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φ-fixed point results for nonlinear contractions with an application

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Abstract

In this paper, using the notion of w-distance in metric spaces, we introduce two types of nonlinear contractions, $(\iota,\psi,\varphi,\phi,p)$ and rational-$(\iota,\psi,\varphi,\phi,p)$ contractions. Based on these contractions, we prove the existence and uniqueness of a φ-fixed point for the corresponding contraction. We also provide some examples to demonstrate the correctness and practicability of our results, along with a numerical experiment. Finally, we apply the obtained results to linear matrix equations and nonlinear Fredholm integral equations.

Keywords: Metric spaces; φ-fixed point; w-distance; Nonlinear contractions

1 Introduction

As part of a new study of fixed point theory, the φ-fixed point is a hot topic, which has attracted the interest of many researches. In 2014, Jleli et al. [1] firstly employed a control function $F:[0,\infty)^3\to[0,\infty)$ satisfying suitable properties in (partial) metric spaces, (F,φ)-contraction mappings, and φ-fixed point results for such mappings. In 2016, Kumrod et al. [2] introduced the notionss of (F,φ,θ)-contraction mapping and (F,φ,θ)-weak contraction mapping in metric spaces by adding an auxiliary function θ and established the existence of φ-fixed points for such mappings, which extended the results by Jleli et al. [1]. In 2017, Asadi et al. [3] improved the control function F by replacing the continuity condition on F with more weaker condition and proved results similar to those by Kumrod et al. [2]. In 2019, Gopal et al. [4] proved the existence and uniqueness of a φ-fixed point for an (F,φ,θ)-contraction mapping in a complete metric space with a binary relation, which generalized the results of [2]. In the same year, by α-admissible mappings, Imdad et al. [5] introduced the notions of $(F,\varphi,\alpha\text{-}\psi)$-contractions and $(F,\varphi,\alpha\text{-}\psi)$-weak contractions in metric spaces and presented some results on the existence and uniqueness of φ-fixed points. In 2022, Sun et al. [6] introduced $(\gamma,\psi,\varphi,\phi)$ contractions and rational-$(\gamma,\psi,\varphi,\phi)$ contractions in metric spaces by some auxiliary functions and presented some results on φ-fixed points in metric spaces, which generalized some φ-fixed point results in [1, 2].

On the other hand, it is worth noting that the study of asymmetric structure and its application in mathematics are of great significance. In 1996, Kada et al. [7] firstly introduced a class of asymmetric structures in metric spaces in trms of w-distance. In 1997, using a w-distance, Suzuki [8] obtained some fixed point results for classical contractions such as in [9, 10]. In 2013, Hossein et al. [11] proved fixed points results for (φ,ψ,p)-weakly contractive mappings in metric spaces with w-distance on the basis of the (φ,ψ)-contraction in [12]. In 2020, Kari et al. [13] introduced the notion of a $(\theta\text{-}F)$ contraction in metric spaces with w-distance and obtained the related fixed point results. In 2022, Rossafi et al. [14] presented the notion of θ-ϕ-contraction on a complete metric space equipped with w-distance to prove some fixed point theorems. More references on the w-distance can be found in [15–18].

Based on the above works, employing a w-distance, we introduce two kinds of nonlinear contractions, $(\iota,\psi,\varphi,\phi,p)$ and rational-$(\iota,\psi,\varphi,\phi,p)$ contractions. Meanwhile, we establish some φ-fixed point results. Furthermore, we give some examples and applications to show the effectiveness of our results.

2 Preliminaries

Let $\aleph$ be a nonempty set, let $\varphi :\to [0,\infty)$ be a given function, and let $\hbar : \aleph \to \aleph$ be a mapping. We denote by $\mathbb{N}$ the set of all nonnegative integers, the set of all fixed points of $\hbar$ by

$$F(\hbar) := \{\varrho \in \aleph : \hbar\varrho = \varrho\},$$

and the set of all zeros of the function φ by

$$Z_\varphi := \bigl\{\varrho \in \aleph : \varphi(\varrho) = 0\bigr\}.$$

The notions of a φ-fixed point, φ-Picard mapping, and control function $\iota : [0,\infty)^3 \to [0,\infty)$ are introduced by Jleli et al. [1].

Definition 2.1 [1] Let $\varphi : \aleph \to [0,\infty)$. If $\hbar\varrho = \varrho$ and $\varphi(\varrho) = 0$, then ϱ is called a φ-fixed point of the mapping $\hbar$.

Definition 2.2 [1] Let $(\aleph, d)$ be a metric space, and let $\varphi : \aleph \to [0,\infty)$. A mapping $\hbar : \aleph \to \aleph$ is said to be a φ-Picard mapping if $F_\hbar \cap Z_\varphi = \{\varrho\}$ with $\varrho \in \aleph$ and $\lim_{n\to\infty} \hbar^n\varrho = \varrho$ for each $\varrho \in \aleph$.

Definition 2.3 [1] A continuous function $\iota : [0,\infty)^3 \to [0,\infty)$ satisfying the conditions

($\iota 1$) $\max\{\varrho,\sigma\} \le \iota(\varrho,\sigma,\rho)$ and

($\iota 2$) $\iota(0,0,0) = 0$

is called a control function.

For more details of the control function ι, we refer to [1].

Theorem 2.1 [1] *Let $(\aleph, d)$ be a complete metric space, and let $\hbar : \aleph \to \aleph$. If $\hbar$ satisfies*

$$\iota\bigl(d(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr) \le \kappa\iota\bigl(d(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr),$$

φ is lower semicontinuous, and $0 < \kappa < 1$, then $\hbar$ is a φ-Picard mapping.

Using φ-Picard mappings introduced by Jleli et al., some other scholars also generalized many existing fixed point results. For example, Proinov [19] studied the problem of finding (sufficient) conditions on functions ψ and φ to ensure that T has a unique fixed point. Their results generalized those by Jleli et al. [1], Wardowski et al. [20], and Piri et al. [21].

Theorem 2.2 [19] *Let* $(\aleph, d)$ *be a complete metric space, and let a mapping* $\hbar : \aleph \to \aleph$ *be such that*

$$d(\hbar\varrho, \hbar\sigma) > 0 \quad \Rightarrow \quad \psi\big(d(\hbar\varrho, \hbar\sigma)\big) \le \phi\big(d(\varrho, \sigma)\big)$$

for all $\varrho, \sigma \in \aleph$. *Then* $\hbar$ *has a unique fixed point if* $\psi, \phi : (0, \infty) \to (-\infty, +\infty)$ *satisfy the following conditions*:

(i) ψ *is nondecreasing*;

(ii) $\phi(\varrho) < \psi(\varrho)$ *for all* $\varrho > 0$;

(iii) $\limsup_{\varrho \to \varepsilon^+} \phi(\varrho) < \psi(\varepsilon^+)$ *for all* $\varepsilon > 0$.

Definition 2.4 [7] Let $(\aleph, d)$ be a metric space. Assume that a function $p : \aleph \times \aleph \to [0, \infty)$ satisfies the following conditions:

(a) $p(\varrho, \rho) \le p(\varrho, \sigma) + p(\sigma, \rho)$ for all $\varrho, \sigma, \rho \in \aleph$;

(b) $p(\varrho, \cdot) : \aleph \to [0, \infty)$ is lower semicontinuous for all $\varrho \in \aleph$;

(c) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(\rho, \varrho) \le \delta$ and $p(\rho, \sigma) \le \delta$ imply $d(\varrho, \sigma) \le \varepsilon$.

Then p is called a w-distance on $\aleph$.

For more details related to w-distances, we refer to [7, 22].

Lemma 2.1 [8] *Let* $(\aleph, d)$ *be a metric space equipped with a w-distance p, and let* $\{\varrho_n\}$ *and* $\{\sigma_n\}$ *be two sequences in* $\aleph$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \beta_n = 0$. *If* $p(\varrho_n, \sigma) \le \alpha_n$ *and* $p(\varrho_n, \rho) \le \beta_n$ *for* $\varrho, \sigma, \rho \in \aleph$ *and* $n \in N$, *then* $\sigma = \rho$. *In particular, if* $p(\varrho, \sigma) = 0$ *and* $p(\varrho, \rho) = 0$, *then* $\sigma = \rho$.

Lemma 2.2 [7] *Suppose that* $(\aleph, d)$ *is a metric space, p is a w-distance on* $\aleph$, $\{\varrho_n\}$, $\{\sigma_n\}$, *and* $\{\rho_n\}$ *are three sequences in* $\aleph$, *and* $\varrho, \sigma, \rho \in \aleph$.

(i) *If* $p(\varrho_n, \sigma) \to 0$ *and* $p(\varrho_n, \rho) \to 0$, *then* $\sigma = \rho$. *In particular, if* $p(\varrho, \sigma) = 0$ *and* $p(\varrho, \rho) = 0$, *then* $\sigma = \rho$.

(ii) *If* $p(\varrho_n, \sigma_n) \to 0$ *and* $p(\varrho_n, \rho) \to 0$, *then* σ_n *converges to* ρ.

(iii) *If* $p(\varrho_n, \sigma_n) \to 0$ *and* $p(\varrho_n, \rho_n) \to 0$, *then* $d(\sigma_n, \rho_n)$ *converges to* 0.

Lemma 2.3 [7] *Assume that* $(\aleph, d)$ *is a metric space equipped with a w-distance p and* $\{\varrho_n\}$ *is a sequence in* $\aleph$ *such that for each* $\varepsilon > 0$, *there exists* $N_\varepsilon \in N$ *such that* $p(\varrho_n, \varrho_m) < \varepsilon$ *(or* $\lim_{m,n\to\infty} p(\varrho_n, \varrho_m) = 0$*) for* $m > n > N_\varepsilon$. *Then* $\{\varrho_n\}$ *is a Cauchy sequence.*

Definition 2.5 [6] A mapping $\hbar : \aleph \to \aleph$ is called a $(\iota, \psi, \phi, \varphi)$ contraction in a metric space $(\aleph, d)$ if for all $\varrho, \sigma \in \aleph$ such that $\iota(d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)) > 0$,

$$\psi\big(\iota\big(d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) \le \phi\big(\iota\big(d(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\big)\big),$$

where ψ, ϕ satisfy the conditions of Theorem 2.2.

