



$W^{2,p}$ -solutions of parabolic SPDEs in general domains

Kai Du¹

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China

Received 5 May 2018; received in revised form 4 December 2018; accepted 12 December 2018

Available online xxxx

Abstract

The Dirichlet problem for a class of stochastic partial differential equations is studied in Sobolev spaces. The existence and uniqueness result is proved under certain compatibility conditions that ensure the finiteness of $L^p(\Omega \times (0, T), W^{2,p}(G))$ -norms of solutions. The Hölder continuity of solutions and their derivatives is also obtained by embedding.

© 2018 Elsevier B.V. All rights reserved.

Keywords: Stochastic partial differential equations; Dirichlet problem; L^p estimates; Compatibility conditions; Unbounded domains

1. Introduction

Given a domain $G \subset \mathbb{R}^n$ and a sequence of independent Wiener processes w^k , let us consider the following stochastic partial differential equation (SPDE)

$$du = (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + f) dt + (\sigma^{ik}u_{x^i} + v^k u + g^k) dw_t^k \quad (1.1)$$

with $(t, x, \omega) \in (0, T] \times G \times \Omega$, where the leading coefficients $a^{ij}(t, x, \omega)$ and $\sigma^{ik}(t, x, \omega)$ satisfy the *strong parabolicity condition*: there are positive numbers κ and K such that

$$\kappa |\xi|^2 + \sigma^{ik} \sigma^{jk} \xi^i \xi^j \leq 2a^{ij} \xi^i \xi^j \leq K |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } (t, x, \omega). \quad (1.2)$$

Einstein summation convention is used in this paper with $i, j = 1, \dots, n$ and $k = 1, 2, \dots$. Such equations arise in many applications such as nonlinear filtering, statistical physics, and so on (see [3] and references therein). The countable sum of stochastic integrals in (1.1) is instrumental in treating equations driven by cylindrical white noise (cf. [15,27]).

E-mail address: kdu@fudan.edu.cn.

¹ Research of K. Du was partially supported by NSF of China (No. 11801084).

The main goal of this paper is to obtain the solvability of parabolic SPDEs in the space $L^p(\Omega \times (0, T), W^{2,p}(G))$ with natural structural conditions, where $W^{2,p}(G)$ is a standard Sobolev space with $p \geq 2$. To explain our interest in this problem, let us recall some well-known results from SPDE theory in this aspect. Under the framework of Hilbert spaces $H^m(G) = W^{m,2}(G)$, Krylov and Rozovsky [20] proved the existence and uniqueness of weak solutions for a large class of parabolic SPDEs, and then they proved the smoothness of solutions when $G = \mathbb{R}^n$. So far, the theory for the Cauchy problem is rather complete and satisfactory: a comprehensive L^p -theory of parabolic SPDEs in the whole space was developed by Krylov [14] in Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ (equivalent to $W^{s,p}(\mathbb{R}^n)$ when s is a natural number), and a solvability theory in Hölder classes was constructed by Mikulevicius [24], Du and Liu [6]. As far as general domains G are concerned, one of the greatest difficulties is how to handle the “bad” behaviour of derivatives of solutions near the boundary. Indeed, unless certain compatibility conditions are fulfilled, the derivatives of the solutions may blow up near the boundary even in the one-dimensional case. As an example, let us take a look at the following finding from Krylov [16, Theorem 5.3].

Lemma 1.1 (Krylov [16]). *There exists a $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0)$ and the function u with $u(t, 0) = 0$ for all t and $u(0, \cdot) \in C_0^\infty(0, \infty)$ satisfies the equation*

$$du = u_{xx} dt + \sqrt{2 - \lambda} u_x dW_t \quad \text{on } (0, \infty)^2,$$

then there exists a dense subset $S \subset (0, \infty)$ such that for all $s \in S$ and $\alpha > e^{-\frac{1}{2\lambda}}$, it holds almost surely (a.s.) that $\lim_{x \downarrow 0} x^{-\alpha} u(s, x) = \infty$; consequently, $\limsup_{x \downarrow 0} |u_x(s, x)| = \infty$.

Flandoli [7] proved the existence and uniqueness of solutions of parabolic SPDEs in the Hilbert space $H^{2m+1}(G)$ under a long series of compatibility conditions (see theorem 4.1 there). Brzeniak [2] solved the equations in the Besov space $B_{p,2}^1(G)$ (whose elements have first-order weak derivatives) requiring σ to be sufficiently small. Both of them used a semigroup method, and the leading coefficients of their equations were deterministic. Applying PDE techniques, Krylov [12] developed a $W^{m,2}$ -theory of linear SPDEs in general smooth domains, where the equations could have random coefficients; instead of the compatibility conditions, he introduced Sobolev spaces with weights to control the blow-up of derivatives of solutions near the boundary. This idea was adopted to develop a weighted L^p -theory for parabolic SPDEs in general domains, see [10–12, 19] among others. For more aspects of regularity theory for quasilinear SPDEs in domains, we refer to Denis et al. [5], Zhang [28], Van Neerven et al. [25, 26], Debussche et al. [4], Gerencsér [8] and the references therein.

By relaxing the requirement on derivatives of solutions, the weighted L^p -theory of SPDEs is successful in dealing with equations under very general assumptions on the coefficients. Nevertheless, it is still interesting enough to ask under which circumstances the solutions of SPDEs lie in the normal Sobolev spaces, especially the space $W^{2,p}$ in which the solutions found are called *strong solutions* in classical PDE theory (cf. [22]). This question seems not to be answered by the weighted L^p -theory of SPDEs. To find natural conditions, let us start with two examples as follows, which show that, if there is no restriction on the boundary values of coefficient σ , the second-order derivatives need not be square integrable.

Example 1.2. Let $u(t, x)$ with $t \in (0, \infty)$ and $x \in G = (0, 1)$ be a solution of the equation:

$$\begin{aligned} du(t, x) &= (u_{xx}(t, x) + f(t)) dt + \sigma(x) u_x(t, x) dW_t, \\ u(0, x) &= u(t, 0) = u(t, 1) = 0, \end{aligned}$$

where W is a one-dimensional Wiener process, $\sigma \in C^2(\bar{G})$ with $\sup_G |\sigma| < 2$, and $f \in L^2(0, \infty)$ are not identically zero. From L^2 -theory of SPDEs (cf. [20]), the equation has a unique (nonzero) solution $u \in L^2(\Omega \times (0, \infty), H_0^1(G))$. However, if $\sigma(0)\sigma(1) \neq 0$, one can see that $u_{xx} \notin L^2(\Omega \times (0, T) \times G)$ for any $T > 0$. Indeed, if not so, then by embedding u_x is continuous on $[0, T] \times \bar{G}$ for some $T > 0$, and we can claim: $\sigma(0)u_x(t, 0) = \sigma(1)u_x(t, 1) = 0$ for all $t \in [0, T]$. Then $v := u_x \in H_0^1(G)$ satisfies (in the sense of distribution)

$$dv = v_{xx}dt + (\sigma v_x + \sigma_x v)dW_t, \quad v(0, x) = v(t, 0) = v(t, 1) = 0,$$

which implies $u_x = v = 0$, and furthermore, $u = 0$ by the boundary condition, yielding a contradiction. To prove the claim, one can take a nonnegative function $\varphi \in C_b^2(\mathbb{R})$ such that $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(0) = 1$, and smooth even functions $\zeta_n \geq 0$ supported in $[-1/n, 1/n]$ that converges to the Dirac function as $n \rightarrow \infty$. By Itô's formula, one has that

$$\begin{aligned} \mathbf{E} \int_0^1 \zeta_n(x) \varphi(u(T, x)) dx &= \mathbf{E} \int_0^T \int_0^1 \zeta_n(x) \left\{ \varphi'(u(t, x)) [u_{xx}(t, x) + f(t)] \right. \\ &\quad \left. + \frac{1}{2} \varphi''(u(t, x)) |\sigma(x)u_x(t, x)|^2 \right\} dx dt. \end{aligned}$$

Using the boundary condition $u(t, 0) = 0$ and the continuity of u and u_x , one can obtain that $\mathbf{E}|\sigma(0)u_x(t, 0)|^2 = 0$ for all $t \in [0, T]$ from the above identity by sending n to infinity, so $\sigma(0)u_x(t, 0) = 0$. Similarly, one has $\sigma(1)u_x(t, 1) = 0$, so the claim is proved.

Example 1.3. With $\sigma \in (-2, 2) \setminus \{0\}$, $T > 0$ and $G = (0, 1)$ the following equation

$$du = (u_{xx} + x/\sigma)dt + (\sigma u_x - t)dW_t, \quad u(0, x) = u(t, 0) = u(t, 1) = 0$$

has a unique solution $u \in L^2(\Omega \times (0, T), H_0^1(G))$ from L^2 -theory of SPDEs. Suppose $u_{xx} \in L^2(\Omega \times (0, T), L^2(G))$. Then by a similar argument as in the previous example, we have that $v = u_x \in H^1(G)$ satisfies

$$dv = (v_{xx} + 1/\sigma)dt + \sigma v_x dW_t, \quad v(0, x) = 0, \quad v(t, 0) = v(t, 1) = t/\sigma.$$

Solving this equation we have $u_x(t, x) = v(t, x) = t/\sigma$, which is impossible given $u(t, 0) = u(t, 1) = 0$. Therefore, $u \notin L^2(\Omega \times (0, T), H^2(G))$.

From the above, certain compatibility condition on the coefficient σ must be involved to ensure the second-order derivatives of solutions lie in $L^p(\Omega \times (0, T), W^{2,p}(G))$. This issue was first addressed by Flandoli [7], where $p = 2$ and the coefficients of equations depended only on x . For $p > 2$ there seems be no result in the literature. In this note we propose the following condition.

Assumption 1.4. The vectors $\sigma^{-k} = (\sigma^{1k}, \dots, \sigma^{nk})$ restricted on ∂G are tangent to ∂G , namely,

$$\mathbf{n}(x) \cdot \sigma^{-k}(t, x, \omega) = 0, \quad k = 1, 2, \dots \quad (1.3)$$

for all $x \in \partial G$ and all (t, ω) , where $\mathbf{n}(x)$ is a unit normal vector of ∂G at x .

When considering zero boundary conditions, this assumption is quite necessary for our goal according to our examples, and technically, it gives the least condition on σ to ensure that $\sigma^{ik}u_{x_i}$ vanishes on the boundary for all $u \in W^{2,p}(G) \cap W_0^{1,p}(G)$ and all k . Meanwhile, the free term g must equal zero on the boundary consequently, otherwise the second-order derivatives of solutions of Eq. (1.1) may still blow up near the boundary, as was illustrated in [12, Example 1.2].

Assumption 1.4 and the boundary value restriction of g are all we need additionally to achieve our goal. Indeed, the main result of this paper, **Theorem 2.5**, yields that, under **Assumption 1.4** along with other standard conditions on the coefficients and on the domain, SPDE (1.1) with zero initial-boundary condition has a unique solution u in the space $L^p(\Omega \times (0, T), \mathcal{P}, W^{2,p}(G))$ for any given $f \in L^p(\Omega \times (0, T), \mathcal{P}, L^p(G))$ and $g \in L^p(\Omega \times (0, T), \mathcal{P}, W_0^{1,p}(G; \ell^2))$, where \mathcal{P} is the predictable σ -field. The requirement on the boundary value of g is attracted into the space. By embedding the solution and its derivatives are globally Hölder continuous as long as $p > n + 2$.

