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RESEARCH

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The optimization of the number of deployed antennas in large-scale CL-DAS for energy efficiency

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Abstract
We consider the deployment of a large-scale circular layout-distributed antenna system. To maximize energy efficiency, we define the optimization problem with the approximated analysis and find the number of deployed antennas as a closed-form solution. In simulation results, we verify that the analysis based on approximation is accurate and the closed-form solution can achieve near-optimal energy efficiency without an exhaustive search method.

Keywords: Large-scale circular layout-distributed antenna system, Antenna deployment, Optimization, Energy efficiency

1 Introduction

A massive multiple-input multiple-output (MIMO) system, also known as a large-scale MIMO system, is considered a key technology to guarantee the requirements of next-generation communications [1, 2]. Especially, a centralized massive MIMO system can improve overall performances including spectral efficiency (SE) and link reliability by adopting a large number of antennas at the base station. However, the spatial correlation of co-located antennas at the base station limits the performance of a massive MIMO system. Moreover, a centralized massive MIMO system is not efficient to provide high-quality service for cell-edge users.

To handle these problems, the distributed antenna system (DAS) was introduced [3–7]. In DAS, the antennas are geometrically separated from each other within a cell and each distributed antenna (DA) is controlled by a central unit (CU) via dedicated channels. Therefore, the average distance between DAs and users is shorter than the centralized MIMO system and DAS can achieve enhanced performance by mitigating the effects of spatial correlation and path loss. Adopting a large number of antennas has advantages in both a centralized MIMO system and DAS, especially, in view of the SE [8]. However, as the energy consumption used by the antennas rapidly increases, the energy efficiency (EE), which is defined as the total data rate divided by the overall power consumption, of the massive MIMO system decreases [9].

The downlink capacity of multicell DAS is analyzed in [3] and a comprehensive overview of DAS in high-speed trains is introduced in [4]. The authors in [5] proposed an antenna selection and user clustering method to maximize the EE of large-scale DAS. In [6], the power allocation method for circular layout-DAS (CL-DAS) to maximize the EE was proposed. The authors in [7] analyzed the achievable data rate of large-scale CL-DAS and optimized the location of DAs to maximize the average data rate. Note that the existing studies mainly focused on resource allocation with numerical approaches, and large-scale DAS and EE were rarely considered at the same time. In [5], methods for maximizing the EE of large-scale DAS were presented, but they were heuristic algorithms without analysis. Moreover, even though an antenna deployment of the large-scale DAS significantly affects the EE of the system, not enough study on that subject has been done yet.

In this paper, we optimize an antenna deployment of large-scale CL-DAS by finding the number of deployed antennas. Especially, the optimization problem to maximize the EE of the system is defined and we perform an analysis based on an approximation to express the EE by the system parameters, explicitly. With the explicit expression, we solve the approximated optimization problem and obtain the number of deployed antennas as a closed-form solution. Simulation results show the validity of the approximated analysis. We also demonstrate that the closed-form solution of the optimization problem can attain almost the same EE compared with the optimal EE which is obtained by an exhaustive search method.

2 System model

Figure 1 illustrates the architecture of a multi-user downlink large-scale CL-DAS. The system consists of one CU, M DAs which are perfectly controlled by CU, and $K(\leq M)$ users with one receive antenna each. A CU coordinates the operation of DAs and

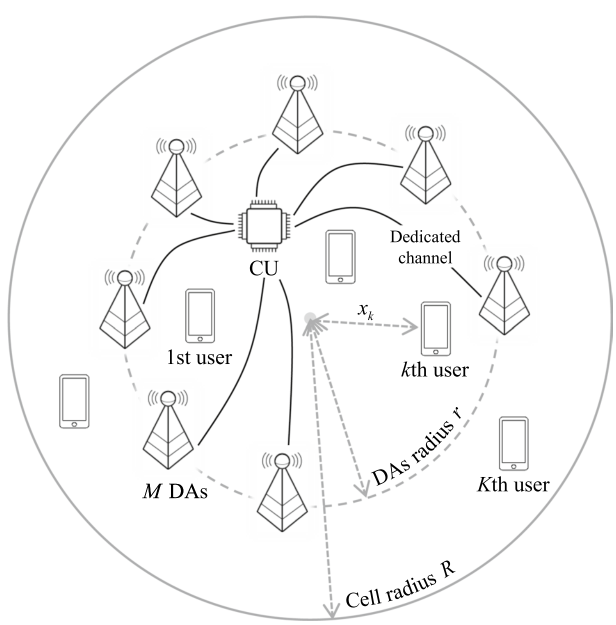

Fig. 1 A multi-user downlink large-scale CL-DAS

exchanges limited information between a CU and DAs such as configuration parameters, control parameters, and traffic load information [10]. We assume that DAs are uniformly located on the circle with a radius of r and its center is the same as that of the cell. Moreover, K users are uniformly distributed within the cell coverage radius R and the distance from the center of the cell to the kth user is denoted as x_k. Then, the downlink channel can be expressed as

