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Approximations of stochastic Navier–Stokes equations

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Abstract

In this paper we show that solutions of two-dimensional stochastic Navier–Stokes equations driven by Brownian motion can be approximated by stochastic Navier–Stokes equations forced by pure jump noise/random kicks.

MSC: primary 60H15; secondary 93E20; 35R60

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1. Introduction

Stochastic Navier–Stokes equations (SNSEs) are now a widely accepted model for fluid motion with random perturbations. In this paper, we consider the two-dimensional stochastic Navier–Stokes equations with Dirichlet boundary conditions on a bounded domain, which is given as follows:

$$\begin{cases} du - \kappa \Delta u\, dt + (u \cdot \nabla) u\, dt + \nabla \mathfrak{P}\, dt = F(u)\, dt + \sum_{i=1}^{m} \sigma^i(u)\, dW^i, & \text{in } \mathcal{O} \times (0, T], \\ \operatorname{div} u = 0 & \text{in } \mathcal{O} \times (0, T], \\ u = 0 & \text{in } \partial\mathcal{O} \times (0, T], \\ u(0) = h & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a bounded domain of $\mathbb{R}^2$ with boundary $\partial\mathcal{O}$ of class $\mathcal{C}^3$. $u = (u_1, u_2)$ and $\mathfrak{P}$ represent the random velocity and modified pressure, respectively. κ is the kinematic viscosity,

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for simplicity, we let $\kappa = 1$ in this paper. $W = (W^1(t), \dots, W^m(t))$ is a m-dimensional standard Brownian motion. The fluid is driven by external force $F(u)\,dt$ and the random noise $\sum_{i=1}^m \sigma^i(u)\,dW^i$.

Stochastic Navier–Stokes equations have been studied by many people. There is a great amount of literature. Let us mention a few. SNSEs driven by white noise in time were first studied by Bensoussan and Temam in [2]. The existence and uniqueness of solutions of 2-D SNSEs driven by Lévy noise were obtained in [4], large deviation and moderate deviation principles were established in [6,14]. The ergodic properties and invariant measures of the 2-D SNSEs were studied in [9] and [8].

The aim of this paper is to study the approximations of SNSEs in (1.1) by SNSEs forced by Poisson random measures. One of the motivations is to shed some light on numerical simulations of SNSEs driven by pure jump noise. Recently, Nunno and Zhang in [5] obtained such an approximation for a general class of SPDEs. However, the results in [5] could not cover the stochastic Navier–Stokes equations, an important model in fluid dynamics. The difficulty lies in establishing the tightness of the approximating equations in the space of Hilbert space-valued right continuous paths with left limits. To overcome this difficulty, we first assume that the initial value has higher regularity, the external force and the coefficients of the jump noise take values in a more regular space, so that we can derive a uniform estimate of the stronger norm of the approximating solutions. With these estimates, we are able to prove the tightness of the approximating equations by Aldou's criterion, then through martingale characterization we show that the limit of the solutions of approximating equations is the solution of the SNSE driven by Brownian motion. We emphasize that the method of establishing the tightness here is different and simpler than that used in [5]. In the second step, we are able to remove the regularity restrictions on the coefficients and the initial condition by using finite dimensional approximations and establishing some uniform convergence in probability of the approximating solutions. In the final part of the paper, we provide several illustrating examples.

The rest of the paper is organized as follows. In Section 2 we lay down the precise framework. The main part is Section 3, where the approximations are established. In Section 4 some examples are provided.

2. Framework

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $\nu^i(dx), i = 1, \dots, m$ denote σ-finite measures on the measurable space $(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Let $N^i, i = 1, \dots, m$ be mutually independent $\mathcal{F}_t$-Poisson random measures on $[0, T] \times \mathbb{R}_0$ with intensity measure $dt \times \nu^i(dz)$ respectively. For $U \in \mathcal{B}(\mathbb{R}_0)$ with $\nu^i(U) < \infty$, we write

$$\widetilde{N}^i((0, t] \times U) := N^i((0, t] \times U) - t\nu^i(U), \quad t \geq 0,$$

for the corresponding compensated Poisson random measures on $[0, T] \times \Omega \times \mathbb{R}_0$. See [10] for the details on Poisson random measures.

We introduce the following standard space

$$V = \{u \in H^1(\mathcal{O})^2 : \nabla \cdot u = 0, u|_{\partial\mathcal{O}} = 0\},$$

with the norm $\|u\|_V := (\int_{\mathcal{O}} |\nabla u|^2 dx)^{1/2}$ and the inner product $((\cdot, \cdot))$. Denote by H the closure of V in the L^2-norm $\|u\|_H := (\int_{\mathcal{O}} |u|^2 dx)^{1/2}$. The inner product on H will be denoted by $(\cdot, \cdot)$.

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Identifying the Hilbert space H with its dual space H^*, via the Riesz representation, we consider the system (1.1) in the framework of Gelfand triple:

$$V \subset H \cong H^* \subset V^*.$$

We also denote by $\langle \cdot, \cdot \rangle$ the dual pair between V^* and V from now on.

Define the Stokes operator by

$$Au := -P_H \Delta u, \quad u \in D(A) := H^2(\mathcal{O})^2 \cap V, \tag{2.1}$$

where $P_H : L^2(\mathcal{O})^2 \longrightarrow H$ is the usual Helmholtz–Leray projection. Actually, the map A is an isomorphism between V and V^*, and

$$\langle Au, v \rangle = \langle u, Av \rangle = ((u, v)), \quad \forall\, u, v \in V. \tag{2.2}$$

Note that $\|Au\|_H$ is a norm on $V \cap H^2(\mathcal{O})^2$ which is equivalent to the Sobolev norm in $H^2(\mathcal{O})^2$(for simplicity denoted by H^2 from now on), see Lemma III.3.7 in [13]. It is known that there exist an orthonormal basis $\{e_i, i \in \mathbb{N}\}$ in H and corresponding eigenvalues $0 < \lambda_i \uparrow < \infty$, that is

$$Ae_i = \lambda_i e_i, \quad i \in \mathbb{N}. \tag{2.3}$$

Since the boundary $\partial\mathcal{O}$ is of class $\mathcal{C}^3$, it follows from Chapter I.2.6 in [13] that

$$e_i \in H^3(\mathcal{O}). \tag{2.4}$$

Set

$$b(u, v, w) := \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \partial_i v_j w_j \, dx, \quad u, v, w \in V. \tag{2.5}$$

Using integration by parts, it is easy to see that

$$b(u, v, w) = -b(u, w, v), \ \ b(u, v, v) = 0, \quad u, v, w \in V. \tag{2.6}$$

Throughout the paper, we will denote various generic positive constants by the same letter C, although the constants may differ from line to line. We now list some well-known estimates for b which will be used in the sequel (see [13] for example):

$$|b(u, v, w)| \le 2\|u\|_H^{\frac{1}{2}} \|u\|_V^{\frac{1}{2}} \|w\|_H^{\frac{1}{2}} \|w\|_V^{\frac{1}{2}} \|v\|_V, \quad u, v, w \in V, \tag{2.7}$$

$$|b(u, u, v)| \le C\|u\|_{H^2}^{\frac{1}{2}} \|u\|_V \|u\|_H^{\frac{1}{2}} \|v\|_H, \quad u \in V \cap H^2, \quad v \in H. \tag{2.8}$$

For $u, v \in V$, we denote by $B(u, v)$ the element of V^* defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \ \forall\, w \in V. \tag{2.9}$$

Therefore,

$$\|B(u, v)\|_{V^*} = \sup_{\|w\|_V \le 1} |b(u, v, w)| \le 2\|u\|_H^{\frac{1}{2}} \|u\|_V^{\frac{1}{2}} \|v\|_H^{\frac{1}{2}} \|v\|_V^{\frac{1}{2}}; \tag{2.10}$$

hence

$$\|B(u, u)\|_{V^*} \le 2\|u\|_H \|u\|_V. \tag{2.11}$$

We will often use the short notation $B(u) := B(u, u)$. On the other hand, the nonlinear operator $P_H((u \cdot \nabla)v)$ is well defined whenever u, v are such that $(u \cdot \nabla)v$ belongs to L^2. One can show

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that $P_H((u\cdot\nabla)v)$ can be linearly extended to $V\times V\longrightarrow V^*$, and actually coincides with the previous $B(u,v)$.

It is known that the system (1.1) can be reformulated as follows:

$$\begin{cases} du(t)=-Au(t)dt-B(u(t),u(t))dt+F(u(t))dt+\sum_{i=1}^m\sigma^i(u(t))dW^i(t),\\ u(0)=h. \end{cases} \tag{2.12}$$

Let $F,\sigma^i, i=1,\dots,m$ be measurable mappings from H into H. We introduce the following condition:

(H.1) $F(\cdot),\sigma^i(\cdot):H\to H$ are globally Lipschitz maps, i.e., there exists a constant $C<\infty$ such that

$$\|F(u_1)-F(u_2)\|_H^2+\sum_{i=1}^m\|\sigma^i(u_1)-\sigma^i(u_2)\|_H^2\le C\|u_1-u_2\|_H^2,\quad \forall u_1,u_2\in H. \tag{2.13}$$

Definition 2.1. A continuous H-valued $(\mathcal{F}_t)$-adapted process $u=(u(t))_{t\ge0}$ is said to be a solution to Eq. (2.12) if for any $T>0$, $X\in L^2([0,T]\times\Omega,dt\times P,V)$ and for any $t\ge0$, the following equation holds in V^*, P-a.s.:

$$u(t)=h-\int_0^t Au(s)ds-\int_0^t B(u(s))ds+\int_0^t F(u(s))ds+\sum_{i=1}^m\int_0^t\sigma^i(u(s))dW^i(s). \tag{2.14}$$

Under the assumption (H.1) and $h\in H$, it is known that Eq. (2.12) admits a unique solution (see e.g. [4]).

3. Approximations of SNSEs by pure jump type SNSEs

For $\varepsilon>0$, let $\sigma^{i,\varepsilon}:H\times\mathbb{R}_0\to H$ be given measurable maps. Consider the following SNSE driven by pure jump noise:

$$\begin{aligned} u^\varepsilon(t)=&h-\int_0^t Au^\varepsilon(s)ds-\int_0^t B(u^\varepsilon(s))ds+\int_0^t F(u^\varepsilon(s))\\ &+\sum_{i=1}^m\int_0^t\int_{\mathbb{R}_0}\sigma^{i,\varepsilon}(u^\varepsilon(s-),z)\widetilde{N}^i(dzds). \end{aligned} \tag{3.1}$$

We impose the following conditions on $\sigma^{i,\varepsilon}$.