Theorem 2.3 [6] *If $\hbar : \aleph \to \aleph$ is a $(\iota, \psi, \phi, \varphi)$ contraction in a complete metric space $(\aleph, d)$ and φ is lower semicontinuous, then $\hbar$ is a φ-Picard mapping.*

Definition 2.6 [6] Let $(\aleph, d)$ be a metric space. Suppose that for all $\varrho, \sigma \in \aleph$ such that $\iota(d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)) > 0$, a function $\hbar : \aleph \to \aleph$ satisfies

$$\psi\big(\iota\big(d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) \le \phi\big(\iota\big(N(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\big)\big),$$

where $N(\varrho, \sigma) = \max\{d(\varrho, \sigma), \frac{d(\varrho,\hbar\varrho)(1+d(\sigma,\hbar\sigma))}{1+d(\hbar\varrho,\hbar\sigma)}\}$, and ψ, ϕ satisfy conditions (i), (ii), and (iii) of Theorem 2.2. Then $\hbar : \aleph \to \aleph$ is called a rational-$(\iota, \psi, \phi, \varphi)$ contraction.

Theorem 2.4 [6] *Let $(\aleph, d)$ be a complete metric space. If $\hbar$ is a rational-$(\iota, \psi, \phi, \varphi)$ contraction and φ is lower semicontinuous, then $\hbar$ is a φ-Picard mapping.*

3 Main results

In this part, we get some novel results through some new contractions.

Definition 3.1 A mapping $\hbar : \aleph \to \aleph$ is called a $(\iota, \psi, \varphi, \phi, p)$ contraction in a metric space $(\aleph, d)$ equipped with a w-distance p if there exist functions ψ and ϕ satisfying condition (i), (ii), and (iii) of Theorem 2.2 and the inequality

$$\psi\big(\iota\big(p(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) \le \phi\big(\iota\big(p(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\big)\big) \tag{1}$$

for all $\varrho, \sigma \in \aleph$ such that $\min\{\iota(p(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)), \iota(p(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma))\} > 0$.

Example 3.1 Consider a complete metric space $\aleph = [0, 1]$ under the usual metric $d(\varrho, \sigma) = 2|\varrho - \sigma|$ for $\varrho, \sigma \in \aleph$. Then $p(\varrho, \sigma) = \frac{1}{2}\sigma$ is a w-distance on $(\aleph, d)$. Let $\hbar : \aleph \to \aleph$ be given by

$$\hbar\varrho = \frac{1}{3}\varrho^2 \quad \text{for } \varrho \in [0, 1].$$

Let the functions ϕ, φ, ψ, ι be defined by $\psi(\varrho) = \varrho$, $\phi(\varrho) = \frac{1}{3}\varrho$, $\varphi(\varrho) = 0$, and $\iota(\varrho, \sigma, \rho) = \max\{\varrho, \sigma, \rho\}$.

Then for $\sigma \in [0, 1]$, we have

$$\begin{aligned}
\max\big\{p(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big\} &= \frac{1}{6}\sigma^2 \\
&\le \frac{1}{6}\sigma \\
&= \frac{1}{3}\max\big\{p(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\big\}.
\end{aligned}$$

So $\hbar$ is a $(\iota, \psi, \varphi, \phi, p)$ contraction and satisfies Definition 3.1.

Furthermore, it is easy to observe that $\hbar$ does not satisfy the contractive condition

$$\psi\big(\iota\big(d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) \le \phi\big(r\big(d(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\big)\big)$$

for any distinct $\varrho, \sigma \in \aleph$. Indeed, letting $\varrho = \frac{1}{3}$ and $\sigma = \frac{3}{4}$, we can derive that

$$\begin{aligned}\max\{d(\hbar\varrho, \hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\} &= \frac{2}{3}\left|\varrho^2 - \sigma^2\right| \\ &> \frac{1}{3}d(\varrho, \sigma) \\ &= \frac{1}{3}\max\{d(\varrho, \sigma), \varphi(\varrho), \varphi(\sigma)\}.\end{aligned}$$

So Definition 2.5 is invalid in this example.

Remark 3.1 If $p(\varrho, \sigma) = d(\varrho, \sigma)$, then a $(\iota, \psi, \varphi, \phi, p)$ contraction reduces to a $(\iota, \psi, \varphi, \phi)$ contraction.

Theorem 3.1 *Let $(\aleph, d)$ be a complete metric space equipped with a w-distance p, and let $\hbar : \aleph \to \aleph$ be a $(\iota, \psi, \varphi, \phi, p)$ contraction. If φ is lower semicontinuous, then $\hbar$ is a φ-Picard mapping.*

Proof **Step 1.** We have to prove that $F(\hbar) \subseteq Z_\varphi$, so we first assume that there exists $\varrho \in F(\hbar)$ such that $\varphi(\varrho) \neq 0$. Set $\sigma = \varrho$ in (1). Then

$$\begin{aligned}\psi\left(\iota\big(p(\varrho, \varrho), \varphi(\varrho), \varphi(\varrho)\big)\right) &= \psi\left(\iota\big(p(\hbar\varrho, \hbar\varrho), \varphi(\hbar\varrho), \varphi(\hbar\varrho)\big)\right) \\ &\le \phi\left(\iota\big(p(\varrho, \varrho), \varphi(\varrho), \varphi(\varrho)\big)\right) \\ &< \psi\left(\iota\big(p(\varrho, \varrho), \varphi(\varrho), \varphi(\varrho)\big)\right).\end{aligned}$$

This is a contraction, so $F(\hbar) \subseteq Z_\varphi$.

Step 2. We have to check that

$$\lim_{n\to\infty} p(\varrho_n, \varrho_{n+1}) = 0, \qquad \lim_{n\to\infty} p(\varrho_{n+1}, \varrho_n) = 0, \quad \text{and} \quad \lim_{n\to\infty} \varphi(\varrho_n) = 0.$$

Let $\varrho_0 \in \aleph$ and define $\{\varrho_n\}$ by $\varrho_{n+1} = \hbar\varrho_n$ for each $n \in \mathbb{N}$. We consider two cases.

Case 1: There exists $n_0 \in \mathbb{N}$ such that $p(\varrho_{n_0}, \varrho_{n_0+1}) = 0$.

If $p(\varrho_{n_0+1}, \varrho_{n_0+2}) = \delta \neq 0$, then we have

$$p(\varrho_{n_0}, \varrho_{n_0+2}) \le p(\varrho_{n_0}, \varrho_{n_0+1}) + p(\varrho_{n_0+1}, \varrho_{n_0+2}) = \delta.$$

By Definition 2.4 we get $0 \le d(\varrho_{n_0+1}, \varrho_{n_0+2}) \le \epsilon$, and taking the limit as $n \to \infty$ at both sides of the inequality, we get $d(\varrho_{n_0+1}, \varrho_{n_0+2}) = 0$, so that $\varrho_{n_0+1} = \varrho_{n_0+2} = \hbar\varrho_{n_0+1}$. When $n \ge n_0 + 1$, the sequence $\{\varrho_n\}$ is a constant sequence, and each of these terms is ϱ_{n_0+1} or ϱ_{n_0+2}, so that

$$p(\varrho_{n_0+1}, \varrho_{n_0+2}) = p(\varrho_{n_0+1}, \varrho_{n_0+1}) = p(\varrho_{n_0+2}, \varrho_{n_0+2}) = \delta > 0.$$

By (1) we have

$$\begin{aligned}&\psi\left(\iota\big(p(\varrho_{n_0+2}, \varrho_{n_0+2}), \varphi(\varrho_{n_0+2}), \varphi(\varrho_{n_0+2})\big)\right) \\ &\quad = \psi\left(\iota\big(p(\hbar\varrho_{n_0+1}, \hbar\varrho_{n_0+1}), \varphi(\hbar\varrho_{n_0+1}), \varphi(\hbar\varrho_{n_0+1})\big)\right)\end{aligned}$$

$$\begin{aligned}
&\le \phi\bigl(\iota\bigl(p(\varrho_{n_0+1},\varrho_{n_0+1}),\varphi(\varrho_{n_0+1}),\varphi(\varrho_{n_0+1})\bigr)\bigr)\\
&< \psi\bigl(\iota\bigl(p(\varrho_{n_0+1},\varrho_{n_0+1}),\varphi(\varrho_{n_0+1}),\varphi(\varrho_{n_0+1})\bigr)\bigr)\\
&= \psi\bigl(\iota\bigl(p(\varrho_{n_0+2},\varrho_{n_0+2}),\varphi(\varrho_{n_0+2}),\varphi(\varrho_{n_0+2})\bigr)\bigr).
\end{aligned}$$

This is contradiction, so $p(\varrho_{n_0+1},\varrho_{n_0+2}) = 0$. Continuing this process inductively, we obtain

$$\lim_{n\to\infty} p(\varrho_n,\varrho_{n+1}) = 0, \tag{2}$$

and $\varphi(\varrho_n) = 0$ for when $n \ge n_0 + 1$. So

$$\lim_{n\to\infty} \varphi(\varrho_n) = 0. \tag{3}$$

Now it suffices to prove that

$$p(\varrho_{n_0+1},\varrho_{n_0}) = 0 \quad \text{implies that} \quad p(\varrho_{n_0+2},\varrho_{n_0+1}) = 0. \tag{4}$$

Suppose that $p(\varrho_{n_0+2},\varrho_{n_0+1}) = \delta$. Then $p(\varrho_{n_0+2},\varrho_{n_0}) \le p(\varrho_{n_0+2},\varrho_{n_0+1}) + p(\varrho_{n_0+1},\varrho_{n_0}) = \delta$ yields $\varrho_{n_0} = \varrho_{n_0+1} = \hbar\varrho_{n_0}$. Therefore for $n > n_0$, both are equal to ϱ_{n_0} or ϱ_{n_0+1}. Then $p(\varrho_{n_0+2},\varrho_{n_0+1}) = p(\varrho_{n_0+1},\varrho_{n_0}) = 0$. Keeping up this process inductively, we obtain

$$\lim_{n\to\infty} p(\varrho_{n+1},\varrho_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(\varrho_n) = 0. \tag{5}$$

Case 2: $p(\varrho_n,\varrho_{n+1}) > 0$.
Let $\varrho = \varrho_n$ and $\sigma = \varrho_{n+1}$ in (1). We have

$$\begin{aligned}
\psi\bigl(\iota\bigl(p(\varrho_{n+1},\varrho_{n+2}),\varphi(\varrho_{n+1}),\varphi(\varrho_{n+2})\bigr)\bigr) &= \psi\bigl(\iota\bigl(p(\hbar\varrho_n,\hbar\varrho_{n+1}),\varphi(\hbar\varrho_n),\varphi\bigl(\hbar(\varrho_{n+1})\bigr)\bigr)\bigr)\\
&\le \phi\bigl(\iota\bigl(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1})\bigr)\bigr)\\
&< \psi\bigl(\iota\bigl(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1})\bigr)\bigr).
\end{aligned} \tag{6}$$

