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New results for the upper bounds of the distance between adjacent zeros of first-order differential equations with several variable delays

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Abstract

The distance between consecutive zeros of a first-order differential equation with several variable delays is studied. Here, we show that the distribution of zeros of differential equations with variable delays is not an easy extension of the case of constant delays. We obtain new upper bounds for the distance between zeros of all solutions of a differential equation with several delays, which extend and improve some existing results. Two illustrative examples are given to show the advantages of the proposed results over the known ones.

Mathematics Subject Classification: 34K11; 34K06

Keywords: Differential equations; Variable delays; Distance between zeros; Oscillation

1 Introduction

Consider the differential equation with several variable delays

$$x'(t) + \sum_{j=1}^{n} a_j(t)x\big(g_j(t)\big) = 0, \quad t \geq t_0, \tag{1}$$

where $a_j, g_j \in C([t_0,\infty),[0,\infty))$, $g_j(t)$ is a strictly increasing function such that $g_j(t) \leq t$, $\lim_{t\to\infty} g_j(t) = \infty$, $j = 1,2,\dots,n$. We make use of the following notation:

$$h_i(t) = \max_{1\leq j\leq i} g_j(t), \qquad w_i(t) = \min_{1\leq j\leq i} g_j(t), \quad i = 1,2,\dots,n.$$

Therefore,

$$h_j^{-k}(t) \geq h_i^{-k}(t) \quad \text{and} \quad w_j^{-k}(t) \leq w_i^{-k}(t), \quad i \geq j, i,j = 1,2,\dots,n, k = 1,2,\dots,$$

where $h_j^{-1}(t)$ and $w_j^{-1}(t)$ are the inverse of the functions $h_j(t)$ and $w_j(t)$, $j = 1,2,\dots,n$.

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Consequently,

$$\max_{1\le j\le i} w_j^{-k}(t) = w_i^{-k}(t) \quad \text{and} \quad \max_{1\le j\le i} h_j^{-k}(t) = h_1^{-k}(t), \quad i = 1,2,\dots,n, k = 1,2,\dots.$$

Let $t^* \ge t_0$ and $x(t)$ be a continuous function on $[t^*,\infty)$. The function $x(t)$ is said to be a solution of Eq. (1) on $[t^*,\infty)$ if $x(t)$ is continuously differentiable on $[w_n^{-1}(t^*),\infty)$ and satisfying Eq. (1) for $t \ge w_n^{-1}(t^*)$. Any solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory; otherwise, it is called nonoscillatory.

The oscillation theory of delay differential equations has received a great deal of attention in recent years; see the monographs [1, 2, 13–15] and the papers [3–12, 16–27] for more details. Many efforts have been made to establish sufficient and/or necessary oscillation criteria for Eq. (1); see [1, 3, 9, 11, 13, 15, 17]. In oscillation theory, the distribution of zeros of delay differential equations has always been an important problem. In this topic, not only is the existence of zeros demonstrated, but efforts are also being made to determine their locations. In fact, the study of the distribution of zeros raises many challenges. This explains the few studies that concern the distance between zeros compared to the oscillation.

Many upper bounds for the distance between consecutive zeros of the delay differential equations

$$x'(t) + a(t)x(t-\sigma) = 0, \quad t \ge t_0, \tag{2}$$

and

$$x'(t) + a(t)x\bigl(g(t)\bigr) = 0, \quad t \ge t_0, \tag{3}$$

where $\sigma > 0$, $a, g \in C([t_0,\infty),[0,\infty))$, $g(t)$ is a strictly increasing function such that $\lim_{t\to\infty} g(t) = \infty$, have been obtained by [6–11, 17, 18, 20–27]. Further, some results concerning the lower bounds for the distance between consecutive zeros of all solutions of Eqs. (2) and (3) were investigated in [6–10, 17]. For example, Barr [6] showed that the lower bound of the distance between zeros of an oscillatory solution of Eq. (3) goes to infinity when $t - g(t)$ is not bounded. Therefore, we will restrict our attention to the case when $t - g_j(t) < \infty$, $j = 1,2,\dots,n$. In this work, we obtain new upper bounds for the distance between consecutive zeros of all solutions of Eq. (1), which would improve the abovementioned ones. We conclude by providing two illustrative examples to show the applicability and importance of some of our findings.

