

RESEARCH

Open Access



# Some spectral domain in approximate point-spectrum-preserving maps on $\mathcal{B}(\mathcal{X})$

Ying Wang<sup>1\*</sup>

\*Correspondence:  
[eaglesoars@126.com](mailto:eaglesoars@126.com)

<sup>1</sup>China Academy of Space  
Technology (Xi'an), Xi'an, 710100,  
China

## Abstract

Let  $\mathcal{X}$  be an infinite-dimensional complex Banach space,  $\mathcal{B}(\mathcal{X})$  the algebra of all bounded linear operators on  $\mathcal{X}$ . Denote the spectral domain by  $\sigma_{\gamma}(T) = \{\lambda \in \sigma_a(T) : T \text{ that is semi-Fredholm and } \text{asc}(T - \lambda I) < \infty\}$ . In this paper, we characterize the structure of additive surjective maps  $\varphi : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{X})$  with  $\sigma_{\gamma}(\varphi(T)) = \sigma_{\gamma}(T)$  for all  $T \in \mathcal{B}(\mathcal{X})$ .

**Mathematics Subject Classification:** Primary 47B48; secondary 47A10

**Keywords:** Spectral domain; Approximate point spectrum; Additive preserver

## 1 Introduction

The study of preserver problems has a long history and has established many remarkable results in past decades. Preserver problems aim to characterize those linear or nonlinear maps on operator algebras preserving certain properties, subsets or relations ([1–6, 9, 12, 15, 16, 22]). One of the most famous problems in this area is Kaplansky's problem [12] asking whether every surjective unital invertibility preserving linear map between two semisimple Banach algebras is a Jordan homomorphism. This problem was solved in some special cases of semisimple Banach algebras ([10, 13, 23]). Then, Aupetit in [2] solved it for von Neumann algebras. It is known that a spectrum is a very fundamental and key concept in operator theory. Some results about linear or additive maps preserving the spectrum as well as certain parts of the spectrum have been established by many authors ([2, 9, 20, 21]). Recently, many authors are interested in nonadditive preserver problems related to spectral domains of operators. For example, Hajighasemi and Hejazian in [11] characterized the nonlinear surjective maps on  $\mathcal{B}(\mathcal{X})$  preserving the semi-Fredholm domain, the Fredholm domain, and the Weyl domain respectively. In [4], Bouramdane and Ech-Chérif El Kettani investigated the form of maps preserving some spectral domains of the skew product of operators. As is known, certain parts of a spectrum of operators are introduced to analyze the structure of operators, such as various spectra in the Weyl-type theorem. The Weyl-type theorem can reflect the connections between several spectra, which has been studied for more than one hundred years. There have been numerous significant results in terms of this. Note that some spectral domains play a key role in the research of the Weyl-type theorem and its perturbation [8, 17–19]. Moreover, these spectral domains are at most countable and very “small” subsets of a spectrum in general, such

© The Author(s) 2023. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

as normal eigenvalues, the semi-Fredholm domain in a spectrum, and so on. Thus, how may these spectral domains influence the structure of automorphisms on the algebra of all bounded linear operators on a Banach space? In [20], the authors characterized additive surjective maps  $\varphi$  on  $\mathcal{B}(\mathcal{X})$  that preserve the semi-Fredholm domain in a spectrum, and showed that such a map is an automorphism or an antiautomorphism on  $\mathcal{B}(\mathcal{X})$ . In [7], Cao discussed the linear surjective maps preserving upper semi-Weyl operators, and showed that their induced maps on the Calkin algebra are Jordan automorphisms. In this paper, we combine the approximate point spectrum with a semi-Fredholm domain of operators, and consider an additive map that preserves the intersection of a semi-Fredholm domain with finite ascent and approximate point spectrum. How does the spectral domain influence the structure of automorphisms on the algebra of all bounded linear operators on a Banach (or Hilbert) space?