Because ψ is nondecreasing, $\{\iota(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1}))\}$ is a nonincreasing sequence with a lower bound. So there exists $\varepsilon \ge 0$ such that $\lim_{n\to\infty}\iota_n = \varepsilon$, where $\iota_n = \iota(p(\varrho_n,\varrho_{n+1}), \varphi(\varrho_n),\varphi(\varrho_{n+1}))$. For $\varepsilon > 0$, taking the right upper limits on both sides of (6), by (iii) in Theorem 2.2, we derive

$$\psi\bigl(\varepsilon^+\bigr) = \limsup_{\iota_{n+1}\to\varepsilon^+} \psi(\iota_{n+1}) \le \limsup_{\iota_n\to\varepsilon^+} \phi(\iota_n) \le \limsup_{\varrho\to\varepsilon^+} \phi(\varrho) < \psi\bigl(\varepsilon^+\bigr),$$

a contradiction, so $\varepsilon = 0$. By $(\iota 1)$ in Definition 2.3 we get

$$\max\bigl\{p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n)\bigr\} \le \iota\bigl(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1})\bigr),$$

which yields that

$$\begin{cases} p(\varrho_n,\varrho_{n+1}) \le \iota(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1})),\\ \varphi(\varrho_n) \le \iota(p(\varrho_n,\varrho_{n+1}),\varphi(\varrho_n),\varphi(\varrho_{n+1})). \end{cases} \tag{7}$$

Taking the limits of (7), we derive

$$\begin{cases} 0 \le \lim_{n\to\infty} p(\varrho_n, \varrho_{n+1}) \le \lim_{n\to\infty} \iota(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1})) = 0, \\ 0 \le \lim_{n\to\infty} \varphi(\varrho_n) \le \lim_{n\to\infty} \iota(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1})) = 0, \end{cases}$$

that is,

$$\begin{cases} \lim_{n\to\infty} p(\varrho_n, \varrho_{n+1}) = 0, \\ \lim_{n\to\infty} \varphi(\varrho_n) = 0. \end{cases} \tag{8}$$

When $p(\varrho_{n+1}, \varrho_n) > 0$, let $\varrho = \varrho_{n+1}$, $\sigma = \varrho_n$ in (1). By the same method we can derive that

$$\lim_{n\to\infty} p(\varrho_{n+1}, \varrho_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(\varrho_n) = 0. \tag{9}$$

Step 3. We want to prove that $\lim_{m,n\to\infty} p(\varrho_n, \varrho_m) = 0$.

Suppose on the contrary that there exist $\varepsilon > 0$ and two sequences $\{\varrho_{n(k)}\}$ and $\{\varrho_{m(k)}\}$ such that

$$p(\varrho_{n(k),}\varrho_{m(k)}) \ge \varepsilon \quad \text{and} \quad p(\varrho_{n(k)}, \varrho_{m(k)-1}) < \varepsilon,$$

where $m_{(k)} > n_{(k)}$. The by the triangle inequality

$$\varepsilon \le p(\varrho_{n(k)}, \varrho_{m(k)}) \le p(\varrho_{n(k)}, \varrho_{m(k)-1}) + p(\varrho_{m(k)-1}, \varrho_{m(k)}). \tag{10}$$

Taking the limits in (10), by the squeeze theorem we derive that

$$\lim_{k\to\infty} p(\varrho_{n(k)}, \varrho_{m(k)}) = \varepsilon.$$

Now

$$\begin{aligned} \varepsilon &\le p(\varrho_{n(k)}, \varrho_{m(k)}) \\ &\le p(\varrho_{n(k)}, \varrho_{m(k)-1}) + p(\varrho_{m(k)-1}, \varrho_{m(k)}) \\ &\le p(\varrho_{n(k)}, \varrho_{n(k)-1}) + p(\varrho_{n(k)-1}, \varrho_{m(k)-1}) + p(\varrho_{m(k)-1}, \varrho_{m(k)}) \\ &\le p(\varrho_{n(k)}, \varrho_{n(k)-1}) + p(\varrho_{n(k)-1}, \varrho_{n(k)}) + p(\varrho_{n(k)}, \varrho_{m(k)-1}) + p(\varrho_{m(k)-1}, \varrho_{m(k)}) \\ &< p(\varrho_{n(k)}, \varrho_{n(k)-1}) + p(\varrho_{n(k)-1}, \varrho_{n(k)}) + p(\varrho_{n(k)}, \varrho_{m(k)-1}) + \varepsilon. \end{aligned} \tag{11}$$

Taking the limits on both sides of (11), we derive

$$\lim_{k\to\infty} p(\varrho_{n(k)-1}, \varrho_{m(k)-1}) = \varepsilon^+.$$

Letting $\varrho = \varrho_{n(k)-1}$ and $\sigma = \varrho_{m(k)-1}$ in (1), we have

$$\psi(\iota_{n,m}) = \psi\bigl(\iota\bigl(p(\hbar\varrho_{n(k)-1}, \hbar\varrho_{m(k)-1}), \varphi(\hbar\varrho_{n(k)-1}), \varphi(\hbar\varrho_{m(k)-1})\bigr)\bigr) \le \phi(\iota_{n-1,m-1}), \tag{12}$$

where $\iota_{n,m} = \iota(p(\varrho_{n(k)}, \varrho_{m(k)}), \varphi(\varrho_{n(k)}), \varphi(\varrho_{m(k)}))$. Taking the right upper limits on both sides of (12), by (iii) in Theorem 2.2 we obtain

$$\psi(\iota(\varepsilon^+, 0, 0) = \lim_{n,m\to\infty} \psi(\iota_{n,m}) \le \lim_{n,m\to\infty} \phi(\iota_{n-1,m-1}) \le \lim_{\varrho\to\iota(\varepsilon^+,0,0)} \phi(\varrho) < \psi\big(\iota\big(\varepsilon^+, 0, 0\big)\big),$$

a contradiction. So

$$\lim_{n,m\to\infty} p(\varrho_n, \varrho_m) = 0. \tag{13}$$

In view of Lemma 2.3, $\{\varrho_n\}$ is a Cauchy sequence.

Step 4. Next, we need to verify that ϱ^* is a φ-fixed point of $\hbar$.

Since $\{\varrho_n\}$ is a Cauchy sequence, there exists $\varrho^* \in \aleph$ such that $\varrho_n \to \varrho^*(n \to \infty)$. Since φ is lower semicontinuous, we get

$$\varphi\big(\varrho^*\big) \le \lim_{n\to\infty} \inf \varphi(\varrho_{n+1}) \le \lim_{n\to\infty} \varphi(\varrho_{n+1}) = 0,$$

so $\varphi(\varrho^*) = 0$.

Since $\lim_{m,n\to\infty} p(\varrho_n, \varrho_m) = 0$, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $p(\varrho_n, \varrho_m) < \varepsilon$ for all $n, m > N_\varepsilon$. As $\varrho_n \to \varrho^*$ and $p(\varrho, \cdot)$ is lower semicontinuous, we get

$$0 \le p\big(\varrho_n, \varrho^*\big) \le \liminf_{m\to\infty} p(\varrho_n, \varrho_m) \le \varepsilon. \tag{14}$$

Taking the limits on both sides of (14), by (13) we obtain

$$\lim_{n\to\infty} p\big(\varrho_n, \varrho^*\big) = 0. \tag{15}$$

Next, we prove that $\lim_{n\to\infty} p(\varrho_n, \hbar\varrho^*) = 0$. We consider the following two cases.

Case 1: There exists $N \in \mathbb{N}$ such that $p(\varrho_{n+1}, \hbar\varrho^*) > 0$ for all $n > N$. Suppose that there exists $n_0 > N$ such that $p(\varrho_{n_0+1}, \varrho^*) = 0$ and $p(\varrho_{n_0+1}, \hbar\varrho^*) = \delta > 0$. Then we can conclude that $p(\varrho_{n_0+1}, \varrho^*) \le p(\varrho_{n_0+1}, \hbar\varrho^*)$ and $p(\varrho_{n_0+1}, \hbar\varrho^*) \le \delta$. So by (c) in Definition 2.4 we get $\varrho^* = \hbar\varrho^*$. The proof is completed. If $p(\varrho_n, \varrho^*) > 0$ for all $n > N$, then by (ii) in Theorem 2.2 and (1) we get

$$\begin{aligned}
\psi\big(\iota\big(p\big(\varrho_{n+1}, \hbar\varrho^*\big), \varphi(\varrho_{n+1}), \varphi\big(\hbar\varrho^*\big)\big)\big) &= \psi\big(\iota\big(p\big(\hbar\varrho_n, \hbar\varrho^*\big), \varphi(\hbar\varrho_n), \varphi\big(\hbar\varrho^*\big)\big)\big) \\
&\le \phi\big(\iota\big(p\big(\big(\varrho_n, \varrho^*\big), \varphi(\varrho_n), \varphi\big(\varrho^*\big)\big)\big)\big) \\
&< \psi\big(\iota\big(p\big(\varrho_n, \varrho^*\big), \varphi(\varrho_n), \varphi\big(\varrho^*\big)\big)\big).
\end{aligned}$$

Since ψ is nondecreasing,

$$0 \le \iota\big(p\big(\varrho_{n+1}, \hbar\varrho^*\big), \varphi(\varrho_{n+1}), \varphi\big(\hbar\varrho^*\big)\big) < \iota\big(p\big(\varrho_n, \varrho^*\big), \varphi(\varrho_n), \varphi\big(\varrho^*\big)\big),$$

and by (ι_1) we obtain

$$0 \le p\big(\varrho_{n+1}, \hbar\varrho^*\big) \le \iota\big(p\big(\varrho_{n+1}, \hbar\varrho^*\big), \varphi(\varrho_{n+1}), \varphi\big(\hbar\varrho^*\big)\big) < \iota\big(p\big(\varrho_n, \varrho^*\big), \varphi(\varrho_n), \varphi\big(\varrho^*\big)\big).$$

Taking the limits on both sides of the inequality, it follows that

$$\lim_{n\to\infty} p\big(\varrho_{n+1}, \hbar\varrho^*\big) = 0.$$

By the triangle inequality this yields that

$$p\big(\varrho_n, \hbar\varrho^*\big) \le p(\varrho_n, \varrho_{n+1}) + p\big(\varrho_{n+1}, \hbar\varrho^*\big),$$

and therefore

$$\lim_{n\to\infty} p\big(\varrho_n, \hbar\varrho^*\big) = 0. \tag{16}$$

Hence by (i) in Lemma 2.2 and (15)–(16) we can conclude that $\varrho^* = \hbar\varrho^*$.