2 Main results

Let $t_1 \ge t_0$ and $D_{t_1}(x)$ be the upper bound of the distance between consecutive zeros of all solutions of Eq. (1) on the interval $[t_1,\infty)$. Throughout this paper, it is assumed that

$$\sup_{t\ge t_1}\bigl\{t - g_j(t)\bigr\} < \infty \quad \text{for } j = 1,2,\dots,n.$$

Let $r \in \{1, 2, \dots, n\}$ and the sequence $\{R^k(\eta_r)\}_{k\geq 0}$ be defined by $R^0(\eta_r) = 1$ and

$$\begin{aligned} R^1(\eta_r) &= \frac{1}{1-\eta_r}, \\ R^k(\eta_r) &= \frac{1}{1-\eta_r - \frac{1}{2}\eta_r^2 R^{k-1}(\eta_r)}, \quad k = 2, 3, \dots, \end{aligned} \tag{4}$$

where

$$\int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\, ds \geq \eta_r \quad \text{for } t \geq h_r^{-1}(t_1).$$

Lemma 2.1 *Let $k \in \mathbb{N}_0$, $r \in \{1, 2, \dots, n\}$ and $x(t)$ be a solution of Eq.* (1) *such that $x(t) > 0$ on $[T_0, T_1]$, $T_0 \geq t_1$, $T_1 \geq h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0)))$. Then*

$$\frac{x(h_r(t))}{x(t)} \geq R^k(\eta_r) \quad \text{for } t \in \bigl[h_r^{-(k-1)}\bigl(w_r^{-k}\bigl(w_n^{-1}(T_0)\bigr)\bigr), T_1\bigr], \tag{5}$$

where $w_r^0(T_0) = T_0$.

Proof Since $x(t) > 0$ on $[T_0, T_1]$, it follows from Eq. (1) that $x'(t) \leq 0$ on $[w_n^{-1}(T_0), T_1]$, and hence

$$\frac{x(h_r(t))}{x(t)} \geq 1 = R^0(\eta_r) \quad \text{for } t \in \bigl[h_r^{-1}\bigl(w_n^{-1}(T_0)\bigr), T_1\bigr].$$

In view of Eq. (1) and the positivity of $x(t)$ on $[T_0, T_1]$, we have

$$x'(t) + \sum_{j=1}^{r} a_j(t) x\bigl(g_j(t)\bigr) \leq 0 \quad \text{for } t \in \bigl[w_n^{-1}(T_0), T_1\bigr]. \tag{6}$$

Integrating from $h_r(t)$ to t, we get

$$x(t) - x\bigl(h_r(t)\bigr) + \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) x\bigl(g_j(s)\bigr)\, ds \leq 0 \quad \text{for } t \in \bigl[h_r^{-1}\bigl(w_n^{-1}(T_0)\bigr), T_1\bigr]. \tag{7}$$

Since $h_r(t) \geq g_j(t)$, so $h_r(t) \geq g_j(s)$ for $h_r(t) \leq s \leq t$, $j = 1, 2, \dots, r$, it follows from (6) that

$$x\bigl(g_j(s)\bigr) \geq x\bigl(h_r(t)\bigr) + \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1) x\bigl(g_{j_1}(s_1)\bigr)\, ds_1, \quad t \in \bigl[h_r^{-1}\bigl(w_r^{-1}\bigl(w_n^{-1}(T_0)\bigr)\bigr), T_1\bigr].$$

