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Hölder continuous densities of solutions of SDEs with measurable and path dependent drift coefficients

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Abstract

We consider a process given as the solution of a one-dimensional stochastic differential equation with irregular, path dependent and time-inhomogeneous drift coefficient and additive noise. Hölder continuity of the density at any given time is achieved using a different approach than the classical ones in the literature. Namely, the Hölder regularity is obtained via a control problem by identifying the equation with the worst global Hölder constant. Then we generalise our findings to a larger class of diffusions. The novelty of this method is that it is not based on a variational calculus and it is suitable for non-Markovian processes.
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1. Introduction

The examination of densities of random variables has been an active area of research during the last decades for its applications and in its own interest. To find criteria for the Lebesgue density of an absolutely continuous random variable to be regular has been the aim over the past

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years. P. Malliavin in [19] was interested in providing a probabilistic proof of L. Hörmander's theorem, see L. Hörmander [15], which, in short, is about a sufficient condition for an operator to be *hypoelliptic*. He believed that Hörmander's condition implies that the finite dimensional Lebesgue densities of a solution of a stochastic differential equation (SDE) are smooth. For this reason, he developed a stochastic calculus of variations in order to provide the concept of derivative of a random variable in a certain sense. For instance, a classical result on this matter is that if the coefficients of an SDE are smooth with bounded derivatives and the so-called Hörmander's condition mentioned above holds, then the solution is smooth in the Malliavin sense at any time. Then, it is shown in [19] that Malliavin smoothness together with a non-degeneracy condition implies that the laws of the solutions are absolutely continuous with respect to the Lebesgue measure and the densities are smooth. Classical results based on analytic methods can be found in [11,9].

A different approach is credited to N. Bouleau and F. Hirsch [5] where they show absolute continuity of the finite dimensional laws of solutions to SDEs based on a stochastic calculus of variations in finite dimensions utilising a limit argument. Also, as a motivation of [5], D. Nualart and M. Zakai [21] found related results on the existence and smoothness of conditional densities of Malliavin differentiable random variables.

Further research has been carried out, let us mention some achievements on this topic. S. De Marco [8] shows local smoothness of densities on an open domain under the usual condition of ellipticity and that the coefficients are smooth on such domain. Also, V. Bally and A. Kohatsu-Higa [2] show that the densities of a type of a two-dimensional degenerated SDE are bounded and they provide both upper and lower bounds, for this case, it is assumed that the coefficients are five times differentiable with bounded derivatives. Let us also mention the advances of V. Bally and L. Caramellino [1] where an integration by parts formula (IPF) is derived and the integrability of the weight obtained in the formula gives the desired regularity of the density. As a consequence of the aforementioned result D. Baños and T. Nilssen [4] give a condition to obtain regularity of densities of solutions to SDEs according to how regular the drift is. The technique is also based on Malliavin calculus and a sharper estimate on the moments of the derivative of the flow associated to the solution. This result is a slight improvement of a very similar condition obtained by S. Kusuoka and D. Stroock in [18] when the diffusion coefficient is constant and the drift may be unbounded. Last but not least, we also cite the results by A. Kohatsu-Higa and A. Makhlouf [17] where the authors show smoothness of the density for smooth coefficients that may also depend on an external process whose drift coefficient is irregular. D. Nualart and L. Quer-Sardanyons provide in [20] bounds for the densities of a class of interesting SPDEs such as stochastic heat equation. There the drift is assumed to be continuously differentiable with bounded derivative. Optimality of bounds is also discussed.

It appears to be impossible to deduce (optimal) regularity properties of densities of solutions of SDEs with irregular coefficients, e.g. non-Lipschitz drift coefficient. Nevertheless, some results in this direction have been obtained. For example, M. Hayashi, A. Kohatsu-Higa and G. Yûki in [13] prove that SDEs with bounded Hölder continuous drift and smooth elliptic diffusion coefficients admit Hölder continuous densities at any time. Their method is also based on Malliavin calculus in connection with IPFs and estimates on the characteristic function of the solution.

We remark that all preceding works are based upon Malliavin calculus in connection with IPFs or Fourier analysis.

On the contrary, as a recent new technique, originally developed in [10], there are related results where Malliavin calculus is not directly employed. A. Debussche and N. Fournier [7]

prove that the finite dimensional densities of a solution of an SDE with jumps lies in a certain (low regular) Besov space when the drift is Hölder continuous. The very related result by Hayashi, Kohatsu-Higa and Yûki [14] shows that the density of the marginals of an SDE are Hölder continuous of some degree which depends on the Fourier regularity of the drift coefficient. The result requires that the SDE has a Markovian structure and the drift is bounded with its Fourier transform belonging to some Sobolev space. Their methodology is essentially based on finding estimates for the Fourier–Stieltjes transform of the finite dimensional laws of the solution. This paper improves the results in [14] in the one-dimensional case in two ways. First, we only need boundedness of the drift coefficient while Hayashi et al. [14] need additional Fourier regularity. Second, we do allow for path-dependence in the drift or even dependence on any other adapted process.

It is therefore important to emphasise that in this paper we do *not* use Malliavin calculus or any other type of variational calculus. We show that Itô processes with additive noise and merely bounded and measurable drift admit Hölder continuous densities of any order strictly less than one. In other words, we look at

$$X_u(t) := x + \int_0^t u(s)ds + W(t), \quad t \in [0, T], \; x \in \mathbb{R}, \tag{1}$$

where W is a standard Brownian motion and u is a progressively measurable stochastic process bounded by some constant $C > 0$. Note that this includes SDEs where the drift may depend on the solution in a non-Markovian manner.