(H.2) There exist constants $C>0$ and $\varepsilon_0>0$ such that

$$\|F(u)\|_H^2+\sup_{\varepsilon\le\varepsilon_0}\sum_{i=1}^m\int_{\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u,z)\|_H^2\nu^i(dz)\le C(1+\|u\|_H^2), \tag{3.2}$$

$$\sup_{\varepsilon\le\varepsilon_0}\sum_{i=1}^m\int_{\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u,z)\|_H^4\nu^i(dz)\le C(1+\|u\|_H^4), \tag{3.3}$$

$$\begin{aligned} &\|F(u_1)-F(u_2)\|_H^2+\sup_{\varepsilon\le\varepsilon_0}\sum_{i=1}^m\int_{\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u_1,z)-\sigma^{i,\varepsilon}(u_2,z)\|_H^2\nu^i(dz)\\ &\quad\le C\|u_1-u_2\|_H^2. \end{aligned} \tag{3.4}$$

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Denote by $D([0,T],H)$ the space of all càdlàg paths from $[0,T]$ into H equipped with the Skorohod topology.

Definition 3.1. An H-valued $(\mathcal{F}_t)$-adapted process $u^\varepsilon=(u^\varepsilon(t))_{t\geq 0}$ is said to be a solution to Eq. (3.1) if

(i) for any $T>0$, $u^\varepsilon\in D([0,T],H)\cap L^2([0,T]\times\Omega, dt\times P, V)$;

(ii) for every $t\geq 0$, (3.1) holds in V^*, P-a.s.

Under the assumption (H.2) and $h\in H$, it is known that for $\varepsilon\leq\varepsilon_0$, Eq. (3.1) admits a unique solution (see e.g. [4]).

Consider the following conditions.

(H.3) (i) For each $i\in\{1,\dots,m\}$, $\forall M>0$,

$$\sup_{\|u\|_H\leq M}\sup_{z\in\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u,z)\|_H\xrightarrow{\varepsilon\to 0}0. \tag{3.5}$$

(ii) For each $i\in\{1,\dots,m\}$ and each $k,j\in\mathbb{N}$, $u\in H$,

$$\int_{\mathbb{R}_0}(\sigma^{i,\varepsilon}(u,z),e_k)(\sigma^{i,\varepsilon}(u,z),e_j)\nu^i(dz)\xrightarrow{\varepsilon\to 0}(\sigma^i(u),e_k)(\sigma^i(u),e_j). \tag{3.6}$$

(H.4) For each $i\in\{1,\dots,m\}$ and every $u\in H$,

$$\int_{\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u,z)\|_H^2\nu^i(dz)\xrightarrow{\varepsilon\to 0}\|\sigma^i(u)\|_H^2. \tag{3.7}$$

Remark 3.2. In order to approximate Brownian motion by pure jump noise, intuitively, the jump height of all jumps must converge to zero, which motivate us to introduce condition (i) of (H.3). Applying Ito's formula to $\|\cdot\|_H^2$, we introduce (H.4) such that the H-norm of the solutions of (3.1) approximates to the H-norm of the solution of (2.12) in some sense. Condition (ii) of (H.3) is introduced to justify the limit of the solutions of (3.1) is a probabilistic weak solution of (2.12) through the associated martingale problem.

From condition (i) of (H.3) and (H.4), it can be seen that the jump measures ν^i, $i=1,\dots,m$ must have infinite volume, otherwise, (H.4) contradicts condition (i) of (H.3) by the dominated convergence theorem.

(H.5) The maps $F,\sigma^{i,\varepsilon}$ take the space V into itself and there exist constants $C>0$ and $\varepsilon_0>0$ such that

$$\|F(u)\|_V^2+\sup_{\varepsilon\leq\varepsilon_0}\sum_{i=1}^m\int_{\mathbb{R}_0}\|\sigma^{i,\varepsilon}(u,z)\|_V^2\nu^i(dz)\leq C(1+\|u\|_V^2). \tag{3.8}$$

3.1. Preliminary estimates

We first prepare some preliminary results needed for the proofs of the main results. In the rest of the paper, for simplicity of the exposition, we let $m=1$ and omit the superscript i of $\sigma^i,\widetilde{N}^i,\nu^i$. The case of $m>1$ does not cause extra difficulties.

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Lemma 3.3. *Assume (H.2) and $h \in H$, let u^ε be the solution of Eq. (3.1), then we have*

$$\sup_{\varepsilon \le \varepsilon_0} \left\{ E \sup_{0 \le t \le T} \|u^\varepsilon(t)\|_H^4 + E\Big(\int_0^T \|u^\varepsilon(s)\|_V^2 ds \Big)^2 \right\} < \infty. \tag{3.9}$$

Remark 3.4. If we assume (H.1) and $h \in H$, then using similar methods, it can be shown that the following norm estimate holds for the solution u of Eq. (2.12),

$$E \sup_{0 \le t \le T} \|u(t)\|_H^4 + E\Big(\int_0^T \|u(s)\|_V^2 ds \Big)^2 < \infty.$$

Proof. By Itô's formula and (2.6), we have

$$\begin{aligned} \|u^\varepsilon(t)\|_H^2 =& \|h\|_H^2 - 2\int_0^t \langle Au^\varepsilon(s), u^\varepsilon(s)\rangle ds + 2\int_0^t (F(u^\varepsilon(s)), u^\varepsilon(s)) ds \\ &+ M(t) + \int_0^t \int_{\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(s-), z)\|_H^2 \nu(dz) ds, \end{aligned} \tag{3.10}$$

where

$$M(t) := \int_0^t \int_{\mathbb{R}_0} \Big(\|\sigma^\varepsilon(u^\varepsilon(s-), z)\|_H^2 + 2\big(\sigma^\varepsilon(u^\varepsilon(s-), z), u^\varepsilon(s-)\big)\Big) \widetilde{N}(dzds). \tag{3.11}$$

Using Burkholder's inequality and the assumption (H.2), we have

$$\begin{aligned} & E \sup_{0 \le r \le t} |M(r)|^2 \\ \le& CE \int_0^t \int_{\mathbb{R}_0} \Big(\|\sigma^\varepsilon(u^\varepsilon(s-), z)\|_H^2 + 2\big(\sigma^\varepsilon(u^\varepsilon(s-), z), u^\varepsilon(s-)\big)\Big)^2 \nu(dz) ds \\ \le& CE \int_0^t (1 + \|u^\varepsilon(s)\|_H^4) ds. \end{aligned} \tag{3.12}$$

By (2.2), it follows from (3.10) that for $t \le T$,

$$\|u^\varepsilon(t)\|_H^4 + \Big(\int_0^t \|u^\varepsilon(s)\|_V^2 ds \Big)^2 \le C\|h\|_H^4 + C\int_0^t (1 + \|u^\varepsilon(s)\|_H^4) ds + CM(t)^2. \tag{3.13}$$

Take supremum over the interval $[0, t]$ in (3.13), and use (3.12) to get

$$E \sup_{0 \le s \le t} \|u^\varepsilon(s)\|_H^4 + E\left(\int_0^t \|X_s^\varepsilon\|_V^2 ds \right)^2 \le C\|h\|_H^4 + CE\int_0^t (1 + \|u^\varepsilon(s)\|_H^4) ds. \tag{3.14}$$

Applying Gronwall's inequality completes the proof of the lemma. ■

Lemma 3.5. *Assume (H.2), (H.5) and $h \in V$. For any constant $M > 0$, define*

$$\tau_M^\varepsilon := T \wedge \inf\{t \ge 0 : \int_0^t \|u^\varepsilon(s)\|_V^2 ds > M\} \wedge \inf\{t \ge 0 : \|u^\varepsilon(t)\|_H^2 > M\}, \tag{3.15}$$

where we set $\inf\{\emptyset\} = \infty$. *Then we have*

$$\sup_{\varepsilon \le \varepsilon_0} \left\{ E \sup_{0 \le t \le \tau_M^\varepsilon} \|u^\varepsilon(t)\|_V^2 + E\Big(\int_0^{\tau_M^\varepsilon} \|u^\varepsilon(s)\|_{H^2}^2 ds \Big)^2 \right\} < \infty. \tag{3.16}$$

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Proof. Through Galerkin approximations, it can be shown that for $\varepsilon \leq \varepsilon_0$, the solution $u^\varepsilon \in L^\infty([0,T],V) \cap L^2([0,T],H^2)$ with probability one (see e.g. Proposition 2.2 in [12]). Apply Itô's formula to $\|u^\varepsilon(t)\|_V^2$ to get

$$\begin{aligned}\|u^\varepsilon(t)\|_V^2 =& \|h\|_V^2 - 2\int_0^t \|Au^\varepsilon(s)\|_H^2 ds - 2\int_0^t \big(B(u^\varepsilon(s)), Au^\varepsilon(s)\big)ds \\ &+ 2\int_0^t \big(\big(F(u^\varepsilon(s)), u^\varepsilon(s)\big)\big)ds + M_1(t) + M_2(t),\end{aligned} \tag{3.17}$$

where

$$M_1(t) := 2\int_0^t \int_{\mathbb{R}_0} \big(\big(\sigma^\varepsilon(u^\varepsilon(s-),z), u^\varepsilon(s-)\big)\big)\widetilde{N}(dzds), \tag{3.18}$$

$$M_2(t) := \int_0^t \int_{\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(s-),z)\|_V^2 N(dzds). \tag{3.19}$$

Use (2.8) and Young's inequality to obtain

$$|(B(u^\varepsilon(s)), Au^\varepsilon(s))| \leq C\|u^\varepsilon\|_{H^2}^{\frac{3}{2}}\|u^\varepsilon\|_V\|u^\varepsilon\|_H^{\frac{1}{2}} \leq \|u^\varepsilon\|_{H^2}^2 + C\|u^\varepsilon\|_V^4\|u^\varepsilon\|_H^2. \tag{3.20}$$

Therefore, by (3.20) and (H.5), we obtain

$$\begin{aligned}&\|u^\varepsilon(t)\|_V^2 + \int_0^t \|u^\varepsilon(s)\|_{H^2}^2 ds \\ \leq& \|h\|_V^2 + C\int_0^t \|u^\varepsilon(s)\|_V^4\|u^\varepsilon(s)\|_H^2 ds + C\int_0^t (1+\|u^\varepsilon(s)\|_V^2)ds + M_1(t) + M_2(t).\end{aligned} \tag{3.21}$$

Applying Gronwall's inequality yields that

$$\begin{aligned}&\|u^\varepsilon(t)\|_V^2 + \int_0^t \|u^\varepsilon(s)\|_{H^2}^2 ds \\ \leq& \big(C_T + \|h\|_V^2 + \sup_{0\leq t\leq \tau_M^\varepsilon}|M_1(t)| + M_2(\tau_M^\varepsilon)\big) \\ &\times \exp\Big(C_T + C\int_0^t \|u^\varepsilon(s)\|_V^2\|u^\varepsilon(s)\|_H^2 ds\Big), \quad t\in[0,\tau_M^\varepsilon].\end{aligned} \tag{3.22}$$