Throughout this paper let  $\mathcal{X}$  be a complex infinite-dimensional Banach space and  $\mathcal{B}(\mathcal{X})$  the algebra of all bounded linear operators on  $\mathcal{X}$ . For  $T \in \mathcal{B}(\mathcal{X})$ , we denote by  $\mathcal{X}^*$ ,  $T^*$ ,  $\mathcal{N}(T)$ , and  $\mathcal{R}(T)$  the dual space of  $\mathcal{X}$ , the conjugate operator, the null space, and the range of  $T$ , respectively. Let  $\dim \mathcal{N}(T)$  be the dimension of  $\mathcal{N}(T)$ , and  $\text{codim} \mathcal{R}(T)$  be the codimension of  $\mathcal{R}(T)$ . An operator  $T$  is called Fredholm if  $\mathcal{R}(T)$  is closed,  $\dim \mathcal{N}(T) < \infty$  and  $\text{codim} \mathcal{R}(T) < \infty$ . Also, an operator  $T$  is called semi-Fredholm if  $\mathcal{R}(T)$  is closed and  $\dim \mathcal{N}(T) < \infty$  or  $\text{codim} \mathcal{R}(T) < \infty$ . If  $T$  is a semi-Fredholm operator, then the index of  $T$  is denoted by  $\text{ind}(T) = \dim \mathcal{N}(T) - \text{codim} \mathcal{R}(T)$ . An operator  $T$  is Weyl if it is Fredholm of index zero. Recall that a bounded operator  $T$  is said to be bounded below if it is injective and has closed range. For an operator  $T$ , the ascent of  $T$  is defined by

$$\text{asc}(T) = \inf \{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}.$$

If the infimum does not exist, then the  $\text{asc}(T)$  is defined as  $\infty$ . It is known that

$$\text{asc}(T) \leq p \Leftrightarrow \mathcal{R}(T^p) \cap \mathcal{N}(T) = \{0\}$$

for some  $p \geq 0$ .

Denote the spectrum, the point spectrum, and the approximate point spectrum of  $T$ , respectively, by

$$\begin{aligned}\sigma(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}, \\ \sigma_p(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}, \\ \sigma_a(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.\end{aligned}$$

In [24, 25], Cao considered property  $(R)$ , which is a variant of Weyl's theorem. Also, there is a spectral domain that plays an important role in the study of property  $(R)$ . Now, we define the spectral domain by

$$\sigma_\gamma(T) = \{\lambda \in \sigma_a(T) : T \text{ is semi-Fredholm and } \text{asc}(T - \lambda I) < \infty\}.$$

Note that if  $\lambda \in \sigma_\gamma(T)$ , then  $T - \lambda I$  is a semi-Fredholm operator. Thus,  $T - \lambda I$  has closed range. It follows from  $\lambda \in \sigma_a(T)$  that  $\mathcal{N}(T - \lambda I) \neq \{0\}$ . This implies that  $\sigma_\gamma(T) \subseteq \sigma_p(T)$ .

Moreover, since  $\text{asc}(T - \lambda I) < \infty$ , we have that there exists  $\epsilon > 0$  such that  $T - \lambda I$  is bounded below for all  $\lambda$  with  $0 < |\lambda - \lambda_0| < \epsilon$ . Then,  $\lambda \in \text{iso}\sigma_a(T)$ , which induces that

$$\sigma_\gamma(T) \subseteq \text{iso}\sigma_a(T) \subseteq \sigma(T).$$

Thus, the spectral domain  $\sigma_\gamma(T)$  is at most countable and is a very “small” subset of a spectrum in general.

In this paper, we will characterize additive surjective maps  $\varphi$  on  $\mathcal{B}(\mathcal{X})$  preserving the spectral domain  $\sigma_\gamma(T)$  in both directions. The main result is the following Theorem:

**Theorem 1.1** *Let  $\varphi : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{X})$  be a surjective additive map. If  $\sigma_\gamma(\varphi(T)) = \sigma_\gamma(T)$  for all  $T \in \mathcal{B}(\mathcal{X})$ , then one of the following assertions holds:*