$$\mathbf{G} = \mathbf{L} \circ \mathbf{H}, \tag{1}$$

where $\mathbf{G} = \left[\mathbf{g}_1^T \cdots \mathbf{g}_k^T \cdots \mathbf{g}_K^T\right]^T$ denotes a $K \times M$ channel matrix whose (k, m) element $g_{k,m}$ is a complex channel coefficient associated with the mth DA and the receive antenna of the kth user, $\mathbf{g}_k$ is the kth user's channel vector of $1 \times M$ size, $\mathbf{L}$ and $\mathbf{H}$ are a large-scale fading and a small-scale fading matrix, respectively, and "$\circ$" denotes the Hadamard product. Note that $g_{k,m} = \beta_{k,m} h_{k,m}$ where $\beta_{k,m}$ and $h_{k,m}$ are the (k, m) elements of $\mathbf{L}$ and $\mathbf{H}$. We assume that the large-scale fading factor $\beta_{k,m}$ is mainly composed of shadow fading factor ζ and path loss factor $d_{k,m}^{\alpha}$ where ζ is a random variable with log-normal distribution with standard deviation of σ_{sh}^2, $d_{k,m}$ is the distance between the mth DA and the kth user, and α is a path loss exponent, respectively. Therefore, $\beta_{k,m}$ can be modeled as $\beta_{k,m} = \sqrt{\frac{\zeta}{d_{k,m}^{\alpha}}}$. In the case of a small-scale fading, $h_{k,m}$ is zero mean circularly symmetric complex Gaussian random variable [11, 12]. Defining a received signal of kth user as y_k, we can represent a received signal vector of K users, $\mathbf{y} = [y_1 \cdots y_k \cdots y_K]^T$, as

$$\mathbf{y} = \mathbf{GVPs} + \mathbf{n}, \tag{2}$$

where $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_k \cdots \mathbf{v}_K]$ denotes an $M \times K$ precoding matrix, $\mathbf{v}_k$ is the kth user's precoding vector of $M \times 1$ size, $\mathbf{P}$ is a $K \times K$ diagonal matrix for power allocation whose kth diagonal term is given by $\sqrt{P_k}$, $\mathbf{s} = [s_1 \cdots s_k \cdots s_K]^T$ is a transmitted signal vector, s_k is the kth user's transmit symbol, and $\mathbf{n} = [n_1 \cdots n_k \cdots n_K]^T$ is a noise vector. We assume that $E\left[\mathbf{ss}^H\right] = \mathbf{I}_K$ and $E\left[\mathbf{nn}^H\right] = N_0 \mathbf{I}_K$ where $\mathbf{I}_K$ is the identity matrix of size K and N_0 is the noise spectral density. Then, the total data rate of the system, R_{tot}, is given by

$$R_{tot} = \sum_{k=1}^{K} R_k = W \sum_{k=1}^{K} \log_2 \left(1 + \frac{P_k \left|\mathbf{g}_k \mathbf{v}_k\right|^2}{\sum\limits_{j \neq k} P_j \left|\mathbf{g}_k \mathbf{v}_j\right|^2 + W N_0} \right), \tag{3}$$

where R_k is the kth user's data rate and W is the system bandwidth. To define the EE, we assume that the overall power consumption of the system, P_{tot}, is given by

$$P_{tot} = \eta_{PA} P_T + M P_R + P_C + \sum_{m=1}^{M} P_{BH,m}, \tag{4}$$

where η_{PA} is the inefficiency of power amplifier, $P_T = \sum\limits_{k=1}^{K} P_k$, P_R and P_C denote the power consumption of RF chains and circuits, respectively, and $P_{BH,m}$ represents the power consumption for the mth backhaul link of the dedicated channels [13, 14]. Then, the EE of the system can be represented as

$$\mathrm{EE} = \frac{R_{tot}}{P_{tot}} = \frac{W\sum_{k=1}^{K}\log_2\left(1+\frac{P_k|\mathbf{g}_k\mathbf{v}_k|^2}{\sum\limits_{j\neq k}P_j|\mathbf{g}_k\mathbf{v}_j|^2+WN_0}\right)}{\eta_{PA}P_T+MP_R+P_C+\sum_{m=1}^{M}P_{BH,m}}. \tag{5}$$