Our main technique is based on a worst-case study by employing optimal stochastic control. First we restrict to the class of controlled processes like (1) whose drift coefficient is bounded by some constant $C > 0$ and we look at a time $t > 0$, no further regularity is assumed. Then, for given $t > 0$ one seeks to find a *worst-case* process which maximises the global Hölder constant of the density of $X_u(t)$ at time t among all members in our class. Hence, one is reducing the overall problem to studying a specific case, namely the process in (1) associated to the optimal control. If one is able to show that this *optimal* process admits marginal Hölder continuous densities at any time $t > 0$, then this implies that any other process $X_u(t), t > 0$, has Hölder continuous densities of any order $\alpha \in (0, 1)$, as well. Nevertheless, solving the proposed stochastic control problem is not an easy task. In order to circumvent this difficulty we proceed in a slightly different way. We simply pick a specific control $\widetilde{u}$ in the class of allowed controls and compute *how well* it performs compared to any other control, including the optimal one.

This idea is inspired by a previous work, see [3], to find the optimal lower and upper bounds for densities of Itô type processes with additive noise. Our method is robust since no well-behaviour on the drift is needed other than merely boundedness and no Markovianity of the system is assumed, hence overcoming limitations of Malliavin calculus.

Moreover, it is known that the density of $X(t), t > 0$, given by

$$X(t) = -\int_0^t \operatorname{sgn}(X(s))ds + W(t), \quad t \geq 0,$$

where W is a standard Brownian motion is globally Lipschitz continuous but not more, meaning that the regularity obtained is almost optimal, see e.g. [3, Theorem 3.5].

This paper is organised as follows. In Section 2 we summarise our main results with some generalisations to non-trivial diffusion coefficients. In Section 3 we pose the stochastic control problem of interest, then we provide an *error* estimate for the performance of a selected control compared to any other admissible control. Finally, we prove the Hölder continuity of the density

of $X_u(t)$ at any given time $t > 0$. To conclude the paper, we summarise in two appendices the technical results needed in Section 3.

1.1. Notations

For a differentiable function $f : (0, T) \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto f(t, x)$, we denote by $\partial_1 f$, resp. $\partial_2 f$, the derivatives with respect to its first argument, resp. its second argument. For an open subset $U \subset \mathbb{R}$ and a function $f : (0, T) \times U \to \mathbb{R}, (t, x) \mapsto f(t, x)$, we say that f is of class $C^{1,2}$ if $f \in C^{1,2}((0, T) \times U)$ being $C^{1,2}((0, T) \times U)$ the space of continuous functions on $(0, T) \times U$ such that the partial derivatives $\partial_1 f, \partial_2 f$ and $\partial_2^2 f$ exist and are continuous. The notation $C^{1,2}([0, T) \times U)$ means that $f \in C^{1,2}((0, T) \times U)$ and the indicated partial derivatives have continuous extensions to $[0, T) \times U$. We denote the *signum* function by $\mathrm{sgn}(x) := 1_{\{x>0\}} - 1_{\{x \leq 0\}}$ for any $x \in \mathbb{R}$. Finally, we denote by φ, respectively, Φ, the density, respectively distribution function, of a standard normal random variable.

Further notations are used as in [16].

2. Main results

In this section we present our main result and some of its consequences. In particular, we will show that densities of the finite dimensional laws of the solution of an SDE with additive noise in the one-dimensional case are Hölder continuous of order $\alpha \in (0, 1)$ and give some extensions to more general diffusion coefficients.

Throughout this section let $(\Omega, \mathfrak{A}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space with the usual assumptions on the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, i.e. $\mathcal{F}_0$ contains all P-null sets, $\mathcal{F}_0$ contains only sets of full or zero measure and $\mathcal{F}$ is right-continuous. Let W be a one-dimensional standard $\mathcal{F}$-Brownian motion.

The next results constitute the core result of this paper and will be proven in detail in the next section.

Theorem 2.1. *Let $x_0 \in \mathbb{R}$, β be a bounded progressively measurable process with values in $\mathbb{R}$ and $X(t) := x_0 + \int_0^t \beta(s)ds + W(t), t \geq 0$.*

Then $X(T)$ has α-Hölder continuous density for any $\alpha \in (0, 1)$, $T > 0$. Furthermore, if β is bounded by $K > 0$, then the Hölder-continuity constant is at most

$$
C_\alpha^K(T) := K^{1+\alpha} \left(\frac{1}{\sqrt{2\pi e^\alpha} (TK^2)^{\frac{1+\alpha}{2}}} \right.
$$
$$
\left. + \frac{4}{\sqrt{2\pi e^\alpha}} \int_0^{TK^2} \left(\frac{\varphi(\sqrt{s})}{\sqrt{s}} + \Phi\left(\sqrt{s}\right) \right) \frac{1}{(TK^2 - s)^{\frac{1+\alpha}{2}}} ds \right).
$$

Proof. Let K be the uniform bound for β. Define

$$
B(t) := KW(t/K^2),
$$
$$
u(t) := \beta(t/K^2)/K,
$$
$$
Y(t) := KX(t/K^2)
$$
$$
= KX(0) + \int_0^t u(s)ds + B(t).
$$

Then $Y(TK^2)$ admits the assertions of Proposition 3.6 and hence $Y(TK^2)$ has α-Hölder continuous density. Clearly, $X(T) = Y(TK^2)/K$ has α-Hölder continuous density as well with the constant given above. $\quad \square$

We now focus on two simple extensions of our main result which utilise Itô's formula to allow for a non-constant diffusion coefficient. In the next corollary one could allow for a path-dependent function σ instead by using the pathwise Itô-formula, cf. [6, Theorem 4.1]. However, in order to not introduce heavy notation we rely on the classical Itô-formula.