(1) *there is an invertible operator  $A \in \mathcal{B}(\mathcal{X})$  such that*

$$\varphi(T) = ATA^{-1} \quad \text{for all } T \in \mathcal{B}(\mathcal{X});$$

(2) *there is a bounded invertible linear operator  $C : \mathcal{X}^* \rightarrow \mathcal{X}$  such that*

$$\varphi(T) = CT^*C^{-1} \quad \text{for all } T \in \mathcal{B}(\mathcal{X}).$$

*In this case,  $\mathcal{X}$  must be a reflexive space.*

## 2 Preliminaries

Let  $z \in \mathcal{X}$  and  $f \in \mathcal{X}^*$ , we denote by  $z \otimes f$  the bounded linear rank-one operator if both  $z$  and  $f$  are nonzero. The rank-one operator  $z \otimes f$  is defined by  $(z \otimes f)x = f(x)z$  for all  $x \in \mathcal{X}$ . For a subset  $\mathcal{M}$  of  $\mathcal{X}$ ,  $\bigvee \{\mathcal{M}\}$  denotes the closed subspace spanned by  $\mathcal{M}$ . We first establish some useful results that are needed for the proof of our main Theorem.

**Proposition 2.1** *Let  $T \in \mathcal{B}(\mathcal{X})$  be such that  $0 \in \sigma_\gamma(T)$ . If  $K \in \mathcal{B}(\mathcal{X})$  is a rank-one operator such that  $0 \in \sigma_\gamma(T + 2K)$ , then  $0 \in \sigma_\gamma(T + K)$  or  $0 \in \sigma_\gamma(T - K)$ .*

*Proof* From the fact that the class of Fredholm operators is invariant under compact perturbations and  $0 \in \sigma_\gamma(T)$ , we derive that the operators  $T + K$ ,  $T - K$  are Fredholm.

Let  $K = x \otimes f$ , where  $x \in \mathcal{X}$  and  $f \in \mathcal{X}^*$ . Then, it follows from  $0 \in \sigma_\gamma(T + 2K) \subseteq \sigma_p(T + 2K)$  that  $\mathcal{N}(T + 2K) \neq \{0\}$ . Moreover, note that  $0 \in \sigma_\gamma(T) \subseteq \sigma_p(T)$ , we also have  $\mathcal{N}(T) \neq \{0\}$ . We claim that

$$\mathcal{N}(T + K) \neq \{0\} \quad \text{and} \quad \mathcal{N}(T - K) \neq \{0\}.$$

Note that  $\mathcal{N}(T) \cap \mathcal{N}(f) \subseteq (\mathcal{N}(T + K) \cap \mathcal{N}(T - K))$ . Thus, we assume that  $\mathcal{N}(T) \cap \mathcal{N}(f) = \{0\}$ . Then, there exist nonzero vectors  $u \in \mathcal{N}(T)$  and  $v \in \mathcal{N}(T + K)$  such that  $f(u) = f(v) = 1$ . Take  $w = u + v$  and  $z = 3u - v$ , then  $f(w) = f(z) = 2$ . Thus, we obtain that  $w, z$  are nonzero vectors such that

$$w \in \mathcal{N}(T + K) \quad \text{and} \quad z \in \mathcal{N}(T - K).$$

Thus, we have that the operators  $T - K$  and  $T + K$  are Fredholm with  $\mathcal{N}(T - K) \neq \{0\}$  and  $\mathcal{N}(T + K) \neq \{0\}$ .

Moreover, note that  $0 \in \sigma_\gamma(T)$ , then  $\text{asc}(T) < \infty$ . By the Proposition 2.7 in [14], we obtain that  $\text{asc}(T - K) < \infty$  or  $\text{asc}(T + K) < \infty$ . This implies that  $0 \in \sigma_\gamma(T + K)$  or  $0 \in \sigma_\gamma(T - K)$ .  $\square$

The following result will give a necessary condition for operators with the rank not less than one.