Case 2: If $\lim_{n\to\infty} p(\varrho_n, \varrho^*) = 0$, then there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $p(\varrho_n, \varrho^*) < \delta$ for all $n > N$. If there exists $n_0 > N$ such that $p(\varrho_{n_0+1}, \hbar\varrho^*) = 0$, then we have $\hbar\varrho^* = \varrho^*$.

Step 5. We have to prove that ϱ^* is a unique fixed point of $\hbar$.

Let $\varrho^*, \sigma \in F(\hbar)$. Assume that $p(\varrho^*, \sigma) \neq 0$. Letting $\varrho = \varrho^*$ in (1), we have

$$\begin{aligned}
\psi\big(\iota\big(p\big(\varrho^*, \sigma\big), 0, 0\big)\big) &= \psi\big(\iota\big(p\big(\varrho^*, \sigma\big), \varphi\big(\varrho^*\big), \varphi(\sigma)\big)\\
&= \psi\big(\iota\big(p\big(\hbar\varrho^*, \hbar\sigma\big)\big), \varphi\big(\hbar\varrho^*\big), \varphi(\hbar\sigma)\big)\\
&\le \phi\big(r\big(p\big(\varrho^*, \sigma\big), \varphi\big(\varrho^*\big), \varphi(\sigma)\big)\\
&= \phi\big(\iota\big(p\big(\varrho^*, \sigma\big), 0, 0\big)\big)\\
&< \psi\big(\iota\big(p\big(\varrho^*, \sigma\big), 0, 0\big)\big),
\end{aligned}$$

a contradiction. Therefore

$$p\big(\varrho^*, \sigma\big) = 0.$$

In the same way, we get $p(\varrho^*, \varrho^*) = 0$. So $\varrho^* = \sigma$. This completes the proof of the theorem. □

Example 3.2 Let $\aleph = [0, 2]$ be endowed with the usual metric $d(\varrho, \sigma) = |\varrho - \sigma|$, and let $p(\varrho, \sigma) = 2d(\varrho, \sigma)$ for $\varrho, \sigma \in \aleph$. Then $(\aleph, d)$ is a complete metric space. Consider the mapping $\hbar : \aleph \to \aleph$ defined by

$$\hbar(\varrho) = \begin{cases} \frac{1}{4}\varrho & \text{for } \varrho \in [0, 2), \\ \frac{1}{8} & \text{otherwise.} \end{cases}$$

Let the functions ϕ, φ, ψ, ι be defined by $\psi(\varrho) = \varrho$, $\phi(\varrho) = \frac{1}{2}\varrho$, $\varphi(\varrho) = \varrho$, and $\iota(u, v, \varrho) = u + v + \varrho$.

Case 1: $\varrho = \sigma \in \aleph - \{0, 2\}$. Then

$$2d(\hbar\varrho, \hbar\sigma) + \varphi(\hbar\varrho) + \varphi(\hbar\sigma) \le \frac{1}{2}\big(2d(\varrho, \sigma) + \varphi(\varrho) + \varphi(\sigma)\big).$$

Case 2: $\varrho \in [0,2)$, $\sigma = 2$. Then

$$\begin{aligned}2d(\hbar\varrho,\hbar\sigma)+\varphi(\hbar\varrho)+\varphi(\hbar\sigma)&=2\left|\frac{1}{8}-\frac{1}{4}\varrho\right|+\frac{1}{4}\varrho+\frac{1}{8}\\&\le\max\left\{\frac{3}{8}-\frac{1}{4}\varrho,\frac{3}{4}\varrho-\frac{1}{8}\right\}\\&\le 2\\&\le\frac{1}{2}\bigl(2d(\varrho,\sigma)+\varphi(\varrho)+\varphi(\sigma)\bigr).\end{aligned}$$

Case 3: $\sigma \in [0,2)$, $\varrho = 2$. Then

$$\begin{aligned}2d(\hbar\varrho,\hbar\sigma)+\varphi(\hbar\varrho)+\varphi(\hbar\sigma)&=2\left|\frac{1}{4}\varrho-\frac{1}{4}\sigma\right|+\frac{1}{4}\varrho+\frac{1}{4}\sigma\\&\le\max\left\{\frac{3}{4}\varrho-\frac{1}{4}\sigma,\frac{3}{4}\sigma-\frac{1}{4}\varrho\right\}\\&\le\frac{1}{2}\bigl(2d(\varrho,\sigma)+\varphi(\varrho)+\varphi(\sigma)\bigr).\end{aligned}$$

Then $\hbar$ is a $(\iota,\psi,\varphi,\phi,p)$ contraction and satisfies all conditions of Theorem 3.1. By Theorem 3.1 we know that $\hbar$ has a unique fixed point $\varrho = 0$ such that $\hbar(0) = 0$ and $\varphi(0) = 0$.

Example 3.3 Let $\aleph = [0,\infty)$ be endowed with the metric

$$d(\varrho,\sigma)=\begin{cases}0 & \text{if } \varrho=\sigma,\\ 1 & \text{if } \varrho\neq\sigma,\end{cases}$$

and let $p(\varrho,\sigma) = \sigma$ for $\varrho,\sigma \in \aleph$. Then $(\aleph, d)$ is a complete metric space. Consider the mapping $\hbar : \aleph \to \aleph$ defined by

$$\hbar(\varrho)=\begin{cases}\frac{1}{8}\varrho & \text{for } \varrho\ge 1,\\ 0 & \text{for } \varrho<1.\end{cases}$$

Let the functions ϕ, φ, ψ, ι be defined by $\psi(\varrho) = \varrho$, $\phi(\varrho) = \frac{1}{2}\varrho$, and $\varphi(\varrho) = \varrho$, $\iota(\varrho,\sigma,\rho) = \max\{\varrho,\sigma\}$. We consider the following cases (the iterates of Picard iterations and the convergence behavior in Example 3.3, see Figs. 1 and 2).

Case 1: $\varrho,\sigma \in [1,\infty)$. Then

$$\begin{aligned}\psi\bigl(\iota\bigl(p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr)\bigr)&=\max\bigl\{p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho)\bigr\}\\&=\max\left\{\frac{1}{8}\sigma,\frac{1}{8}\varrho\right\}\\&\le\frac{1}{2}\max\{\sigma,\varrho\}\\&=\frac{1}{2}\max\bigl\{p(\varrho,\sigma),\varphi(\varrho)\bigr\}\\&=\phi\bigl(\iota\bigl(p(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr)\bigr).\end{aligned}$$

	$x_0 = 10$	$x_0 = 20$	$x_0 = 30$	$x_0 = 40$	$x_0 = 50$
x_1	1.25000	2.50000	3.75000	5.00000	6.25000
x_2	0.15625	0.3125	0.46875	0.62500	0.78125
x_3	0.00000	0.00000	0.00000	0.00000	0.00000
x_4	0.00000	0.00000	0.00000	0.00000	0.00000
x_5	0.00000	0.00000	0.00000	0.00000	0.00000
…	…	…	…	…	…

Figure 1 Picard iteration

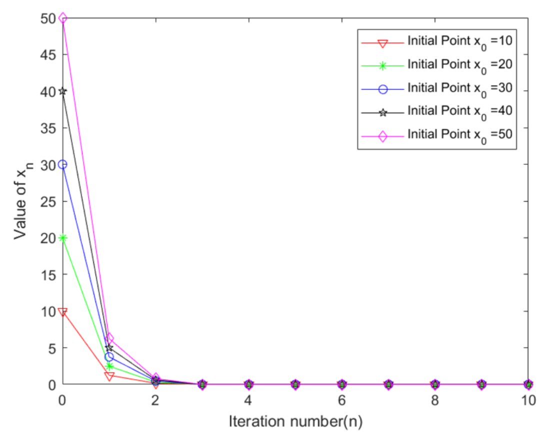

Figure 2 Convergence behavior

Case 2: $\varrho, \sigma \in [0,1)$. Then

$$\begin{aligned}\max\{p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho)\} &= 0\\ &\leq \frac{1}{2}\max\{\sigma,\varrho\}\\ &= \frac{1}{2}\max\{p(\varrho,\sigma),\varphi(\varrho)\}.\end{aligned}$$

Case 3: $x \in [1,\infty)$, $y \in [0,1)$. Then

$$\begin{aligned}\max\{p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho)\} &= \max\left\{0,\frac{1}{8}\varrho\right\}\\ &\leq \frac{1}{2}\max\{\sigma,\varrho\}\\ &= \frac{1}{2}\max\{p(\varrho,\sigma),\varphi(\varrho)\}.\end{aligned}$$

Case 4: $x \in [0,1)$, $y \in [1,\infty)$. Then

$$\begin{aligned}\max\{p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho)\} &= \max\left\{0,\frac{1}{8}\sigma\right\}\\ &\leq \frac{1}{2}\max\{\sigma,\varrho\}\\ &= \frac{1}{2}\max\{p(\varrho,\sigma),\varphi(\varrho)\}.\end{aligned}$$

Then $\hbar$ is a $(\iota,\psi,\varphi,\phi,p)$ contraction and satisfies all conditions of Theorem 3.1. By Theorem 3.1 we know that $\hbar$ has a unique fixed point $\varrho = 0$ such that $\hbar(0) = 0$ and $\varphi(0) = 0$.

Example 3.4 Consider the complete metric space $\aleph = [0,1]$ under the usual metric $d(\varrho,\sigma) = |\varrho - \sigma|$ for all $\varrho,\sigma \in \aleph$. Defining $p(\varrho,\sigma) = \sigma$, we obtain that p is a w-distance on $(\aleph,d)$. Let $\hbar : \aleph \to \aleph$ be given by

$$\hbar\varrho = \frac{4}{5}\varrho^2 \quad \text{for } \varrho \in [0,1].$$

Let the functions ϕ, φ, ψ, ι be defined by $\psi(\varrho) = \varrho$, $\phi(\varrho) = \frac{4}{5}\varrho$, $\varphi(\varrho) = 0$, and $\iota(\varrho,\sigma,\rho) = \max\{\varrho,\sigma,\rho\}$.

