = 0, \\
&\left.\frac{\partial^2\widetilde{C}(\theta,\varphi)}{\partial\tilde{\theta}_i\partial\tilde{\varphi}_j}\right|_{\theta=\varphi} = -\left.\frac{\partial^2\widetilde{C}(\theta,\varphi)}{\partial\tilde{\theta}_i\partial\tilde{\theta}_j}\right|_{\theta=\varphi} = C'\delta_{ij}, \\
&\left.\frac{\partial^4\widetilde{C}(\theta,\varphi)}{\partial\tilde{\theta}_i\partial\tilde{\theta}_j\partial\tilde{\varphi}_k\partial\tilde{\varphi}_l}\right|_{\theta=\varphi} = C''(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + C'\delta_{ij}\delta_{kl},
\end{aligned}$$

where

$$C' = \sum_{n=1}^{\infty} a_n\left(\frac{d}{dx}P_n^\lambda(x)|_{x=1}\right) \quad \textit{and} \quad C'' = \sum_{n=2}^{\infty} a_n\left(\frac{d^2}{dx^2}P_n^\lambda(x)|_{x=1}\right). \tag{3.11}$$

Now we can formulate the expected Euler characteristic of non-centered Gaussian fields on the sphere as follows.

Theorem 3.11. *Let $X = \{X(t),\ t \in \mathbb{S}^N\}$ be a Gaussian random field such that $X(t) = Z(t) + m(t)$, where Z is a centered unit-variance isotropic Gaussian random field on $\mathbb{S}^N$ with* *covariance (3.10) and m is the mean function of X. If X satisfies conditions* (H1) *and* (H2′)*, then*

$$\begin{aligned}
&\mathbb{E}\{\chi(A_u(X,\mathbb{S}^N))\} \\
&= \frac{1}{(2\pi)^{(N+1)/2}} \int_\Theta \phi(\theta)d\theta \int_u^\infty dx \exp\left\{-\frac{1}{2}\left[(x-\widetilde{m}(\theta))^2 + (C')^{-1}\|(\nabla\widetilde{m}(\theta))\|^2\right]\right\} \\
&\quad\times \left[\sum_{j=0}^{N} \frac{(-1)^j}{(N-j)!}\left(\sum_{i=0}^{\lfloor j/2\rfloor} \frac{(-1)^i(N-j+2i)!}{i!2^i}(C')^{\frac{N}{2}-j+i}\right.\right. \\
&\quad\left.\left.\times(C'-1)^i S_{j-2i}\left(\nabla^2\widetilde{m}(\theta)\right)\right)x^{N-j}\right],
\end{aligned} \tag{3.12}$$

where $\Theta = [0,\pi]^{N-1}\times[0,2\pi)$, $\phi(\theta) = \prod_{i=1}^{N-1}(\sin\theta_i)^{N-i}$, *and* C' *and* $S_{j-2i}(\cdot)$ *are defined respectively in (3.11) and Lemma 3.2.*

Proof. Since $\mathbb{S}^N$ is a smooth and compact manifold without boundary, it follows from (2.25), Lemma 3.10 and the Kac–Rice metatheorem that

$$\begin{aligned}
&E\{\chi(A_u(X,\mathbb{S}^N))\} \\
&= (-1)^N \int_\Theta \phi(\theta)d\theta \int_u^\infty dx p_{\nabla\widetilde{X}(\theta)}(0)p_{\widetilde{X}(\theta)}(x)\mathbb{E}\{\det\nabla^2\widetilde{X}(\theta)|\widetilde{X}(\theta)=x\} \\
&= \frac{(-1)^N}{(2\pi)^{(N+1)/2}(C')^{N/2}} \int_\Theta \phi(\theta)d\theta \int_u^\infty dx \\
&\quad\times \exp\left\{-\frac{1}{2}\left[(x-\widetilde{m}(\theta))^2 + C'^{-1}\|(\nabla\widetilde{m}(\theta))\|^2\right]\right\} \\
&\quad\times \mathbb{E}\{\det\nabla^2\widetilde{X}(\theta)|\widetilde{X}(\theta)=x\}.
\end{aligned} \tag{3.13}$$

We only need to compute $\mathbb{E}\{\det\nabla^2\widetilde{X}(\theta)|\widetilde{X}(\theta)=x\}$.