**Proposition 2.2** *Let  $T \in \mathcal{B}(\mathcal{X})$  and  $\dim R(T) \geq 2$ . Then, there exists an operator  $S$  such that  $0 \in \sigma_\gamma(S)$  and  $\sigma_\gamma(S + 2T)$ , but  $0 \notin \sigma_\gamma(S - T)$  and  $0 \notin \sigma_\gamma(S + T)$ .*

*Proof* Assume that  $\dim R(T) \geq 2$ . Then, there exist two linearly independent vectors  $x_1, x_2$  such that  $Tx_1, Tx_2$  are linearly independent. Let  $\mathcal{L} = \bigvee \{x_1, x_2\}$ ,  $\mathcal{M} = \bigvee \{Tx_1, Tx_2\}$ . Then, there are two infinite-dimensional closed subspaces  $\mathcal{K}, \mathcal{R}$  such that  $\mathcal{X} = \mathcal{L} \oplus \mathcal{K} = \mathcal{M} \oplus \mathcal{R}$ , with respect to this decomposition, the operator  $T$  can be expressed as follows:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

Note that  $\dim \mathcal{L} = \dim \mathcal{M} = 2$ . This implies that

$$\mathcal{R} \cong \mathcal{X}/\mathcal{L} \cong \mathcal{X}/\mathcal{M} \cong \mathcal{K},$$

where “ $\cong$ ” denotes isomorphism. Then, the two closed subspaces  $\mathcal{K}, \mathcal{R}$  are isomorphic. Thus, we can find an invertible operator  $A : \mathcal{R} \rightarrow \mathcal{K}$ . For the operator  $T_{22}A : \mathcal{R} \rightarrow \mathcal{R}$ , there exists a complex number  $\mu \in \mathbb{C}$  such that the operators  $\mu I + T_{22}A$ ,  $\mu I + 2T_{22}A$ , and  $\mu I - 2T_{22}A$  are invertible. It follows that the operators  $\mu A^{-1} + T_{22}$ ,  $\mu A^{-1} + 2T_{22}$  and  $\mu A^{-1} - 2T_{22}$  are invertible. With respect to the decomposition  $\mathcal{X} = \mathcal{L} \oplus \mathcal{K} = \mathcal{M} \oplus \mathcal{R}$ , we now define an operator  $S \in \mathcal{B}(\mathcal{X})$  by

$$S = \begin{pmatrix} S_{11} & 0 \\ 0 & \mu A^{-1} \end{pmatrix},$$

where

$$\begin{cases} S_{11}x_1 = 0, \\ S_{11}x_2 = -2Tx_2. \end{cases}$$

Then,

$$S + 2T = \begin{pmatrix} S_{11} + 2T_{11} & 2T_{12} \\ 0 & \mu A^{-1} + 2T_{22} \end{pmatrix}.$$

We can verify that  $0 \in \sigma_\gamma(S)$  and  $0 \in \sigma_\gamma(S + 2T)$ . Moreover,

$$S \pm T = \begin{pmatrix} S_{11} \pm T_{11} & \pm T_{12} \\ 0 & \mu A^{-1} \pm T_{22} \end{pmatrix}.$$

Note that  $(S_{11} \pm T_{11})x_1 = \pm Tx_1$ ,  $(S_{11} + T_{11})x_2 = -Tx_2$  and  $(S_{11} - T_{11})x_2 = -3Tx_2$ . This implies that the operators  $S + T$  and  $S - T$  are invertible operators. Therefore,

$$0 \notin \sigma_\gamma(S + T) \quad \text{and} \quad 0 \notin \sigma_\gamma(S - T). \quad \square$$

The next Proposition gives a criterion of two operators being equal by rank-one operators and the spectral domain  $\sigma_\gamma(\bullet)$ .