For each $\sigma \in (0,1]$, we have

$$\begin{aligned} \max\bigl\{p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr\} &= \frac{4}{5}\sigma^2 \\ &\le \frac{4}{5}\sigma \\ &= \frac{4}{5}\max\bigl\{p(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr\}. \end{aligned}$$

So $\hbar$ is a $(\iota,\psi,\varphi,\phi,p)$ contraction and satisfies the conditions of Theorem 3.1, so that $\hbar$ has a unique φ-fixed point $\varrho = 0$.

Furthermore, it is easy to observe that $\hbar$ does not satisfy the contractive condition

$$\psi\bigl(\iota\bigl(d(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr)\bigr) \le \phi\bigl(r\bigl(d(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr)\bigr)$$

for distinct $\varrho,\sigma \in \aleph$. Indeed, let $\varrho = \frac{1}{2}$ and $\sigma = \frac{2}{3}$, we derive that

$$\begin{aligned} \max\bigl\{d(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr\} &= \frac{4}{5}\bigl|\varrho^2 - \sigma^2\bigr| \\ &> \frac{4}{5}d(\varrho,\sigma) \\ &= \frac{4}{5}\max\bigl\{d(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr\}. \end{aligned}$$

So Theorem 2.3 is not valid in this example.

Definition 3.2 A mapping $\hbar : \aleph \to \aleph$ is called a rational-$(\iota,\psi,\varphi,\phi,p)$ contraction in a metric space $(\aleph,d)$ equipped with a w-distance p if there are functions ψ and ϕ satisfying conditions (i), (ii), and (iii) of Theorem 2.2 such that for all $\varrho,\sigma \in \aleph$ with $\iota(d(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)) > 0$,

$$\psi\bigl(\iota\bigl(p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\bigr)\bigr) \le \phi\bigl(\iota\bigl(M(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\bigr)\bigr), \tag{17}$$

where $M(\varrho,\sigma) = \max\{p(\varrho,\sigma), \frac{p(\varrho,\hbar\varrho)(1+p(\sigma,\hbar\sigma))}{1+p(\hbar\varrho,\hbar\sigma)}\}$.

Example 3.5 Consider the complete metric space $\aleph = [0,1]$ under the usual metric $d(\varrho,\sigma) = |\varrho - \sigma|$ for $\varrho,\sigma \in \aleph$. Defining $p(\varrho,\sigma) = \sigma$, we obtain that p is a w-distance on

$(\aleph, d)$. Let $\hbar : \aleph \to \aleph$ be given by

$$\hbar\varrho = \frac{4}{5}\varrho^2 \quad \text{for } \varrho \in [0,1].$$

Let the functions ϕ, φ, ψ, ι be defined by $\varphi(\varrho) = 0$, $\iota(\varrho,\sigma,\rho) = \varrho + \sigma$, and $\psi(\varrho) = \varrho$, $\phi(\varrho) = \frac{4}{5}\varrho$. Then

$$\begin{aligned} p(\hbar\varrho, \hbar\sigma) &= \frac{4}{5}\sigma^2 \\ &\le \frac{4}{5}\max\left\{\sigma, \frac{4}{5}\varrho^2\right\} \\ &= \frac{4}{5}\max\left\{p(\varrho,\sigma), \frac{p(\varrho,\hbar\varrho)(1+p(\sigma,\hbar\sigma))}{1+p(\hbar\varrho,\hbar\sigma)}\right\}. \end{aligned}$$

So $\hbar$ is a rational-$(\iota,\psi,\varphi,\phi,p)$ contraction and satisfies Definition 3.2.

Furthermore, it is easy to observe that $\hbar$ does not satisfy the contractive condition

$$\psi\big(\iota\big(d(\hbar\varrho,\hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) \le \phi\big(\iota\big(M(\varrho,\sigma), \varphi(\varrho), \varphi(\sigma)\big)\big),$$

where $M(\varrho,\sigma) = \max\{d(\varrho,\sigma), \frac{d(\varrho,\hbar\varrho)(1+d(\sigma,\hbar\sigma))}{1+d(\hbar\varrho,\hbar\sigma)}\}$ for distinct $\varrho,\sigma \in \aleph$. Indeed, letting $\varrho = \frac{1}{2}$ and $\sigma = 1$, we derive that

$$\begin{aligned} \psi\big(\iota\big(d(\hbar\varrho,\hbar\sigma), \varphi(\hbar\varrho), \varphi(\hbar\sigma)\big)\big) &= d(\hbar\varrho,\hbar\sigma) \\ &= \frac{4}{5}\left|\varrho^2 - \sigma^2\right| \\ &= \frac{3}{5} \\ &> \frac{2}{5} \\ &= \frac{4}{5}\max\left\{d(\varrho,\sigma), \frac{d(\varrho,\hbar\varrho)(1+d(\sigma,\hbar\sigma))}{1+d(\hbar\varrho,\hbar\sigma)}\right\}. \end{aligned}$$

So Definition 2.6 is not valid in this example.

Remark 3.2 If $p(\varrho,\sigma) = d(\varrho,\sigma)$, then a rational-$(\iota,\psi,\varphi,\phi,p)$ contraction reduces to a rational-$(\iota,\psi,\varphi,\phi)$ contraction.

Theorem 3.2 *Let $(\aleph, d)$ be a complete metric space equipped with a w-distance p, and let $\hbar$ be a rational-$(\iota,\psi,\varphi,\phi,p)$ contraction. If φ is lower semicontinuous, then $\hbar$ is a φ-Picard mapping.*

Proof **Step 1.** We have to prove that $F(\hbar) \subseteq Z_\varphi$. We suppose that there exists $\varrho_0 \in F(\hbar)$ such that $\varphi(\varrho_0) \ne 0$. Set $\sigma = \varrho_0$ in (17). Then

$$\begin{aligned} \psi\big(\iota\big(p(\varrho,\varrho_0), \varphi(\varrho), \varphi(\varrho_0)\big)\big) &= \psi\big(\iota\big(p(\hbar\varrho,\hbar\varrho_0), \varphi(\hbar\varrho), \varphi(\hbar\varrho_0)\big)\big) \\ &\le \phi\big(\iota\big(M(\varrho,\varrho_0), \varphi(\varrho), \varphi(\varrho_0)\big)\big) \\ &< \psi\big(\iota\big(p(\varrho,\varrho_0), \varphi(\varrho), \varphi(\varrho_0)\big)\big). \end{aligned}$$

This is a contradiction, so $F(\hbar) \subseteq Z_\varphi$.

Step 2. It suffices to prove that

$$\lim_{n\to\infty} p(\varrho_n,\varrho_{n+1}) = 0, \lim_{n\to\infty} p(\varrho_{n+1},\varrho_n) = 0 \text{ and } \lim_{n\to\infty} \varphi(\varrho_n) = 0.$$

Suppose $\varrho_0 \in \aleph$ and the sequence $\{\varrho_n\}$ is defined by $\varrho_{n+1} = \hbar\varrho_n$ for $n \geq 0$. We discuss two cases.

Case 1: There exists $n_0 \in N$ such that $p(\varrho_{n_0},\varrho_{n_0+1}) = 0$.

If $p(\varrho_{n_0+1},\varrho_{n_0+2}) = \delta \neq 0$, then we have

$$p(\varrho_{n_0},\varrho_{n_0+2}) \leq p(\varrho_{n_0},\varrho_{n_0+1}) + p(\varrho_{n_0+1},\varrho_{n_0+2}) = \delta.$$

So $\varrho_{n_0+1} = \varrho_{n_0+2} = \hbar\varrho_{n_0+1}$. When $n \geq n_0 + 1$, $\{\varrho_n\}$ is a constant sequence, and each of these terms is ϱ_{n_0+1} or ϱ_{n_0+2}, so that

$$p(\varrho_{n_0+1},\varrho_{n_0+2}) = p(\varrho_{n_0+1},\varrho_{n_0+1}) = p(\varrho_{n_0+2},\varrho_{n_0+2}) = \delta > 0.$$

By (17) and

$$\begin{aligned} M(\varrho_{n_0+1},\varrho_{n_0+1}) &= \max\left\{p(\varrho_{n_0+1},\varrho_{n_0+1}), \frac{p(\varrho_{n_0+1},\varrho_{n_0+1})(1+p(\varrho_{n_0+1},\varrho_{n_0+1}))}{1+p(\varrho_{n_0+1},\varrho_{n_0+1})}\right\} \\ &= p(\varrho_{n_0+1},\varrho_{n_0+1}) \end{aligned}$$

we have

$$\begin{aligned} &\psi\bigl(\iota\bigl(p(\varrho_{n_0+2},\varrho_{n_0+2}),\varphi(\varrho_{n_0+2}),\varphi(\varrho_{n_0+2})\bigr)\bigr) \\ &\quad = \psi\bigl(\iota\bigl(p(\hbar\varrho_{n_0+1},\hbar\varrho_{n_0+1}),\varphi(\hbar\varrho_{n_0+1}),\varphi(\hbar\varrho_{n_0+1})\bigr)\bigr) \\ &\quad \leq \phi\bigl(\iota\bigl(M(\varrho_{n_0+1},\varrho_{n_0+1}),\varphi(\varrho_{n_0+1}),\varphi(\varrho_{n_0+1})\bigr)\bigr) \\ &\quad < \psi\bigl(\iota\bigl(M(\varrho_{n_0+1},\varrho_{n_0+1}),\varphi(\varrho_{n_0+1}),\varphi(\varrho_{n_0+1})\bigr)\bigr) \\ &\quad = \psi\bigl(\iota\bigl(p(\varrho_{n_0+2},\varrho_{n_0+2}),\varphi(\varrho_{n_0+2}),\varphi(\varrho_{n_0+2})\bigr)\bigr). \end{aligned}$$

This is contradiction, so $p(\varrho_{n_0+1},\varrho_{n_0+2}) = 0$. Continuing this process inductively, we obtain

$$\lim_{n\to\infty} p(\varrho_n,\varrho_{n+1}) = 0, \tag{18}$$

and when $n \geq n_0 + 1$, we have $\varphi(\varrho_n) = 0$. So

$$\lim_{n\to\infty} \varphi(\varrho_n) = 0. \tag{19}$$

Next, we have to prove

$$p(\varrho_{n_0+1},\varrho_{n_0}) = 0 \quad \text{implies that} \quad p(\varrho_{n_0+2},\varrho_{n_0+1}) = 0.$$

Suppose that $p(\varrho_{n_0+2},\varrho_{n_0+1}) = \delta$. Then

$$p(\varrho_{n_0+2},\varrho_{n_0}) \leq p(\varrho_{n_0+2},\varrho_{n_0+1}) + p(\varrho_{n_0+1},\varrho_{n_0}) = \delta,$$

which implies that $\varrho_{n_0} = \varrho_{n_0+1} = \hbar\varrho_{n_0}$. So when $n > n_0$, both are equal to ϱ_{n_0} or ϱ_{n_0+1}. Then $p(\varrho_{n_0+2}, \varrho_{n_0+1}) = p(\varrho_{n_0+1}, \varrho_{n_0}) = 0$. Continuing this process inductively, we obtain

$$\begin{cases} \lim_{n\to\infty} p(\varrho_{n+1}, \varrho_n) = 0, \\ \lim_{n\to\infty} \varphi(\varrho_n) = 0. \end{cases} \tag{20}$$

Case 2: $p(\varrho_n, \varrho_{n+1}) > 0$.