Corollary 2.2. *Let X be an $\mathcal{F}$-adapted, $\mathbb{R}$-valued stochastic process such that*

$$X(t) = x_0 + \int_0^t \beta(s)ds + \int_0^t \sigma(s, X(s))dW(s), \quad t \geq 0$$

where $x_0 \in \mathbb{R}$, β is progressively measurable process, $\sigma : \mathbb{R}_+ \times \mathbb{R} \to (0, \infty)$ is a continuously differentiable function and assume that

$$u(t) := \frac{\beta(t)}{\sigma(t, X(t))} - \frac{\partial_2\sigma(t, X(t))}{2} - \int_0^{X(t)} \frac{\partial_1\sigma(t, x)}{\sigma^2(t, x)}dx, \quad t \geq 0$$

is a bounded process.
    Then $X(T)$ has locally α-Hölder continuous density for any $\alpha \in (0, 1)$, $T > 0$.
    Moreover, if $\sigma(t, x) \geq \epsilon$ for some $\epsilon > 0$ and any $t \geq 0, x \in \mathbb{R}$ and $\partial_2\sigma$ is bounded, then $X(T)$ has globally α-Hölder continuous density for any $\alpha \in (0, 1)$, $T > 0$.

Proof. Define $F(t, x) := \int_0^x \frac{1}{\sigma(t,z)}dz$ for any $x \in \mathbb{R}, t \geq 0$ and $Y(t) := F(t, X(t))$ for any $t \geq 0$. Then, Itô's formula yields

$$Y(t) = F(0, x_0) + \int_0^t u(s)ds + W(t).$$

Hence, $Y(T)$ has Hölder-continuous density by Theorem 2.1. The claim follows because for $T > 0$ we have $X(T) = (F(T, \cdot))^{-1}(Y(T))$ and $F(T, \cdot)$ is an invertible and continuously differentiable function. $\quad \square$

Sometimes, processes live on a half line or an interval. If the coefficients behave nicely enough, then our result still applies.

Corollary 2.3. *Let $-\infty \leq c < d \leq \infty, x_0 \in (c, d), b : (c, d) \to \mathbb{R}$ be measurable, $\sigma : (c, d) \to (0, \infty)$ be Lipschitz continuous and assume that there is a constant $K > 0$ such that $|b(x)| \leq K\sigma(x)$ for any $x \in (c, d)$. Assume that X is an $\mathcal{F}$-adapted, (c, d)-valued process with*

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s), \quad t \geq 0.$$

Then $X(T)$ has locally α-Hölder continuous density for any $\alpha \in (0, 1)$, $T > 0$.
    Moreover, if $(c, d) = \mathbb{R}$ and $\sigma(x) \geq \epsilon$ for some $\epsilon > 0$ and any $x \in \mathbb{R}$, then $X(T)$ has globally α-Hölder continuous density.

Proof. Define $F(y) := \int_{x_0}^y \frac{1}{\sigma(z)}dz$ for any $y \in (c, d)$, $Y(t) := F(X(t))$ and denote a bounded version of the absolutely continuous derivative of σ by σ'. Let $t \geq 0$. Then, Itô–Tanaka's

formula [23, Theorem VI.1.1] yields

$$Y(t) = F(x_0) + \int_0^t \left(\frac{b(X(s))}{\sigma(X(s))} - \frac{1}{2}\sigma'(X(s)) \right) ds + W(t), \quad t \geq 0.$$

The process $u(t) := \frac{b(X(t))}{\sigma(X(t))} - \frac{1}{2}\sigma'(X(t))$ is bounded by assumptions and hence Theorem 2.1 yields that $Y(t)$ has α-Hölder continuous density $\rho_{Y(T)}$ for any $\alpha \in (0, 1)$. We have

$$\rho_{X(T)}(x) = \rho_{Y(T)}(F(x))F'(x)$$

where $\rho_{X(T)}$ denotes the density of $X(T)$ and hence $X(T)$ has locally α-Hölder continuous density for any $\alpha \in (0, 1)$. $\quad \square$

3. A control problem

Throughout this section, let $(\Omega, \mathfrak{A}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space with the usual assumptions on the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, i.e. $\mathcal{F}_0$ contains all P-null sets, $\mathcal{F}_0$ contains only sets with zero or full measure and $\mathcal{F}$ is right-continuous. Let W be a one-dimensional $\mathcal{F}$-Brownian motion and define the process class

$$\mathcal{A} := \{u : u \text{ is an } \mathbb{R}\text{-valued progressively measurable process bounded by } 1\}.$$

We want to study processes of the form

$$X_u^x(t) := x + \int_0^t u(s)ds + W(t), \quad t \in [0, T], \ x \in \mathbb{R}, \tag{2}$$

where $u \in \mathcal{A}$ and $T > 0$ is some fixed time horizon. Since the process u in (2) is bounded a simple application of Girsanov's theorem in connection with Novikov's condition guarantees existence of a density $\rho_{u,t,x} : \mathbb{R} \to \mathbb{R}$ of $X_u^x(t)$ at time $t \in (0, T]$ and starting point $x \in \mathbb{R}$ for any $u \in \mathcal{A}$.