Case 1: $C'>1$. By Lemma 3.10, similarly to the proof of Theorem 3.5, we get

$$\begin{aligned}
&\mathbb{E}\{\det\nabla^2\widetilde{X}(\theta)|\widetilde{X}(\theta)=x\} = \mathbb{E}\{\det[\nabla^2\widetilde{Z}(\theta)+\nabla^2\widetilde{m}(\theta)]|\widetilde{X}(\theta)=x\} \\
&= (C'^2-C')^{N/2}\mathbb{E}\{\det[(C'^2-C')^{-1/2}\nabla^2\widetilde{Z}(\theta)+(C'^2-C')^{-1/2}\nabla^2\widetilde{m}(\theta)]|\widetilde{X}(\theta)=x\} \\
&= (C'^2-C')^{N/2}\mathbb{E}\{\det[\Delta+(C'^2-C')^{-1/2}\nabla^2\widetilde{m}(\theta)-C'(C'^2-C')^{-1/2}xI_N]\},
\end{aligned}$$

where $\Delta = (\Delta_{ij})_{1\le i,j\le N}$ and Δ_{ij} are centered Gaussian variables satisfying

$$\begin{aligned}
\mathbb{E}\{\Delta_{ij}\Delta_{kl}\} &= (C'^2-C')^{-1}\mathbb{E}\{\widetilde{Z}_{ij}(\theta)\widetilde{Z}_{kl}(\theta)|\widetilde{X}(\theta)=x\} \\
&= (C'^2-C')^{-1}[C''(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+C'\delta_{ij}\delta_{kl}-C'^2\delta_{ij}\delta_{kl}] \\
&= \mathcal{E}(i,j,k,l)-\delta_{ij}\delta_{kl},
\end{aligned}$$

and $\mathcal{E}$ is a symmetric function of i,j,k,l. It then follows from Proposition 3.3 that

$$\begin{aligned}
&\mathbb{E}\{\det\nabla^2\widetilde{X}(\theta)|\widetilde{X}(\theta)=x\} \\
&= (C'^2-C')^{N/2}\sum_{j=0}^{N}\frac{(-1)^{N-j}}{(N-j)!}\left(\sum_{i=0}^{\lfloor j/2\rfloor}\frac{(-1)^i(N-j+2i)!}{i!2^i}S_{j-2i}\left(\frac{\nabla^2\widetilde{m}(\theta)}{\sqrt{C'^2-C'}}\right)\right) \\
&\quad\times\left(\frac{C'x}{\sqrt{C'^2-C'}}\right)^{N-j}.
\end{aligned} \tag{3.14}$$

Case 2: $C' < 1$. It follows from similar discussions in the previous case and a slightly revised version of Proposition 3.3 that

$$\begin{aligned}&\mathbb{E}\{\det \nabla^2 \widetilde{X}(\theta)|\widetilde{X}(\theta)=x\}\\&\quad=(C'-C'^2)^{N/2}\sum_{j=0}^{N}\frac{(-1)^{N-j}}{(N-j)!}\left(\sum_{i=0}^{\lfloor j/2\rfloor}\frac{(N-j+2i)!}{i!2^i}S_{j-2i}\left(\frac{\nabla^2\widetilde{m}(\theta)}{\sqrt{C'-C'^2}}\right)\right)\\&\qquad\times\left(\frac{C'x}{\sqrt{C'-C'^2}}\right)^{N-j}.\end{aligned}\tag{3.15}$$

Case 3: $C' = 1$. By Lemma 3.10 again,

$$\begin{aligned}\mathbb{E}\{\det \nabla^2 \widetilde{X}(\theta)|\widetilde{X}(\theta)=x\}&=\mathbb{E}\{\det(\nabla^2\widetilde{Z}(\theta)+\nabla^2\widetilde{m}(\theta))|\widetilde{X}(\theta)=x\}\\&=\mathbb{E}\{\det(\Xi+\nabla^2\widetilde{m}(\theta)-xI_N)\},\end{aligned}$$

where $\Xi=(\Xi_{ij})_{1\le i,j\le N}$ and Ξ_{ij} are centered Gaussian variables satisfying

$$\mathbb{E}\{\Xi_{ij}\Xi_{kl}\}=\mathbb{E}\{\widetilde{Z}_{ij}(\theta)\widetilde{Z}_{kl}(\theta)|\widetilde{X}(\theta)=x\}=C''(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})=\mathcal{F}(i,j,k,l),$$

and $\mathcal{F}$ is a symmetric function of i, j, k, l. It then follows from Proposition 3.3 that

$$\mathbb{E}\{\det \nabla^2 \widetilde{X}(\theta)|\widetilde{X}(\theta)=x\}=\sum_{j=0}^{N}(-1)^{N-j}S_j\left(\nabla^2\widetilde{m}(\theta)\right)x^{N-j}.\tag{3.16}$$

Plugging respectively (3.14)–(3.16) into (3.13), we see that the expected Euler characteristic for all three cases above can be formulated by the same expression (3.12). □