**Proposition 2.3** *Let  $A, B \in \mathcal{B}(\mathcal{X})$ . If  $\sigma_\gamma(A + F) = \sigma_\gamma(B + F)$  for all rank-one operator  $F \in \mathcal{B}(\mathcal{X})$ . Then,  $A = B$ .*

*Proof* For any nonzero vector  $x \in \mathcal{X}$ , let  $\mathcal{G} = \{f \in \mathcal{X}^* | f(x) = 1\}$ . We choose a scalar  $\alpha \in \mathbb{C}$  such that  $\alpha > \|A\| + \|B\|$ . For any  $f \in \mathcal{G}$ , we define an operator

$$F_f = (A - \alpha I)x \otimes f.$$

Then,  $F_fx = Ax - \alpha x$ . Thus,  $\alpha \in \sigma_p(A - F_f)$ . Note that

$$\|A - F_x\| \geq \alpha > \|A\| \geq \|A\|_e = \|A - F_x\|_e,$$

where  $\|A\|_e$  is the essential norm of  $A$ . We derive that  $\alpha \in \sigma_\gamma(A - F_f)$ , and so  $\alpha \in \sigma_\gamma(B - F_f)$ . It follows that  $\alpha \in \sigma_p(B - F_f)$ . Then, there exists a nonzero vector  $y_f \in \mathcal{X}$  such that  $(B - F_f)y_f = \alpha y_f$ . We can obtain

$$y_f = f(y_f)(B - \alpha I)^{-1}(A - \alpha I)x.$$

Putting  $y = (B - \alpha I)^{-1}(A - \alpha I)x$ , it follows from  $(B - F_f)y_f = \alpha y_f$  that  $(B - F_f)y = \alpha y$  for any  $f \in \mathcal{G}$ . We claim that  $x$  and  $y$  are linearly dependent. Indeed, if  $x$  and  $y$  are linearly independent, then there exists some  $f' \in \mathcal{G}$  such that  $f'(y) = 0$ . This implies that  $By = \alpha y$ . This is a contradiction with the fact  $\alpha > \|A\| + \|B\|$ . It follows that  $(B - F_f)x = \alpha x$ . Therefore, we obtain  $Ax = Bx$ . From the arbitrariness of  $x$ , we have  $A = B$ .  $\square$

### 3 Proof of main result

In the following, we will give the proof of the main theorem and show the result in four steps.

*Step 1.*  $\varphi$  is injective.

Let  $\varphi(T) = 0$ . If  $T \neq 0$ , then we can choose  $z_0 \in \mathcal{X}$  such that  $Tz_0 \neq 0$ . It follows that there is  $f \in \mathcal{X}^*$  such that  $f(z_0) = 1$  and  $f(Tz_0) \neq 0$ . Choose a scalar  $\mu_0 \in \mathbb{C}$  such that  $|\mu_0| > \|T\|$ . Now, we define a rank-one operator  $U \in \mathcal{B}(\mathcal{X})$  as follows

$$U = (Tz_0 + \mu_0 z_0) \otimes f.$$

Then, we have  $(U - T)(z_0) = \mu_0 z_0$ . This implies that  $\mu_0 \in \sigma_\gamma(U - T)$ . Thus,

$$\mu_0 \in \sigma_\gamma(\varphi(U - T)) = \sigma_\gamma(\varphi(U)) = \sigma_\gamma(U).$$

However,  $\sigma_\gamma(U) = \{\mu_0 + f(Tz_0)\}$ , which is a contradiction. Thus,  $\varphi$  is injective.

*Step 2.*  $\varphi$  preserves rank-one operators in both directions.

Let  $T \in \mathcal{B}(\mathcal{X})$  be such that  $\dim R(T) \geq 2$ . From Proposition 2.2, we derive that there exists an operator  $S$  such that

$$0 \in \sigma_\gamma(S) \quad \text{and} \quad 0 \in \sigma_\gamma(S + 2T).$$

However,

$$0 \notin \sigma_\gamma(S - T) \quad \text{and} \quad 0 \notin \sigma_\gamma(S + T).$$

Then,

$$0 \in \sigma_\gamma(\varphi(S)) \quad \text{and} \quad 0 \in \sigma_\gamma(\varphi(S) + 2\varphi(T)).$$

Moreover,

$$0 \notin \sigma_\gamma(\varphi(S) + \varphi(T)) \quad \text{and} \quad 0 \notin \sigma_\gamma(\varphi(S) - \varphi(T)).$$

Then, by Proposition 2.1, we have that  $\dim R(\varphi(T)) \geq 2$ . Since  $\varphi$  is bijective and  $\varphi^{-1}$  has the same property as  $\varphi$ , it follows that  $\varphi$  preserves the set of operators of rank one in both directions.