Take supremum over the interval $[0,\tau_M^\varepsilon]$, remember the definition of τ_M^ε and take expectations to get

$$\begin{aligned}&E\sup_{0\leq t\leq \tau_M^\varepsilon}\|u^\varepsilon(t)\|_V^2 + E\int_0^{\tau_M^\varepsilon}\|u^\varepsilon(s)\|_{H^2}^2 ds \\ \leq& \Big(C_T + \|h\|_V^2 + E\sup_{0\leq t\leq \tau_M^\varepsilon}|M_1(t)| + EM_2(\tau_M^\varepsilon)\Big)\exp(C_T + CM^2).\end{aligned} \tag{3.23}$$

By Burkholder's inequality, (H.5) and Young's inequality, we have for $\delta > 0$,

$$\begin{aligned}E\sup_{0\leq t\leq \tau_M^\varepsilon}|M_1(t)| &\leq 2E\left[\int_0^{\tau_M^\varepsilon}\int_{\mathbb{R}_0}\big(\big(\sigma^\varepsilon(u^\varepsilon(s-),z), u^\varepsilon(s-)\big)\big)^2\nu(dz)ds\right]^{\frac{1}{2}} \\ &\leq 2E\left[\int_0^{\tau_M^\varepsilon} C\|u^\varepsilon(s)\|_V^2(1+\|u^\varepsilon(s)\|_V^2)ds\right]^{\frac{1}{2}}\end{aligned}$$

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$$
\begin{aligned}
&\le 2(CT+CM)^{\frac{1}{2}} E \sup_{0\le t\le \tau_M^\varepsilon} \|u^\varepsilon(t)\|_V \\
&\le \delta E \sup_{0\le t\le \tau_M^\varepsilon} \|u^\varepsilon(t)\|_V^2 + \frac{1}{\delta}(CT+CM).
\end{aligned}
\tag{3.24}
$$

By (H.5) and (3.9), we have

$$
\begin{aligned}
EM_2(\tau_M^\varepsilon) =& E\int_0^{\tau_M^\varepsilon}\int_{\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(s-),z)\|_V^2 N(dzds) \\
=& E\int_0^{\tau_M^\varepsilon}\int_{\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(s-),z)\|_V^2 \nu(dz)ds \\
\le& E\int_0^{\tau_M^\varepsilon} C(1+\|u^\varepsilon(s)\|_V^2)ds \le C < \infty.
\end{aligned}
\tag{3.25}
$$

Combining (3.23), (3.24) and (3.25) and choosing sufficiently small δ, we obtain

$$
E \sup_{0\le t\le \tau_M^\varepsilon} \|u^\varepsilon(t)\|_V^2 + E\int_0^{\tau_M^\varepsilon} \|u^\varepsilon(s)\|_{H^2}^2 ds \le C_{T,M}\|h\|_V^2 + C_{T,M} \tag{3.26}
$$

completing the proof of (3.16). ■

Proposition 3.6. *Assume (H.2), (H.5) and $h \in V$. Then the family $\{u^\varepsilon, \varepsilon \le \varepsilon_0\}$ is tight in the space $D([0,T],H)$.*

Proof. Note that V is compactly embedded into H. Thus, by Aldou's tightness criterion (see Theorem 1 in [1]), it suffices to show that:

(i) for any $0 < \eta < 1$, there exists $L_\eta > 0$ such that

$$
\sup_{\varepsilon\le\varepsilon_0} P\left(\sup_{0\le t\le T} \|u^\varepsilon(t)\|_V > L_\eta\right) < \eta; \tag{3.27}
$$

(ii) for any stopping time $0 \le \zeta^\varepsilon \le T$ with respect to the natural filtration generated by $\{u^\varepsilon(s), s \le t\}$, and any $\eta > 0$,

$$
\lim_{\delta\to 0}\sup_{\varepsilon\le\varepsilon_0} P(\|u^\varepsilon(\zeta^\varepsilon+\delta) - u^\varepsilon(\zeta^\varepsilon)\|_H > \eta) = 0, \tag{3.28}
$$

where we set $\zeta^\varepsilon + \delta := T \wedge (\zeta^\varepsilon + \delta)$.

Note that (3.9) implies

$$
\begin{aligned}
&\sup_{\varepsilon\le\varepsilon_0} P\left(\tau_M^\varepsilon < T\right) \\
\le& \sup_{\varepsilon\le\varepsilon_0} P\left(\int_0^T \|u^\varepsilon(s)\|_V^2 ds > M\right) + \sup_{\varepsilon\le\varepsilon_0} P\left(\sup_{0\le s\le T} \|u^\varepsilon(s)\|_H^2 > M\right) \\
\le& \frac{1}{M}\sup_{\varepsilon\le\varepsilon_0} E\int_0^T \|u^\varepsilon(s)\|_V^2 ds + \frac{1}{M}\sup_{\varepsilon\le\varepsilon_0} E\sup_{0\le t\le T} \|u^\varepsilon(s)\|_H^2 \\
\le& \frac{C}{M}.
\end{aligned}
\tag{3.29}
$$

For any $L > 0$, by (3.29) and (3.16), we have

$$
\begin{aligned}
&\sup_{\varepsilon\le\varepsilon_0} P\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|_V > L\right)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|_V > L, \tau_M^\varepsilon = T\right) + \sup_{\varepsilon\le\varepsilon_0} P\left(\tau_M^\varepsilon < T\right)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\left(\sup_{0\le t\le \tau_M^\varepsilon}\|u^\varepsilon(t)\|_V > L\right) + \frac{C}{M}\\
&\le \frac{1}{L^2}\sup_{\varepsilon\le\varepsilon_0} E \sup_{0\le t\le \tau_M^\varepsilon}\|u^\varepsilon(t)\|_V^2 + \frac{C}{M}\\
&\le \frac{C_M}{L^2} + \frac{C}{M}.
\end{aligned}
\tag{3.30}
$$

Given any $\eta > 0$, we can first take sufficiently large constant M, and then choose the constant L so that the right hand side of (3.30) will be smaller than η. Hence (i) is satisfied.

Now, we come to verify (ii). For any $\eta > 0$,

$$
\begin{aligned}
&\sup_{\varepsilon\le\varepsilon_0} P(\|u^\varepsilon(\zeta^\varepsilon+\delta) - u^\varepsilon(\zeta^\varepsilon)\|_H > \eta)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\Big\|\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} Au^\varepsilon(s)ds\Big\|_H > \frac{\eta}{4}\Big)\\
&\quad + \sup_{\varepsilon\le\varepsilon_0} P\Big(\Big\|\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} B(u^\varepsilon(s))ds\Big\|_H > \frac{\eta}{4}\Big)\\
&\quad + \sup_{\varepsilon\le\varepsilon_0} P\Big(\Big\|\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} F(u^\varepsilon(s))ds\Big\|_H > \frac{\eta}{4}\Big)\\
&\quad + \sup_{\varepsilon\le\varepsilon_0} P\Big(\Big\|\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0} \sigma^\varepsilon(u^\varepsilon(s-), z)\widetilde{N}(dzds)\Big\|_H > \frac{\eta}{4}\Big)\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.31}
$$

By Hölder's inequality and Chebyshev's inequality, it follows from (3.16) and (3.29) that for $M > 0$,

$$
\begin{aligned}
I_1 &\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\delta\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} \|Au^\varepsilon(s)\|_H^2 ds > \frac{\eta^2}{16}\Big)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\delta\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} \|Au^\varepsilon(s)\|_H^2 ds > \frac{\eta^2}{16}, \tau_M^\varepsilon = T\Big) + \sup_{\varepsilon\le\varepsilon_0} P(\tau_M^\varepsilon < T)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\delta\int_0^{\tau_M^\varepsilon} \|Au^\varepsilon(s)\|_H^2 ds > \frac{\eta^2}{16}\Big) + \frac{C}{M}\\
&\le \frac{16}{\eta^2}\delta \sup_{\varepsilon\le\varepsilon_0} E\int_0^{\tau_M^\varepsilon} \|Au^\varepsilon(s)\|_H^2 ds + \frac{C}{M}\\
&\le \frac{C_M}{\eta^2}\delta + \frac{C}{M}.
\end{aligned}
\tag{3.32}
$$

By (2.8), we have $\|B(u^\varepsilon(s))\|_H \le C\|u^\varepsilon(s)\|_{H^2}^{\frac{1}{2}}\|u^\varepsilon(s)\|_V\|u^\varepsilon(s)\|_H^{\frac{1}{2}}$. Using (3.16) and (3.29), we have

$$\begin{aligned}
I_2 &\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\|B(u^\varepsilon(s))\|_H ds > \frac{\eta}{4}\Big)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\|u^\varepsilon(s)\|_{H^2}^{\frac{1}{2}}\|u^\varepsilon(s)\|_V\|u^\varepsilon(s)\|_H^{\frac{1}{2}} ds > \frac{\eta}{4C}\Big)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\|u^\varepsilon(s)\|_{H^2}^{\frac{1}{2}}\|u^\varepsilon(s)\|_V\|u^\varepsilon(s)\|_H^{\frac{1}{2}} ds > \frac{\eta}{4C}, \tau_M^\varepsilon = T\Big)\\
&\quad + \sup_{\varepsilon\le\varepsilon_0} P(\tau_M^\varepsilon < T)\\
&\le \sup_{\varepsilon\le\varepsilon_0} P\Big(\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_M^\varepsilon}\|u^\varepsilon(s)\|_{H^2}^{\frac{1}{2}}\|u^\varepsilon(s)\|_V\|u^\varepsilon(s)\|_H^{\frac{1}{2}} ds > \frac{\eta}{4C}\Big) + \frac{C}{M}\\
&\le \frac{4C}{\eta}\sup_{\varepsilon\le\varepsilon_0}\Big[\Big(E\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_M^\varepsilon}\|u^\varepsilon(s)\|_H^2 ds\Big)^{\frac{1}{4}}\Big(E\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_M^\varepsilon}\|u^\varepsilon(s)\|_{H^2}^2 ds\Big)^{\frac{1}{4}}\\
&\quad \times\Big(E\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_M^\varepsilon}\|u^\varepsilon(s)\|_V^2 ds\Big)^{\frac{1}{2}}\Big] + \frac{C}{M}\\
&\le \frac{C_M}{\eta}\delta^{\frac{3}{4}}\sup_{\varepsilon\le\varepsilon_0}\Big(E\sup_{0\le s\le T}\|u^\varepsilon(s)\|_H^2\Big)^{\frac{1}{4}}\times\sup_{\varepsilon\le\varepsilon_0}\Big(E\int_0^{\tau_M^\varepsilon}\|u^\varepsilon(s)\|_{H^2}^2 ds\Big)^{\frac{1}{4}}\\
&\quad \times\sup_{\varepsilon\le\varepsilon_0}\Big(E\sup_{0\le s\le\tau_M^\varepsilon}\|u^\varepsilon(s)\|_V^2\Big)^{\frac{1}{2}} + \frac{C}{M}\\
&\le \frac{C_M}{\eta}\delta^{\frac{3}{4}} + \frac{C}{M}.
\end{aligned} \tag{3.33}$$