Substituting into (7), we obtain

$$\begin{aligned} & x(t) - x\bigl(h_r(t)\bigr) + x\bigl(h_r(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\, ds \\ & \quad + \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1) x\bigl(g_{j_1}(s_1)\bigr)\, ds_1\, ds \leq 0 \end{aligned} \tag{8}$$

for $t \in [h_r^{-1}(w_r^{-1}(w_n^{-1}(T_0))), T_1]$. Therefore,

$$x(t) - x\bigl(h_r(t)\bigr) + x\bigl(h_r(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\,ds \le 0 \quad \text{for } t \in \bigl[h_r^{-1}\bigl(w_r^{-1}\bigl(w_n^{-1}(T_0)\bigr)\bigr), T_1\bigr].$$

That is,

$$\begin{aligned} \frac{x(h_r(t))}{x(t)} &\ge \frac{1}{1 - \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\,ds} \ge \frac{1}{1-\eta_r} \\ &= R^1(\eta_r) \quad \text{for } t \in \bigl[h_r^{-1}\bigl(w_r^{-1}\bigl(w_n^{-1}(T_0)\bigr)\bigr), T_1\bigr]. \end{aligned} \tag{9}$$

Also, since $h_r^2(t) \ge g_{j_1}(s_1)$ for $h_r(t) \le s \le t$, $g_j(s) \le s_1 \le h_r(t)$, $j, j_1 = 1, 2, \dots, r$. Then

$$x\bigl(g_{j_1}(s_1)\bigr) \ge x\bigl(h_r^2(t)\bigr) + \int_{g_{j_1}(s_1)}^{h_r^2(t)} \sum_{j_2=1}^{r} a_{j_2}(s_2) x\bigl(g_{j_2}(s_2)\bigr)\,ds_2, \quad g_{j_1}(s_1) \le s_2 \le h_r^2(t)$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$. From this and (8), it follows that

$$\begin{aligned} &x(t) - x\bigl(h_r(t)\bigr) + x\bigl(h_r(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\,ds \\ &\quad + x\bigl(h_r^2(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1)\,ds_1\,ds \\ &\quad + \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1) \int_{g_{j_1}(s_1)}^{h_r^2(t)} \sum_{j_2=1}^{r} a_{j_2}(s_2) x\bigl(g_{j_2}(s_2)\bigr)\,ds_2\,ds_1\,ds \le 0 \end{aligned} \tag{10}$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$. Using the positivity of $x(t)$ on $[T_0, T_1]$, we have

$$\begin{aligned} &x(t) - x\bigl(h_r(t)\bigr) + x\bigl(h_r(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\,ds \\ &\quad + x\bigl(h_r^2(t)\bigr) \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1)\,ds_1\,ds \le 0 \end{aligned}$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$. Therefore,

$$\frac{x(h_r(t))}{x(t)} \ge \frac{1}{1 - \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s)\,ds - \frac{x(h_r^2(t))}{x(h_r(t))} \int_{h_r(t)}^{t} \sum_{j=1}^{r} a_j(s) \int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} a_{j_1}(s_1)\,ds_1\,ds} \tag{11}$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$. Clearly,

$$\begin{aligned}\int_{h_r(t)}^{t}\sum_{j=1}^{r}a_j(s)\int_{g_j(s)}^{h_r(t)}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds &\geq \int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\int_{g_j(s)}^{s}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds\\ &\quad-\int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\int_{h_r(t)}^{s}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds\\ &\geq \eta_r^2-\int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\int_{h_r(t)}^{s}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds, \qquad (12)\end{aligned}$$

where $\bar{t} \in (h_r(t), t]$ such that $\int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\,ds = \eta_r$. It is easy to see that (see [13, Lemma 2.1.3])

$$\int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\int_{h_r(t)}^{s}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds = \frac{1}{2}\left(\int_{h_r(t)}^{\bar{t}}\sum_{j=1}^{r}a_j(s)\,ds\right)^2 = \frac{1}{2}\eta_r^2.$$