It is worth noting that **Assumption 1.4** has local impact on the regularity of solutions; in other words, if (1.3) is satisfied only on a portion of ∂G , then the solutions possess $W^{2,p}$ -regularity and continuity near this portion. This property is elaborated in **Theorem 2.7** in the next section.

This paper is organized as follows. In the next section the main results are stated after introducing some notation and assumptions. Section 3 is devoted to the proof of **Theorem 2.5**, consisting of four subsections: in Section 3.1 we obtain the existence, uniqueness and estimates of the solution of a model equation in the half space; in Section 3.2 we derive a priori estimates for general equations in C^2 domains; the existence and uniqueness of solutions in the general case is proved in Section 3.3 with the help of the method of continuity and the Banach fixed-point theorem; and in Section 3.4 we prove the continuity of solutions and their derivatives. **Theorem 2.7** is proved in the final section.

2. Notation and main results

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a complete filtered probability space carrying a sequence of independent Wiener process w^k , and \mathcal{P} the predictable σ -field generated by \mathcal{F}_t . Let \mathbb{R}^n be an n -dimensional Euclidean space of points $x = (x^1, \dots, x^n)$, and

$$\mathbb{R}_+^n = \{x = (x^1, x') : x^1 > 0, x' = (x^2, \dots, x^n) \in \mathbb{R}^{n-1}\}.$$

Denote $B_\rho(x) = \{y \in \mathbb{R}^n : |x - y| < \rho\}$ and $B_\rho = B_\rho(0)$. Let G be a domain in \mathbb{R}^n . The following definition is taken from Krylov [17, Page 165].

Definition 2.1. We write $G \in \mathcal{C}^2$ if there are positive constants K_0 and ρ_0 such that for each $z \in \partial G$ there exists a one-to-one map ψ from $B_{\rho_0}(z)$ to a domain $U^z \subset \mathbb{R}^n$ such that

1. $\psi(z) = 0$ and $U_+^z := \psi(B_{\rho_0}(z) \cap G) \subset \mathbb{R}_+^n$,
2. $\psi(B_{\rho_0}(z) \cap \partial G) = U^z \cap \{y \in \mathbb{R}^n : y^1 = 0\}$,
3. $\psi \in C^2(\bar{B}_{\rho_0}(z))$, $\psi^{-1} \in C^2(\bar{U}^z)$, and $\|\psi\|_{C^2} + \|\psi^{-1}\|_{C^2} \leq K_0$.

We say that the diffeomorphism ψ flattens the boundary near z .

Fix real numbers $T > 0$ and $p \geq 2$ in this paper.

For $m \geq 0$ we let $W^{m,p}(G)$ and $W_0^{1,p}(G)$ be the usual Sobolev spaces (cf. [1]), and $W^{m,p}(G; \ell^2)$ and $W_0^{1,p}(G; \ell^2)$ the corresponding spaces of ℓ^2 -valued functions. Denote

$$W_\circ^{m,p}(G) = W^{m,p}(G) \cap W_0^{1,p}(G), \quad m \geq 1,$$

and $W_\circ^{0,p}(\cdot) = W^{0,p}(\cdot) = L^p(\cdot)$. Denote by $W_{\text{loc}}^{m,p}(G)$ the space of all functions u such that $u \in W^{m,p}(G')$ for any $G' \subset G$ with $\text{dist}(G', \partial G) > 0$.

For random functions, we define

$$\begin{aligned} \mathbb{W}^{m,p}(G, \tau) &= L^p(\Omega \times (0, \tau), \mathcal{P}, W^{m,p}(G)), & \mathbb{W}^{m,p}(G) &= \mathbb{W}^{m,p}(G, T), \\ \mathbb{W}_\circ^{m,p}(G, \tau) &= L^p(\Omega \times (0, \tau), \mathcal{P}, W_\circ^{m,p}(G)), & \mathbb{W}_\circ^{m,p}(G) &= \mathbb{W}_\circ^{m,p}(G, T), \end{aligned}$$

and analogously, $\mathbb{W}^{m,p}(G; \tau; \ell^2)$, $\mathbb{W}^{m,p}(G; \ell^2)$, $\mathbb{W}_o^{m,p}(G; \ell^2)$, $\mathbb{W}_{\text{loc}}^{m,p}(G)$, etc. Denote $\mathbb{L}^p(\cdot) = \mathbb{W}^{0,p}(\cdot) = \mathbb{W}_o^{0,p}(\cdot)$.

The understanding of solutions of SPDEs is implied in the following definition of a functional space for solutions (cf. [15]).

Definition 2.2. For a positive integer m , by $\mathcal{W}_o^{m,p}(G)$ we denote the space of all functions $u \in \mathbb{W}_o^{m,p}(G)$ such that

$$u(0, \cdot) \in L^p(\Omega, \mathcal{F}_0, W_o^{m-2/p,p}(G))$$

and for some $u_D \in \mathbb{W}^{m-2,p}(G)$ and $u_S \in \mathbb{W}_o^{m-1,p}(G; \ell^2)$, the equation $du = u_D dt + u_S^k dw_t^k$ holds in the sense of distributions, namely, for all $\phi \in C_0^\infty(G)$,

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (u_D(s, \cdot), \phi) ds + \int_0^t (u_S^k(s, \cdot), \phi) dw_s^k$$

for all $t \leq T$ with probability 1.

Now we consider the following semilinear equation

$$du = (a^{ij} u_{x_i x_j} + f(t, x, u)) dt + (\sigma^{ik} u_{x_i} + g^k(t, x, u)) dw_t^k \quad (2.1)$$

with the initial-boundary condition

$$\begin{cases} u(t, x) = 0, & x \in \partial G, t \geq 0; \\ u(0, x) = u_0(x), & x \in G. \end{cases} \quad (2.2)$$

The following conditions on the given data are quite standard (cf. [15]).

Assumption 2.3. The functions $a^{ij} = a^{ji}$ and σ^{ik} are real valued and $\mathcal{P} \times \mathcal{B}(G)$ -measurable and satisfy the strong parabolicity condition (1.2), and there are a number $L > 0$ and a continuous and increasing function $\varpi(\cdot)$ with $\varpi(0) = 0$ such that

$$|a^{ij}(t, x) - a^{ij}(t, y)| \leq \varpi(|x - y|), \quad \|\sigma^{ik}(t, x) - \sigma^{ik}(t, y)\|_{\ell^2} \leq L|x - y|,$$

for all (t, ω) , all $x, y \in \bar{G}$, and all $i, j = 1, \dots, n$.

Assumption 2.4. (a) For any $u \in W_o^{2,p}(G)$, the functions $f(\cdot, \cdot, u)$ and $g(\cdot, \cdot, u)$ are predictable as functions taking values in $L^p(G)$ and $W_o^{1,p}(G; \ell^2)$, respectively.

(b) $f(\cdot, \cdot, 0) \in \mathbb{L}^p(G)$ and $g(\cdot, \cdot, 0) \in \mathbb{W}_o^{1,p}(G; \ell^2)$.

(c) For any $\varepsilon > 0$, there is a $K_\varepsilon \geq 0$ such that for any $u, v \in W_o^{2,p}(G)$, t, ω , we have

$$\begin{aligned} \|f(t, \cdot, u) - f(t, \cdot, v)\|_{L^p(G)} + \|g(t, \cdot, u) - g(t, \cdot, v)\|_{W^{1,p}(G; \ell^2)} \\ \leq \varepsilon \|u - v\|_{W^{2,p}(G)} + K_\varepsilon \|u - v\|_{L^p(G)}. \end{aligned}$$

The main result of this paper is the following theorem.

Theorem 2.5. Let $G \in \mathcal{C}^2$ and Assumptions 1.4, 2.3 and 2.4 be satisfied. Then we have that

(i) for any $u_0(\cdot) \in L^p(\Omega, \mathcal{F}_0, W_o^{2-2/p,p}(G))$ Dirichlet problem (2.1)–(2.2) admits a unique solution $u \in W_o^{2,p}(G)$;

(ii) the solution satisfies the estimate

$$\|u\|_{W^{2,p}(G)}^p \leq C (\|f(\cdot, \cdot, 0)\|_{\mathbb{L}^p(G)}^p + \|g(\cdot, \cdot, 0)\|_{\mathbb{W}^{1,p}(G; \ell^2)}^p + \mathbf{E} \|u_0\|_{W^{2-2/p,p}(G)}^p), \quad (2.3)$$

where the constant C depends only on $\kappa, K, n, p, T, K_0, \rho_0, L$, and the functions $\varpi(\cdot)$ and K_ε ;

(iii) when $p > \max\{2, (n+2)/2\}$, $u \in L^p(\Omega, C^{\alpha/2, \alpha}([0, T] \times \bar{G}))$ for any $\alpha \in (0, \frac{2p-n-2}{2p})$, and when $p > n+2$, $u_x \in L^p(\Omega, C^{\beta/2, \beta}([0, T] \times \bar{G}))$ for any $\beta \in (0, \frac{p-n-2}{2p})$.

The Hölder space $C^{\alpha/2, \alpha}([0, T] \times \bar{G})$ is defined in the standard way (cf. [13]), which contains all continuous functions $u : [0, T] \times \bar{G} \rightarrow \mathbb{R}$ such that

$$\|u\|_{C^{\alpha/2, \alpha}([0, T] \times \bar{G})} = \sup_{[0, T] \times \bar{G}} |u| + \sup_{(t, x) \neq (s, y)} \frac{|u(t, x) - u(s, y)|}{|t - s|^{\alpha/2} + |x - y|^{\alpha}} < \infty.$$

In the literature the domain G was usually assumed to be bounded (unless it is the whole space or a half space), but here it can be unbounded, and a detailed argument will be presented in our proof to address unbounded domains (see Section 3.2). Moreover, the above theorem still holds true if the terminal time T is replaced by any stopping time $\tau \leq T$ as in [10, 15]. A simple way to do this is to zero extend the functions f and g after time τ until T and solve the problem in the time period $[0, T]$.

By interpolation it is easily checked that the linear equation (1.1) fits the assumptions of Theorem 2.5 provided that $f \in \mathbb{L}^p(G)$ and $g \in \mathbb{W}_0^{1, p}(G; \ell^2)$ along with the following condition.

Assumption 2.6. The functions b^i, c, v^k are real valued and $\mathcal{P} \times \mathcal{B}(G)$ -measurable, and $|b^i|, |c|, \|v\|_{\ell^2}, \|v_x\|_{\ell^2}$ are uniformly bounded on $[0, T] \times \bar{G} \times \Omega$.