Let $\varrho = \varrho_n$ and $\sigma = \varrho_{n+1}$ in (17). By $M(\varrho_n, \varrho_{n+1}) = \max\{p(\varrho_n, \varrho_{n+1}), \frac{p(\varrho_n,\varrho_{n+1})(1+p(\varrho_{n+1},\varrho_{n+2}))}{1+p(\varrho_{n+1},\varrho_{n+2})}\} = p(\varrho_n, \varrho_{n+1})$ we have

$$\begin{aligned} \psi\bigl(\iota\bigl(p(\varrho_{n+1}, \varrho_{n+2}), \varphi(\varrho_{n+1}), \varphi(\varrho_{n+2})\bigr)\bigr) &= \psi\bigl(\iota\bigl(p(\hbar\varrho_n, \hbar\varrho_{n+1}), \varphi(\hbar\varrho_n), \varphi\bigl(\hbar(\varrho_{n+1})\bigr)\bigr)\bigr) \\ &\le \phi\bigl(\iota\bigl(M(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1})\bigr)\bigr) \\ &< \psi\bigl(\iota\bigl(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1})\bigr)\bigr). \end{aligned} \tag{21}$$

Because ψ is nondecreasing, $\{\iota(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1}))\}$ is a nonincreasing sequence with a lower bound. So there exists $\varepsilon \ge 0$ such that $\lim_{n\to\infty} \iota_n = \varepsilon$, where $\iota_n = \iota(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1}))$. If $\varepsilon > 0$, then taking the right upper limits on both sides of (21), by (iii) in Theorem 2.2 we derive that

$$\psi\bigl(\varepsilon^+\bigr) = \limsup_{\iota_{n+1}\to\varepsilon^+} \psi(\iota_{n+1}) \le \limsup_{\iota_n\to\varepsilon^+} \phi(\iota_n) \le \limsup_{\varrho\to\varepsilon^+} \phi(\varrho) < \psi\bigl(\varepsilon^+\bigr),$$

a contradiction, so $\varepsilon = 0$. By (r1) in Definition 2.3, letting $\iota(p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n), \varphi(\varrho_{n+1})) = R$, we derive that

$$\max\bigl\{p(\varrho_n, \varrho_{n+1}), \varphi(\varrho_n)\bigr\} \le R.$$

This implies that

$$\begin{cases} p(\varrho_n, \varrho_{n+1}) \le R, \\ \varphi(\varrho_{n+1}) \le R. \end{cases} \tag{22}$$

Taking the limits in (22), we have

$$\begin{cases} 0 \le \lim_{n\to\infty} p(\varrho_n, \varrho_{n+1}) \le R = 0, \\ 0 \le \lim_{n\to\infty} \varphi(\varrho_{n+1}) \le R = 0, \end{cases}$$

that is,

$$\begin{cases} \lim_{n\to\infty} p(\varrho_n, \varrho_{n+1}) = 0, \\ \lim_{n\to\infty} \varphi(\varrho_{n+1}) = 0. \end{cases} \tag{23}$$

When $p(\varrho_{n+1}, \varrho_n) > 0$, letting $\varrho = \varrho_{n+1}$, $\sigma = \varrho_n$ in (17), by the same method we have

$$\lim_{n\to\infty} p(\varrho_{n+1}, \varrho_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(\varrho_n) = 0. \tag{24}$$

Step 3. We want to prove that $\lim_{m,n\to\infty} p(\varrho_n,\varrho_m) = 0$.

Suppose on the contrary that there exist $\varepsilon > 0$ and two sequences $\{\varrho_{n(l)}\}$ and $\{\varrho_{m(l)}\}$ such that

$$p(\varrho_{n(l)},\varrho_{m(l)}) \geq \varepsilon \quad \text{and} \quad p(\varrho_{n(l)},\varrho_{m(l)-1}) < \varepsilon,$$

where $m_{(l)} > n_{(l)}$, and

$$\varepsilon \leq p(\varrho_{n(l)},\varrho_{m(l)}) \leq p(\varrho_{n(l)},\varrho_{m(l)-1}) + p(\varrho_{m(l)-1},\varrho_{m(l)}). \tag{25}$$

Taking the limits on both sides of (25), we have

$$\lim_{l\to\infty} p(\varrho_{n(l)},\varrho_{m(l)}) = \varepsilon^+. \tag{26}$$

Now

$$\begin{aligned}
\varepsilon &\leq p(\varrho_{n(l)},\varrho_{m(l)}) \\
&\leq p(\varrho_{n(l)},\varrho_{m(l)-1}) + p(\varrho_{m(l)-1},\varrho_{m(l)}) \\
&\leq p(\varrho_{n(l)},\varrho_{n(l)-1}) + p(\varrho_{n(l)-1},\varrho_{m(l)-1}) + p(\varrho_{m(l)-1},\varrho_{m(l)}) \\
&\leq p(\varrho_{n(l)},\varrho_{n(l)-1}) + p(\varrho_{n(l)-1},\varrho_{n(l)}) + p(\varrho_{n(l)},\varrho_{m(l)-1}) + p(\varrho_{m(l)-1},\varrho_{m(l)}) \\
&< p(\varrho_{n(l)},\varrho_{n(l)-1}) + p(\varrho_{n(l)-1},\varrho_{n(l)}) + p(\varrho_{n(l)},\varrho_{m(l)-1}) + \varepsilon.
\end{aligned} \tag{27}$$

Taking the limits on both sides of (27), we obtain

$$\lim_{l\to\infty} p(\varrho_{n(l)-1},\varrho_{m(l)-1}) = \varepsilon^+. \tag{28}$$

Letting $\varrho = \varrho_{n(l)-1}$ and $\sigma = \varrho_{m(l)-1}$ in (17), we have

$$\begin{aligned}
\psi(\iota_{n,m}) &= \psi\bigl(\iota\bigl(p(\hbar\varrho_{n(l)-1},\hbar\varrho_{m(l)-1}),\varphi(\hbar\varrho_{n(l)-1}),\varphi(\hbar\varrho_{m(l)-1})\bigr)\bigr) \\
&\leq \phi\bigl(\iota\bigl(M(\varrho_{n(l)-1},\varrho_{m(l)-1}),\varphi(\varrho_{n(l)-1}),\varphi(\varrho_{m(l)-1})\bigr)\bigr),
\end{aligned} \tag{29}$$

where $M(\varrho_{n(l)-1},\varrho_{m(l)-1}) = \max\{p(\varrho_{n(l)-1},\varrho_{m(l)-1}), \frac{p(\varrho_{n(l)-1},\hbar\varrho_{n(l)-1})(1+p(\varrho_{m(l)-1},\hbar\varrho_{m(l)-1}))}{1+p(\hbar\varrho_{n(l)-1},\hbar\varrho_{m(l)-1})}\}$. By (28) we obtain

$$\lim_{l\to\infty} M(\varrho_{n(l)-1},\varrho_{m(l)-1}) = \varepsilon^+.$$

Taking the right upper limits on both sides of (29), by (iii) in Theorem 2.2 and (26) we obtain

$$\begin{aligned}
\psi\bigl(r\bigl(\varepsilon^+,0,0\bigr)\bigr) &= \lim_{n,m\to\infty} \psi(\iota_{n,m}) \\
&\leq \lim_{n,m\to\infty} \phi\bigl(M(\varrho_{n(l)-1},\varrho_{m(l)-1}),\varphi(\varrho_{n(l)-1}),\varphi(\varrho_{m(l)-1})\bigr)\bigr)
\end{aligned}$$

$$\begin{aligned}&\leq \lim_{\varrho\to\iota(\varepsilon^+,0,0)}\phi(\varrho)\\&<\psi\big(\iota\big(\varepsilon^+,0,0\big)\big),\end{aligned}$$

a contradiction, so

$$\lim_{n,m\to\infty}p(\varrho_n,\varrho_m)=0. \tag{30}$$

In view of Lemma 2.3, $\{\varrho_n\}$ is a Cauchy sequence.

Step 4. Next, we need to verify that ϱ^* is a φ-fixed point of $\hbar$.

Since $\{\varrho_n\}$ is a Cauchy sequence, there exists $\varrho^*\in\aleph$ such that $\varrho_n\to\varrho^*(n\to\infty)$. Since φ is lower semicontinuous, we get

$$\varphi\big(\varrho^*\big)\leq\lim_{n\to\infty}\inf\varphi(\varrho_{n+1})\leq\lim_{n\to\infty}\varphi(\varrho_{n+1})=0,$$

so $\varphi(\varrho^*)=0$.

Since $\lim_{m,n\to\infty}p(\varrho_n,\varrho_m)=0$, for each $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that $p(\varrho_n,\varrho_m)<\varepsilon$ for $n,m>N_\varepsilon$. Since $\{\varrho_n\}\to\varrho^*$ and $p(\varrho,\cdot)$ is lower semicontinuous, we get

$$0\leq p\big(\varrho_n,\varrho^*\big)\leq\liminf_{m\to\infty}p(\varrho_n,\varrho_m)\leq\varepsilon. \tag{31}$$

Taking the limits on both sides of (31), by (30) we obtain

$$\lim_{n\to\infty}p\big(\varrho_n,\varrho^*\big)=0. \tag{32}$$

Next, we prove that $\lim_{n\to\infty}p(\varrho_n,\hbar\varrho^*)=0$. We consider two cases.