On the other hand, by (H.2) and (3.9) we have

$$\begin{aligned}
I_3 &\le \frac{4}{\eta}\sup_{\varepsilon\le\varepsilon_0} E\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\|F(u^\varepsilon(s))\|_H ds\\
&\le \frac{4}{\eta}\sup_{\varepsilon\le\varepsilon_0} E\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} C(1+\|u^\varepsilon(s)\|_H) ds\\
&\le \frac{C}{\eta}\delta\Big(1+\sup_{\varepsilon\le\varepsilon_0} E\sup_{0\le s\le T}\|u^\varepsilon(s)\|_H\Big)\\
&\le \frac{C}{\eta}\delta.
\end{aligned} \tag{3.34}$$

Similarly,

$$\begin{aligned}
I_4 &\le \frac{16}{\eta^2}\sup_{\varepsilon\le\varepsilon_0} E\Big\|\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0}\sigma^\varepsilon(u^\varepsilon(s-),z)\widetilde{N}(dzds)\Big\|_H^2\\
&\le \frac{16}{\eta^2}\sup_{\varepsilon\le\varepsilon_0} E\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0}\|\sigma^\varepsilon(u^\varepsilon(s-),z)\|_H^2\nu(dz)ds
\end{aligned}$$

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$$\leq \frac{16}{\eta^2} \sup_{\varepsilon \leq \varepsilon_0} E \int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta} C(1+\|u^\varepsilon(s)\|_H^2)ds$$
$$\leq \frac{C}{\eta^2}\delta. \tag{3.35}$$

Combine (3.32)–(3.35) together, first let $\delta \to 0$, then let $M \to \infty$ to obtain (3.28). Thus (ii) is verified, which completes the proof. ■

3.2. The weak convergence

Denote by μ_ε, μ respectively the laws of u^ε and u on the spaces $D([0,T],H)$ and $C([0,T],H)$. We will establish the weak convergence by two stages. We first obtain the weak convergence in Theorem 3.7 under stronger conditions, and then we remove the extra assumptions and get the general convergence result in Theorem 3.8.

Theorem 3.7. *Assume (H.1), (H.2), (H.3), (H.5) and $h \in V$. Then, for any $T > 0$, μ_ε converges weakly to μ, as $\varepsilon \to 0$, on the space $D([0,T],H)$ equipped with the Skorohod topology.*

Proof. By Proposition 3.6, the family $\{\mu_\varepsilon, \varepsilon \leq \varepsilon_0\}$ is tight in $D([0,T],H)$. Let μ_0 be the weak limit of any convergent subsequence $\{\mu_{\varepsilon_n}\}$. We will show that $\mu_0 = \mu$. The rest of the proof is divided into three steps. In step 1, we show that μ_0 is supported on the space $C([0,T],H)$. In step 2, we prove that μ_0 is a solution of a martingale problem. In step 3, we show that μ_0 is the law of a weak solution of SNSE (2.12), hence complete the proof.

Step 1. For any $\eta > 0$, $M > 0$, we have

$$P\left(\sup_{0\leq t\leq T} \|u^\varepsilon(t) - u^\varepsilon(t-)\|_H \geq \eta\right)$$
$$\leq P\left(\sup_{0\leq t\leq T}\sup_{z\in\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(t-),z)\|_H \geq \eta\right)$$
$$\leq P\left(\sup_{0\leq t\leq T}\sup_{z\in\mathbb{R}_0} \|\sigma^\varepsilon(u^\varepsilon(t),z)\|_H > \eta, \sup_{0\leq t\leq T}\|u^\varepsilon(t)\| \leq M\right)$$
$$+ P\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\| > M\right)$$
$$\leq P\Big(\sup_{\|x\|_H\leq M}\sup_{z\in\mathbb{R}_0} \|\sigma^\varepsilon(x,z)\|_H > \eta\Big) + \frac{1}{M^2}\sup_{\varepsilon\leq\varepsilon_0} E\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|_H^2. \tag{3.36}$$

By (3.9) and (3.5), we first let $\varepsilon \to 0$ and then $M \to \infty$ to see that

$$\sup_{0<t\leq T}\|u^\varepsilon(t) - u^\varepsilon(t-)\|_H \xrightarrow{\varepsilon\to 0} 0 \quad \text{in probability.} \tag{3.37}$$

Therefore, it follows from Theorem 13.4 in [3] that μ_0 is supported on the space $C([0,T],H)$. As a consequence, the finite dimensional distributions of μ_{ε_n} converge to that of μ_0.

Step 2. For $k, j \in \mathbb{N}$, let $f(x) = (x, e_k)(x, e_j)$, $x \in H$. The gradient of f (denoted by ∇f) and the operator (denoted by f'') associated with the second derivatives of f are respectively given by

$$\nabla f(x) = (x, e_j)e_k + (x, e_k)e_j, \tag{3.38}$$

$$f''(x) = e_j \otimes e_k + e_k \otimes e_j. \tag{3.39}$$

Set

$$\begin{aligned} L^\varepsilon f(x) :=& -(A\nabla f(x), x) - \langle B(x), \nabla f(x)\rangle + (F(x), \nabla f(x)) \\ &+ \int_{\mathbb{R}_0} \big[f(x+\sigma^\varepsilon(x,z)) - f(x) - (\nabla f(x), \sigma^\varepsilon(x,z))\big]\nu(dz), \end{aligned} \tag{3.40}$$

$$Lf(x) := -(A\nabla f(x), x) - \langle B(x), \nabla f(x)\rangle + (F(x), \nabla f(x)) + \frac{1}{2}(f''(x)\sigma(x), \sigma(x)). \tag{3.41}$$

By Itô's formula,

$$\begin{aligned} & f(u^\varepsilon(t)) - f(h) - \int_0^t L^\varepsilon f(u^\varepsilon(s))ds \\ &= \int_0^t \int_{\mathbb{R}_0} \Big[f\big(u^\varepsilon(s-) + \sigma^\varepsilon(u^\varepsilon(s-), z)\big) - f\big(u^\varepsilon(s-)\big)\Big]\widetilde{N}(dzds) \end{aligned} \tag{3.42}$$

is a martingale. Denote by $X_t(\omega) := \omega(t)$, $\omega \in D([0,T], H)$ the coordinate process on $D([0,T], H)$. By the above martingale property, for any $m \in \mathbb{N}$, $0 \le s_0 < s_1 < \cdots < s_m \le s < t$ and $f_0, f_1, \ldots, f_m \in C_b(H)$(the collection of bounded continuous functions on H), it holds that

$$E^{\mu_\varepsilon}\left[\left(f(X_t) - f(X_s) - \int_s^t L^\varepsilon f(X_r)dr\right) f_0(X_{s_0})\ldots f_m(X_{s_m})\right] = 0. \tag{3.43}$$

Let

$$G_\varepsilon(x) := \left|\int_{\mathbb{R}_0} (\sigma^\varepsilon(x,z), e_k)(\sigma^\varepsilon(x,z), e_j)\nu(dz) - (\sigma(x), e_k)(\sigma(x), e_j)\right|, \tag{3.44}$$

$x \in H$. By (3.40) and (3.41), we have

$$|L^\varepsilon f(X_r) - Lf(X_r)| = G_\varepsilon(X_r). \tag{3.45}$$

We claim that

$$\lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\left[\int_s^t |L^{\varepsilon_n} f(X_r) - Lf(X_r)|dr\right] = 0. \tag{3.46}$$

Note that

$$E^{\mu_{\varepsilon_n}}\left[\int_s^t |L^{\varepsilon_n} f(X_r) - Lf(X_r)|dr\right] = \int_s^t EG_{\varepsilon_n}(u^{\varepsilon_n}(r))dr, \tag{3.47}$$

$$\sup_{\varepsilon\le\varepsilon_0} G_\varepsilon(x) \le C(1 + \|x\|_H^2). \tag{3.48}$$

By the dominated convergence theorem and (3.9), to prove (3.46), it suffices to prove that for every $r \in [0,T]$,

$$\lim_{n\to\infty} EG_{\varepsilon_n}(u^{\varepsilon_n}(r)) = 0. \tag{3.49}$$

Now, we take any $r \in [0,T]$ and fix it. Since the finite dimensional distributions of μ_{ε_n} converge weakly to that of μ_0, by Skorohod's representation theorem, in order not to introduce more notations, we can assume that $u^{\varepsilon_n}(r)$ converges almost surely to an H-valued random variable u^0. In view of (3.9), $\{\|u^{\varepsilon_n}(r)\|_H^2\}_{n\geq 1}$ is uniformly integrable, and therefore we can further deduce that $u^0 \in L^2(\Omega, H)$ and

$$\lim_{n\to\infty} E\|u^{\varepsilon_n}(r) - u^0\|_H^2 = 0. \tag{3.50}$$

By the dominated convergence theorem, it follows from (3.6) and (3.48) that

$$\lim_{n\to\infty} EG_{\varepsilon_n}(u^0) = 0. \tag{3.51}$$

Hence to prove (3.49), it suffices to prove

$$\lim_{n\to\infty} E|G_{\varepsilon_n}(u^{\varepsilon_n}(r)) - G_{\varepsilon_n}(u^0)| = 0. \tag{3.52}$$

We have

$$\begin{aligned}
&E|G_{\varepsilon_n}(u^{\varepsilon_n}(r)) - G_{\varepsilon_n}(u^0)|\\
\leq &E\Big|\int_{\mathbb{R}_0}\big(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),e_k\big)\big(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),e_j\big)\nu(dz)\\
&\quad-\int_{\mathbb{R}_0}\big(\sigma^{\varepsilon_n}(u^0,z),e_k\big)\big(\sigma^{\varepsilon_n}(u^0,z),e_j\big)\nu(dz)\Big|\\
&+\ E|(\sigma(u^{\varepsilon_n}(r)),e_k)(\sigma(u^{\varepsilon_n}(r)),e_j)-(\sigma(u^0),e_k)(\sigma(u^0),e_j)|\\
:=&I_1+I_2.
\end{aligned} \tag{3.53}$$