3 Optimization of the number of deployed antennas for energy efficiency

To maximize the EE, we optimize the antenna deployment of large-scale CL-DAS. Considering that DAs are uniformly located on the circle, the optimization of the antenna deployment is equivalent to finding the number of DAs. Therefore, the optimization problem can be defined as

$$M^* = \arg\max_{M} \mathrm{EE}, \tag{6}$$

where M^* denotes the optimal number of deployed antennas. However, taking into account that R_{tot} corresponding to the numerator of the EE is not an explicit function of M, we cannot solve the optimization problem in (6), directly. Furthermore, the Hadamard product of $\mathbf{G}$ makes it hard to find the closed-form solution. Therefore, we will solve the optimization problem with approximation.

3.1 Approximated optimization problem

As already explained, we cannot easily analyze the channel matrix $\mathbf{G}$ of large-scale CL-DAS because it is defined by the Hadamard product. To handle this problem, we approximate $\mathbf{G} = \mathbf{L} \circ \mathbf{H}$ as $\bar{\boldsymbol{G}} = \bar{\boldsymbol{L}}\mathbf{H}$ which is based on matrix multiplication. With sufficiently large M, we can assume that an approximated large-scale fading matrix $\bar{\boldsymbol{L}}$ is a diagonal matrix and it can be expressed as [12]

$$\bar{\boldsymbol{L}} = \begin{bmatrix} \beta_{1,avg} & 0 & \cdots & 0 \\ 0 & \beta_{2,avg} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{K,avg} \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} \beta_{k,avg} &= \lim_{M\to\infty}\sum_{m=1}^{M}\frac{\beta_{k,m}}{M} \\ &= \left(r^2+x_k^2\right)^{-\alpha/4} {}_2F_1\left(\frac{\alpha}{8},\frac{4+\alpha}{8};1;\frac{4r^2x_k^2}{\left(r^2+x_k^2\right)^2}\right), \end{aligned} \tag{8}$$

r is the radius of large-scale CL-DAS, x_k is the distance from the center of the cell to the kth user, and α is a path loss exponent. Note that ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ in (8) is the Euler type of the hypergeometric function and defined as [15]

$$\begin{aligned} &{}_2F_1(a,b;c;z) \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \end{aligned} \tag{9}$$

where $B(\cdot,\cdot)$ indicates the beta function. Therefore, assuming $\bar{\boldsymbol{G}} = \bar{\boldsymbol{L}}\mathbf{H}$, we can approximate R_{tot} as

$$\tilde{R}_{tot} = W\sum_{k=1}^{K}\log_2\left(1+\frac{\beta_{k,avg}^2 P_k|\mathbf{h}_k\mathbf{v}_k|^2}{\sum\limits_{j\neq k}\beta_{k,avg}^2 P_j\left|\mathbf{h}_k\mathbf{v}_j\right|^2+WN_0}\right), \tag{10}$$

where $\mathbf{h}_k$ is the kth row vector of $\mathbf{H}$. Compared with (3), $\tilde{R}_{tot}$ is easy to analyze because $\mathbf{g}_k$ based on the Hadamard product is replaced by $\mathbf{h}_k$. Utilizing the statistical characteristics of $\mathbf{h}_k$, we can express (10) in terms of M according to the type of precoder and power allocation.

From now on, to focus on the effect of the antenna deployment, we assume a zero-forcing beamforming (ZFBF) with equal power allocation. Therefore, the precoding matrix $\mathbf{V}$ is given by $\mathbf{V} = \mathbf{H}^H\left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{D}$ and $P_k = P_T/K$. Note that the power normalization matrix $\mathbf{D}$ is a diagonal matrix whose kth diagonal element $D_k = \frac{1}{\sqrt{\left[\left(\mathbf{H}\mathbf{H}^H\right)^{-1}\right]_{k,k}}}$ [16]. Since ZFBF eliminates inter-user interference, we only need to calculate the average desired channel gain of ZFBF, $E\left[|\mathbf{h}_k\mathbf{v}_k|^2\right]$, in (10). Considering that $|\mathbf{h}_k\mathbf{v}_k|^2 = D_k^2$ is a Chi-square random variable whose distribution is defined by $f(x) = \frac{e^{-x}x^{M-K}}{(M-K)!}$ [17], by applying the integration by parts, the average desired channel gain of ZFBF, $E\left[|\mathbf{h}_k\mathbf{v}_k|^2\right]$, can be calculated as