Case 1: There exists $N\in\mathbb{N}$ such that $p(\varrho_{n+1},\hbar\varrho^*)>0$ for all $n>N$. Suppose that there exists $n_0>N$ such that $p(\varrho_{n_0+1},\varrho^*)=0$ and $p(\varrho_{n_0+1},\hbar\varrho^*)=\delta>0$. Then we have $p(\varrho_{n_0+1},\varrho^*)\leq p(\varrho_{n_0+1},\hbar\varrho^*)=\delta$ and $p(\varrho_{n_0+1},\hbar\varrho^*)\leq\delta$, so $\varrho^*=\hbar\varrho^*$ by (c) in Definition 2.4. The proof is completed. If $p(\varrho_n,\varrho^*)>0$ for all $n>N$, by (17) we get

$$\begin{aligned}\psi\big(\iota\big(p\big(\varrho_{n+1},\hbar\varrho^*\big),\varphi(\varrho_{n+1}),\varphi\big(\hbar\varrho^*\big)\big)\big)&=\psi\big(\iota\big(p\big(\hbar\varrho_n,\hbar\varrho^*\big),\varphi(\hbar\varrho_n),\varphi\big(\hbar\varrho^*\big)\big)\big)\\&\leq\phi\big(\iota\big(M\big(\big(\varrho_n,\varrho^*\big),\varphi(\varrho_n),\varphi\big(\varrho^*\big)\big)\big)\big)\\&<\psi\big(\iota\big(M\big(\varrho_n,\varrho^*\big),\varphi(\varrho_n),\varphi\big(\varrho^*\big)\big)\big).\end{aligned}$$

Since ψ is nondecreasing, we have

$$0\leq\iota\big(p\big(\varrho_{n+1},\hbar\varrho^*\big),\varphi(\varrho_{n+1}),\varphi\big(\hbar\varrho^*\big)\big)<\iota\big(M\big(\varrho_n,\varrho^*\big),\varphi(\varrho_n),\varphi\big(\varrho^*\big)\big).$$

By (ι_1) we can derive that

$$\begin{aligned}&0\leq p\big(\varrho_{n+1},\hbar\varrho^*\big)\leq\iota\big(p\big(\varrho_{n+1},\hbar\varrho^*\big),\varphi(\varrho_{n+1}),\varphi\big(\hbar\varrho^*\big)\big)\\&\quad<\iota\big(M\big(\varrho_n,\varrho^*\big),\varphi(\varrho_n),\varphi\big(\varrho^*\big)\big).\end{aligned} \tag{33}$$

In addition, $\lim_{n\to\infty}M(\varrho_n,\varrho^*)=\lim_{n\to\infty}\max\{p(\varrho_n,\varrho^*),\frac{p((\varrho_n,\hbar\varrho_n)(1+p((\varrho^*,\hbar\varrho^*)))}{1+p(\hbar\varrho_n,\hbar\varrho^*)}\}=0$.

Taking the limits on both sides of (33), it follows that

$$\lim_{n\to\infty} p(\varrho_{n+1}, \hbar\varrho^*) = 0.$$

By the triangle inequality it follows that

$$p(\varrho_n, \hbar\varrho^*) \le p(\varrho_n, \varrho_{n+1}) + p(\varrho_{n+1}, \hbar\varrho^*),$$

and therefore

$$\lim_{k\to\infty} p(\varrho_n, \hbar\varrho^*) = 0. \tag{34}$$

Hence by (i) in Lemma 2.2, (32), and (34) we can conclude that $\varrho^* = \hbar\varrho^*$.

Case 2: $\lim_{n\to\infty} p(\varrho_n, \varrho^*) = 0$. Then there exist $N \in \mathbb{N}$ and $\delta > 0$ such that $p(\varrho_n, \varrho^*) < \delta$ for all $n > N$. If there exists $n_0 > N$ such that $p(\varrho_{n_0+1}, \hbar\varrho^*) = 0$, then $\hbar\varrho^* = \varrho^*$.

Step 5. We claim that ϱ^* is a unique fixed point of $\hbar$.

If $\varrho^* \in F(\hbar)$, then assuming that $p(\varrho^*, \varrho^*) \neq 0$ and letting $\varrho = \sigma = \varrho^*$ (17), we have

$$\begin{aligned}
\psi\bigl(\iota(p(\varrho^*, \varrho^*), 0, 0)\bigr) &= \psi\bigl(\iota(p(\varrho^*, \varrho^*), \varphi(\varrho^*), \varphi(\varrho^*)\bigr) \\
&= \psi\bigl(\iota(p(\hbar\varrho^*, \hbar\varrho^*))\varphi(\hbar\varrho^*), \varphi(\hbar\varrho^*)\bigr) \\
&\le \phi\bigl(\iota(M(\varrho^*, \varrho^*), \varphi(\varrho^*), \varphi(\varrho^*)\bigr) \\
&= \phi\bigl(\iota(M(\varrho^*, \varrho^*), 0, 0)\bigr) \\
&< \psi\bigl(\iota(p(\varrho^*, \varrho^*), 0, 0)\bigr).
\end{aligned}$$

This leads to a contradiction, so $p(\varrho^*, \varrho^*) = 0$.

If $\varrho^*, \sigma \in F(\hbar)$, then assuming that $p(\varrho^*, \sigma) \neq 0$ and letting $\varrho = \varrho^*$ in (17), we have

$$\begin{aligned}
\psi\bigl(\iota(p(\varrho^*, \sigma), 0, 0)\bigr) &= \psi\bigl(\iota(p(\varrho^*, \sigma), \varphi(\varrho^*), \varphi(\sigma)\bigr) \\
&= \psi\bigl(\iota(p(\hbar\varrho^*, \hbar\sigma))\varphi(\hbar\varrho^*), \varphi(\hbar\sigma)\bigr) \\
&\le \phi\bigl(\iota(M(\varrho^*, \sigma), \varphi(\varrho^*), \varphi(\sigma)\bigr) \\
&= \phi\bigl(\iota(M(\varrho^*, \sigma), 0, 0)\bigr) \\
&< \psi\left(\iota\left(\max\left\{p(\varrho^*, \sigma), \frac{p(\varrho^*, \hbar\varrho^*)(1 + p(\sigma, \hbar\sigma))}{1 + p(\hbar\varrho^*, \hbar\sigma)}\right\}, 0, 0\right)\right) \\
&= \psi\bigl(\iota(p(\varrho^*, \sigma), 0, 0)\bigr),
\end{aligned}$$

a contradiction, and therefore $p(\varrho^*, \sigma) = 0$. So we deduce that $\varrho^* = \sigma$. The proof is completed. □

Example 3.6 Let $\aleph = 0 \cup \{\frac{1}{8^n} : n \ge 1\}$, where $(\aleph, d)$ is a complete metric space with a metric d. We define $p : \aleph \times \aleph \to [0, \infty)$ by $p(\varrho, \sigma) = \sigma$. Let $\phi(\varrho) = \frac{1}{4}\varrho$, $\iota(\varrho, \sigma, \rho) = \varrho + \sigma + \rho$, $\varphi(\varrho) = 2\varrho$, and $\psi(\varrho) = \frac{1}{2}\varrho$. Let $\hbar : \aleph \to \aleph$ be defined by $\hbar\varrho = \frac{\varrho}{32}$ for $\varrho \in \aleph$.

For $\varrho \neq 0$ or $\sigma \neq 0$, we have

$$\begin{aligned}\psi\big(\iota\big(p(\hbar\varrho,\hbar\sigma),\varphi(\hbar\varrho),\varphi(\hbar\sigma)\big)\big) &= \psi\left(\iota\left(\frac{\varrho+\sigma}{32}\right),\frac{\varrho}{16},\frac{\sigma}{16}\right)) \\ &= \frac{3\varrho+3\sigma}{64} \\ &\leq \frac{1}{4}(3\varrho+3\sigma) \\ &= \phi\left(\iota\left(\max(\varrho+\sigma),\frac{\frac{33\varrho+33\sigma}{32}}{1+\frac{\varrho+\sigma}{32}}\right),2\varrho,2\sigma\right) \\ &= \phi\big(\iota\big(M(\varrho,\sigma),\varphi(\varrho),\varphi(\sigma)\big)\big).\end{aligned}$$

So $\hbar$ is a rational-$(\iota,\psi,\varphi,\phi,p)$ contraction and satisfies all conditions of Theorem 3.2. So $\hbar$ has a unique φ-fixed point $\varrho = 0$.

4 Application

Application 1 For convenience, we first give the following notations. We denote the set of all complex number matrices of order m by M^m, the set of all Hermitian matrices of order m by H^m, and P^m and H_+^m represent the sets of all $m \times n$ positive matrices and $m \times m$ positive semidefinite matrices, respectively. Clearly, $P^m \subseteq H^m \subseteq M^m$ and $H_+^m \subseteq H^m$. Here $A_1 \succ O$ (O represents the null matrix of the same order) and $A_1 \succeq O$ mean that $A_1 \in P^m$ and $A_1 \in H_+^m$, respectively; for $A_1 - A_2 \succeq O$ and $A_1 - A_2 \succ O$, we will use $A_1 \succeq A_2$ and $A_1 \succ A_2$, respectively.

In this section, we study the existence of solutions for the following linear matrix equation:

$$U = G + \sum_{i=1}^{m} A_i^* U A_i + \sum_{i=1}^{m} B_i^* U B_i, \tag{35}$$

where $G \in P^m, A_i, B_i$ are arbitrary $m \times m$ matrices for each i. We use the metric $d(A,B) = \| A - B \|_{\mathrm{tr},R} = \| R^{\frac{1}{2}}(A-B)R^{\frac{1}{2}} \|_{\mathrm{tr}}$, which is induced by the norm $\| A \|_{\mathrm{tr}} = \sum_{i=1}^{n} \sigma_i(A)$, where $R \in P^m, A, B \in H^m$, and $\sigma_i(A), i = 1,2,\dots,n$, are the eigenvalues of $A \in M^m$. Clearly, the set H^m equipped with the metric d is a complete metric space. Then (H^m, d) is a complete extended rectangular b-metric space with respect to $\xi = 3$.

Define the mapping $\hbar : H^m \to H^m$ by

$$\hbar(U) = G + \sum_{i=1}^{m} A_i^* U A_i + \sum_{i=1}^{m} B_i^* U B_i$$

for $A, B \in H^m$.