In view of (3.2) and (3.4), we have

$$\begin{aligned}
I_1 \leq &E\int_{\mathbb{R}_0}\Big|\big(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),e_k\big)\big(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z),e_j\big)\Big|\nu(dz)\\
&+\ E\int_{\mathbb{R}_0}\Big|\big(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z),e_k\big)\big(\sigma^{\varepsilon_n}(u^0,z),e_j\big)\Big|\nu(dz)\\
\leq&\Big[E\int_{\mathbb{R}_0}\big\|\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)\big\|_H^2\nu(dz)\Big]^{\frac12}\Big[E\int_{\mathbb{R}_0}\big\|\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z)\big\|_H^2\nu(dz)\Big]^{\frac12}\\
&+\ \Big[E\int_{\mathbb{R}_0}\big\|\sigma^{\varepsilon_n}(u^0,z)\big\|_H^2\nu(dz)\Big]^{\frac12}\Big[E\int_{\mathbb{R}_0}\big\|\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z)\big\|_H^2\nu(dz)\Big]^{\frac12}\\
\leq&C\Big[(1+E\|u^0\|_H^2)^{\frac12}+\sup_{\varepsilon_n}(1+E\|u^{\varepsilon_n}(r)\|_H^2)^{\frac12}\Big]\big(E\|u^{\varepsilon_n}(r)-u^0\|_H^2\big)^{\frac12}.
\end{aligned} \tag{3.54}$$

This yields that $I_1 \to 0$ taking into account (3.9) and (3.50). A similar argument leads to $I_2 \to 0$. Therefore, (3.52) holds. Hence the claim (3.46) is proved.

Next we prove that

$$M_{k,j}(t) := f(X_t) - f(h) - \int_0^t Lf(X_r)dr \tag{3.55}$$

is a martingale under μ_0. This is equivalent to proving that

$$E^{\mu_0}\Big[\Big(f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr\Big) f_0(X_{s_0})\ldots f_m(X_{s_m})\Big] = 0. \tag{3.56}$$

Since the finite dimensional distributions of μ_{ε_n} converge to that of μ_0, noticing that $\|f(x)\|_H \leq \|x\|_H^2$ and (3.9), it follows from Theorem 1.6.8 in [7] that

$$E^{\mu_0}\Big[f(X_t)f_0(X_{s_0})\cdots f_m(X_{s_m})\Big] = \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\Big[f(X_t)f_0(X_{s_0})\ldots f_m(X_{s_m})\Big]. \tag{3.57}$$

In view of (2.4), we have

$$|\langle B(x,x), e_k\rangle| = |\langle B(x,e_k), x\rangle| \leq C\|x\|_H^2 \|\nabla e_k\|_{L^\infty} \leq C\|e_k\|_{H^3}\|x\|_H^2. \tag{3.58}$$

Thus, $Lf(x)$ is a continuous function on H and

$$|Lf(x)| \leq C(1+\|x\|_H^3). \tag{3.59}$$

Therefore, for the same reason as (3.57), we have for every $r \in [s,t]$,

$$E^{\mu_0}\big[\big(Lf(X_r)\big) f_0(X_{s_0})\dots f_m(X_{s_m})\big] = \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\big[\big(Lf(X_r)\big) f_0(X_{s_0})\dots f_m(X_{s_m})\big]. \tag{3.60}$$

By the Fubini theorem and the dominate convergence theorem, we obtain

$$\begin{aligned}
&E^{\mu_0}\Big[\Big(\int_s^t Lf(X_r)dr\Big) f_0(X_{s_0})\dots f_m(X_{s_m})\Big]\\
&= \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\Big[\Big(\int_s^t Lf(X_r)dr\Big) f_0(X_{s_0})\dots f_m(X_{s_m})\Big].
\end{aligned} \tag{3.61}$$

Using (3.57), (3.61), (3.46) and (3.43), we have

$$\begin{aligned}
&E^{\mu_0}\Big[\Big(f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr\Big) f_0(X_{s_0})\dots f_m(X_{s_m})\Big]\\
&= \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\Big[\Big(f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr\Big) f_0(X_{s_0})\dots f_m(X_{s_m})\Big]\\
&= \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\Big[\Big(f(X_t) - f(X_s) - \int_s^t L^{\varepsilon_n} f(X_r)dr\Big) f_0(X_{s_0})\dots f_m(X_{s_m})\Big]\\
&=0.
\end{aligned} \tag{3.62}$$

Hence $M_{k,j}(t)$ in (3.55) is a martingale under μ_0.

For $k \in \mathbb{N}$, let $g(x) = (x, e_k)$, $x \in H$. By a similar argument, we can show that

$$\begin{aligned}
&M_k(t) := g(X_t) - g(h) - \int_0^t Lg(X_r)dr\\
&=(X_t, e_k) - (h, e_k) + \int_0^t (Ae_k, X_s)ds + \int_0^t \langle B(X_s), e_k\rangle ds - \int_0^t \big(F(X_s), e_k\big)ds
\end{aligned} \tag{3.63}$$

is a martingale under μ_0.

Step 3. (3.55) and (3.63) together with Itô's formula yield that

$$\langle M_k, M_j\rangle(t) = \int_0^t (\sigma(X_s), e_k)(\sigma(X_s), e_j)ds, \tag{3.64}$$

where $\langle M_k, M_j\rangle$ stands for the sharp bracket of the two martingales. Now by Lemma A.1 in the Appendix, there exists a probability space $(\Omega', \mathcal{F}', P')$ with a filtration $\mathcal{F}'_t$ such that on the standard extension

$$(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathcal{F}_t \times \mathcal{F}'_t, \mu_0 \times P')$$

of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ there exists a one-dimensional Brownian motion $W(t), t \geq 0$ such that

$$M_k(t) = \int_0^t (\sigma(X_s), e_k)dW(s), \tag{3.65}$$

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namely,

$$\begin{aligned}(X_t, e_k) - (h, e_k) = &- \int_0^t (Ae_k, X_s)ds - \int_0^t \langle B(X_s), e_k\rangle ds \\ &+ \int_0^t (F(X_s), e_k)ds + \int_0^t (\sigma(X_s), e_k)dW(s) \end{aligned} \tag{3.66}$$

for every $k \geq 1$. Thus, under μ_0, $\{X_t, t \geq 0\}$ is a solution to SNSE (2.12). By the uniqueness of the SNSE, we conclude that $\mu_0 = \mu$ completing the proof of the theorem. ■

In the next theorem, we will remove the restrictions placed on the coefficients and the initial value h.

Theorem 3.8. *Assume (H.1), (H.2), (H.3), (H.4) and $h \in H$. Then, for any $T > 0$, μ_ε converges weakly to μ, as $\varepsilon \to 0$, on the space $D([0, T], H)$ equipped with the Skorohod topology.*

Proof. For each $n \in \mathbb{N}$, let $h^n, F_n(u), \sigma_n(u), \sigma_n^\varepsilon(u, z)$ denote the corresponding orthogonal projections of $h, F(u), \sigma(u), \sigma^\varepsilon(u, z)$ into the n-dimensional space span$\{e_1, \dots, e_n\}$. Then, for each $n \in \mathbb{N}$, $\{\sigma_n^\varepsilon\}_{\varepsilon\leq\varepsilon_0}$ and F_n satisfy (H.2)—(H.5). Moreover, there is a constant C independent of n such that for every $u, u_1, u_2 \in H$,

$$\sup_{n\in\mathbb{N}} \|F_n(u)\|_H^2 + \sup_{n\in\mathbb{N}} \|\sigma_n(u)\|_H^2 + \sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0} \int_{\mathbb{R}_0} \|\sigma_n^\varepsilon(u, z)\|_H^2 \nu(dz) \leq C(1 + \|u\|_H^2), \tag{3.67}$$

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \|F_n(u_1) - F_n(u_2)\|_H^2 + \sup_{n\in\mathbb{N}} \|\sigma_n(u_1) - \sigma_n(u_2)\|_H^2 \\ &+ \sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0} \int_{\mathbb{R}_0} \|\sigma_n^\varepsilon(u_1, z) - \sigma_n^\varepsilon(u_2, z)\|_H^2 \nu(dz) \leq C\|u_1 - u_2\|_H^2. \end{aligned} \tag{3.68}$$

Let $u^{n,\varepsilon}, u^n$ be the solutions of the SNSEs:

$$\begin{aligned} u^{n,\varepsilon}(t) =& h^n - \int_0^t Au^{n,\varepsilon}(s)ds - \int_0^t B(u^{n,\varepsilon}(s))ds + \int_0^t F_n(u^{n,\varepsilon}(s))ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \sigma_n^\varepsilon(u^{n,\varepsilon}(s-), z)\widetilde{N}(dzds), \end{aligned} \tag{3.69}$$

$$\begin{aligned} u^n(t) =& h^n - \int_0^t Au^n(s)ds - \int_0^t B(u^n(s))ds + \int_0^t F_n(u^n(s))ds \\ &+ \int_0^t \sigma_n(u^n(s))dW(s). \end{aligned} \tag{3.70}$$

By Theorem 3.7, we have for each $n \in \mathbb{N}$,

$$u^{n,\varepsilon} \xrightarrow{\varepsilon\to 0} u^n \quad \text{in distribution on the space } D([0, T], H). \tag{3.71}$$

Moreover, as in the proof of (3.9), using (3.67) we can show that

$$\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0} \left\{ E \sup_{0\leq t\leq T} \|u^{n,\varepsilon}(t)\|_H^4 + E\Big(\int_0^T \|u^{n,\varepsilon}(s)\|_V^2 ds\Big)^2 \right\} < \infty, \tag{3.72}$$

$$\sup_{n\in\mathbb{N}} \left\{ E \sup_{0\leq t\leq T} \|u^n(t)\|_H^4 + E\Big(\int_0^T \|u^n(s)\|_V^2 ds\Big)^2 \right\} < \infty. \tag{3.73}$$

We claim that for any $\delta > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T}\|u^n(t)-u(t)\|_H > \delta\right) = 0, \tag{3.74}$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} P\left(\sup_{0\le t\le T}\|u^{n,\varepsilon}(t)-u^\varepsilon(t)\|_H > \delta\right) = 0. \tag{3.75}$$