We would like to find the process $u^* \in \mathcal{A}$ such that $X_{u^*}^x(T)$ has the density $\rho_{u^*,T,x}$ with the *worst* Hölder constant among all densities $\rho_{u,T,x}, u \in \mathcal{A}$ which is a value in $[0, \infty]$. Then proving that $\rho_{u^*,T,x}$ has a Hölder continuous density yields that all $\rho_{u,T,x}$ are Hölder continuous as well for any $u \in \mathcal{A}$.

An application of Proposition A.4 allows us to restate the question for Hölder continuous densities in form of the function

$$J_{h,k}(x) := 1_{\{x \in [k, k+h]\}} - 1_{\{x \in [0, h]\}}, \quad x \in \mathbb{R}, \ 0 < h \leq k.$$

Girsanov's theorem [23, Theorem VIII.1.4] yields that $X_u^x(T)$ has absolutely continuous distribution function F. The next lemma connects Hölder continuity of F' to properties of the random variable $X_u^x(T)$.

Lemma 3.1. *Let $u \in \mathcal{A}$ and $\alpha \in (0, 1]$. Then $X_u^x(T)$ has α-Hölder continuous density if and only if there is $C > 0$ such that*

$$\frac{|\mathrm{E}[J_{h,k}(X_u^x(T))]|}{hk^\alpha} \leq C$$

for any $0 < h \leq k, x \in \mathbb{R}$.

Proof. This is immediate from Proposition A.4. $\quad \square$

In the following argumentation we seek to maximise the expression $E[J_{h,k}(X_u^x(T))]$ over $x \in \mathbb{R}$ and $u \in \mathcal{A}$ for fixed $0 < h \le k$. In other words, we try to analyse the control problem

$$\sup_{x \in \mathbb{R}, u \in \mathcal{A}} E[J_{h,k}(X_u^x(T))] = E[J_{h,k}(X_{u^*}^{x^*}(T))] \tag{3}$$

for some optimal starting location x^* and optimal control u^* where $0 < h \le k$ are fixed parameters. First note that

$$E[J_{h,k}(X_u^x(T))] = -E[J_{h,k}(X_{-u}^{k+h-x}(T))]$$

and, hence,

$$\sup_{x \in \mathbb{R}, u \in \mathcal{A}} E[J_{h,k}(X_u^x(T))] = \sup_{x \in \mathbb{R}, u \in \mathcal{A}} |E[J_{h,k}(X_u^x(T))]|.$$

We like to mention, although not used in this article, that existence of an optimal control and optimal starting location can be shown by applying general theory. For the general theory of control problems we relate to Øksendal and Sulem [22].

However, it appears to be difficult to solve the control problem in (3) explicitly. This is why, we proceed differently and we simply pick a trivial control, namely $u = -1$, and compare its performance to arbitrary controls.

Before going to the main theorem of this section we need a couple of intermediate results which will be used in the proof. The first provides explicit bounds for the density of processes like (2) and the proof can be found in [3]. The second is a key estimate for the quantity $E[J_{h,k}(X_u^x(T))]$ in terms of a bang–bang Markovian control $\widetilde{u} \in \mathcal{A}$, i.e. with $\widetilde{u}(t) = v(t, X_{\widetilde{u}}^x(t))$ where $v : \mathbb{R}_+ \times \mathbb{R} \to \{-1, 1\}$ is some measurable function.

Proposition 3.2. *Let $C > 0$ and $u \in \mathcal{A}$. Then $X(t) := \int_0^t Cu(s)ds + W(t)$ has Lebesgue density and one of its versions is given by*

$$\rho_{X(t)}(x) := \limsup_{\epsilon \to 0} \frac{P(|X(t) - x| \le \epsilon)}{2\epsilon}, \quad x \in \mathbb{R}.$$

Moreover, $\rho_{X(t)}$ satisfies

$$0 < \alpha_{t,C}(x) \le \rho_{X(t)}(x) \le \beta_{t,C}(x) \le \beta_{t,C}(0)$$

for any $t > 0$, $x \in \mathbb{R}$ where

$$\alpha_{t,C}(0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) - C\Phi\left(-C\sqrt{t}\right), \quad and$$

$$\beta_{t,C}(0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) + C\Phi\left(C\sqrt{t}\right)$$

and Φ denotes the distribution function of the standard normal law and φ its density function. For $x \in \mathbb{R}\backslash\{0\}$ we have

$$\alpha_{t,C}(x) = \int_0^{tC^2} C\alpha_{tC^2-s,C}(0)\rho_{\theta_0^{Cx}}(s)ds \quad and$$

$$\beta_{t,C}(x) = \int_0^{tC^2} C\beta_{tC^2-s,C}(0)\rho_{\tau_0^{Cx}}(s)ds$$

where

$$\rho_{\tau_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|-s)^2}{2s}} \quad and$$

$$\rho_{\theta_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|+s)^2}{2s}}$$

for any $s > 0$.