*Step 3.*  $\varphi$  preserves idempotents of rank one and their linear spans in both directions.

Let  $x \otimes f$  be an idempotent of rank one and  $\varphi(x \otimes f) = y \otimes g$ . Then,  $\sigma_\gamma(x \otimes f) = \{1\}$ , and so  $\sigma_\gamma(y \otimes g) = \{1\}$ . This implies that  $\sigma(y \otimes g) = \{0, 1\}$ . Then,  $y \otimes g$  is an idempotent of rank one. That is,  $g(y) = 1$ . Thus,  $\varphi$  preserves idempotents of rank one in both directions. In the following, we will prove  $\varphi(\mathbb{C}x \otimes f) \subseteq \mathbb{C}y \otimes g$ . We choose a nonzero vector  $z \in \mathcal{X}$  and a linear function  $h \in \mathcal{X}^*$  such that

$$g(z) = 0, \quad h(y) = 0 \quad \text{and} \quad h(z) = 1.$$

For the operator  $y \otimes h$  and  $z \otimes g$ , since  $\varphi$  is surjective, there are two rank-one operators  $y_0 \otimes h_0$  and  $z_0 \otimes g_0$  such that

$$\varphi(y_0 \otimes h_0) = y \otimes h \quad \text{and} \quad \varphi(z_0 \otimes g_0) = z \otimes g.$$

Note that

$$\varphi(x \otimes f + y_0 \otimes h_0) = y \otimes g + y \otimes h, \quad \varphi(x \otimes f + z_0 \otimes g_0) = y \otimes g + z \otimes g.$$

Then,  $x \otimes f + y_0 \otimes h_0$  and  $x \otimes f + z_0 \otimes g_0$  are two rank-one operators. Thus, for any nonzero  $\lambda \in \mathbb{C}$ ,  $\lambda x \otimes f + y_0 \otimes h_0$  and  $\lambda x \otimes f + z_0 \otimes g_0$  are also rank-one operators. Fix a nonzero complex number  $\lambda$ , we let  $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$ , where  $y_\lambda \otimes g_\lambda$  is a rank-one idempotent. Then,  $\varphi(\lambda x \otimes f + y_0 \otimes h_0) = \lambda y_\lambda \otimes g_\lambda + y \otimes h$  is also rank one. We obtain that  $y_\lambda$  and  $y$  are linearly dependent or the same is true for  $g_\lambda$  and  $h$ . We claim that  $y_\lambda$  and  $y$  are linearly dependent. Indeed, suppose on the contrary, we can obtain  $g_\lambda$  and  $h$  are linearly dependent. Then, there exists a nonzero  $\alpha_\lambda \in \mathbb{C}$  such that  $g_\lambda = \overline{\alpha_\lambda} h$ . Thus,

$$\varphi(\lambda x \otimes f + z_0 \otimes g_0) = \alpha_\lambda \lambda y_\lambda \otimes h + z \otimes g.$$

Since  $h$  and  $g$  are linearly independent, we choose  $\beta_\lambda \in \mathbb{C}$  such that  $y_\lambda = \beta_\lambda z$ , and so  $\varphi(\lambda x \otimes f) = \alpha_\lambda \beta_\lambda \lambda z \otimes h$ , where  $z \otimes h$  is a rank-one idempotent. Note that  $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$ , where  $y_\lambda \otimes g_\lambda$  is a rank-one idempotent. Thus,  $\alpha_\lambda \beta_\lambda \lambda = \lambda$ . It follows that

$$\varphi(\lambda x \otimes f) = \lambda z \otimes h.$$

Since  $\varphi$  is surjective, there is a rank-one operator  $e_0 \otimes k_0$  such that  $\varphi(e_0 \otimes k_0) = z \otimes h$  and  $k_0(e_0) = 1$ . Since  $y \otimes g + z \otimes h$  is a projection of rank two, we have  $x \otimes f + e_0 \otimes k_0$  is a rank-two operator. Thus,  $\lambda x \otimes f + e_0 \otimes k_0$  is also a rank-two operator. However,