Because of similarity, we only prove (3.75) here. Applying Itô's formula, we have

$$\begin{aligned}
&e^{-\gamma\int_0^t \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(t)-u^\varepsilon(t)\|_H^2\\
=&\|h^n-h\|_H^2 - \gamma\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2\|u^\varepsilon(s)\|_V^2 ds\\
&-2\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\langle A(u^{n,\varepsilon}(s)-u^\varepsilon(s)), u^{n,\varepsilon}(s)-u^\varepsilon(s)\rangle ds\\
&-2\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\langle B(u^{n,\varepsilon}(s))-B(u^\varepsilon(s)), u^{n,\varepsilon}(s)-u^\varepsilon(s)\rangle ds\\
&+2\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\big(F_n(u^{n,\varepsilon}(s))-F(u^\varepsilon(s)), u^{n,\varepsilon}(s)-u^\varepsilon(s)\big) ds\\
&+2\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\times\\
&\qquad\big(\sigma_n^\varepsilon(u^{n,\varepsilon}(s-),z)-\sigma^\varepsilon(u^\varepsilon(s-),z), u^{n,\varepsilon}(s-)-u^\varepsilon(s-)\big)\widetilde{N}(dzds)\\
&+\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s-),z)-\sigma^\varepsilon(u^\varepsilon(s-),z)\|_H^2 N(dzds)\\
:=&\sum_{k=1}^7 I_k^{n,\varepsilon}(t).
\end{aligned}\tag{3.76}$$

By (2.6) and (2.7) we have

$$\begin{aligned}
&2|\langle B(u^{n,\varepsilon}(s))-B(u^\varepsilon(s)), u^{n,\varepsilon}(s)-u^\varepsilon(s)\rangle| = 2|\langle B(u^{n,\varepsilon}(s)-u^\varepsilon(s)), u^\varepsilon(s)\rangle|\\
\le&4\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H\|u^\varepsilon(s)\|_V\\
\le&\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V^2 + 4\|u^\varepsilon(s)\|_V^2\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2.
\end{aligned}\tag{3.77}$$

Therefore, by (2.2) and (3.77) we obtain that

$$\begin{aligned}
\sum_{k=2}^4 I_k^{n,\varepsilon}(t) \le& \int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\Big[-\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V^2\\
&\qquad + (4-\gamma)\|u^\varepsilon(s)\|_V^2\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2\Big]ds\\
\le& -\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V^2 ds,
\end{aligned}\tag{3.78}$$

if we take $\gamma \ge 4$. Using the Lipschitz continuity of F, we have

$$\begin{aligned}
&E\sup_{0\le s\le t}|I_5^{n,\varepsilon}(s)|\\
\le&E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|F_n(u^{n,\varepsilon}(s)) - F(u^\varepsilon(s))\|_H^2 ds\\
\le& E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2 ds\\
&+ 2E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|F_n(u^{n,\varepsilon}(s)) - F(u^{n,\varepsilon}(s))\|_H^2 ds\\
&+ 2E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|F(u^{n,\varepsilon}(s)) - F(u^\varepsilon(s))\|_H^2 ds\\
\le& CE\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2 ds\\
&+ 2E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|F_n(u^{n,\varepsilon}(s)) - F(u^{n,\varepsilon}(s))\|_H^2 ds. \qquad (3.79)
\end{aligned}
$$

By Burkholder's inequality, we get

$$
\begin{aligned}
&E\sup_{0\le s\le t}|I_6^{n,\varepsilon}(s)|\\
\le& 2E\Big[\int_0^t\int_{\mathbb{R}_0} e^{-2\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^\varepsilon(s),z)\|_H^2\times\\
&\qquad \|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2\nu(dz)ds\Big]^{\frac12}\\
\le& 2E\Big[\sup_{0\le s\le t} e^{-\frac{\gamma}{2}\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H\times\\
&\quad\Big(\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^\varepsilon(s),z)\|_H^2\nu(dz)ds\Big)^{\frac12}\Big]\\
\le& \frac12 E\sup_{0\le s\le t} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2\\
&+ 2E\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^\varepsilon(s),z)\|_H^2\nu(dz)ds\\
\le& \frac12 E\sup_{0\le s\le t} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2\\
&+ CE\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s) - u^\varepsilon(s)\|_H^2 ds\\
&+ 4E\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2\nu(dz)ds, \qquad (3.80)
\end{aligned}
$$

where the uniform Lipschitz constant of σ^ε has been used. Similar to (3.79), we have

$$
\begin{aligned}
&E\sup_{0\le s\le t}|I_7^{n,\varepsilon}(s)|\\
=& E\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^\varepsilon(s),z)\|_H^2\nu(dz)ds
\end{aligned}
$$

$$\leq CE\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2 ds$$
$$+\ 2E\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2\nu(dz)ds. \tag{3.81}$$

Combining (3.76), (3.78)–(3.81) together yields that for $t\leq T$,

$$E\sup_{0\leq s\leq t} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2$$
$$+\ 2E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V^2 ds$$
$$\leq 2\|h^n-h\|_H^2+CE\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2 ds$$
$$+\ 4E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|F_n(u^{n,\varepsilon}(s))-F(u^{n,\varepsilon}(s))\|_H^2 ds$$
$$+\ 12E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2\nu(dz)ds. \tag{3.82}$$

Applying Gronwall's inequality we obtain for $t\in[0,T]$,

$$E\sup_{0\leq s\leq t} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2$$
$$+\ E\int_0^t e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_V^2 ds$$
$$\leq C\times\Big[\|h^n-h\|_H^2+E\int_0^t\|F_n(u^{n,\varepsilon}(s))-F(u^{n,\varepsilon}(s))\|_H^2 ds$$
$$+E\int_0^t\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2\nu(dz)ds\Big]. \tag{3.83}$$

We claim that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}E\int_0^T\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2\nu(dz)ds=0, \tag{3.84}$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}E\int_0^T\|F_n(u^{n,\varepsilon}(s))-F(u^{n,\varepsilon}(s))\|_H^2 ds=0. \tag{3.85}$$

Suppose the above claims are proved. Then we conclude from (3.83) that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}E\sup_{0\leq s\leq T} e^{-\gamma\int_0^s \|u^\varepsilon(\rho)\|_V^2 d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2=0. \tag{3.86}$$

Let us only prove (3.84). The proof of (3.85) is similar and simpler. Let

$$G_n^\varepsilon(x):=\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(x,z)-\sigma^\varepsilon(x,z)\|_H^2\nu(dz),\qquad x\in H. \tag{3.87}$$

Note that

$$\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0} G_n^\varepsilon(x)\leq C(1+\|x\|_H^2). \tag{3.88}$$

By (3.72) and the dominated convergence theorem, to prove (3.84), it suffices to show that for each $s \in [0, T]$,

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} EG_n^\varepsilon(u^{n,\varepsilon}(s)) = 0. \tag{3.89}$$

Obviously, (3.89) will follow if the following three equalities are proved.

$$\lim_{\varepsilon\to 0} EG_n^\varepsilon(u^{n,\varepsilon}(s)) = \lim_{\varepsilon\to 0} EG_n^\varepsilon(u^n(s)), \quad \forall n \in \mathbb{N}, \tag{3.90}$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} EG_n^\varepsilon(u^n(s)) = \lim_{n\to\infty}\lim_{\varepsilon\to 0} EG_n^\varepsilon(u(s)), \tag{3.91}$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} EG_n^\varepsilon(u(s)) = 0. \tag{3.92}$$

We first prove (3.90). Since u^n is a continuous process, due to (3.71), we see that for each $n \in \mathbb{N}$, $s \in [0, T]$,

$$u^{n,\varepsilon}(s) \xrightarrow{\varepsilon\to 0} u^n(s) \quad \text{in distribution.} \tag{3.93}$$

Therefore, to prove (3.90), we can use Skorohod's representation theorem to assume that $\|u^{n,\varepsilon}(s)-u^n(s)\|_H \to 0$ almost surely as $\varepsilon \to 0$. In view of (3.72), $\{\|u^{n,\varepsilon}(s)\|_H^2\}_{\varepsilon\le\varepsilon_0}$ is uniformly integrable, and therefore, we can further deduce that

$$\lim_{\varepsilon\to 0} E\|u^{n,\varepsilon}(s) - u^n(s)\|_H^2 = 0. \tag{3.94}$$

On the other hand,

$$\begin{aligned}
&E|G_n^\varepsilon(u^{n,\varepsilon}(s)) - G_n^\varepsilon(u^n(s))|\\
\le& E\int_{\mathbb{R}_0}\Big|\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2 - \|\sigma_n^\varepsilon(u^n(s),z) - \sigma^\varepsilon(u^n(s),z)\|_H^2\Big|\nu(dz)\\
\le& E\int_{\mathbb{R}_0}\Big(\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma_n^\varepsilon(u^n(s),z)\|_H + \|\sigma^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^n(s),z)\|_H\Big)\\
&\times\Big(\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H + \|\sigma_n^\varepsilon(u^n(s),z) - \sigma^\varepsilon(u^n(s),z)\|_H\Big)\nu(dz)\\
\le&\Big[2E\int_{\mathbb{R}_0}\Big(\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma_n^\varepsilon(u^n(s),z)\|_H^2\\
&\qquad + \|\sigma^\varepsilon(u^{n,\varepsilon}(s),z) - \sigma^\varepsilon(u^n(s),z)\|_H^2\Big)\nu(dz)\Big]^{\frac{1}{2}}\\
&\times\Big[4E\int_{\mathbb{R}_0}\Big(\|\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2 + \|\sigma^\varepsilon(u^{n,\varepsilon}(s),z)\|_H^2 + \|\sigma_n^\varepsilon(u^n(s),z)\|_H^2\\
&\qquad + \|\sigma^\varepsilon(u^n(s),z)\|_H^2\Big)\nu(dz)\Big]^{\frac{1}{2}}\\
:=& I_1^\varepsilon \times I_2^\varepsilon.
\end{aligned}\tag{3.95}$$

By (3.67), (3.2), (3.72) and (3.73), we deduce that

$$\sup_{\varepsilon\le\varepsilon_0}|I_2^\varepsilon|^2 \le C \sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0} E(1 + \|u^{n,\varepsilon}(s)\|_H^2 + \|u^n(s)\|_H^2) < \infty. \tag{3.96}$$

(3.4), (3.68) and (3.94) imply

$$|I_1^\varepsilon|^2 \le CE\|u^{n,\varepsilon}(s) - u^n(s)\|_H^2 \xrightarrow{\varepsilon\to 0} 0. \tag{3.97}$$

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Therefore, (3.90) follows from (3.95), (3.96) and (3.97). In view of (3.74), a similar argument leads to

$$\lim_{n\to\infty}\sup_{\varepsilon\le\varepsilon_0} E|G_n^\varepsilon(u^n(s))-G_n^\varepsilon(u(s))|=0. \tag{3.98}$$

Hence (3.91) holds. Note that (H.4) and the condition (ii) of (H.3) imply

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(x,z)-\sigma^\varepsilon(x,z)\|_H^2\nu(dz)\\
&=\lim_{n\to\infty}\lim_{\varepsilon\to 0}\Big[\int_{\mathbb{R}_0}\|\sigma^\varepsilon(x,z)\|_H^2\nu(dz)-\int_{\mathbb{R}_0}\|\sigma_n^\varepsilon(x,z)\|_H^2\nu(dz)\Big]\\
&=\|\sigma(x)\|_H^2-\lim_{n\to\infty}\|\sigma_n(x)\|_H^2=0,\qquad\forall x\in H.
\end{aligned}\tag{3.99}$$

Therefore, (3.92) immediately follows from (3.99) and (3.88) by the dominated convergence theorem. Thus, (3.84) is proved, and so is (3.86).