Proof. See [3, Theorems 2.1, 2.2]. □

We now state an error estimate for bang–bang controls.

Proposition 3.3. *Let $0 < h \leq k$ be fixed and $v : \mathbb{R}_+ \times \mathbb{R} \to \{-1, 1\}$ be measurable. Let $Z^{s,x}$ be the unique strong solution to the stochastic differential equation*

$$Z^{s,x}(t) := x + \int_s^t v(r, Z^{s,x}(r))dr + W(t) - W(s)$$

for $x \in \mathbb{R}, 0 \leq s \leq t \leq T$, cf. [23, Theorem IX.3.5], and assume that the function

$$\widetilde{V}(s, x) := E[J_{h,k}(Z^{s,x}(T))]$$

is of class $C^{1,2}$ on $[0, T) \times \mathbb{R}$.
    Then for any $x \in \mathbb{R}, u \in \mathcal{A}$ with

$$\int_0^T E[(\partial_2 \widetilde{V}(s, X_u^x(s)))^2]ds < \infty$$

we have

$$E[J_{h,k}(X_u^x(T))] \leq \widetilde{V}(0, x) + 2 \int_0^T \int_{\mathbb{R}} \beta_{s,1}(0) 1_{\{v(s,z) \neq \text{sgn}(\partial_2 \widetilde{V}(s,z))\}} |\partial_2 \widetilde{V}(s, z)|dzds, \quad (4)$$

where $\beta_{s,1}(0) = \frac{\varphi(\sqrt{s})}{\sqrt{s}} + \Phi(\sqrt{s})$ is the uniform bound given in Proposition 3.2.

Proof. Let $u \in \mathcal{A}, x \in \mathbb{R}$ such that

$$\int_0^T E[(\partial_2 \widetilde{V}(s, X_u^x(s)))^2]ds < \infty.$$

Then, $M(t) := \int_0^t \partial_2 \widetilde{V}(s, X_u^x(s))dW(s)$ for $t \in [0, T]$ is a martingale.
    The function $\widetilde{V}$ satisfies the Kolmogorov backward equation

$$\partial_1 \widetilde{V}(s, z) + \partial_2 \widetilde{V}(s, z)v(s, z) + \frac{1}{2}\partial_2^2 \widetilde{V}(s, z) = 0 \quad (5)$$

for any $s \in [0, T)$ and $z \in \mathbb{R}$.
    On the other hand, observe that

$$E[J_{h,k}(X_u^x(T))] = E[\widetilde{V}(T, X_u^x(T))].$$

Since $\widetilde{V} \in C^{1,2}([0,T) \times \mathbb{R})$ we can apply Itô's formula to express $E[\widetilde{V}(T-, X_u^x(T-))]$ as

$$
\begin{aligned}
E[\widetilde{V}&(T-, X_u^x(T-))] \\
&= \widetilde{V}(0,x) + \int_0^T E\left[\partial_1 \widetilde{V}(t, X_u^x(t)) + \partial_2 \widetilde{V}(t, X_u^x(t)) u(t) + \frac{1}{2} \partial_2^2 \widetilde{V}(t, X_u^x(t)) \right] dt \\
&= \widetilde{V}(0,x) + \int_0^T E\left[\partial_1 \widetilde{V}(t, X_u^x(t)) + \partial_2 \widetilde{V}(t, X_u^x(t)) v(t, X_u^x(t)) + \frac{1}{2} \partial_2^2 \widetilde{V}(t, X_u^x(t)) \right] dt \\
&\quad + \int_0^T E\left[\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) \right] dt \\
&= \widetilde{V}(0,x) + \int_0^T E\left[\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) \right] dt,
\end{aligned}
$$

where we used the martingale property of M in the first equality and in the last equality we have used relation (5). Clearly, we have $E[\widetilde{V}(T-, X_u^x(T-))] = E[\widetilde{V}(T, X_u^x(T))]$.

We continue to estimate the last integrand $E\left[\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) \right]$ for $t \in [0,T]$. On $A := \{ v(t, X_u^x(t)) \mathrm{sgn}(\partial_2 \widetilde{V}(t, X_u^x(t))) = 1 \}$ we have

$$
\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) \le 0
$$

and hence we get

$$
\begin{aligned}
E\left[\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) \right] &\le E\left[\partial_2 \widetilde{V}(t, X_u^x(t)) \left(u(t) - v(t, X_u^x(t)) \right) 1_{A^c} \right] \\
&\le 2 E\left[|\partial_2 \widetilde{V}(t, X_u^x(t))| 1_{A^c} \right].
\end{aligned}
$$

Inserting the density for $X_u^x(t)$ together with Proposition 3.2 yields

$$
E[\widetilde{V}(T, X_u^x(T))] \le \widetilde{V}(0,x) + 2 \int_0^T \int_{\mathbb{R}} \beta_{t,1}(0) |\partial_2 \widetilde{V}(t,z)| 1_{\{ v(t,z) \neq \mathrm{sgn}(\partial_2 \widetilde{V}(t,z)) \}} dz \, dt. \qquad \square
$$

Lemma 3.4. *Let $0 < h \le k$, $v : \mathbb{R}_+ \times \mathbb{R} \to \{-1,1\}$, $(t,x) \mapsto -1$ and $\widetilde{V}$ as in Proposition 3.3. Then, $\widetilde{V}$ is infinitely differentiable on $[0,T) \times \mathbb{R}$ and*

$$
\int_0^T \mathrm{E}[(\partial_2 \widetilde{V}(s, X_u^x(s)))^2] ds < \infty
$$

for any $x \in \mathbb{R}$, $u \in \mathcal{A}$.