Even if the compatibility condition (1.3) is satisfied only on a portion of ∂G , it is still possible to obtain local regularity of solutions near this portion. The main issue here is that the solution may not lie in $\mathcal{W}_0^{2, p}(G)$ or even not exist. Fortunately, when G is bounded (or a half space) and the equation is linear, the Dirichlet problem can be solved in the weighted Sobolev space $\mathfrak{H}_{p, \theta}^2(G)$ by means of the main results in [11]; the space $\mathfrak{H}_{p, \theta}^2(G)$ that $\mathcal{W}_0^{2, p}(G)$ can be embedded to was introduced by Krylov and Lototsky [19], Lototsky [23], based on delicately selected weights. With this observation, we formulate the local regularity result into the following theorem by assuming the existence of solutions without thorough verification of conditions, and for simplicity but without loss of essence, we consider the linear equation (1.1).

Theorem 2.7. Let Γ be an open subset of ∂G and (1.3) satisfied at each point $x \in \Gamma$. Let Assumptions 2.3 and 2.6 be satisfied and $|a_x^{ij}|$ dominated by the constant L . Suppose that $u \in \mathbb{L}^p(G) \cap \mathbb{W}_{\text{loc}}^{2, p}(G)$ with $u(0, \cdot) \in L^p(\Omega, \mathcal{F}_0, \mathcal{W}_0^{2-2/p, p}(G))$ satisfies Eq. (1.1) with $u|_{\partial G} = 0$ for given $f \in \mathbb{L}^p(G)$ and $g \in \mathbb{W}^{1, p}(G)$ with $g|_{\Gamma} = 0$. Then for any bounded domain $G' \subset G$ with $\text{dist}(G', \partial G \setminus \Gamma) > 0$, we have $u \in \mathbb{W}^{2, p}(G')$. Moreover, $u \in L^p(\Omega, C^{\alpha/2, \alpha}([0, T] \times \bar{G}'))$ for any $\alpha \in (0, \frac{2p-n-2}{2p})$ when $p > \max\{2, (n+2)/2\}$, and $u_x \in L^p(\Omega, C^{\beta/2, \beta}([0, T] \times \bar{G}'))$ for any $\beta \in (0, \frac{p-n-2}{2p})$ when $p > n+2$.

In the above theorem the assumption that the solution lies in $\mathbb{L}^p(G) \cap \mathbb{W}_{\text{loc}}^{2, p}(G)$ is not restrictive: on the one hand, a function in the space $\mathfrak{H}_{p, \theta}^2(G)$ naturally belongs to $\mathbb{W}_{\text{loc}}^{2, p}(G)$; on the other hand, the property $u \in \mathbb{L}^p(G)$ can be derived from the other assumptions of the theorem with the help of Itô's formula, at least when G is bounded. Requiring $a^{ij}(t, \cdot) \in C^{0, 1}(G)$ allows us to write the equation into the divergence form that helps us prove $u \in \mathbb{W}^{1, p}(G')$ as an important intermediate step. We remark that $u \in \mathfrak{H}_{p, \theta}^2(G)$ does not always imply $u \in \mathbb{W}^{1, p}(G)$ (cf. [10]).

3. Proof of Theorem 2.5

3.1. Model equations in a half space

Let $G = \mathbb{R}_+^n$ in this subsection. In the first step we consider the equations on $[0, T] \times \mathbb{R}_+^n$ with coefficients independent of x .

Proposition 3.1. *Let $a^{ij} = a^{ij}(t)$ and $\sigma^{ik} = \sigma^{ik}(t)$ be predictable processes and satisfy (1.2). Assume that*

$$\sigma^{1\cdot} = (\sigma^{11}, \sigma^{12}, \dots) \equiv 0, \quad \forall(t, \omega) \in [0, T] \times \Omega.$$

Consider the Dirichlet problem

$$\begin{cases} du = (a^{ij}u_{x^i x^j} + f)dt + (\sigma^{ik}u_{x^i} + g^k)dw_t^k, \\ u|_{t=0} = 0, \quad u|_{x^1=0} = 0. \end{cases} \quad (3.1)$$

Then, (i) for $f \in \mathbb{L}^p(\mathbb{R}_+^n)$ and $g \in \mathbb{W}_\circ^{1,p}(\mathbb{R}_+^n; \ell^2)$, (3.1) has a unique solution $u \in \mathcal{W}_\circ^{2,p}(\mathbb{R}_+^n)$, and

$$\|u\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} \leq C(\|f\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n; \ell^2)}); \quad (3.2)$$

(ii) if $g \in \mathbb{L}^p(\mathbb{R}_+^n; \ell^2)$ and $f = f^0 + c^i F_{x^i}$ with $f^0, F \in \mathbb{L}^p(\mathbb{R}_+^n)$ and $c^i \in \mathbb{L}^\infty(G)$, then (3.1) has a unique solution $u \in \mathcal{W}_\circ^{1,p}(\mathbb{R}_+^n)$, and

$$\|u\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n)} \leq C(\|(f^0, F)\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{L}^p(\mathbb{R}_+^n; \ell^2)}); \quad (3.3)$$

where the constant C depends only on κ, K, n, p, T , and additionally on $\|c^i\|_{\mathbb{L}^\infty}$ for (ii).

Proof. The proofs of (i) and (ii) are quite similar, so we only present the proof of (i) in detail. Consider the following equation

$$d\hat{u} = K \Delta \hat{u} dt + (\sigma^{ik} \hat{u}_{x^i} + g^k) dw_t^k \quad \text{on } (0, T] \times \mathbb{R}_+^n \quad (3.4)$$

with zero initial–boundary condition. Obviously, this equation is also strongly parabolic. Define the odd continuation of g , i.e.,

$$g(x^1, x') := -g(-x^1, x'), \quad \forall x^1 < 0, x' \in \mathbb{R}^{n-1}. \quad (3.5)$$

As $g \in \mathbb{W}_\circ^{1,p}(\mathbb{R}_+^n; \ell^2)$, the continued function g belongs to $\mathbb{W}_\circ^{1,p}(\mathbb{R}^n; \ell^2)$. By Theorem 5.1 in [15], there exists a unique solution $\hat{u} \in \mathcal{W}_\circ^{2,p}(\mathbb{R}^n)$ of (3.4) considered in the whole \mathbb{R}^n with zero initial condition. From the uniqueness, $\hat{u}(t, x) = \hat{u}(t, x^1, x')$ is odd with respect to x^1 , so $\hat{u}(t, x) = 0$ for $x^1 = 0$, which means that \hat{u} restricted in \mathbb{R}_+^n satisfies (3.4) with zero initial–boundary condition, and $\hat{u} \in \mathcal{W}_\circ^{2,p}(\mathbb{R}_+^n)$. Also from Theorem 5.1 in [15], we have the following estimate

$$\|\hat{u}\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} \leq C\|g\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n; \ell^2)}, \quad (3.6)$$

where the constant C depends only on κ, K, n, p and T .

Now we adopt an idea from Krylov [15] to convert a stochastic PDE to a family of PDEs. Define a stochastic process $\xi_t = (0, \xi_t^2, \dots, \xi_t^n)$ with

$$\xi_t^i = \int_0^t \sigma^{ik}(s) dw_s^k, \quad i = 2, \dots, n.$$

It is easily seen that for each $x \in \mathbb{R}_+^n$ the process $x \pm \xi_t$ always stays in \mathbb{R}_+^n . Moreover, for any given $\tilde{f} \in \mathbb{L}^p(\mathbb{R}_+^n)$, the random translation $\tilde{f}(t, x - \xi_t)$ as a function of (t, x, ω) also lies in $\mathbb{L}^p(\mathbb{R}_+^n)$, and

$$\|\tilde{f}(\omega)\|_{L^p((0,T) \times \mathbb{R}_+^n)} = \|\tilde{f}(\cdot, \cdot - \xi(\omega), \omega)\|_{L^p((0,T) \times \mathbb{R}_+^n)}.$$

Consider the following random partial differential equation (PDE)

$$\begin{aligned} \partial_t v &= \left(a^{ij} - \frac{1}{2} \sigma^{ik} \sigma^{jk} \right) v_{x^i x^j} + \tilde{f}(t, x - \xi_t), \quad \text{on } (0, T] \times \mathbb{R}_+^n, \\ v|_{t=0} &= 0, \quad v|_{x^1=0} = 0. \end{aligned} \quad (3.7)$$

Due to (1.2), this PDE is strongly parabolic. Moreover, $\tilde{f}(t, x - \xi_t(\omega), \omega)$ as a function of (t, x) belongs to $L^p((0, T) \times \mathbb{R}_+^n)$ for almost every ω . So by the classical PDE theory (cf. [22, Theorem 7.32]), problem (3.7) has a unique strong solution

$$v(\cdot, \cdot, \omega) \in L^p((0, T), W_{\circ}^{2,p}(\mathbb{R}_+^n)) \times C([0, T], L^p(\mathbb{R}_+^n))$$

for almost every ω , and $v(\cdot, \cdot, \omega)$ satisfies the estimates

$$\|v(\cdot, \cdot, \omega)\|_{L^p((0,T), W^{2,p}(\mathbb{R}_+^n))}^p \leq C \|\tilde{f}(\cdot, \cdot, \omega)\|_{L^p((0,T) \times \mathbb{R}_+^n)}^p,$$

where the constant C depends only on κ, K, n, p and T , but not on ω . Thus, one has $v \in \mathcal{W}_{\circ}^{2,p}(\mathbb{R}_+^n)$, and taking mathematical expectation on the above estimate yields

$$\|v\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)}^p \leq C \|\tilde{f}\|_{\mathbb{L}^p(\mathbb{R}_+^n)}^p. \quad (3.8)$$

Now applying the Itô–Wentzell formula obtained in [18] to $\tilde{u}(t, x) := v(t, x + \xi_t)$, one can check that $\tilde{u} \in \mathcal{W}_{\circ}^{2,p}(\mathbb{R}_+^n)$, and it solves the problem

$$d\tilde{u} = (a^{ij} \tilde{u}_{x^i x^j} + \tilde{f}) dt + \sigma^{ik} \tilde{u}_{x^i} dw_t^k, \quad \tilde{u}|_{t=0} = 0, \quad \tilde{u}|_{x^1=0} = 0, \quad (3.9)$$

and satisfies the estimate

$$\|\tilde{u}\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} = \|v\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} \leq C \|\tilde{f}\|_{\mathbb{L}^p(\mathbb{R}_+^n)}. \quad (3.10)$$

On the other hand, we remark that $\tilde{u} \in \mathcal{W}_{\circ}^{2,p}(\mathbb{R}_+^n)$ is a solution to (3.9) if and only if $v(t, x) = \tilde{u}(t, x - \xi_t)$ is the solution to (3.7); as the latter has a unique solution, the solution of (3.9) is also unique.

Define $u = \hat{u} + \tilde{u} \in \mathcal{W}_{\circ}^{2,p}(\mathbb{R}_+^n)$. It follows from Eqs. (3.4) and (3.9) that

$$du = [a^{ij} u_{x^i x^j} + (K \delta^{ij} - a^{ij}) \hat{u}_{x^i x^j} + \tilde{f}] dt + (\sigma^{ik} u_{x^i} + g^k) dw_t^k,$$

where δ^{ij} is the Kronecker delta. With

$$\tilde{\tilde{f}} = f - (K \delta^{ij} - a^{ij}) \hat{u}_{x^i x^j}, \quad (3.11)$$

it is seen that u is a solution to problem (3.1), so the existence part is proved. Moreover, from estimates (3.6) and (3.10), the obtained u satisfies

$$\begin{aligned} \|u\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} &\leq \|\hat{u} + \tilde{u}\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} \\ &\leq C (\|g\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n; \ell^2)} + \|f - (K \delta^{ij} - a^{ij}) \hat{u}_{x^i x^j}\|_{\mathbb{L}^p(\mathbb{R}_+^n)}) \\ &\leq C (\|f\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n; \ell^2)} + \|\hat{u}_{xx}\|_{\mathbb{L}^p(\mathbb{R}_+^n)}) \\ &\leq C (\|f\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n; \ell^2)}), \end{aligned} \quad (3.12)$$

where $C = C(\kappa, K, n, p, T)$.