Next, we proceed with the proof of (3.75). For any given $\delta_1>0$, in view of (3.9), we can choose a positive constant M_1 such that

$$\begin{aligned}
&\sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0} P\left(\sup_{0\le t\le T}\|u^{n,\varepsilon}(t)-u^\varepsilon(t)\|_H>\delta,\int_0^T\|u^\varepsilon(s)\|_V^2ds>M_1\right)\\
&\le\sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0} P\left(\int_0^T\|u^\varepsilon(s)\|_V^2ds>M_1\right)\le\delta_1.
\end{aligned}\tag{3.100}$$

On the other hand, by (3.86), we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0} P\left(\sup_{0\le t\le T}\|u^{n,\varepsilon}(t)-u^\varepsilon(t)\|_H>\delta,\int_0^T\|u^\varepsilon(s)\|_V^2ds\le M_1\right)\\
&\le\lim_{n\to\infty}\lim_{\varepsilon\to 0} P\left(\sup_{0\le s\le T} e^{-\gamma\int_0^s\|u^\varepsilon(\rho)\|_V^2d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2\ge e^{-\gamma M_1}\delta^2\right)\\
&\le e^{\gamma M_1}\frac{1}{\delta^2}\lim_{n\to\infty}\lim_{\varepsilon\to 0}E\sup_{0\le s\le T}e^{-\gamma\int_0^s\|u^\varepsilon(\rho)\|_V^2d\rho}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H^2=0.
\end{aligned}\tag{3.101}$$

Combining (3.100) and (3.101) together yields

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}P\left(\sup_{0\le t\le T}\|u^{n,\varepsilon}(t)-u^\varepsilon(t)\|_H>\delta\right)\le\delta_1. \tag{3.102}$$

Since δ_1 is arbitrary, (3.75) is proved.

Finally we prove that μ^ε converges weakly to μ. Let μ_n^ε, μ_n denote respectively the laws of $u^{n,\varepsilon}$ and u^n on $S:=D([0,T],H)$. Let G be any given bounded, uniformly continuous function on S. For any $n\ge 1$, we write

$$\begin{aligned}
&\int_S G(w)\mu^\varepsilon(dw)-\int_S G(w)\mu(dw)\\
&=\int_S G(w)\mu^\varepsilon(dw)-\int_S G(w)\mu_n^\varepsilon(dw)+\int_S G(w)\mu_n^\varepsilon(dw)-\int_S G(w)\mu_n(dw)\\
&\quad+\int_S G(w)\mu_n(dw)-\int_S G(w)\mu(dw)
\end{aligned}$$

$$=E[G(u^\varepsilon)-G(u^{n,\varepsilon})]+\left(\int_S G(w)\mu_n^\varepsilon(dw)-\int_S G(w)\mu_n(dw)\right)$$
$$+\ E[G(u^n)-G(u)]. \tag{3.103}$$

Give any $\delta>0$. Since G is uniformly continuous, there exists $\delta_1>0$ such that

$$\left|E\left[G(u^\varepsilon)-G(u^{n,\varepsilon});\ \sup_{0\le s\le T}\|u^{n,\varepsilon}(s)-u^\varepsilon(s)\|_H\le\delta_1\right]\right|\le\frac{\delta}{4} \tag{3.104}$$

for all $n\ge1$, $\varepsilon>0$. In view of (3.75) and (3.74), there exists n_1 and then ε_{n_1} such that

$$\sup_{\varepsilon\le\varepsilon_{n_1}}\left|E\left[G(u^\varepsilon)-G(u^{n_1,\varepsilon});\ \sup_{0\le s\le T}\|u^{n_1,\varepsilon}(s)-u^\varepsilon(s)\|_H>\delta_1\right]\right|$$
$$\le C\sup_{\varepsilon\le\varepsilon_{n_1}}P\left(\sup_{0\le s\le T}\|u^{n_1,\varepsilon}(s)-u^\varepsilon(s)\|_H>\delta_1\right)\le\frac{\delta}{4}, \tag{3.105}$$

and

$$|E[G(u^{n_1})-G(u)]|\le\frac{\delta}{4}. \tag{3.106}$$

On the other hand, by (3.71), there exists $\varepsilon_1>0$ such that for $\varepsilon\le\varepsilon_1$,

$$\left|\int_S G(w)\mu_{n_1}^\varepsilon(dw)-\int_S G(w)\mu_{n_1}(dw)\right|\le\frac{\delta}{4}. \tag{3.107}$$

Putting (3.103)–(3.107) together, we obtain that for $\varepsilon\le\min\{\varepsilon_{n_1},\varepsilon_1\}$,

$$\left|\int_S G(w)\mu^\varepsilon(dw)-\int_S G(w)\mu(dw)\right|\le\delta. \tag{3.108}$$

Since $\delta>0$ is arbitrarily small, we deduce that

$$\lim_{\varepsilon\to0}\int_S G(w)\mu^\varepsilon(dw)=\int_S G(w)\mu(dw) \tag{3.109}$$

completing the proof of the Theorem. ■

4. Examples

In this section, we give some examples of $\{\sigma^\varepsilon\}$ which satisfy the Hypotheses in Section 3.

Proposition 4.1. *Let σ be a global Lipschitz mapping from H into H. For each $\varepsilon>0$, let*

$$\sigma^\varepsilon(u,z)=\sigma(\theta_\varepsilon(z)u)h_\varepsilon(z),\qquad u\in H,z\in\mathbb{R}_0, \tag{4.1}$$

where $\{\theta_\varepsilon(\cdot)\}$, $\{h_\varepsilon(\cdot)\}$ are two families of real-valued functions on $\mathbb{R}_0$. Assume that $\{\theta_\varepsilon\}$ satisfies

$$\sup_{z\in\mathbb{R}_0}|\theta_\varepsilon(z)-1|\xrightarrow{\varepsilon\to0}0, \tag{4.2}$$

and $\{h_\varepsilon\}$ satisfies

$$\int_{\mathbb{R}_0}|h_\varepsilon(z)|^2\nu(dz)\xrightarrow{\varepsilon\to0}1, \tag{4.3}$$

$$\sup_{z\in\mathbb{R}_0}|h_\varepsilon(z)|\xrightarrow{\varepsilon\to0}0. \tag{4.4}$$

Then $\{\sigma^\varepsilon\}$ satisfies (H.2)–(H.4).

Proof. By (4.2), there exists a constant ε_1 such that

$$\sup_{\varepsilon\le\varepsilon_1}\sup_{z\in\mathbb{R}_0}|\theta_\varepsilon(z)|\le 2. \tag{4.5}$$

By (4.3), there exists a constant ε_2 such that

$$\sup_{\varepsilon\le\varepsilon_2}\int_{\mathbb{R}_0}|h_\varepsilon(z)|^2\nu(dz)\le 2. \tag{4.6}$$

By (4.4), there exists a constant ε_3 such that

$$\sup_{\varepsilon\le\varepsilon_3}\sup_{z\in\mathbb{R}_0}|h_\varepsilon(z)|\le 1. \tag{4.7}$$

Let $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ and assume $\varepsilon\le\varepsilon_0$ in the following calculation. The linear growth condition for σ together with (4.5) and (4.4) yield

$$\begin{aligned}\sup_{\|x\|_H\le M}\sup_{z\in\mathbb{R}_0}\|\sigma^\varepsilon(x,z)\|_H&=\sup_{\|x\|_H\le M}\sup_{z\in\mathbb{R}_0}\|\sigma(\theta_\varepsilon(z)x)\|_H|h_\varepsilon(z)|\\&\le\sup_{\|x\|_H\le M}\sup_{z\in\mathbb{R}_0}C(1+|\theta_\varepsilon(z)|\|x\|_H)\sup_{z\in\mathbb{R}_0}|h_\varepsilon(z)|\\&\le C(1+2M)\sup_{z\in\mathbb{R}_0}|h_\varepsilon(z)|\xrightarrow{\varepsilon\to0}0.\end{aligned} \tag{4.8}$$

Thus, (i) of (H.3) is satisfied. By the Lipschitz condition of σ, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}_0}\big(\sigma(\theta_\varepsilon(z)x),e_k\big)\big(\sigma(\theta_\varepsilon(z)x),e_j\big)|h_\varepsilon(z)|^2\nu(dz)\\&\quad-\int_{\mathbb{R}_0}\big(\sigma(x),e_k\big)\big(\sigma(x),e_j\big)|h_\varepsilon(z)|^2\nu(dz)\Big|\\&\le\int_{\mathbb{R}_0}\Big|\big(\sigma(\theta_\varepsilon(z)x)-\sigma(x),e_k\big)\big(\sigma(\theta_\varepsilon(z)x),e_j\big)\Big||h_\varepsilon(z)|^2\nu(dz)\\&\quad+\int_{\mathbb{R}_0}\Big|\big(\sigma(x),e_k\big)\big(\sigma(\theta_\varepsilon(z)x)-\sigma(x),e_j\big)\Big||h_\varepsilon(z)|^2\nu(dz)\\&\le\Big[\int_{\mathbb{R}_0}\|\sigma(\theta_\varepsilon(z)x)-\sigma(x)\|_H^2|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\\&\qquad\times\Big[\int_{\mathbb{R}_0}\|\sigma(\theta_\varepsilon(z)x)\|_H^2|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\\&\quad+\Big[\int_{\mathbb{R}_0}\|\sigma(\theta_\varepsilon(z)x)-\sigma(x)\|_H^2|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\\&\qquad\times\Big[\int_{\mathbb{R}_0}\|\sigma(x)\|_H^2|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\\&\le C\sup_{z\in\mathbb{R}_0}|\theta_\varepsilon(z)-1|\|x\|_H\Big[\int_{\mathbb{R}_0}|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\times\Big\{\Big[\int_{\mathbb{R}_0}(1+|\theta_\varepsilon(z)|^2\|x\|_H^2)\\&\qquad\times|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}+\Big[\int_{\mathbb{R}_0}(1+\|x\|_H^2)|h_\varepsilon(z)|^2\nu(dz)\Big]^{\frac12}\Big\}\\&\xrightarrow{\varepsilon\to0}0,\end{aligned} \tag{4.9}$$

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where we have used (4.5), (4.6) and (4.2). On the other hand, (4.3) gives

$$\int_{\mathbb{R}_0} \big(\sigma(x), e_k\big)\big(\sigma(x), e_j\big)|h_\varepsilon(z)|^2\nu(dz) \xrightarrow{\varepsilon\to 0} \big(\sigma(x), e_k\big)\big(\sigma(x), e_j\big). \tag{4.10}$$

Combining (4.9) with (4.10), condition (ii) of (H.3) is obtained. (H.2) and (H.4) can be similarly verified, we omit the details. ■

Example 4.2. Here we give some examples of θ_ε and h_ε in Proposition 4.1. One can take θ_ε to be any family of functions converging to 1 uniformly, such as

$$\theta_\varepsilon(z) = 1,\ \ 1+\varepsilon\cos z,\ \ 1-\frac{\varepsilon}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\varepsilon^2 z^2}{2}},\ \ldots. \tag{4.11}$$

And the following examples of h_ε satisfy (4.3) and (4.4).