From this and (12), we get

$$\int_{h_r(t)}^{t}\sum_{j=1}^{r}a_j(s)\int_{g_j(s)}^{h_r(t)}\sum_{j_1=1}^{r}a_{j_1}(s_1)\,ds_1\,ds \geq \frac{1}{2}\eta_r^2.$$

Substituting into (11), we have

$$\frac{x(h_r(t))}{x(t)} \geq \frac{1}{1-\eta_r-\frac{1}{2}\eta_r^2\frac{x(h_r^2(t))}{x(h_r(t))}} \quad \text{for } t \in \big[h_r^{-1}\big(w_r^{-2}\big(w_n^{-1}(T_0)\big)\big), T_1\big]. \qquad (13)$$

In view of (9), we have

$$\frac{x(h_r^2(t))}{x(h_r(t))} \geq R^1(\eta_r) \quad \text{for } t \in \big[h_r^{-2}\big(w_r^{-1}\big(w_n^{-1}(T_0)\big)\big), T_1\big].$$

This together with (13) implies that

$$\frac{x(h_r(t))}{x(t)} \geq \frac{1}{1-\eta_r-\frac{1}{2}\eta_r^2R^1(\eta_r)} = R^2(\eta_r) \quad \text{for } t \in \big[h_r^{-1}\big(w_r^{-2}\big(w_n^{-1}(T_0)\big)\big), T_1\big].$$

Therefore,

$$\frac{x(h_r^2(t))}{x(h_r(t))} \geq R^2(\eta_r) \quad \text{for } t \in \big[h_r^{-2}\big(w_r^{-2}\big(w_n^{-1}(T_0)\big)\big), T_1\big].$$

From this and (13), we get

$$\frac{x(h_r(t))}{x(t)} \geq \frac{1}{1-\eta_r-\frac{1}{2}\eta_r^2R^2(\eta_r)} = R^3(\eta_r)$$

for $t \in [h_r^{-1}(w_r^{-3}(w_n^{-1}(T_0))), T_1] \subseteq [h_r^{-2}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$.

Repeating this procedure k times, we obtain (5). The proof is complete. □

Let $r \in \{1,2,\dots,n\}$ and the sequence $\{B^i_{j,r}(s,t)\}_{i\geq 1}$, $j = 1,2,\dots,r$, be defined by

$$B^1_{j,r}(s,t) = a_j(s), \quad h_r(t) \leq s \leq t \text{ for } t \geq w_r^{-1}(t_1)$$
$$B^i_{j,r}(s,t) = a_j(s)\int_{g_j(s)}^{h_r(t)} \sum_{j_1=1}^{r} B^{i-1}_{j_1,r}\bigl(s_1, h_r(t)\bigr)\,ds_1, \quad h_r(t) \leq s \leq t, i = 2,3,\dots$$

for $t \geq w_r^{-i}(t_1)$.

Theorem 2.1 *Assume that $k \in \mathbb{N}$ and $r \in \{1,2,\dots,n\}$. If*

$$\sum_{i=1}^{k}\prod_{j=2}^{i} R^{k+1-j}(\eta_r)\int_{h_r(t)}^{t}\sum_{j=1}^{r} B^i_{j,r}\bigl(s,h_r(t)\bigr)\,ds \geq 1 \quad \textit{for all } t \geq w_r^{-k-1}(t_1),$$

then Eq. (1) oscillates and $D_{t_1}(x) \leq \sup_{t\geq t_1}\{h_r^{-1}(w_r^{-k}(w_n^{-1}(t))) - t\}$.