$$\varphi(\lambda x \otimes f + e_0 \otimes k_0) = (\lambda + 1)z \otimes h.$$

This contradicts the fact that  $\varphi$  preserves the set of operators of rank one in both directions. Therefore, we have that  $y_\lambda$  and  $y$  are linearly dependent. Then, there exists a nonzero  $\gamma_\lambda \in \mathbb{C}$  such that  $y_\lambda = \gamma_\lambda y$ . Since  $\varphi(\lambda x \otimes f + z_0 \otimes g_0) = \gamma_\lambda \lambda y \otimes g_\lambda + z \otimes g$  and the two vectors  $y$  and  $z$  are linearly independent, we can find a nonzero  $\mu_\lambda \in \mathbb{C}$  such that  $g_\lambda = \overline{\mu_\lambda} g$ . Then,  $\varphi(\lambda x \otimes f) = \gamma_\lambda \mu_\lambda \lambda y \otimes g$ , where  $y \otimes g$  is a rank-one idempotent. We know that  $\varphi(\lambda x \otimes f) = \lambda y_\lambda \otimes g_\lambda$ , where  $y_\lambda \otimes g_\lambda$  is also a rank-one idempotent. Thus,  $\gamma_\lambda \mu_\lambda \lambda = \lambda$ . It follows that

$$\varphi(\lambda x \otimes f) = \lambda y \otimes g.$$

Therefore,  $\varphi$  preserves idempotents of rank one and their linear spans in both directions.

From the main result of [16] it gives that

(1) There is an invertible operator  $A \in \mathcal{B}(\mathcal{X})$  such that  $\varphi(T) = AFA^{-1}$  for all finite-rank operators  $F \in \mathcal{B}(\mathcal{X})$ , or

(2) There is a bounded invertible linear operator  $C : \mathcal{X}^* \rightarrow \mathcal{X}$  such that  $\varphi(T) = CF^*C^{-1}$  for all finite-rank operators  $F \in \mathcal{B}(\mathcal{X})$ . In this case  $\mathcal{X}$  must be a reflexive space.

*Step 4.*  $\varphi$  takes the desired form.

Assume that (1) holds. Let  $T \in \mathcal{B}(\mathcal{X})$  and for any rank-one operator  $F$ , we have

$$\begin{aligned}\sigma_\gamma(T + F) &= \sigma_\gamma(\varphi(T) + \varphi(F)) \\ &= \sigma_\gamma(\varphi(T) + AFA^{-1}) \\ &= \sigma_\gamma(A(A^{-1}\varphi(T)A + F)A^{-1}) \\ &= \sigma_\gamma(A^{-1}\varphi(T)A + F).\end{aligned}$$

Then, we obtain that  $T = A^{-1}\varphi(T)A$  by the Proposition 2.3. Consequently,  $\varphi(T) = ATA^{-1}$  for all  $T \in \mathcal{B}(\mathcal{X})$ .

If (2) holds, then we similarly have that  $\varphi(T) = CT^*C^{-1}$  for all  $T \in \mathcal{B}(\mathcal{X})$ .  $\square$

#### Funding

This research was supported by the National Natural Science Foundation of China (11701351) and the Natural Science Basic Research Plan in Shaanxi Province of China (2023-JC-YB-050).

#### Availability of data and materials

Not applicable.

## Declarations

### Competing interests

The authors declare no competing interests.

### Author contributions

The author characterize the structure of additive surjective maps on  $B(X)$  preserving the spectral domain in both directions.