To prove the uniqueness, we let $u^* \in \mathcal{W}_+^{2,p}(\mathbb{R}_+^n)$ be any solution of (3.1), \hat{u} be the solution of (3.4) determined by g . Then $u^* - \hat{u}$ satisfies (3.9) with \tilde{f} given by (3.11), so by uniqueness (for problem (3.9)) we have $u^* - \hat{u} = \tilde{u}$, which means $u = u^*$. The uniqueness part is also proved, and the estimate (3.2) follows from (3.12) immediately. The proof is complete. \square

3.2. A priori estimates

In the following result we obtain a priori estimates for linear equations in general domains. We adapt the technique of straightening (the boundary) and partitioning (the domain) from PDE theory (cf. [9,17]). The new difficulties here are due to the compatibility conditions and the (possible) unbounded domains. Recall that $u \in \mathcal{W}_+^{2,p}(G)$ implies $u(t, \cdot) = 0$ on the boundary ∂G .

Proposition 3.2. *Let $G \in \mathcal{C}^2$ and Assumptions 1.4 and 2.3 be satisfied. Suppose that $u \in \mathcal{W}_+^{2,p}(G)$ with $u(0, \cdot) = 0$ satisfies the equation*

$$du = (a^{ij}u_{x^i x^j} + f)dt + (\sigma^{ik}u_{x^i} + g^k)dw_t^k \quad (3.13)$$

for some $f \in \mathbb{L}^p(G)$ and $g \in \mathbb{W}_+^{1,p}(G)$. Then we have

$$\|u\|_{\mathbb{W}^{2,p}(G)} \leq C(\|f\|_{\mathbb{L}^p(G)} + \|g\|_{\mathbb{W}^{1,p}(G;\ell^2)}), \quad (3.14)$$

where the constant C depends only on $\kappa, K, n, p, T, K_0, \rho_0, L$, and the functions $\varpi(\cdot)$.

Let us do some preparation before proving the above result.

Fix a $z \in \partial G$ and take the objects associated with z from Definition 2.1. For a function h defined in $B_{\rho_0}(z) \cap G$, we introduce

$$\tilde{h}(y) = h \circ \psi^{-1}(y) = h(\psi^{-1}(y)) \quad \forall y \in U_+^z.$$

Obviously, $h(x) = \tilde{h} \circ \psi(x)$. In what follows, we keep the relation

$$y = \psi(x) \quad \text{for } x \in B_{\rho_0}(z) \cap G,$$

which implies that $h(x) = \tilde{h}(y)$.

For the sake of convenience, we denote $h_i = \partial_i h$ in this subsection to be the partial derivative of a function u with respect to the i th spatial variable. Then for $h \in W^{2,p}(B_{\rho_0}(z) \cap G)$ we have

$$\begin{aligned} h_i(x) &= \psi_i^r(x) \tilde{h}_r(y), \\ h_{ij}(x) &= \psi_i^r(x) \psi_j^s(x) \tilde{h}_{rs}(y) + \psi_{ij}^r(x) \tilde{h}_r(y). \end{aligned}$$

The following result is taken from Lemma 8.3.4 in [17].

Lemma 3.3. *$h \in W^{k,p}(B_{\rho_0}(z) \cap G)$ if and only if $\tilde{h} \in W^{k,p}(U_+^z)$ for $k = 0, 1, 2$. Moreover,*

$$C^{-1} \|h\|_{W^{k,p}(B_{\rho_0}(z) \cap G)} \leq \|\tilde{h}\|_{W^{k,p}(U_+^z)} \leq C \|h\|_{W^{k,p}(B_{\rho_0}(z) \cap G)}$$

with $C = C(n, p, K_0)$.

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $|x| \leq \rho_0/2$ and $\eta(x) = 0$ for $|x| \geq 3\rho_0/4$. With $y = \psi(x)$ (only for $x \in B_{\rho_0}(z) \cap G$) we define

$$\begin{aligned} \eta^z(x) &= \eta(x - z), \quad \tilde{\eta}^z(y) = \eta^z(x), \\ \tilde{a}^{rs}(t, y) &= a^{ij}(t, x) \psi_i^r(x) \psi_j^s(x) \tilde{\eta}^z(y) + K \delta^{rs} [1 - \tilde{\eta}(y)], \\ \tilde{\sigma}^{rk}(t, y) &= \sigma^{ik}(t, x) \psi_i^r(x) \tilde{\eta}(y). \end{aligned}$$

Formally speaking, $\tilde{a}^{rs}(t, y)$ and $\tilde{\sigma}^{rk}(t, y)$ are not defined for $y \notin \bar{U}^z$, but we may set $\tilde{\eta}^z(y) = 0$ for those y and the corresponding terms to be zero, then $\tilde{a}^{rs}(t, y)$ and $\tilde{\sigma}^{rk}(t, y)$ are well-defined for all $y \in \mathbb{R}_+^n$. From Lemma 8.3.6 in [17], we have

Lemma 3.4. (i) For any $y, y_1, y_2 \in \mathbb{R}_+^n$ and $(t, \omega) \in [0, T] \times \Omega$,

$$\begin{aligned} |\tilde{a}^{rs}(t, y)| + \|\tilde{\sigma}^{r\cdot}(t, y)\|_{\ell^2} &\leq \tilde{K}(n, K, K_0), \\ |\tilde{a}^{rs}(t, y_1) - \tilde{a}^{rs}(t, y_2)| &\leq \tilde{\omega}(|y_1 - y_2|), \\ \|\tilde{\sigma}^{r\cdot}(t, y_1) - \tilde{\sigma}^{r\cdot}(t, y_2)\|_{\ell^2} &\leq \tilde{L}(n, K, K_0, L), \end{aligned}$$

where $\tilde{\omega}(\cdot)$ is a modulus of continuity determined only by $\varpi(\cdot)$, n , K and K_0 .

(ii) There is a constant $\tilde{\kappa} = \tilde{\kappa}(n, \kappa, K_0) > 0$ such that

$$(2\tilde{a}^{rs}(t, y) - \tilde{\sigma}^{rk}(t, y)\tilde{\sigma}^{sk}(t, y))\xi^i\xi^j \geq \tilde{\kappa}|\xi|^2$$

for all $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$ and all $(t, y) \in [0, T] \times \mathbb{R}_+^n$.

Now we are in a position to prove Proposition 3.2.

Proof of Proposition 3.2. Let $\rho \in (0, \rho_0 \wedge 1]$ be a constant to be specified later, and take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 1$ for $|x| \leq \rho/4$ and $\zeta(x) = 0$ for $|x| \geq \rho/2$. Set

$$\zeta^z(x) = \zeta(x - z) \quad \text{and} \quad \tilde{\zeta}^z(y) = \zeta^z(x) = \zeta^z(\psi^{-1}(y)). \quad (3.15)$$

It is easily checked that $\rho|\zeta_x| + \rho^2|\zeta_{xx}| \leq C(n)$.

Let $u \in \mathcal{W}_o^{2,p}(G)$ be a solution to Eq. (3.13) with $u(0, \cdot) = 0$. Define

$$\tilde{u}^z(t, y) = \begin{cases} \tilde{\zeta}^z(y)u(t, x) & \text{for } y \in \bar{U}_+^z, \\ 0 & \text{elsewhere.} \end{cases} \quad (3.16)$$

A direct computation gives that the function $\tilde{u}^z \in \mathcal{W}_o^{2,p}(\mathbb{R}_+^n)$, whose support lies in $\psi(B_{\rho/2}(z)) \cap \bar{U}_+^z$, satisfies the following equation

$$\begin{aligned} d\tilde{u}^z(t, y) &= (\tilde{a}^{rs}(t, 0)\tilde{u}_{rs}^z(t, y) + \hat{f}^z(t, y))dt + (\tilde{\sigma}^{rk}(t, 0)\tilde{u}_r^z(t, y) + \hat{g}^{z,k}(t, y))dw_t^k \\ \tilde{u}^z|_{t=0} &= 0, \quad \tilde{u}^z|_{y^1=0} = 0, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} \hat{f}^z(t, y) &= [\tilde{a}^{rs}(t, y) - \tilde{a}^{rs}(t, 0)]\tilde{u}_{rs}^z(t, y) + \tilde{\zeta}^z(y)f(t, x) - \tilde{a}^{rs}(t, y)\tilde{\zeta}_s^z(y)\tilde{u}(t, y) \\ &\quad - \tilde{a}^{rs}(t, y)\tilde{\zeta}_s^z(y)\tilde{u}_r(t, y) + a^{ij}(t, x)\psi_{ij}^r(x)\tilde{\zeta}^z(y)\tilde{u}_r(t, y), \\ \hat{g}^{z,k}(t, y) &= [\tilde{\sigma}^{rk}(t, y) - \tilde{\sigma}^{rk}(t, 0)]\tilde{u}_r^z(t, y) + \tilde{\zeta}^z(y)g^k(t, x) - \tilde{\sigma}^{rk}(t, y)\tilde{\zeta}_r^z(y)\tilde{u}^z(t, y). \end{aligned} \quad (3.18)$$

To apply Proposition 3.2 to Eq. (3.17), we need to verify the following conditions:

$$\tilde{\sigma}^{1\cdot}(t, 0, y') = 0 \quad \forall y' \in \mathbb{R}_+^{n-1}, \quad (3.19)$$

$$\hat{f}^z \in \mathbb{L}^p(\mathbb{R}_+^n) \quad \text{and} \quad \hat{g}^z \in \mathbb{W}_o^{1,p}(\mathbb{R}_+^n). \quad (3.20)$$

To check (3.19), we notice that, from Definition 2.1, the equation of the surface $B_{\rho_0}(z) \cap \partial G$ is $\psi^1(x) = 0$, so $\partial\psi^1(x)$ is a normal vector of ∂G at $x \in B_{\rho_0}(z) \cap \partial G$. Thanks to Assumption 1.4, one has that for $x \in B_{\rho_0}(z) \cap \partial G$,

$$0 = \partial\psi^1(x) \cdot \sigma^{\cdot k}(t, x) = \sigma^{rk}(t, x)\partial_r\psi^1(x) = \tilde{\sigma}^{1k}(t, \psi(x)). \quad (3.21)$$

Also notice that $\tilde{\sigma}^{1k}(t, \cdot) = 0$ outside \bar{U}_+^z . So (3.19) is valid.