Proof Suppose the contrary, let $x(t)$ be a positive solution of Eq. (1) on $[T_0, T_1]$, $T_0 \geq t_1$, $T_1 > h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0)))$. Using a similar argument as in the proof of Lemma 2.1, we obtain (10). That is,

$$x(t) - x\bigl(h_r(t)\bigr) + \sum_{i=1}^{2} x\bigl(h_r^i(t)\bigr)\int_{h_r(t)}^{t}\sum_{j=1}^{r} B^i_{j,r}\bigl(s,h_r(t)\bigr)\,ds$$
$$+ \int_{h_r(t)}^{t}\sum_{j=1}^{r} a_j(s)\int_{g_j(s)}^{h_r(t)}\sum_{j_1=1}^{r} a_{j_1}(s_1)\int_{g_{j_1}(s_1)}^{h_r^2(t)}\sum_{j_2=1}^{r} a_{j_2}(s_2)x\bigl(g_{j_2}(s_2)\bigr)\,ds_2\,ds_1\,ds \leq 0,$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$, where $h_r^1(t) = h_r(t)$. It follows that

$$x(t) - x\bigl(h_r(t)\bigr) + \sum_{i=1}^{2}\int_{h_r(t)}^{t} x\bigl(h_r^i(t)\bigr)\sum_{j=1}^{r} B^i_{j,r}\bigl(s,h_r(t)\bigr)\,ds \leq 0$$

for $t \in [h_r^{-1}(w_r^{-2}(w_n^{-1}(T_0))), T_1]$. By repeating this argument k times, we get

$$x(t) - x\bigl(h_r(t)\bigr) + \sum_{i=1}^{k}\int_{h_r(t)}^{t} x\bigl(h_r^i(t)\bigr)\sum_{j=1}^{r} B^i_{j,r}\bigl(s,h_r(t)\bigr)\,ds \leq 0 \tag{14}$$

for $t \in [h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0))), T_1]$. Since

$$x\bigl(h_r^i(t)\bigr) = \left(\prod_{j=2}^{i}\frac{x(h_r^j(t))}{x(h_r^{j-1}(t))}\right)x\bigl(h_r(t)\bigr), \quad i = 1,2,\dots.$$

By using (5) and the fact that

$$h_r^{j-1}(t) \in \bigl[h_r^{-1}\bigl(w_r^{-k+(j-1)}\bigl(w_n^{-1}(T_0)\bigr)\bigr), T_1\bigr]$$

for $t \in [h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0))), T_1]$, we obtain

$$\frac{x(h_r^j(t))}{x(h_r^{j-1}(t))} \geq R^{k+1-j}(\eta_r) \quad \text{for } t \in \left[h_r^{-1}\left(w_r^{-k}\left(w_n^{-1}(T_0)\right)\right), T_1\right].$$

Then

$$x\left(h_r^i(t)\right) \geq \left(\prod_{j=2}^{i} R^{k+1-j}(\eta_r)\right) x\left(h_r(t)\right), \quad i = 1, 2, \ldots.$$

Substituting into (14), we get

$$x(t) - x\left(h_r(t)\right) + x\left(h_r(t)\right) \sum_{i=1}^{k} \prod_{j=2}^{i} R^{k+1-j}(\eta_r) \int_{h_r(t)}^{t} \sum_{j=1}^{r} B_{j,r}^{i}\left(s, h_r(t)\right) ds \leq 0$$

for $t \in [h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0))), T_1]$, that is,

$$x(t) + \left(\sum_{i=1}^{k} \prod_{j=2}^{i} R^{k+1-j}(\eta_r) \int_{h_r(t)}^{t} \sum_{j=1}^{r} B_{j,r}^{i}\left(s, h_r(t)\right) ds - 1\right) x\left(h_r(t)\right) \leq 0$$

for $t \in [h_r^{-1}(w_r^{-k}(w_n^{-1}(T_0))), T_1]$. This contradiction completes the proof. □

Theorem 2.2 *Assume that* $k \in \mathbb{N}_0$. *If*

$$\prod_{i=1}^{n} \left(\prod_{j=1}^{n} \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} R^k(\eta_{j_1}) a_{j_1}(s_1)\, ds_1}\, ds\right)^{\frac{1}{n}} \geq \frac{1}{n^n} \quad \text{for } t \geq h_1^{-1}\left(w_n^{-1}(t_1)\right), \tag{15}$$

then Eq. (1) *oscillates and* $D_{t_1}(x) \leq \sup_{t \geq t_1}\{h_1^{-2}(w_n^{-(k+2)}(t)) - t\}$.