Received: 14 March 2023 Accepted: 17 July 2023 Published online: 16 August 2023

## References

1. Aghasizadeh, T., Hejazian, S.: Maps preserving semi-Fredholm operators on Hilbert  $C^*$ -modules. *J. Math. Anal. Appl.* **354**, 625–629 (2009)
2. Aupetit, B.: Spectrum-preserving linear mapping between Banach algebra or Jordan Banach algebras. *J. Lond. Math. Soc.* **62**, 917–924 (2000)
3. Bendaoud, M., Bourhim, A., Burgos, M.: Linear maps preserving Fredholm and Atkinson elements of  $C^*$ -algebras. *Linear Multilinear Algebra* **57**, 823–838 (2009)
4. Bouramdane, Y., Ech-Chérif El Kettani, M.: Maps preserving some spectral domains of skew products of operators. *Linear Multilinear Algebra* **70**, 6978–6988 (2022)
5. Bourhim, A., Mashreghi, J.: Maps preserving the local spectrum of product of operators. *Glasg. Math. J.* **57**, 709–718 (2015)
6. Bourhim, A., Mashreghi, J.: Maps preserving the local spectrum of triple product of operators. *Linear Multilinear Algebra* **63**, 765–773 (2015)
7. Cao, X.H., Chen, S.Z.: Linear maps between operator algebras preserving certain spectral functions. *Banach J. Math. Anal.* **8**, 39–46 (2014)
8. Cao, X.H., Dong, J., Liu, J.H.: Weyl's theorem and its perturbations for the functions of operators. *Oper. Matrices* **12**(4), 1145–1157 (2018)
9. Cui, J.L., Hou, J.C.: Additive maps on standard operator algebras preserving parts of the spectrum. *J. Math. Anal. Appl.* **282**, 266–278 (2003)
10. Dieudonné, J.: Sur une généralisation du groupe orthogonal à quatre variables. *Arch. Math. (Basel)* **1**, 282–287 (1949)
11. Hajighasemi, S., Hejazian, S.: Maps preserving some spectral domains of operator products. *Linear Multilinear Algebra* **70**, 2367–2381 (2022)
12. Kaplansky, I.: Algebraic and Analytic Aspects of Operators Algebras. Am. Math. Soc., Providence (1970)
13. Marcus, M., Purves, R.: Linear transformations on algebras of matrices: the invariance of the elementary symmetric functions. *Can. J. Math.* **11**, 383–396 (1959)
14. Mbekhta, M., Müller, V., Oudghiri, M.: Additive preservers of the ascent, descent and related subsets. *J. Oper. Theory* **71**, 63–83 (2014)
15. Molnár, L.: Some characterizations of the automorphisms of  $B(H)$  and  $C(X)$ . *Proc. Am. Math. Soc.* **130**, 111–120 (2002)
16. Omladič, M., Šemrl, P.: Additive mappings preserving operators of rank one. *Linear Algebra Appl.* **182**, 239–256 (1993)
17. Shen, J.L., Chen, A.: A note on property  $(gaR)$  and perturbations. *Ukr. Mat. Ž.* **69**(7), 974–983 (2017)
18. Shen, J.L., Ji, G.X.: Property  $(t)$  and perturbations. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.* **113**(2), 1417–1427 (2019)
19. Shi, W.J.: Topological uniform descent and compact perturbations. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.* **113**, 2221–2233 (2019)
20. Shi, W.J., Ji, G.X.: Additive maps preserving the semi-Fredholm domain in spectrum. *Ann. Funct. Anal.* **7**(1), 254–260 (2016)
21. Shi, W.J., Ji, G.X.: Additive maps preserving  $m$ -normal eigenvalues on  $B(H)$ . *Oper. Matrices* **10**, 379–387 (2016)
22. Shi, W.J., Ji, G.X.: Additive maps preserving semi-Fredholm operators with bounded ascent on  $B(X)$ . *Publ. Math. (Debr.)* **96**, 259–279 (2020)
23. Sourour, A.R.: Invertibility preserving linear maps on  $L(H)$ . *Trans. Am. Math. Soc.* **348**, 13–30 (1996)
24. Yang, L.L., Cao, X.H.: Property  $(R)$  for functions of operators and its perturbations. *Mediterr. J. Math.* **1**, 19–25 (2022)
25. Yang, L.L., Cao, X.H.: Perturbations of spectra and property  $(R)$  for upper triangular operator matrices. *J. Math. Anal. Appl.* **519**, 126797 (2023)

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.