To check (3.20), one can use Lemmas 3.3 and 3.4 to obtain that $\hat{f}^z \in \mathbb{L}^p(\mathbb{R}_+^n)$, $\hat{g}^z \in \mathbb{W}^{1,p}(\mathbb{R}_+^n)$, and

$$\begin{aligned} \|\hat{f}^z\|_{\mathbb{L}^p(\mathbb{R}_+^n)} &\leq C\tilde{\omega}(\rho)\|\tilde{u}_{yy}^z\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + C\{\|\tilde{\zeta}^z\tilde{f}\|_{\mathbb{L}^p(\mathbb{R}_+^n)} \\ &\quad + \|(\tilde{\zeta}_{yy}^z\tilde{u}, \tilde{\zeta}_y^z\tilde{u}_y, \tilde{\zeta}^z\tilde{u}_y)\|_{\mathbb{L}^p(\mathbb{R}_+^n)}\} \\ &\leq C\tilde{\omega}(\rho)\|u_{xx}^z\|_{\mathbb{L}^p(G)} + C\{\|\zeta^z f\|_{\mathbb{L}^p(G)} \\ &\quad + \|(\zeta_{xx}^z u, \zeta_x^z u, \zeta^z u, \zeta_x^z u_x, \zeta^z u_x)\|_{\mathbb{L}^p(G)}\} \\ &\leq C\tilde{\omega}(\rho)\|u_{xx}\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + C\{\|f\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} \\ &\quad + \rho^{-2}\|u\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G)}\}, \\ \|\hat{g}^z\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n;\ell^2)} &\leq C\tilde{L}\rho\|\tilde{u}_{yy}^z\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + C\{\|\tilde{\zeta}^z\tilde{g}\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n;\ell^2)} + \|\tilde{\zeta}_y^z\tilde{u}\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n)}\} \\ &\leq C\tilde{L}\rho\|u_{xx}\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + C\{\|g\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G;\ell^2)} \\ &\quad + \rho^{-2}\|u\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G)}\} \end{aligned}$$

with $C = C(K, n, p, K_0, \rho_0, L)$ independent of ρ , where $\tilde{\omega}(\cdot)$ and \tilde{L} are taken from Lemma 3.4. It remains to check $\hat{g}^z \in \mathbb{W}_0^{1,p}(\mathbb{R}_+^n)$. This immediately follows from some basic facts in real analysis:

Lemma 3.5. *Let h and φ be functions defined \mathbb{R}_+^n . Then we have*

- (a) *if $h \in W_0^{1,p}(\mathbb{R}_+^n)$ and $\varphi \in C^{0,1}(\bar{\mathbb{R}}_+^n)$, then $\varphi h \in W_0^{1,p}(\mathbb{R}_+^n)$;*
- (b) *if $h \in W^{1,p}(\mathbb{R}_+^n)$ and $\varphi \in C_0^{0,1}(\bar{\mathbb{R}}_+^n)$, then $\varphi h \in W_0^{1,p}(\mathbb{R}_+^n)$;*
- (c) *if $h \in W_0^{2,p}(\mathbb{R}_+^n)$, then $h_{x_i} \in W_0^{1,p}(\mathbb{R}_+^n)$ for $i = 2, \dots, n$,*

where $C^{0,1}(\bar{\mathbb{R}}_+^n)$ is the space of all uniformly Lipschitz continuous functions defined on $\bar{\mathbb{R}}_+^n$, and its subset $C_0^{0,1}(\bar{\mathbb{R}}_+^n)$ collects those functions that vanish on the boundary $\{x^1 = 0\}$.

Now we use the above lemma to verify $\hat{g}^z \in \mathbb{W}_0^{1,p}(\mathbb{R}_+^n)$. By the assertion (a), it is easily seen the last two terms in the expression (3.18) of \hat{g}^z belong to $\mathbb{W}_0^{1,p}(\mathbb{R}_+^n; \ell^2)$. Assumption 2.3 and the condition $\tilde{\sigma}^1(t, 0, y') = 0$ checked above imply that $\tilde{\sigma}^1(\cdot, \cdot) \in C_0^{0,1}(\bar{\mathbb{R}}_+^n; \ell^2)$ uniformly with respect to (t, ω) , which along with $\tilde{u}_{y_1}^z \in W^{1,p}(\mathbb{R}_+^n)$ yields $[\tilde{\sigma}^1 - \tilde{\sigma}^1(\cdot, 0)]\tilde{u}_1^z \in \mathbb{W}_0^{1,p}(\mathbb{R}_+^n; \ell^2)$ by means of the assertion (b). Moreover, because $\tilde{u}^z \in W_0^{2,p}(\mathbb{R}_+^n; \ell^2)$ and $\tilde{\sigma}^{i\cdot}(t, \cdot) \in C^{0,1}(\bar{\mathbb{R}}_+^n; \ell^2)$, it follows from the assertion (c) that $[\tilde{\sigma}^{i\cdot} - \tilde{\sigma}^{i\cdot}(\cdot, 0)]\tilde{u}_i^z \in \mathbb{W}_0^{1,p}(\mathbb{R}_+^n; \ell^2)$ for $i = 2, \dots, n$. Therefore, we have $\hat{g}^z \in \mathbb{W}_0^{1,p}(\mathbb{R}_+^n)$.

The facts (3.19) and (3.20) along with Lemma 3.4 ensure us to apply Proposition 3.2 to Eq. (3.17) to get the estimate

$$\begin{aligned} \|u\|_{\mathbb{W}^{2,p}(B_{\rho/4}(z)\cap G)} &\leq \|u^z\|_{\mathbb{W}^{2,p}(B_{\rho/2}(z)\cap G)} \leq C\|\tilde{u}^z\|_{\mathbb{W}^{2,p}(\mathbb{R}_+^n)} \\ &\leq C(\|\hat{f}^z\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|\hat{g}^z\|_{\mathbb{W}^{1,p}(\mathbb{R}_+^n;\ell^2)}) \\ &\leq C(\tilde{\omega}(\rho) + \tilde{L}\rho)\|u_{xx}\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + C\rho^{-2}\|u\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G)} \\ &\quad + C\{\|f\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + \|g\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G;\ell^2)}\}, \end{aligned}$$

where $C = C(\kappa, K, n, p, T, K_0, \rho_0, L)$. By interpolation, we have

$$\|u_x\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} \leq \rho^3\|u_{xx}\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + C(n)\rho^{-3}\|u\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)}.$$

Combining the last two inequalities, we obtain

$$\begin{aligned} \|u\|_{\mathbb{W}^{2,p}(B_{\rho/4}(z)\cap G)} &\leq C(\tilde{\omega}(\rho) + \tilde{L}\rho)\|u_{xx}\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + C\rho^{-5}\|u\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} \\ &\quad + C\{\|f\|_{\mathbb{L}^p(B_{\rho/2}(z)\cap G)} + \|g\|_{\mathbb{W}^{1,p}(B_{\rho/2}(z)\cap G;\ell^2)}\}. \end{aligned} \quad (3.22)$$

Now we define the narrow area near the boundary ∂G :

$$G_r = \{x \in G : \text{there is an } \bar{x} \in \partial G \text{ such that } |x - \bar{x}| < r\}.$$

Lemma 3.6. *There exist countable points $z_1, z_2, \dots \in \partial G$ satisfying the following properties:*

1. $|z_i - z_j| \geq \rho/8$ for $i \neq j$, and the whole ∂G is covered by $\cup_i B_{\rho/8}(z_i)$;
2. any $x \in G_{\rho/8}$ lies in at least one $B_{\rho/4}(z_i)$;
3. any $x \in G_{\rho/2}$ is covered by at most $N(n)$ balls from $\{B_{\rho/2}(z_i)\}$, where $N(n)$ is the greatest number of such points in B_1 that any two of them are over $1/4$ apart.

Now we postpone the proof of this lemma to the end of this subsection and move on the proof of Proposition 3.2. From this lemma it follows that

$$\begin{aligned} \|u\|_{\mathbb{W}^{2,p}(G_{\rho/8})} &\leq \sum_i \|u\|_{\mathbb{L}^p(B_{\rho/2}(z_i)\cap G)} \leq \sum_i \|u\|_{\mathbb{W}^{2,p}(B_{\rho/4}(z_i)\cap G)} \\ &\leq N(n)\|u\|_{\mathbb{W}^{2,p}(G_{\rho/2})} \leq N(n)\|u\|_{\mathbb{W}^{2,p}(G)}, \end{aligned}$$

which along with the estimate (3.22) yields that

$$\|u\|_{\mathbb{W}^{2,p}(G_{\rho/8})} \leq C(\tilde{\omega}(\rho) + \tilde{L}\rho)\|u_{xx}\|_{\mathbb{L}^p(G)} + C\{\rho^{-5}\|u\|_{\mathbb{L}^p(G)} + \|f\|_{\mathbb{L}^p(G)} + \|g\|_{\mathbb{W}^{1,p}(G;\ell^2)}\} \quad (3.23)$$

with $C = C(\kappa, K, n, p, T, K_0, \rho_0, L)$.

To obtain the estimate in $G^{\rho/8} := G \setminus G_{\rho/8}$, we write $\bar{\zeta}(x) = \zeta(4x)$ and define a cut-off function $\zeta_0 = \bar{\zeta} * \mathbf{1}_{G^{\rho/8}}$. For a solution $u \in \mathcal{W}_o^{2,p}(G)$ of Eq. (3.13), the function $u^0 = \zeta_0 u \in \mathcal{W}_o^{2,p}(\mathbb{R}^n)$, whose support lies in \bar{G} , satisfies the following equation

$$du^0 = (a^{ij}u_{x^i x^j}^0 + f^0)dt + (\sigma^{ik}u_{x^i}^0 + g^{0,k})dw_t^k, \quad u^0(0, \cdot) = 0,$$

on $(0, T] \times \mathbb{R}^n$, where

$$f^0 = \zeta_0 f - a^{ij}(\zeta_0)_{x^i x^j} u - a^{ij}(\zeta_0)_{x^i} u_{x^j}, \quad g^{0,k} = \zeta_0 g^k - \sigma^{ik}(\zeta_0)_{x^i} u^0.$$

Thanks to the L^p -theory of SPDEs in the whole space (cf. Theorem 5.1 in [15]), we have the estimate

$$\begin{aligned} \|u\|_{\mathbb{W}^{2,p}(G^{\rho/8})} &\leq \|u^0\|_{\mathbb{W}^{2,p}(\mathbb{R}^n)} \leq C(\|f^0\|_{\mathbb{L}^p(\mathbb{R}^n)} + \|g^0\|_{\mathbb{W}^{1,p}(\mathbb{R}^n;\ell^2)}) \\ &\leq C(\rho^{-2}\|u\|_{\mathbb{W}^{1,p}(G)} + \|f^0\|_{\mathbb{L}^p(\mathbb{R}^n)} + \|g^0\|_{\mathbb{W}^{1,p}(\mathbb{R}^n;\ell^2)}) \\ &\leq C\rho\|u_{xx}\|_{\mathbb{L}^p(G)} + C(\rho^{-5}\|u\|_{\mathbb{L}^p(G)} + \|f\|_{\mathbb{L}^p(G)} + \|g\|_{\mathbb{W}^{1,p}(G;\ell^2)}), \end{aligned} \quad (3.24)$$

where $C = C(\kappa, K, n, p, T, L, \varpi)$.