Proof Assume that $x(t)$ is a solution of Eq. (1) such that $x(t) > 0$ on $[T_0, T_1]$, $T_0 \geq t_1$, $T_0 > h_1^{-2}(w_n^{-(k+2)}(T_1))$. Integrating Eq. (1) from $h_i(t)$ to t, $i = 1, 2, \ldots, n$, we get

$$x(t) - x\left(h_i(t)\right) + \int_{h_i(t)}^{t} \sum_{j=1}^{n} a_j(s) x\left(g_j(s)\right) ds = 0 \quad \text{for } t \in \left[h_1^{-1}\left(w_n^{-1}(T_0)\right), T_1\right]. \tag{16}$$

It follows from Eq. (1) and $h_j(t) \geq g_j(s)$, $h_i(t) \leq s \leq t$, $j = 1, 2, \ldots, n$, that

$$x\left(g_j(s)\right) = x\left(h_j(t)\right) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j=1}^{n} a_j(s_1) \frac{x(g_j(s_1))}{x(s_1)}\, ds_1}.$$

Substituting into Eq. (16), we get

$$x(t) - x\left(h_i(t)\right) + \sum_{j=1}^{n} x\left(h_j(t)\right) \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) \frac{x(g_{j_1}(s_1))}{x(s_1)}\, ds_1}\, ds = 0$$
$$\text{for } t \in \left[h_1^{-1}\left(w_n^{-2}(T_0)\right), T_1\right]. \tag{17}$$

By using (5), we have

$$\frac{x(h_j(s_1))}{x(s_1)} \geq \frac{x(g_j(s_1))}{x(s_1)} \geq R^k(\eta_j), \quad g_j(s) \leq s_1 \leq h_j(t), h_i(t) \leq s \leq t, i, j = 1, 2, \dots, n,$$

for $t \in [h_j^{-1}(w_j^{-1}(h_j^{-1}(w_j^{-k}(w_n^{-1}(T_0))))), T_1] \subseteq [h_j^{-2}(w_j^{-(k+1)}(w_n^{-1}(T_0))), T_1]$. This together with (17) leads to

$$x(t) - x(h_i(t)) + \sum_{j=1}^{n} x(h_j(t)) \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) R^k(\eta_{j_1})\, ds_1}\, ds \leq 0$$
$$\text{for } t \in [h_1^{-2}(w_n^{-(k+2)}(T_0)), T_1].$$

That is,

$$x(h_i(t)) > \sum_{j=1}^{n} x(h_j(t)) \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) R^k(\eta_{j_1})\, ds_1}\, ds$$
$$\text{for } t \in [h_1^{-2}(w_n^{-(k+2)}(T_0)), T_1].$$

By using the arithmetic–geometric mean, we obtain

$$x(h_i(t)) > n \left(\prod_{j=1}^{n} x(h_j(t)) \right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) R^k(\eta_{j_1})\, ds_1}\, ds \right)^{\frac{1}{n}}$$

for $t \in [h_1^{-2}(w_n^{-(k+2)}(T_0)), T_1]$. Taking the product of both sides

$$\prod_{j=1}^{n} x(h_j(t)) > n^n \left(\prod_{j=1}^{n} x(h_j(t)) \right) \prod_{i=1}^{n} \left(\prod_{j=1}^{n} \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) R^k(\eta_{j_1})\, ds_1}\, ds \right)^{\frac{1}{n}}$$

for $t \in [h_1^{-2}(w_n^{-(k+2)}(T_0)), T_1]$. Therefore,

$$\prod_{i=1}^{n} \left(\prod_{j=1}^{n} \int_{h_i(t)}^{t} a_j(s) e^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{n} a_{j_1}(s_1) R^k(\eta_{j_1})\, ds_1}\, ds \right)^{\frac{1}{n}} < \frac{1}{n^n} \quad \text{for } t \in [h_1^{-2}(w_n^{-(k+2)}(T_0)), T_1],$$