$$\begin{aligned} E\left[|\mathbf{h}_k\mathbf{v}_k|^2\right] &= E\left[D_k^2\right] = \int_0^\infty x\cdot\frac{e^{-x}x^{M-K}}{(M-K)!}dx \\ &= \frac{1}{(M-K)!}\int_0^\infty e^{-x}\cdot x^{M-K+1}dx \\ &= \frac{(M-K+1)!}{(M-K)!}\int_0^\infty e^{-x}dx = M-K+1. \end{aligned} \tag{11}$$

Therefore, we can rewrite $\tilde{R}_{tot}$ as

$$\tilde{R}_{tot} = W\sum_{k=1}^{K}\log_2\left(1+\beta_{k,avg}^2\frac{P_T}{K}\frac{M-K+1}{WN_0}\right). \tag{12}$$

Finally, approximating $\beta_{k,avg}^2 \approx \beta_{avg}^2 = \left(\sum\limits_{k=1}^{K}\beta_{k,avg}^2\right)/K$, we can obtain an approximated optimization problem with explicit expression of M as

$$M^* = \arg\max_M \mathrm{EE}^{approx}, \tag{13}$$

where

$$\mathrm{EE}^{approx} = \frac{\tilde{R}_{tot}}{P_{tot}} = \frac{WK\log_2\left(1+\beta_{avg}^2\frac{P_T}{K}\frac{M-K+1}{WN_0}\right)}{\eta_{PA}P_T + MP_R + P_C + \sum\limits_{m=1}^{M}P_{BH,m}}. \tag{14}$$

3.2 Closed-form solution

In this subsection, we prove that the objective function, EE^{approx}, of (13) is concave to M and find the solution as a closed-form by using the Lambert-W function.

First, we define EE^{approx} as a function L of M:

$$\mathcal{L}(M) = \frac{WK\log_2\left(1+\beta_{avg}^2\frac{P_T}{K}\frac{M-K+1}{WN_0}\right)}{\eta_{PA}P_T + MP_R + P_C + \sum\limits_{m=1}^{M} P_{BH,m}}. \quad (15)$$

Because the numerator of L is a logarithmic function to M and the denominator of L is a linear function to M, (15) is concave to M. Since the function L of M is concave, we can find a unique global optimal point M^* of an approximated optimization problem by solving the following equation:

$$\left.\frac{\partial L}{\partial M}\right|_{M=M^*} = 0. \quad (16)$$

Calculating (16), we can obtain the closed-form solution for the number of deployed antenna M^* as

$$M^* = (\kappa-1)\left(\frac{WN_0K}{\beta_{avg}^2P_T}\right) + K - 1, \quad (17)$$

where $\kappa = e^{\lambda+1}$, $\lambda = W\left(\frac{\mu}{e}\right)$ and $\mu = \frac{\beta_{avg}^2}{WN_0}\frac{P_T}{K}\left(\frac{\eta_{PA}P_T+P_C+\sum\limits_{m=1}^{M}P_{BH,m}}{P_R}+K-1\right)-1$. Here, W denotes the Lambert-W function, which is defined as $W(z)e^{W(z)}=z$ where z is any complex number [18]. Considering that the closed-form solution (17) is explicitly expressed by K, W, P_T, P_R, P_C, $P_{BH,m}$ and other system parameters, we can easily analyze the effect of each system parameter on the number of deployed antennas. Moreover, by substituting (17) into (14), we can obtain the closed-form expression of EE. Note that since the closed-form solution is optimal for an approximated optimization problem, it cannot guarantee the optimality of the antenna deployment. Therefore, we will verify the accuracy of the closed-form solution in the simulation section.

4 Simulation results and discussion

From the simulation results, we verify the validity of the approximation for the analysis and accuracy of the closed-form solution. K users are uniformly distributed within the cell coverage and there is the minimum separation distance between a user and a DA, guard distance. Other system parameters used in the simulations are described in Table 1. Note that all the simulation results are obtained by averaging over multiple snapshots.