Combining the estimates (3.23) and (3.24), we can choose a small number $\rho = \rho(\kappa, K, n, p, T, L, \varpi) \in (0, \rho_0 \wedge 1]$ such that

$$\|u\|_{\mathbb{W}^{2,p}(G)} \leq \frac{1}{2}\|u_{xx}\|_{\mathbb{L}^p(G)} + C(\|u\|_{\mathbb{L}^p(G)} + \|f\|_{\mathbb{L}^p(G)} + \|g\|_{\mathbb{W}^{1,p}(G;\ell^2)}),$$

which yields

$$\|u\|_{\mathbb{W}^{2,p}(G)} \leq C(\|u\|_{\mathbb{L}^p(G)} + \|f\|_{\mathbb{L}^p(G)} + \|g\|_{\mathbb{W}^{1,p}(G;\ell^2)}). \quad (3.25)$$

It remains to estimate $\|u\|_{\mathbb{L}^p(G)}$. Applying Itô's formula to $e^{-\lambda t}|u(t, x)|^p$ and integrating on $G \times [0, s] \times \Omega$, we have

$$\begin{aligned} & e^{-\lambda T} \mathbf{E} \|u(T, \cdot)\|_{L^p(G)}^p + \lambda \mathbf{E} \int_0^T e^{-\lambda t} \|u(t, \cdot)\|_{L^p(G)}^p dt \\ &= p \mathbf{E} \int_0^T \int_G e^{-\lambda t} |u(t, x)|^{p-2} u(t, x) [a^{ij}(t, x) u_{x^i x^j}(t, x) + f(t, x)] dx dt \\ & \quad + \frac{1}{2} p(p-1) \mathbf{E} \int_0^T \int_G e^{-\lambda t} |u(t, x)|^{p-2} \|\sigma^{i\cdot}(t, x) u_{x^i}(t, x) + g(t, x)\|_{\ell^2}^2 dx dt \\ &\leq \varepsilon \mathbf{E} \int_0^T \|u_{xx}(t, \cdot)\|_{L^p(G)}^p dt + C(\varepsilon, p, K, T) \mathbf{E} \int_0^T e^{-\lambda t} \|u(t, \cdot)\|_{L^p(G)}^p dt \\ & \quad + C(p, T) (\|f\|_{\mathbb{L}^p(G)}^p + \|g\|_{\mathbb{L}^p(G; \ell^2)}^p). \end{aligned} \quad (3.26)$$

Letting $\lambda = 1 + C(\varepsilon, p, K, T)$, one can get that

$$\|u\|_{\mathbb{L}^p(G)}^p \leq \varepsilon C(p, K, T) \|u_{xx}\|_{\mathbb{L}^p(G)}^p + C(\varepsilon, p, K, T) (\|f\|_{\mathbb{L}^p(G)}^p + \|g\|_{\mathbb{L}^p(G; \ell^2)}^p). \quad (3.27)$$

Selecting $\varepsilon > 0$ sufficiently small, the above estimate along with (3.25) yields the desired estimate (3.14), so the proof of Proposition 3.2 is complete. \square

With non-homogeneous initial value condition, we have the following result.

Corollary 3.7. *Let $G \in \mathcal{C}^2$ and Assumptions 1.4 and 2.3 be satisfied. Suppose that for any $f \in \mathbb{L}^p(G)$ and $g \in \mathbb{W}_o^{1,p}(G)$ there exists a unique solution in $\mathcal{W}_o^{2,p}(G)$ to Eq. (3.13) with zero initial-boundary condition. Then for any given $f \in \mathbb{L}^p(G)$, $g \in \mathbb{W}_o^{1,p}(G)$, and*

$$u_0(\cdot) \in L^p(\Omega, \mathcal{F}_0, W_o^{2-2/p,p}(G)),$$

Eq. (3.13) with the initial-boundary condition (2.2) also admits a unique solution $u \in \mathcal{W}_o^{2,p}(G)$, and this solution satisfies

$$\|u\|_{\mathbb{W}^{2,p}(G)}^p \leq C (\|f\|_{\mathbb{L}^p(G)}^p + \|g\|_{\mathbb{W}^{1,p}(G; \ell^2)}^p + \mathbf{E} \|u(0, \cdot)\|_{W^{2-2/p,p}(G)}^p), \quad (3.28)$$

where the constant C depends only on $\kappa, K, n, p, T, K_0, \rho_0, L$, and the functions $\varpi(\cdot)$.

Proof. From Theorem IV.9.1 in [21], the heat equation

$$\partial_t V = \Delta V \text{ on } (0, T] \times G; \quad V(t, \cdot)|_{\partial G} = 0; \quad V(0, \cdot) = u(0, \cdot) \text{ on } G$$

has a unique strong solution $V(\cdot, \cdot, \omega) \in L^p((0, T), W_o^{2,p}(G))$ for each ω , and

$$\|V\| \leq C(n, p, K_0, \rho_0, T) \mathbf{E} \|u(0, \cdot)\|_{W^{2-2/p,p}(G)}^p. \quad (3.29)$$

On the other hand, from the assumptions the following equation

$$\begin{aligned} dU &= [a^{ij} U_{x^i x^j} + f + (a^{ij} - \delta^{ij}) V_{x^i x^j}] dt + (\sigma^{ik} U_{x^i} + g^k + \sigma^{ik} V_{x^i}) dw_t^k, \\ U|_{\partial G} &= U(0, \cdot) = 0 \end{aligned} \quad (3.30)$$

has a unique solution $U \in \mathcal{W}_o^{2,p}(G)$, and from Proposition 3.2 we have

$$\begin{aligned} \|U\|_{\mathbb{W}^{2,p}(G)}^p &\leq C (\|f + (a^{ij} - \delta^{ij}) V_{x^i x^j}\|_{\mathbb{L}^p(G)}^p + \|g + \sigma^{i\cdot} V_{x^i}\|_{\mathbb{W}^{1,p}(G; \ell^2)}^p) \\ &\leq C (\|f\|_{\mathbb{L}^p(G)}^p + \|g\|_{\mathbb{W}^{1,p}(G; \ell^2)}^p + \|V\|_{\mathbb{W}^{2,p}(G)}^p) \\ &\leq C (\|f\|_{\mathbb{L}^p(G)}^p + \|g\|_{\mathbb{W}^{1,p}(G; \ell^2)}^p + \mathbf{E} \|u(0, \cdot)\|_{W^{2-2/p,p}(G)}^p). \end{aligned}$$

Obviously, the function $u = U + V \in \mathcal{W}_o^{2,p}(G)$ solves Eq. (3.13) with condition (2.2), and (3.28) immediately follows from the above estimates for U and V . The uniqueness also holds true, otherwise we can construct different solutions of (3.30) from different solutions of Eq. (3.13) with (2.2) (with the help of V), which contradicts to the assumptions. The proof is complete. \square

Proof of Lemma 3.6. For convenience, we say $\{z_1, z_2, \dots\}$ is a proper $\rho/8$ -covering set of $E \subset \mathbb{R}^n$ if $z_i \in E$, $|z_i - z_j| \geq \rho/8$ for $i \neq j$, and E is covered by $\cup_i B_{\rho/8}(z_i)$.

If G is bounded, then ∂G is a compact subset in \mathbb{R}^n , so there are finite points $\{z_1, \dots, z_N\} \subset \partial G$ such that $\partial G \subset \cup_i B_{\rho/8}(z_i)$, but it is not necessary that $|z_i - z_j| \geq \rho/8$ for any $i \neq j$. Now we adjust the choice of points z_i as follows: in the i th step, we check whether $B_{\rho/8}(z_i) \subset \cup_{j \neq i} B_{\rho/8}(z_j)$: if yes, then remove this z_i from the set; if no, then $E_i := \partial G \setminus \cup_{j \neq i} B_{\rho/8}(z_j)$ is nonempty and covered by $B_{\rho/8}(z_i)$, so we can pick one point $z'_i \in E_i$ or two $z'_i, z''_i \in E_i$ with $|z'_i - z''_i| \geq \rho/8$ such that E_i is covered by $B_{\rho/8}(z'_i)$ or $B_{\rho/8}(z'_i) \cup B_{\rho/8}(z''_i)$, and replace z_i by z'_i or the pair (z'_i, z''_i) . After N steps one obtains a finite proper $\rho/8$ -covering set of ∂G .

If G is unbounded, we fix a large number $R > 0$ and denote $\Gamma_k = \partial G \cap B_{kR}(0)$. Repeating the argument as above one can find a sequence of finite sets A_1, A_2, \dots inductively such that A_1 is a finite proper $\rho/8$ -covering set of Γ_1 , and A_k with $k \geq 2$ is a finite proper $\rho/8$ -covering set of $\Gamma_k \setminus D_{k-1}$, where $D_{k-1} = \cup\{B_{\rho/8}(z) : z \in \cup_{i=1}^{k-1} A_i\}$. It is easily seen that $A := \cup_{i=1}^{\infty} A_i$ is a finite proper $\rho/8$ -covering set of ∂G .

Next we prove that the set A has the second property. For $x \in G_{\rho/8}$ there is an $\bar{x} \in \partial G$ such that $|x - \bar{x}| < \rho/8$. Meanwhile, there is a point $z \in A$ such that $\bar{x} \in B_{\rho/8}(z)$. Hence, $|x - z| \leq |x - \bar{x}| + |\bar{x} - z| \leq \rho/4$, which means $x \in B_{\rho/4}(z)$. Finally, for $x \in G_{\rho/2}$ the ball $B_{\rho/2}(x)$ contains at most $N(n)$ points from the set A according to the definition of $N(n)$, which implies the last property. The proof is complete. \square

3.3. Existence and uniqueness

We start from the solvability of stochastic heat equations. In view of Corollary 3.7, we can just focus on the homogeneous Dirichlet boundary value problem.

Lemma 3.8. Let $G \in \mathcal{C}^2$. Then for given $f \in \mathbb{L}^p(G)$ and $g \in \mathbb{W}_o^{1,p}(G)$, the equation

$$du = (\Delta u + f) dt + g^k dw_t^k \quad \text{on } (0, T] \times G \quad (3.31)$$

with zero initial-boundary condition has a unique solution $u \in \mathcal{W}_o^{2,p}(G)$.

Proof. The uniqueness follows from the estimate (3.14). For the existence we adopt an approximation strategy from the proof of Theorem 2.9 in [11]. It is well-known that $C_0^\infty(G)$ is a dense subset of $W_o^{1,p}(G)$. We can approximate $g = (g^1, g^2, \dots) \in \mathbb{W}_o^{1,p}(G; \ell^2)$ with functions having only finite nonzero entries, bounded on $[0, T] \times G \times \Omega$ along with each derivative of any order, and vanishing near ∂G and the infinity (cf. Theorem 3.17 in [1]). In this case it is known that

$$V(t, x) = \int_0^t g^k(t, x) dw_s^k$$

is infinitely differentiable in x and vanishes near ∂G and the infinity. So we conclude that $V \in \mathcal{W}_o^{2,p}(G)$. Again, from PDE theory, the equation

$$\partial_t U = \Delta U + f + \Delta V, \quad U|_{\partial G} = 0, \quad U(0, \cdot) = 0$$

has a solution U in $\mathcal{W}_\circ^{2,p}(G)$. The solution of (3.31) is then given by $u = U + V \in \mathcal{W}_\circ^{2,p}(G)$. The case of general g can be obtained by approximation by the help of the estimate (3.14). The proof is complete. \square

With the solvability of stochastic heat equation (3.31) and the a priori estimate (3.14) in hand, the existence and uniqueness of solutions to the general linear equation (3.13) immediately follows from the standard method of continuity (cf. [9, Theorem 5.2]). Bearing in mind Corollary 3.7, we have the following result.