(i)

$$h_\varepsilon(z) = \frac{1}{\sqrt{\nu(\{\varepsilon \le |z| \le 1\})}}\mathbf{1}_{\{\varepsilon\le|z|\le 1\}}, \tag{4.12}$$

where the characteristic measure ν satisfies

$$\nu(\{\varepsilon \le |z| \le 1\}) \xrightarrow{\varepsilon\to 0} \infty, \quad \text{i.e.} \quad \nu(\mathbb{R}_0) = \infty. \tag{4.13}$$

(ii)

$$h_\varepsilon(z) = \frac{z}{\sqrt{\int_{1\le|z|\le\frac{1}{\varepsilon}} |z|^2\nu(dz)}}\mathbf{1}_{\{1\le|z|\le\frac{1}{\varepsilon}\}}, \tag{4.14}$$

where the characteristic measure ν satisfies

$$\varepsilon^2\int_{1\le|z|\le\frac{1}{\varepsilon}} |z|^2\nu(dz) \xrightarrow{\varepsilon\to 0} \infty. \tag{4.15}$$

(iii)

$$h_\varepsilon(z) = \frac{z}{\sqrt{\int_{0<|z|\le\varepsilon} |z|^2\nu(dz)}}\mathbf{1}_{\{0<|z|\le\varepsilon\}}, \tag{4.16}$$

where the characteristic measure ν satisfies

$$\frac{1}{\varepsilon^2}\int_{0<|z|\le\varepsilon} |z|^2\nu(dz) \xrightarrow{\varepsilon\to 0} \infty. \tag{4.17}$$

For example, if $\nu_\alpha(dz) = \frac{1}{|z|^{1+\alpha}}dz$, which is the characteristic measure of symmetric α-stable processes, then for each $\alpha \in (0, 2)$, ν_α satisfies (4.13), (4.15) and (4.17).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

The following lemma is an infinite dimensional martingale representation theorem, which is a slight modification of Theorem 18.12 in [11].

Lemma A.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. And let $\{M^i\}_{i=1}^{\infty}$ be a sequence of continuous local $\mathcal{F}_t$-martingale such that for $i, j \in \mathbb{N}$ and $t \geq 0$,*

$$\langle M^i, M^j\rangle(t) = \int_0^t \sum_{k=1}^r \Psi_{ik}(s)\Psi_{jk}(s)ds \tag{A.1}$$

and

$$\sum_{\substack{i=1,2,\ldots \\ k=1,2,\ldots,r}} \int_0^t |\Psi_{ik}(s)|^2 ds < \infty, \quad P-a.s., \tag{A.2}$$

for some real-valued $\mathcal{F}_t$-progressive measurable processes Ψ_{ik}, $i = 1, 2, \ldots$, $k = 1, 2, \ldots, r$. Then on a standard extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{P})$ of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, there exist r independent $\widetilde{\mathcal{F}}_t$-Brownian motions $B^1(t), B^2(t), \ldots, B^r(t)$ such that for all $i \in \mathbb{N}$,

$$M^i(t) = \sum_{k=1}^r \int_0^t \Psi_{ik}(s)dB^k(s), \quad t \geq 0. \tag{A.3}$$

Proof. The proof of this lemma is a slight modification of the proof of Theorem 18.12 in [11]. We first introduce some notations. Let A be a bounded linear operator from Banach space X to Banach space Y, we denote by A^* the conjugate operator of A from the dual space of Y to the dual space of X. If A is a matrix or a vector in Euclidean spaces, then A^* denotes the transpose of A. Define the Hilbert space

$$\ell^2 := \{v = (v_1, v_2, \ldots)^* : \sum_{i=1}^{\infty} |v_i|^2 < \infty\}. \tag{A.4}$$

For any $t \geq 0$, and $x \in \mathbb{R}^r$, we define the mapping $\Psi(t)$ by

$$\Psi(t)x := (\sum_{k=1}^r \Psi_{1k}x_k, \sum_{k=1}^r \Psi_{2k}x_k, \ldots)^*. \tag{A.5}$$

From (A.2), it follows that for a.e. $t \geq 0$, $\Psi(t)$ is a Hilbert–Schmidt operator from $\mathbb{R}^r$ to ℓ^2. Let $N(t)$ and $R(t)$ be the null and range spaces of $\Psi(t)$, and write $N^{\perp}(t)$ and $R^{\perp}(t)$ for their orthogonal complements in $\mathbb{R}^r$ and ℓ^2 respectively. Denote the corresponding orthogonal projections by $\pi_{N(t)}, \pi_{R(t)}, \pi_{N^{\perp}(t)}, \pi_{R^{\perp}(t)}$, respectively. Note that $\Psi(t)$ is a bijection from $N^{\perp}(t)$ to $R(t)$, and write $\Psi^{-1}(t)$ for the inverse mapping from $R(t)$ to $N^{\perp}(t)$. All these mappings are clearly Borel-measurable functions of $\Psi(t)$, and hence again progressive measurable.

We introduce, on a probability space $(\Omega', \mathcal{F}', P')$ equipped with a filtration $\{\mathcal{F}'_t\}$, an r-dimensional $\mathcal{F}'_t$-Brownian motion $B'(t) = (B'^1(t), B'^2(t), \ldots, B'^r(t))^*$, and construct a standard extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{P})$ of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ by $\widetilde{\Omega} := \Omega \times \Omega'$, $\widetilde{\mathcal{F}} := \mathcal{F} \times \mathcal{F}'$, $\widetilde{\mathcal{F}}_t := \mathcal{F}_t \times \mathcal{F}'_t$, $\widetilde{P} := P \times P'$. Thus, Ψ remains $\widetilde{\mathcal{F}}_t$-progressive measurable and the martingale properties of M and B' are still valid for $\widetilde{\mathcal{F}}_t$. In particular, on this extension, we have $\langle M^i, B'^k\rangle(t) = 0$, $\langle B'^k, B'^l\rangle(t) = \delta_{kl}t$ and (A.1).

Consider in $\mathbb{R}^r$ the continuous local $\widetilde{\mathcal{F}}_t$-martingale

$$\begin{aligned} B(t) &= \int_0^t \Psi^{-1}(s)\pi_{R(s)} dM(s) + \int_0^t \pi_{N(s)} dB'(s) \\ &:= \sum_{i=1}^{\infty} \int_0^t \Psi^{-1}(s)\pi_{R(s)} e_i dM^i(s) + \sum_{k=1}^{r} \int_0^t \pi_{N(s)} f_k dB'^k(s), \end{aligned} \tag{A.6}$$

where e_i is the vector in ℓ^2 with a 1 in the ith coordinate and zeros elsewhere, similarly, f_k is the vector in $\mathbb{R}^r$ with a 1 in the kth coordinate and zeros elsewhere. The stochastic integral against M in (A.6) makes sense since

$$\begin{aligned} &E\left|\int_0^t \Psi^{-1}(s)\pi_{R(s)} dM(s)\right|^2 \\ =&E\int_0^t tr\left(\Psi^{-1}(s)\pi_{R(s)}\Psi(s)\Psi^*(s)(\Psi^{-1}(s)\pi_{R(s)})^*\right) ds \\ =&E\int_0^t tr\left(\pi_{N^\perp(s)}(\pi_{N^\perp(s)})^*\right) ds \\ =&E\int_0^t \sum_{k=1}^{r} |\pi_{N^\perp(s)} f_k|^2 ds < \infty, \end{aligned} \tag{A.7}$$

where tr means the trace of an operator, and we have used the equality $\Psi^{-1}(s)\pi_{R(s)}\Psi(s) = \pi_{N^\perp(s)}$. Furthermore,

$$\begin{aligned} \langle B, B\rangle(t) &= \int_0^t \Psi^{-1}(s)\pi_{R(s)}\Psi(s)\Psi^*(s)(\Psi^{-1}(s)\pi_{R(s)})^* ds \\ &\quad + \int_0^t \pi_{N(s)}(\pi_{N(s)})^* ds \\ &= \int_0^t \pi_{N^\perp(s)}(\pi_{N^\perp(s)})^* ds + \int_0^t \pi_{N(s)}(\pi_{N(s)})^* ds \\ &= \int_0^t (\pi_{N^\perp(s)} + \pi_{N(s)}) ds = tI, \end{aligned} \tag{A.8}$$

where I is the identity matrix of order $r \times r$. Hence by Lévy's characterization of Brownian motion, B is an r-dimensional $\widetilde{\mathcal{F}}_t$-Brownian motion. Note that

$$\int_0^t \pi_{R^\perp(s)} dM(s) = 0, \tag{A.9}$$

since

$$\begin{aligned} &\langle \int_0^t \pi_{R^\perp(s)} dM(s), \int_0^t \pi_{R^\perp(s)} dM(s)\rangle \\ =&\int_0^t \pi_{R^\perp(s)}\Psi(s)\Psi^*(s)(\pi_{R^\perp(s)})^* ds = 0, \end{aligned} \tag{A.10}$$

where we have used $\pi_{R^\perp(s)}\Psi(s) = 0$. Now using (A.9), we see that

$$\int_0^t \Psi(s) dB(s) = \int_0^t \Psi(s) \times \Psi^{-1}(s)\pi_{R(s)} dM(s) + \int_0^t \Psi(s) \times \pi_{N(s)} dB'(s)$$

$$
\begin{aligned}
&= \int_0^t \pi_{R(s)} dM(s) \\
&= \int_0^t (\pi_{R(s)} + \pi_{R^{\perp}(s)}) dM(s) = M(t). \quad \blacksquare \qquad (A.11)
\end{aligned}
$$

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