Figure 2 shows true EE value and approximated EE value based on (5) and (14), respectively, according to the number of deployed antennas, M. We performed simulations for various K, from 5 to 40, and the optimization of the number of deployed antennas is not applied. Comparing true EE value and approximated EE value in the

Table 1 System parameters for simulations

Parameter	Notation	Value
Transmit power	P_T	35 [dBm]
Noise spectral density	N_0	−104 [dBm]
Cell radius	R	1000 [m]
Shadow fading standard deviation	σ_{sh}	4 [dB]
Path loss exponent	α	2
System bandwidth	W	10 [MHz]
Antenna gain	ϕ_c	1
Guard distance	g_d	20 [m]
DAs' radius	r	$2R/3$
Power inefficiency of power amplifier	η_{PA}	1/0.12
Circuit power consumption	P_C	45 [W]
RF chain power consumption	P_R	4.5 [W]
mth backhaul link power consumption	$P_{BH,m}$	1.075 [W]

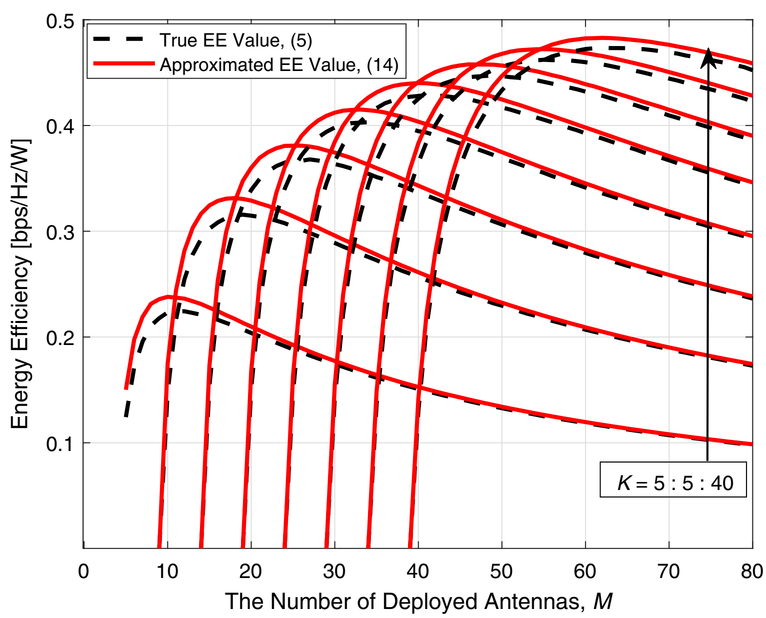

Fig. 2 True and approximated EE value according to various M and K

simulations, we can verify that the analysis based on approximation is valid even though the number of deployed antennas is finite. Moreover, we can confirm that (15), a function L of M, is concave as we explained, regardless of the number of users K.

Figure 3 represents the number of deployed antennas over the various number of users and it was obtained by the closed-form solution (17) and an exhaustive search method. For the simulation results of an exhaustive search method, we calculated EE for all possible candidates and chose the optimal number of deployed antennas maximizing EE for every channel realization. Compared to an exhaustive search method, the number of deployed antennas based on the analysis differs by a maximum of three, which is only about a 5% difference. This gap between the exhaustive search

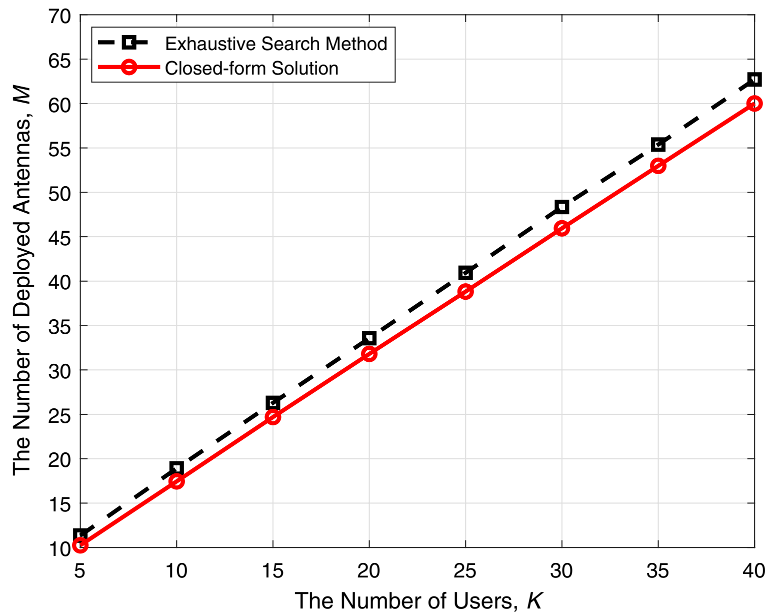

Fig. 3 The number of deployed antennas by the closed-form solution and an exhaustive search method