Corollary 3.9. *Let $G \in \mathcal{C}^2$ and Assumptions 1.4 and 2.3. Then for any given $f \in \mathbb{L}^p(G)$, $g \in \mathbb{W}_\circ^{1,p}(G)$ and $u_0(\cdot) \in L^p(\Omega, \mathcal{F}_0, W_\circ^{2-2/p,p}(G))$, Eq. (3.13) with the initial-boundary condition (2.2) has a unique solution $u \in \mathcal{W}_\circ^{2,p}(G)$.*

Proof of Theorem 2.5(i) and (ii). The argument is similar to the proof of Theorem 6.4 in [15]. From Assumption 2.4, we know that for any $v \in \mathbb{W}_\circ^{2,p}(G)$,

$$f(\cdot, \cdot, v) \in \mathbb{L}^p(G), \quad g(\cdot, \cdot, v) \in \mathbb{W}_\circ^{1,p}(G; \ell^2).$$

So by Corollary 3.9, the equation

$$du = (a^{ij}u_{x_i x_j} + f(t, x, v))dt + (\sigma^{ik}u_{x_i} + g^k(t, x, v))dw_t^k$$

with condition (2.2) has a unique solution $u \in \mathcal{W}_\circ^{2,p}(G)$.

Define a mapping $\mathcal{T}v = u$. Replacing the terminal time T into any $\tau \leq T$, it follows from the estimate (3.14) and Assumption 2.4 that for $v^1, v^2 \in \mathbb{W}_\circ^{2,p}(G)$,

$$\begin{aligned} \|\mathcal{T}v^1 - \mathcal{T}v^2\|_{\mathbb{W}^{2,p}(G,\tau)}^p &\leq C(\|f(\cdot, \cdot, v^1) - f(\cdot, \cdot, v^2)\|_{\mathbb{L}^p(G,\tau)}^p \\ &\quad + \|g(\cdot, \cdot, v^1) - g(\cdot, \cdot, v^2)\|_{\mathbb{W}^{1,p}(G,\tau;\ell^2)}^p) \\ &\leq C\varepsilon^p \|v^1 - v^2\|_{\mathbb{W}^{2,p}(G,\tau)}^p + CK_\varepsilon^p \int_0^\tau \|v^1(s) - v^2(s)\|_{\mathbb{L}^p(G,s)}^p ds. \end{aligned}$$

From the computation (3.26) (with s instead of T) and Assumption 2.4, we can see that

$$\mathbf{E}\|\mathcal{T}v^1(s) - \mathcal{T}v^2(s)\|_{L^p(G)}^p \leq C\|v^1 - v^2\|_{\mathbb{W}^{2,p}(G,s)}^p$$

with C independent of s . Combining the last two inequalities and letting $C\varepsilon^p = 1/4$ and , we have

$$\|\mathcal{T}v^1 - \mathcal{T}v^2\|_{\mathbb{W}^{2,p}(G,\tau)}^p \leq \frac{1}{4}\|v^1 - v^2\|_{\mathbb{W}^{2,p}(G,\tau)}^p + C \int_0^\tau \|v^1 - v^2\|_{\mathbb{W}^{2,p}(G,s)}^p ds,$$

Then by induction we can compute that for positive integer m ,

$$\begin{aligned} \|\mathcal{T}^m v^1 - \mathcal{T}^m v^2\|_{\mathbb{W}^{2,p}(G)}^p &\leq \|v^1 - v^2\|_{\mathbb{W}^{2,p}(G)}^p \sum_{k=0}^m \binom{\alpha}{k} \frac{4^{k-m}}{k!} (CT)^k \\ &\leq 2^{-m} \|v^1 - v^2\|_{\mathbb{W}^{2,p}(G)}^p \max_{k \geq 0} \frac{(4CT)^k}{k!}. \end{aligned}$$

Choose m sufficiently large so that \mathcal{T}^m is a contraction in $\mathbb{W}_\circ^{2,p}(G)$. Then there is a unique $u \in \mathbb{W}_\circ^{2,p}(G)$ such that $\mathcal{T}^m u = u$, and from Corollary 3.9 we have $u \in \mathcal{W}_\circ^{2,p}(G)$.

Now we derive the estimate (2.3). From Corollary 3.9 and Assumption 2.4 (with a proper choice of ε), we can obtain that

$$\|u\|_{\mathbb{W}^{2,p}(G)}^p \leq C(\|u\|_{\mathbb{L}^p(G)}^p + \|f(\cdot, \cdot, 0)\|_{\mathbb{L}^p(G)}^p + \|g(\cdot, \cdot, 0)\|_{\mathbb{W}^{1,p}(G;\ell^2)}^p + \mathbf{E}\|u_0\|_{W^{2-2/p,p}(G)}^p).$$

The term $\|u\|_{\mathbb{L}^p(G)}^p$ can be eliminated just as we got rid of the same one in (3.25). The assertions (i) and (ii) of Theorem 2.5 are proved.

In the proof of Theorem 2.7 we will need the following result concerning the existence and uniqueness of $W^{1,p}$ -solutions of SPDEs of divergence form. We keep the formulation as the most compact form that can be applied comfortably, and leave the general extension to readers.

Proposition 3.10. *Let Assumptions 1.4 and 2.3 be satisfied with $G = \mathbb{R}_+^n$, and let $c^i \in \mathbb{L}^\infty(\mathbb{R}_+^n)$. Then for any $f^0, F \in \mathbb{L}^p(\mathbb{R}_+^n)$ and $g \in \mathbb{L}^p(G; \ell^2)$, the equation*

$$\begin{aligned} du &= [(a^{ij}u_{x^i})_{x^j} + f^0 + c^i F_{x^i}] dt + (\sigma^{ik}u_{x^i} + g^k) dw_t^k, \\ u|_{\partial G} &= 0, \quad u|_{t=0} = 0 \end{aligned}$$

has a unique solution $u \in \mathcal{W}_\circ^{1,p}(\mathbb{R}_+^n)$, and

$$\|u\|_{\mathcal{W}_\circ^{1,p}(\mathbb{R}_+^n)} \leq C(\|(f^0, F)\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{L}^p(\mathbb{R}_+^n; \ell^2)}); \quad (3.32)$$

where the constant C depends only on $\kappa, K, n, p, T, L, \varpi(\cdot)$ and $\|c^i\|_{\mathbb{L}^\infty}$.

Proof. As above the existence and uniqueness of solutions follows from the (a priori) estimate (3.32) by using the method of continuity and the Banach fixed-point theorem. The proof of (3.32) is similar to but much easier than the derivation of estimate (3.14) because one need not straighten the boundary but just do the computation on the original equation, while the auxiliary estimate for model equations is provided by Proposition 3.2 (ii). We suppress the details here to avoid unnecessary repeating. \square

3.4. Embedding for $\mathcal{W}_\circ^{2,p}(G)$

Let us define the following norm for the space $\mathcal{W}_\circ^{2,p}(G)$ (recall Definition 2.2):

$$\|u\|_{\mathcal{W}_\circ^{2,p}(G)} = \|u_{xx}\|_{\mathbb{L}^p(G)} + \|u_D\|_{\mathbb{L}^p(G)} + \|u_S\|_{\mathbb{W}^{1,p}(G; \ell^2)} + (\mathbf{E}\|u_0\|_{\mathbb{W}^{2-2/p,p}(G)}^p)^{1/p}.$$

Following the proof of Theorem 3.7 in [15], one can prove that $\mathcal{W}_\circ^{2,p}(G)$ is a Banach space with the above norm.

The assertion (iii) of Theorem 2.5 is a direct consequence of the following lemma.

Lemma 3.11. *Let $G \in \mathcal{C}^2$ and $p > 2$. Then for $u \in \mathcal{W}_\circ^{2,p}(G)$ we have*

(a) if $\alpha_0 := \frac{2p-n-2}{2p} > 0$, then for any $\alpha \in (0, \alpha_0)$,

$$\mathbf{E}\|u\|_{C^{\alpha/2, \alpha}([0, T] \times \bar{G})}^p \leq C(n, p, \alpha, K_0, \rho_0, T)\|u\|_{\mathcal{W}_\circ^{2,p}(G)}^p;$$

(b) if $\beta_0 := \frac{p-n-2}{2p} > 0$, then for any $\beta \in (0, \beta_0)$,

$$\mathbf{E}\|u_x\|_{C^{\beta/2, \beta}([0, T] \times \bar{G})}^p \leq C(n, p, \beta, K_0, \rho_0, T)\|u\|_{\mathcal{W}_\circ^{2,p}(G)}^p.$$

Proof. When $G = \mathbb{R}^n$ this lemma is a simple consequence of Theorem 7.2 in [15] by means of Sobolev embedding.

For $G = \mathbb{R}_+^n$ it suffices to show that the odd extension of u (see (3.5)) lies in $\mathcal{W}_\circ^{2,p}(\mathbb{R}^n)$. Indeed, we set $f = u_D - \Delta u \in \mathbb{L}^p(\mathbb{R}_+^n)$ and $g = u_S \in \mathbb{W}_\circ^{1,p}(\mathbb{R}_+^n; \ell^2)$, then

$$du = (\Delta u + f) dt + g^k dw_t^k. \quad (3.33)$$

We continue u_0 , f and g to be odd functions of x^1 , and solve the above equation with initial data u_0 in the whole space \mathbb{R}^n . By our solvability results, the solution of the extended equation is the odd continuation of u and belongs to $\mathcal{W}_o^{2,p}(\mathbb{R}^n)$.