Remark 3.12. Let $\widetilde{m}(\theta)\equiv 0$. Let $\omega_j=\frac{2\pi^{(j+1)/2}}{\Gamma((j+1)/2)}$ be the spherical area of the j-dimensional unit sphere. Notice that Hermite polynomials have the following properties:

$$\int_u^\infty H_n(x)e^{-x^2/2}dx=H_{n-1}(u)e^{-u^2/2},$$

$$x^n=n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k!2^k(n-2k)!}H_{n-2k}(x),$$

where $n\ge 0$ and $H_{-1}(x)=\sqrt{2\pi}\,\Psi(x)e^{x^2/2}$. Applying Theorem 3.11, together with the properties above and certain combinatorial tricks, we obtain

$$\begin{aligned}\mathbb{E}\{\chi(A_u(X,\mathbb{S}^N))\}&=\frac{\omega_N}{(2\pi)^{(N+1)/2}}\sum_{n=0}^{\lfloor N/2\rfloor}(C')^{(N-2n)/2}\binom{N}{2n}(2n-1)!!H_{N-2n-1}(u)e^{-u^2/2}\\&=\sum_{j=0}^{N}(C')^{j/2}\mathcal{L}_j(\mathbb{S}^N)\rho_j(u),\end{aligned}$$

where $\rho_0(u)=\Psi(u)$, $\rho_j(u)=(2\pi)^{-(j+1)/2}H_{j-1}(u)e^{-u^2/2}$ for $j\ge 1$ and

$$\mathcal{L}_j(\mathbb{S}^N)=\begin{cases}2\binom{N}{j}\dfrac{\omega_N}{\omega_{N-j}} & \text{if } N-j \text{ is even,}\\ 0 & \text{otherwise}\end{cases}$$

(for $j = 0, 1, \dots, N$) are the Lipschitz–Killing curvatures of $\mathbb{S}^N$ (cf. Eq. (6.3.8) in [2]). This coincides with the formula of the expected Euler characteristic for centered isotropic Gaussian fields on the sphere obtained in [7] via a geometric approach. However, the result in [7] is still more general for studying centered isotropic Gaussian fields on the sphere since it is also applicable when the parameter sets are subsets of $\mathbb{S}^N$.

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Appendix

This appendix contains some auxiliary results.

Lemma A.1. *Let $X = \{X(t),\ t \in T\}$ be a (non-centered) Gaussian random field satisfying* (H1) *and* (H2). *Then for each $u \in \mathbb{R}$,*

$$\begin{aligned}\sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} &\geq \mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} \\ &\geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \left(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}\right) \\ &\quad - \sum_{J\neq J'} \mathbb{E}\{M_u^E(J)M_u^E(J')\}, \end{aligned} \tag{A.1}$$

where $M_u^E(J)$ is defined in (2.4) and the last sum is taken over all pairs of different faces of T.

Proof. The result follows immediately from the same arguments in Section 4.1 in [8] or Section 2 in [12]. □

The following lemma is well-known and is quoted here for reader's convenience.

Lemma A.2. *Let Y and Z be two Gaussian random vectors of dimensions p and q, respectively. Then $Y|Z = z$ is a p-dimensional Gaussian random vector having the following mean and covariance:*

$$\begin{aligned}&\mathbb{E}\{Y|Z=z\} = \mathbb{E}Y + \mathbb{E}\{(Y-\mathbb{E}Y)(Z-\mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}(z-\mathbb{E}Z),\\ &\mathrm{Cov}(Y|Z=z) = \mathrm{Cov}(Y) - \mathbb{E}\{(Y-\mathbb{E}Y)(Z-\mathbb{E}Z)^T\}[\mathrm{Cov}(Z)]^{-1}\mathbb{E}\{(Z-\mathbb{E}Z)(Y-\mathbb{E}Y)^T\}.\end{aligned}$$

In particular, if $p = q = 1$ and $\mathbb{E}Y = \mathbb{E}Z = 0$, then

$$\mathbb{E}\{Y|Z=z\} = \frac{z\mathbb{E}(YZ)}{\mathrm{Var}(Z)}, \qquad \mathrm{Var}(Y|Z=z) = \mathrm{Var}(Y) - \frac{[\mathbb{E}(YZ)]^2}{\mathrm{Var}(Z)}.$$

The following result is a direct consequence of Lemma 4 in [12].

Lemma A.3. *Let $X = \{X(t),\ t \in T\}$ be a Gaussian random field satisfying* (H1) *and* (H2). *Then for any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that for any $J \in \partial_k T$ with $k \geq 1$ and u large enough,*

$$\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\} \leq e^{-u^2/(2\beta_J^2+\varepsilon)} + e^{-u^2/(2-\varepsilon_1)},$$

where $\beta_J^2 = \sup_{t\in J}\sup_{e\in\mathbb{S}^{k-1}} \mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e)$ and $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$.

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