Proof. $\widetilde{V}$ is obviously infinitely differentiable on $[0,T) \times \mathbb{R}$.

Let $x \in \mathbb{R}$, $u \in \mathcal{A}$. We have

$$
\begin{aligned}
\mathrm{E}[(\partial_2 \widetilde{V}(s, X_u^x(s)))^2] &\le \int_{\mathbb{R}} \beta_{s,1}(0)(\partial_2 \widetilde{V}(s,y))^2 dy \\
&= \beta_{s,1}(0) \int_{\mathbb{R}} \frac{1}{T-s} \left[\varphi\left(\frac{h+k-y+T-s}{\sqrt{T-s}} \right) - \varphi\left(\frac{k-y+T-s}{\sqrt{T-s}} \right) \right. \\
&\qquad \left. - \varphi\left(\frac{h-y+T-s}{\sqrt{T-s}} \right) + \varphi\left(\frac{-y+T-s}{\sqrt{T-s}} \right) \right]^2 dy \\
&\le \beta_{s,1}(0) \frac{1}{\sqrt{2(T-s)}} 8 \varphi(0).
\end{aligned}
$$

where we used Proposition 3.2 for the first inequality and Lemma B.3 for the second inequality. The claim follows. □

Proposition 3.5. *Let $0 < h \le k$ and $\alpha \in (0, 1)$. Then we have*

$$|E[J_{h,k}(X_u^x(T))]| \le hk^\alpha C_\alpha$$

where

$$C_\alpha := \frac{1}{\sqrt{2\pi e^\alpha}\, T^{(1+\alpha)/2}} + \frac{4}{\sqrt{2\pi e^\alpha}} \int_0^T \left(\frac{\varphi\left(\sqrt{s}\right)}{\sqrt{s}} + \Phi\left(\sqrt{s}\right) \right) \frac{1}{(T-s)^{(1+\alpha)/2}} ds$$

for any $u \in \mathcal{A}, x \in \mathbb{R}$.

Proof. Let $u \in \mathcal{A}, x \in \mathbb{R}$ and let v and $\widetilde{V}$ be as in Lemma 3.4. Then Lemma 3.4 yields that the requirements of Proposition 3.3 hold. Thus Proposition 3.3 yields

$$E[J_{h,k}(X_u^x(T))] \le \widetilde{V}(0, x) + 2 \int_0^T \int_{\mathbb{R}} \beta_{s,1}(0) 1_{\{1 = \operatorname{sgn}(\partial_2 \widetilde{V}(s,z))\}} \partial_2 \widetilde{V}(s, z) dz ds.$$

Lemma B.2 implies

$$|\widetilde{V}(t, y)| \le \frac{hk^\alpha}{\sqrt{2\pi e^\alpha}(T-t)^{(1+\alpha)/2}}.$$

for any $t \in [0, T), y \in \mathbb{R}$. Moreover, Lemma B.1 yields that there are $a(t), b(t) \in \mathbb{R}$ with $a(t) \le b(t)$ such that $\partial_2 \widetilde{V}(t, \cdot)$ is negative on $\mathbb{R} \setminus [a(t), b(t)]$ and positive on $(a(t), b(t))$ for any $t \in [0, T)$. The functions a, b depend on the choice of h, k but due to the uniformity of the preceding inequality on $\widetilde{V}$ they can be removed easily. We get

$$\int_0^T \int_{\mathbb{R}} \beta_{s,1}(0) 1_{\{1 = \operatorname{sgn}(\partial_2 \widetilde{V}(s,z))\}} \partial_2 \widetilde{V}(s, z) dz ds$$

$$= \int_0^T \beta_{s,1}(0) \int_{a(s)}^{b(s)} \partial_2 \widetilde{V}(s, z) dz ds$$

$$= \int_0^T \beta_{s,1}(0) (\widetilde{V}(s, b(s)) - \widetilde{V}(s, a(s))) ds$$

$$\le 2 \int_0^T \beta_{s,1}(0) \frac{hk^\alpha}{\sqrt{2\pi e^\alpha}(T-s)^{(1+\alpha)/2}} ds$$

$$= 2 \frac{hk^\alpha}{\sqrt{2\pi e^\alpha}} \int_0^T \left(\frac{\varphi\left(\sqrt{s}\right)}{\sqrt{s}} + \Phi\left(\sqrt{s}\right) \right) \frac{1}{(T-s)^{(1+\alpha)/2}} ds.$$

Thus we get

$$E[J_{h,k}(X_u^x(T))] \le hk^\alpha C_\alpha$$

If $E[J_{h,k}(X_u^x(T))] < 0$, then as after Eq. (3) we get

$$0 < -E[J_{h,k}(X_u^x(T))] = E[J_{h,k}(X_{-u}^{h+k-x}(T))] < hk^\alpha C_\alpha$$

and, hence, the claim follows. □

Proposition 3.6. *Let $u \in \mathcal{A}, x \in \mathbb{R}$ and $\alpha \in (0, 1)$. Then $X_u^x(T)$ has α-Hölder continuous density with constant C_α given in Proposition 3.5.*

Proof. Let $F(y) := P(X_u^x(T) \leq y) = P(X_u^{x-y}(T) \leq 0)$ be the distribution function of $X_u^x(T)$. Then F is absolutely continuous and Proposition 3.5 yields that

$$|F(y + h + k) - F(y + k) - F(y + h) + F(y)| \leq hk^\alpha C_\alpha$$

for any $y \in \mathbb{R}$, $0 < h \leq k$. Thus Proposition A.4 yields that F is continuously differentiable and its derivative is α-Hölder continuous with constant C_α. $\square$

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Appendix A. Hölder properties

In this section we gather some results on Hölder continuity. We start by recalling the definition.