method and closed-form solution comes from the large-scale fading approximation and conversion process of a closed-form solution (17) to an integer. However, considering the high computational complexity of an exhaustive search method, we can conclude that the closed-form solution based on approximation is very effective for the real-time operations of practical systems.

Figure 4 describes the optimal EE by an exhaustive search method and achievable EE by the closed-form solution over the various number of users, *K*. As shown in Fig. 4,

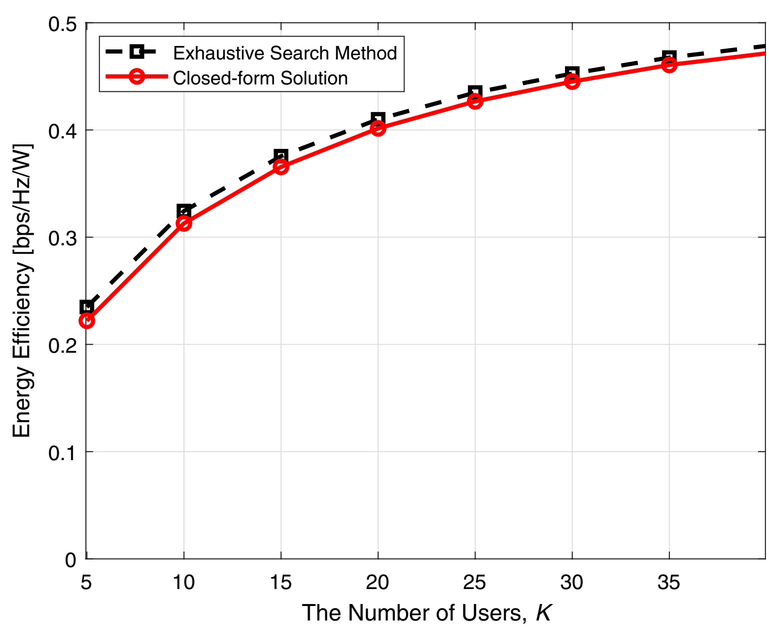

Fig. 4 Optimal EE by an exhaustive search method and achievable EE by the closed-form solution

utilizing the closed-form solution based on the analysis with approximation, we can achieve nearly optimal EE without an exhaustive search method.

5 Conclusion

In this paper, we approximated the function of EE in the large-scale CL-DAS and optimized an antenna deployment to maximize the EE. Especially, by solving the optimization problem, we found the number of deployed antennas as a closed-form solution. Simulation results showed that the analysis based on approximation is valid and nearly optimal EE can be achieved in a closed-form solution compared with an exhaustive search method. However, since the CL-DAS is not general architecture, the results of this study have a limitation. Therefore, in future works, we will extend the research to a generalized system model, large-scale random layout-DAS.

6 Methods/experimental

The purpose of this study is to optimize an antenna deployment of the large-scale CL-DAS in view of EE. The large-scale CL-DAS consists of one CU, M DAs, and K users with one receive antenna each. DAs are uniformly located on the circle with a radius of r and users are uniformly distributed within the cell coverage radius R. The channels between DAs and users are modeled based on Hadamard product of large-scale fading and small-scale fading. For given system parameters including the number of DAs and users, the energy efficiency can be calculated by utilizing the solution of an approximated optimization problem.

Abbreviations

MIMO	Multiple-input multiple-output
SE	Spectral efficiency
DAS	Distributed antenna system
DA	Distributed antenna
CU	Central unit
EE	Energy efficiency
CL-DAS	Circular layout-distributed antenna system
RF	Radio frequency
ZFBF	Zero-forcing beamforming

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Author contributions
Han presented the basic idea, developed the theory, and performed the simulations. Sim verified the analytical methods and wrote the manuscript. All authors discussed the results and approved the final manuscript.

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Availability of data and materials
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Declarations

Competing interests
The authors declare that they have no competing interests.

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