Lemma 4.1 [23] *If* $A, B \in H_+^m$, *then* $0 \leq \mathrm{tr}(AB) \leq \| A \| \,\mathrm{tr}(B)$.

Lemma 4.2 [23] *If* $A \in H^m$ *is such that* $A \prec I_n$, *then* $\| A \| < 1$.

Theorem 4.1 *If $R \in P^m$, $\sum_{i=1}^m A_i^* R A_i \prec \frac{1}{5}R$, and $\sum_{i=1}^m B_i^* R B_i \prec \frac{1}{5}R$, then the mapping $\hbar$ has a fixed point in H^m.*

Proof Suppose that $U, V \in H^m$ and let $p(x,y) = d(x,y)$. Then

$$\begin{aligned}
&P\big(\hbar(U), \hbar(V)\big)\\
&\quad = \| \hbar(U) - \hbar(V) \|_{\mathrm{tr},R}\\
&\quad = \mathrm{tr}\big(R^{\frac{1}{2}}\big(\hbar(U) - \hbar(V)\big)R^{\frac{1}{2}}\big)\\
&\quad = \mathrm{tr}\left(\sum_{i=1}^m \big\{R^{\frac{1}{2}}\big(A_i^*(U-V)A_i + B_i^*(U-V)B_iR^{\frac{1}{2}}\big)\big\}\right)\\
&\quad = \mathrm{tr}\left(\sum_{i=1}^m R^{\frac{1}{2}}A_i^*(U-V)A_iR^{\frac{1}{2}} + \sum_{i=1}^m R^{\frac{1}{2}}B_i^*(U-V)B_iR^{\frac{1}{2}}\right)\\
&\quad = \sum_{i=1}^m \mathrm{tr}\big(R^{\frac{1}{2}}A_i^*(U-V)A_iR^{\frac{1}{2}} + R^{\frac{1}{2}}B_i^*(U-V)B_iR^{\frac{1}{2}}\big)\\
&\quad = \sum_{i=1}^m \mathrm{tr}\big(A_i^*RA_i(U-V)\big) + \sum_{i=1}^m \mathrm{tr}\big(B_i^*RB_i(U-V)\big)\\
&\quad = \sum_{i=1}^m \mathrm{tr}\big(A_i^*RA_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{-\frac{1}{2}}R^{\frac{1}{2}}\big) + \sum_{i=1}^m \mathrm{tr}\big(B_i^*RB_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{-\frac{1}{2}}R^{\frac{1}{2}}\big)\\
&\quad = \sum_{i=1}^m \mathrm{tr}\big(R^{-\frac{1}{2}}A_i^*RA_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{\frac{1}{2}}\big) + \sum_{i=1}^m \mathrm{tr}\big(R^{-\frac{1}{2}}B_i^*RB_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{\frac{1}{2}}\big)\\
&\quad = \mathrm{tr}\left(\sum_{i=1}^m R^{-\frac{1}{2}}A_i^*RA_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{\frac{1}{2}}\right) + \mathrm{tr}\left(\sum_{i=1}^m R^{-\frac{1}{2}}B_i^*RB_iR^{-\frac{1}{2}}R^{\frac{1}{2}}(U-V)R^{\frac{1}{2}}\right)\\
&\quad \le \left\|\sum_{i=1}^m R^{-\frac{1}{2}}A_i^*RA_iR^{-\frac{1}{2}}\right\| \big\|(U-V)\big\|_{tr,R} + \left\|\sum_{i=1}^m R^{-\frac{1}{2}}B_i^*RB_iR^{-\frac{1}{2}}\right\| \big\|(U-V)\big\|_{tr,R}\\
&\quad = \left(\left\|\sum_{i=1}^m R^{-\frac{1}{2}}A_i^*RA_iR^{-\frac{1}{2}}\right\| + \left\|\sum_{i=1}^m R^{-\frac{1}{2}}B_i^*RB_iR^{-\frac{1}{2}}\right\|\right)\big\|(U-V)\big\|_{tr,R}\\
&\quad = k_1\big\|(U-V)\big\|_{tr,R}\\
&\quad \le k_1 N(U,V),
\end{aligned}$$

where $k_1 = (\| \sum_{i=1}^m R^{-\frac{1}{2}}A_i^*RA_iR^{-\frac{1}{2}} \| + \| \sum_{i=1}^m R^{-\frac{1}{2}}B_i^*RB_iR^{-\frac{1}{2}} \|) < \frac{2}{5}$ and

$$\begin{aligned}
N(U,V) &= \max\left\{p(U,V), \frac{p(U,T(U))(1 + P(V,T(V)))}{1 + P(T(U),T(V))}\right\}\\
&\ge p(U,V)\\
&= \|U - V\|_{\mathrm{tr},R}.
\end{aligned}$$

When $\varphi(\varrho) = 0$, $\phi(\varrho) = k_1\varrho$, $\iota(\varrho,\sigma,\rho) = \varrho + \sigma + \rho$, $\psi(\varrho) = \varrho$, T satisfies Theorem 3.2. Therefore $\hbar$ has a unique fixed point, and (35) has a solution. □

Application 2 Here we apply Theorem 3.1 to guarantee the existence of a solution to the following nonlinear Fredholm integral equation:

$$\varrho(t) = \int_a^b G(t,s)f\bigl(t,\varrho(s)\bigr)\,ds, \tag{36}$$

where $G : [a,b] \times [a,b] \to [0,\infty)$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Consider $\aleph = C[a,b]$, the set of all continuous functions from $[a,b]$ into $\mathbb{R}$. Define $\hbar : \aleph \to \aleph$ as

$$\bigl(\hbar\varrho(t)\bigr)(t) = \int_a^b G(t,s)f\bigl(t,\varrho(s)\bigr)\,ds$$

for $\varrho \in \aleph$ and $t \in [a,b]$. Take on $\aleph$ the complete metric

$$d(\varrho,\sigma) = \sup_{t\in[a,b]} \bigl|\varrho(t) - \sigma(t)\bigr|.$$

Consider also on $\aleph$ the w-distance $p : \aleph \times \aleph \to [0,\infty)$ given by

$$p(\varrho,\sigma) = \sup_{t\in[a,b]} \bigl|\varrho(t)\bigr| + \sup_{t\in[a,b]} \bigl|\sigma(t)\bigr|$$

for $\varrho,\sigma \in \aleph$. We will prove that $\hbar$ has a fixed point under the following conditions:

(i) There is $\phi : (0,+\infty) \to (-\infty,+\infty)$ such that for all $t \in [a,b]$ and $\varrho,\sigma \in \mathbb{R}$,

$$\bigl|f(t,\varrho)\bigr| + \bigl|f(t,\sigma)\bigr| \le \phi\left(\max\left\{|\varrho| + |\sigma|, \frac{(|\varrho| + |\hbar\varrho|)(1 + |\sigma| + |\hbar\sigma|)}{1 + (|\hbar\varrho| + |\hbar\sigma|)}\right\}\right);$$

(ii) $\sup_{a\le t\le b} \int_a^b G(t,s)\,ds \le \frac{1}{3}$;

(iii) $\inf\{p(\varrho,\sigma), \frac{p(\varrho,\hbar\varrho)(1+p(\sigma,\hbar\sigma))}{1+p(\hbar\varrho,\hbar\sigma)}\} > 0$ for all $\varrho,\sigma \in \aleph$.

Theorem 4.2 *Under assumptions* (i)–(iii), *equation* (36) *has a solution.*

Proof

$$\begin{aligned}
&\bigl|(\hbar\varrho)(t)\bigr| + \bigl|(\hbar\sigma)(t)\bigr| \\
&\quad\le \int_a^b G(t,s))\bigl|f\bigl(t,\varrho(s)\bigr)\bigr| + \bigl|f\bigl(t,\sigma(s)\bigr)\bigr|\,ds \\
&\quad\le \int_a^b G(t,s)\phi\left(\max\left\{\bigl|\varrho(s)\bigr| + \bigl|\sigma(s)\bigr|, \frac{(|\varrho(s)| + |\hbar\varrho(s)|)(1 + |\sigma(s)| + |\hbar\sigma(s)|)}{1 + (|\hbar\varrho(s)| + |\hbar\sigma(s)|)}\right\}\right)ds \\
&\quad\le \frac{1}{3}\phi\left(\max\left\{\bigl|\varrho(s)\bigr| + \bigl|\sigma(s)\bigr|, \frac{(|\varrho(s)| + |\hbar\varrho(s)|)(1 + |\sigma(s)| + |\hbar\sigma(s)|)}{1 + (|\hbar\varrho(s)| + |\hbar\sigma(s)|)}\right\}\right) \\
&\quad\le \phi\left(\max\left\{\bigl|\varrho(s)\bigr| + \bigl|\sigma(s)\bigr|, \frac{(|\varrho(s)| + |\hbar\varrho(s)|)(1 + |\sigma(s)| + |\hbar\sigma(s)|)}{1 + (|\hbar\varrho(s)| + |\hbar\sigma(s)|)}\right\}\right).
\end{aligned}$$

□

So $\hbar$ satisfies the contraction expression (17), where $\varphi(\varrho) = 0$, $\phi(\varrho) = \frac{1}{2}\varrho$, $\iota(\varrho,\sigma,\rho) = \varrho + \sigma + \rho$, and $\psi(\varrho) = \varrho$. Therefore equation (36) has a solution.

5 Conclusion

In this paper, we introduce two new nonlinear contractions by using the idea of a w-distance in a metric space and establish some new φ-fixed point results in metric spaces with w-distance. Next, we use some simple examples to show the validity of our main results. Then we use our results for investigating the existence and uniqueness of a solution for a nonlinear integral equation and nonlinear matrix equation. There is no doubt about the importance of fixed point theory.

6 Prospective

The two nonlinear contractions proposed in this paper contain some existing nonlinear contraction conditions. By taking some special functions our results develop the results of [1, 2, 6, 11, 13, 14] and so on. At the same time, we apply our results to linear matrix equations and nonlinear Fredholm integral equations. However, this paper has strong constraints on several types of functions in such contraction conditions and may need finding some relatively weak conditions. At the same time the recent generation of some new metric spaces has attracted the research of many scholars.

There are some possible works in the future:

(i) Can the metric space in this paper be replaced with other generalized metric spaces (such as fuzzy metric space, ordered metric space, b_2-distance space, etc.)?

(ii) Is it possible to obtain similar φ-fixed point results by weakening the conditions required for the contraction expression in this paper?

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Declarations

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The authors declare no competing interests.

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