Definition A.1. Let $I \subseteq \mathbb{R}$ be an interval with at least two points and $\alpha \in (0, 1]$. A function $f : I \to \mathbb{R}$ is said to be globally α-Hölder continuous with constant C if

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for any $x, y \in I$.

A function $f : I \to \mathbb{R}$ is locally α-Hölder continuous if $f|_J$ is Hölder continuous for any compact interval $J \subseteq I$ with at least two points. Here, the Hölder-continuity constants may depend on the interval J.

Next we give an elementary interpolation result for Hölder continuity.

Lemma A.2. *Let $I \subseteq \mathbb{R}$ be an interval containing at least two points, $\alpha \in (0, 1]$, $f : I \to \mathbb{R}$ be Lipschitz-continuous and bounded and define*

$$L_f := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in I, x \neq y \right\},$$
$$B_f := \sup\{|f(x) - f(y)| : x, y \in I\}.$$

Then, f is α-Hölder continuous and

$$|f(x) - f(y)| \leq \left(B_f^{1-\alpha} L_f^\alpha \right) |x - y|^\alpha$$

for any $x, y \in I$.

Proof. If $L_f = 0$, then f is constant and the claim obviously holds. Thus, we may assume that $L_f > 0$ and, hence, $B_f > 0$. Define $c := B_f / L_f$. Let $x, y \in I$.
Case 1: $|x - y| \leq c$. Then, we have

$$|f(x) - f(y)| \leq L_f |x - y| \leq L_f c \frac{|x - y|^\alpha}{c^\alpha} = B_f^{1-\alpha} L_f^\alpha |x - y|^\alpha.$$

Case 2: $|x - y| > c$. Then, we have

$$|f(x) - f(y)| \leq B_f \leq B_f \frac{|x - y|^\alpha}{c^\alpha} = B_f^{1-\alpha} L_f^\alpha |x - y|^\alpha. \quad \square$$

Lemma A.2 applied to the normal density yields a relative simple constant.

Corollary A.3. *Let $\alpha \in (0, 1]$. Then φ is α-Hölder continuous with constant $C_\alpha := \frac{1}{\sqrt{2\pi e^\alpha}}$, i.e.*
$$|\varphi(x) - \varphi(y)| \leq \frac{|x-y|^\alpha}{\sqrt{2\pi e^\alpha}} \text{ for any } x, y \in \mathbb{R}.$$

Proof. This is an immediate consequence of Lemma A.2.   □

The next proposition gives an exact condition that ensures that the derivative of a function is Hölder continuous. If X is a random variable, then Hölder-continuity of its density can be expressed in terms of weighted differences of certain probabilities.

Proposition A.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be absolutely continuous and $\alpha \in (0, 1]$. Then the following two statements are equivalent:*

(1) *F is continuously differentiable and F' is α-Hölder continuous.*
(2) *There is $C > 0$ such that*
$$\sup_{x \in \mathbb{R}} \sup_{k \geq h > 0} \left| \frac{F(x + h + k) - F(x + k) - F(x + h) + F(x)}{hk^\alpha} \right| \leq C.$$

Proof. (1) $\Rightarrow$ (2): Assume (1) and let $C > 0$ be the Hölder constant of F'. We have
$$|F(x + h + k) - F(x + k) - F(x + h) + F(x)| = \left| \int_x^{x+h} \left(F'(y + k) - F'(y) \right) dy \right|$$
$$\leq Chk^\alpha$$

for any $x \in \mathbb{R}, k > 0, h > 0$ and we used the fundamental theorem of calculus. Thus, (2) follows.

(2) $\Rightarrow$ (1): Assume (2). Let f be a version of the absolute continuous derivative of F, i.e. $\int_x^{x+h} |f(y)| dy < \infty$ and $\int_x^{x+h} f(y) dy = F(x + h) - F(x)$ for any $x \in \mathbb{R}, h > 0$. By Lebesgue's differentiation theorem [12, Corollary 2.1.17] we have
$$f(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

for any $x \in A$ where $A \subseteq \mathbb{R}$ is Borel measurable and the Lebesgue measure of $\mathbb{R} \backslash A$ equals zero. In particular, A is dense in $\mathbb{R}$.