Finally, we consider the case of general $G \in \mathcal{C}^2$. For $u \in \mathcal{W}_o^{2,p}(G)$, and define \tilde{u}^z , \tilde{u}_D^z and \tilde{u}_S^z in the spirit of (3.16) for any $z \in \partial G$. Evidently, $\tilde{u}^z \in \mathcal{W}_o^{2,p}(\mathbb{R}_+^n)$ with $d\tilde{u}^z = \tilde{u}_D^z dt + \tilde{u}_S^{z,k} dw_t^k$. Bearing in mind the assertion for \mathbb{R}_+^n , a direct computation shows that

$$\begin{aligned} \mathbf{E}\|u\|_{C^{\alpha/2,\alpha}([0,T] \times (\bar{B}_{\rho_0/4}(z) \cap \bar{G}))}^p &\leq \mathbf{E}\|\zeta^z u\|_{C^{\alpha/2,\alpha}([0,T] \times (\bar{B}_{\rho_0/2}(z) \cap \bar{G}))}^p \\ &\leq C(n, p, \alpha, K_0, \rho_0) \mathbf{E}\|\tilde{u}^z\|_{C^{\alpha/2,\alpha}([0,T] \times \mathbb{R}_+^n)}^p \leq C(n, p, \alpha, K_0, \rho_0, T) \|\tilde{u}^z\|_{\mathcal{W}_o^{2,p}(\mathbb{R}_+^n)}^p \\ &\leq C(n, p, \alpha, K_0, \rho_0, T) \|u\|_{\mathcal{W}_o^{2,p}(G)}^p, \\ \mathbf{E}\|u_x\|_{C^{\beta/2,\beta}([0,T] \times (\bar{B}_{\rho_0/4}(z) \cap \bar{G}))}^p &\leq \mathbf{E}\|(\zeta^z u)_x\|_{C^{\beta/2,\beta}([0,T] \times (\bar{B}_{\rho_0/2}(z) \cap \bar{G}))}^p \\ &\leq C(n, p, \beta, K_0, \rho_0) \mathbf{E}\|\partial \tilde{u}^z\|_{C^{\beta/2,\beta}([0,T] \times \mathbb{R}_+^n)}^p \leq C(n, p, \beta, K_0, \rho_0) \|\tilde{u}^z\|_{\mathcal{W}_o^{2,p}(\mathbb{R}_+^n)}^p \\ &\leq C(n, p, \beta, K_0, \rho_0) \|u\|_{\mathcal{W}_o^{2,p}(G)}^p. \end{aligned}$$

For $z \in G^{\rho_0/4} = G \setminus G_{\rho_0/4}$ the estimate is much simpler:

$$\begin{aligned} \mathbf{E}\|u\|_{C^{\alpha/2,\alpha}([0,T] \times \bar{B}_{\rho_0/8}(z))}^p &\leq \mathbf{E}\|\eta^z u\|_{C^{\alpha/2,\alpha}([0,T] \times \mathbb{R}^n)}^p \leq C\|\eta^z u\|_{\mathcal{W}_o^{2,p}(\mathbb{R}^n)}^p \leq C\|u\|_{\mathcal{W}_o^{2,p}(G)}^p, \\ \mathbf{E}\|u_x\|_{C^{\beta/2,\beta}([0,T] \times \bar{B}_{\rho_0/8}(z))}^p &\leq \mathbf{E}\|(\eta^z u)_x\|_{C^{\beta/2,\beta}([0,T] \times \mathbb{R}^n)}^p \leq C\|\eta^z u\|_{\mathcal{W}_o^{2,p}(\mathbb{R}^n)}^p \leq C\|u\|_{\mathcal{W}_o^{2,p}(G)}^p, \end{aligned}$$

where $\eta^z \in C_0^\infty(\mathbb{R}^n)$ such that $\eta^z(x) = 1$ for $|x - z| \leq \rho_0/8$ and $\eta^z(x) = 0$ for $|x - z| \geq \rho_0/4$. Therefore, we have bounded the Höler norms in any $\bar{B}_{\rho_0/8}(z) \cap \bar{G}$ with $z \in \bar{G}$. The desired global estimate follows from the localization property of Höler norms (cf. Theorem 4.1.1 in [13]). The lemma is proved. \square

4. Proof of Theorem 2.7

The interior regularity of the solution is implied in the assumption $u \in \mathbb{W}_{\text{loc}}^{2,p}(G)$. To prove the regularity near $\Gamma' := \Gamma \cap \partial G'$, it suffices to do this in a neighbourhood of any point $z \in \Gamma'$ because G' is bounded (and Γ' is bounded too). In other words, we need prove that $u \in \mathbb{W}^{2,p}(B_\varepsilon(z) \cap G)$, where $\varepsilon > 0$ is a number much smaller than $\text{dist}(G', \partial G \setminus \Gamma)$ and ρ_0 (recall Definition 2.1). In the spirit of the method of straightening boundary as in the proof of Proposition 3.2, the desired result can be converted equivalently to the following lemma.

Lemma 4.1. *The conclusion of Theorem 2.7 holds true for $G = \mathbb{R}_+^n$, $\Gamma = \partial \mathbb{R}_+^n \cap B_{2\varepsilon}(0)$ and $G' = B_\varepsilon(0)$.*

Proof. In view of Corollary 3.7 one can assume that $u(0, \cdot) = 0$. Take a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 1$ for $|x| \leq 3\varepsilon/2$ and $\zeta(x) = 0$ for $|x| \geq 2\varepsilon$. Then $v = \zeta u$ satisfies the following equation

$$dv = [(a^{ij} v_{x_i})_{x_j} + f^0 + c^i F_{x_i}] dt + (\sigma^{ik} v_{x_i} + \tilde{g}^k) dw_t^k,$$

where

$$\begin{aligned} c^i &= b^i \zeta - 2a^{ij} \zeta_{x_j} - a_{x_j}^{ij} \zeta, \quad F = u \\ f^0 &= \zeta f + (c\zeta - a^{ij} \zeta_{x_i x_j} - a_{x_j}^{ij} \zeta_{x_i}) u, \end{aligned}$$

$$\tilde{g}^k = \zeta g^k + (v^k \zeta - \sigma^{ik} \zeta_{x_i})u.$$

From Proposition 3.10 one has $\zeta u = v \in \mathcal{W}_o^{1,p}(\mathbb{R}_+^n)$ and

$$\|u\|_{\mathbb{W}^{1,p}(B_{3\varepsilon/2})} \leq C(\|u\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|f\|_{\mathbb{L}^p(\mathbb{R}_+^n)} + \|g\|_{\mathbb{L}^p(\mathbb{R}_+^n; \ell_2^2)}).$$

Now we let $\tilde{\zeta} \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{\zeta}(x) = 1$ for $|x| \leq \varepsilon$ and $\tilde{\zeta}(x) = 0$ for $|x| \geq 3\varepsilon/2$. Then $\tilde{v} = \tilde{\zeta}u$ satisfies

$$d\tilde{v} = (a^{ij}\tilde{v} + \tilde{f})dt + (\sigma^{ik}\tilde{v} + \tilde{g}^k)dw_t^k,$$

where \tilde{g} is defined above and

$$\tilde{f} = \tilde{\zeta}f + (b\tilde{\zeta} - a^{ij}\tilde{\zeta}_{x_j})u_x + (c\tilde{\zeta} - a^{ij}\tilde{\zeta}_{x_i x_j})u.$$

Since $u \in \mathbb{W}^{1,p}(B_{3\varepsilon/2} \cap \mathbb{R}_+^n)$ and $u = 0$ on $B_{3\varepsilon/2} \cap \partial\mathbb{R}_+^n$, one has $\tilde{f} \in \mathbb{L}^p(\mathbb{R}_+^n)$ and $\tilde{g} \in \mathbb{W}_o^{1,p}(\mathbb{R}_+^n)$. Then from Theorem 2.5 one obtains $\tilde{\zeta}u = \tilde{v} \in \mathcal{W}_o^{2,p}(\mathbb{R}_+^n)$. The continuity property of $\tilde{\zeta}u$ and its derivatives follows from Lemma 3.11. The proof is complete. \square

References

- [1] R.A. Adams, J. Fournier, Sobolev Spaces, Academic press, 2003.
- [2] Z. Brzaniak, Stochastic partial differential equations in M-type 2 Banach spaces, Potential Anal. 4 (1) (1995) 1–45.
- [3] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge university press, 2014.
- [4] A. Debussche, S. de Moor, M. Hofmanová, A regularity result for quasilinear stochastic partial differential equations of parabolic type, SIAM J. Math. Anal. 47 (2) (2015) 1590–1614.
- [5] L. Denis, A. Matoussi, L. Stoica, L^p estimates for the uniform norm of solutions of quasilinear SPDE's, Probab. Theory Related Fields 133 (4) (2005) 437–463.
- [6] K. Du, J. Liu, On the Cauchy problem for stochastic parabolic equations in Hölder spaces, Trans. Am. Math Soc. (2018) (in press).
- [7] Franco Flandoli, Dirichlet boundary value problem for stochastic parabolic equations: compatibility relations and regularity of solutions, Stochastics 29 (3) (1990) 331–357.
- [8] M. Gerencsér, Boundary regularity of stochastic PDEs. arxiv.org/pdf/1705.05364.pdf, 2017.
- [9] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order. Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [10] K.H. Kim, On L_p -theory of stochastic partial differential equations of divergence form in C^1 domains, Probab. Theory Related Fields 130 (4) (2004) 473–492.
- [11] K.H. Kim, On stochastic partial differential equations with variable coefficients in C^1 domains, Stochastic Process. Appl. 112 (2) (2004) 261–283.
- [12] N.V. Krylov, A W_2^n -theory of the Dirichlet problem for SPDEs in general smooth domains, Probab. Theory Related Fields 98 (3) (1994) 389–421.
- [13] N.V. Krylov, Lectures on Elliptic and Parabolic Equations in Hölder Spaces, in: Graduate Studies in Mathematics, vol. 12, American Mathematical Society, Providence, RI, 1996.
- [14] N.V. Krylov, On L_p -theory of stochastic partial differential equations in the whole space, SIAM J. Math. Anal. 27 (2) (1996) 313–340.
- [15] N.V. Krylov, An analytic approach to SPDEs. Pages 185–242 of: Stochastic Partial Differential Equations: Six Perspectives, in: Math. Surveys Monogr, vol. 64, American Mathematical Society, Providence, RI, 1999.
- [16] N.V. Krylov, Brownian trajectory is a regular lateral boundary for the heat equation, SIAM J. Math. Anal. 34 (5) (2003) 1167–1182.
- [17] N.V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, in: Graduate Studies in Mathematics, vol. 96, American Mathematical Society, Providence, RI, 2008.
- [18] N.V. Krylov, On the Itô–Wentzell formula for distribution-valued processes and related topics, Probab. Theory Related Fields 150 (1–2) (2011) 295–319.
- [19] N.V. Krylov, S.V. Lototsky, A Sobolev space theory of SPDEs with constant coefficients on a half space, SIAM J. Math. Anal. 31 (1) (1999) 19–33.
- [20] N.V. Krylov, B. Rozovsky, Stochastic evolution equations, J. Sov. Math. 16 (4) (1981) 1233–1277.

- [21] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, 1988.
- [22] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.
- [23] S.V. Lototsky, Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations, *Methods Appl. Anal.* 7 (1) (2000) 195–204.
- [24] R. Mikulevicius, On the Cauchy problem for parabolic SPDEs in Hölder classes, *Ann. Probab.* 28 (1) (2000) 74–103.
- [25] J. Van Neerven, M. Veraar, L. Weis, Maximal L^p -regularity for Stochastic Evolution Equations, *SIAM J. Math. Anal.* 44 (3) (2012) 1372–1414.
- [26] J. Van Neerven, M. Veraar, L. Weis, Stochastic maximal L^p -regularity, *Ann. Probab.* (2012) 788–812.
- [27] J.B. Walsh, An introduction to stochastic partial differential equations, in: *École d'été de Probabilités de Saint-Flour, XIV—1984, 1986*, pp. 265–439.
- [28] X. Zhang, L^p -theory of semi-linear SPDEs on general measure spaces and applications, *J. Funct. Anal.* 239 (1) (2006) 44–75.