Let $C > 0$ be the given constant, $x \in A$ and $k > 0$ such that $x + k \in A$. We show that $|f(x + k) - f(x)| \leq Ck^\alpha$. Let $\epsilon > 0$ be arbitrary and choose $h \in (0, k]$ such that
$$\left| f(x) - \frac{F(x + h) - F(x)}{h} \right| \leq \epsilon, \quad \text{and}$$
$$\left| f(x + k) - \frac{F(x + h + k) - F(x + k)}{h} \right| \leq \epsilon.$$

Thus, we have
$$|f(x + k) - f(x)| \leq 2\epsilon + \left| \frac{F(x + h + k) - F(x + k) - F(x + h) + F(x)}{h} \right|$$
$$\leq 2\epsilon + Ck^\alpha.$$

Hence, we have $|f(x) - f(y)| \leq C|x - y|^\alpha$ for any $x, y \in A$. Consequently, there is an α-Hölder continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for any $x \in A$. Clearly, g is another

version of the absolutely continuous derivative of F. The fundamental theorem of calculus yields that F is continuously differentiable and $F' = g$. $\square$

Appendix B. Normal estimates

Connected to the last section we find some normal estimates for functions related to Proposition A.4.

Lemma B.1. *Let Z be a normal random variable with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$ and let $0 < h \leq k$. We define*

$$N(x) := P(Z + x \in [k, k + h]) - P(Z + x \in [0, h]), \quad x \in \mathbb{R}.$$

Then N is infinitely differentiable and N' is negative on $(-\infty, -\mu-\sigma)$ and on $(h+k-\mu+\sigma, \infty)$. Moreover, N' is positive if and only if $x \in (a, b)$ where a, b are the two zeros of N' (and N' has exactly two zeros).

Proof. Clearly, N is infinitely differentiable. Thus, we have

$$N(x) = \frac{1}{\sigma^2} \int_0^h \int_0^k \varphi' \left(\frac{y + z - x - \mu}{\sigma} \right) dy \, dz$$

for any $x \in \mathbb{R}$. Hence, by dominated convergence we get

$$N'(x) = \frac{-1}{\sigma^3} \int_0^h \int_0^k \varphi'' \left(\frac{y + z - x - \mu}{\sigma} \right) dy \, dz.$$

Since φ'' is positive on $\mathbb{R} \setminus (-1, 1)$ the first claim follows.

If we show that N' has exactly two zeros, then the second claim follows as well. By scaling and shifting we may assume that $\mu = 0$ and $\sigma = 1$. Then, we have

$$N'(x) = -\varphi(x) + \varphi(x - h) + \varphi(x - k) - \varphi(x - h - k)$$

for any x. We get

$$N'(x) = (\varphi(x - h) - \varphi(x)) + (\varphi(x - k) - \varphi(x - h - k)) > 0$$

for any $x \in [h/2, k + h/2]$. Clearly, there is $x_0 < 0$ and $x_1 > h + k$ such that $N'(x_0) < 0$ and $N'(x_1) < 0$. By the intermediate value theorem there is $a \in [x_0, h/2]$ such that $N'(a) = 0$ and $b \in [k + h/2, x_1]$ such that $N'(b) = 0$.

Moreover, we have

$$N''(x) = -xN'(x) + h\varphi(x - h) + k\varphi(x - k) - (h + k)\varphi(x - h - k)$$

for any $x \in \mathbb{R}$ and for any zero $\tilde{a} \in (-\infty, h/2]$ we have

$$N''(\tilde{a}) = h\varphi(\tilde{a} - h) + k\varphi(\tilde{a} - k) - (h + k)\varphi(\tilde{a} - h - k) > 0.$$

Consequently, N' has exactly one zero in $(-\infty, h/2)$. By symmetry of N' around $(h + k)/2$ we have that N' has exactly one zero in $((k + h)/2, \infty)$, namely $b = (h + k)/2 - a$. $\square$

Lemma B.2. *Assume the requirements of Lemma B.1 and let $\alpha \in (0, 1]$. Then we have*

$$|N(x)| \leq \frac{hk^\alpha}{\sqrt{2\pi e^\alpha} \sigma^{1+\alpha}}.$$

Proof. The fundamental theorem of calculus and Corollary A.3 yield

$$|N(x)| \leq \frac{1}{\sigma} \int_0^h \left| \varphi\left(\frac{y+k-x-\mu}{\sigma} \right) - \varphi\left(\frac{y-x-\mu}{\sigma} \right) \right| dy$$

$$\leq \frac{hk^\alpha}{\sqrt{2\pi e^\alpha} \sigma^{1+\alpha}}. \quad \square$$

Lemma B.3. *Let $\sigma > 0$, $y, z \in \mathbb{R}$. Then we have*

$$\frac{1}{\sigma^2} \int_{\mathbb{R}} \varphi\left(\frac{x-y}{\sigma} \right) \varphi\left(\frac{x-z}{\sigma} \right) dx = \frac{\varphi\left((y-z)/\sqrt{2\sigma^2} \right)}{\sqrt{2\sigma^2}}.$$

Proof. Using the substitution $u = \frac{x-y}{\sigma}$ and the notation $a := \frac{y-z}{\sigma}$ we get

$$\frac{1}{\sigma^2} \int_{\mathbb{R}} \varphi\left(\frac{x-y}{\sigma} \right) \varphi\left(\frac{x-z}{\sigma} \right) dx = \frac{1}{\sigma} \int_{\mathbb{R}} \varphi(u)\varphi(u+a) du$$

$$= \frac{1}{\sigma\sqrt{2}} \varphi\left(a/\sqrt{2} \right)$$

$$= \frac{\varphi\left((y-z)/\sqrt{2\sigma^2} \right)}{\sqrt{2\sigma^2}}. \quad \square$$

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