which contradicts (15). The proof is complete. □

Remark 2.1

(i) It should be noted that $w_n^{-1}(t) - t < \infty$ when $\sup_{t \geq t_1}\{t - g_j(t)\} < \infty$ for $j = 1, 2, \dots, n$. Therefore, all upper bounds of the distance between zeros of all solutions of Eq. (1) obtained in this work are bounded. For example,

$$\begin{aligned} h_r^{-1}(w_r^{-k}(w_n^{-1}(t))) - t &\leq w_n^{-(k+2)}(t) - t \\ &= w_n^{-1}(w_n^{-(k+1)}) - w_n^{-(k+1)}(t) + w_n^{-(k+1)}(t) - \cdots + w_n^{-1}(t) - t \\ &< \infty. \end{aligned}$$

(ii) Since

$$R^k(d) \ge f_k(d), \quad k = 0, 1, \ldots,$$

for some values of d, where

$$\int_{h_n(t)}^{t} \sum_{j=1}^{n} a_j(s)\,ds \ge d \quad \text{for } t \ge h_n^{-1}(t_1),$$

and the sequence $\{R^k(d)\}_{k\ge 1}$ is defined by (4), and

$$f_0(d) = 1, \qquad f_1(d) = \frac{1}{1-d}, \qquad f_k(d) = \frac{f_{k-2}(d)}{f_{k-2}(d) + 1 - \mathrm{e}^{d f_{k-2}(d)}}, \quad k = 2, 3, \ldots.$$

Then, by using a similar argument as in the proof of Lemma 2.1, we can improve [11, Lemma 2.4] and consequently all results that use it, as [11, Theorem 2.23].

3 Numerical examples

This section is devoted to validating the main theoretical findings through several examples. We first begin with the following example:

Example 3.1 Consider the differential equation with multiple delays

$$x'(t) + a_1(t)x\bigl(g_1(t)\bigr) + a_2(t)x\bigl(g_2(t)\bigr) = 0, \quad t \ge 3, \tag{18}$$

where $a_1(t) = \mu$, $a_2(t) = \rho$, $\mu, \rho > 0$,

$$g_1(t) = \begin{cases} t-2 & \text{if } t \in [3i, 3i+1], \\ \frac{1}{4}(5t - 3i - 9) & \text{if } t \in [3i+1, 3i+2], \\ \frac{1}{4}(3t + 3i - 5) & \text{if } t \in [3i+2, 3i+3], \end{cases} \quad i \in \mathbb{N},$$

$g_2(t) = t - \frac{1}{4}$. Clearly,

$$t - 2 \le g_1(t) \le t - \frac{7}{4}.$$

Since $h_1(t) = g_1(t)$ and $h_2(t) = g_2(t)$, so $w_1(t) = g_1(t)$ and $w_2(t) = \min_{1\le j\le 2} g_j(t) = g_1(t)$. It follows that

$$\max_{1\le j\le 2} w_j^{-i}(t) = w_2^{-i}(t) \le t + 2i.$$

Let

$$I(t) = \prod_{i=1}^{2} \left(\prod_{j=1}^{2} \int_{h_i(t)}^{t} a_j(s) \mathrm{e}^{\int_{g_j(s)}^{h_j(t)} \sum_{j_1=1}^{2} R^k(\eta_{j_1}) a_{j_1}(s_1)\,ds_1} \,ds \right)^{\frac{1}{2}}.$$

Then

$$I(t)\geq\left(\mu\rho\int_{h_1(t)}^{t}\mathrm{e}^{(\mu+\rho)(h_1(t)-g_1(s))}\,ds\times\int_{h_1(t)}^{t}\mathrm{e}^{(\mu+\rho)(h_2(t)-g_2(s))}\,ds\right)^{\frac{1}{2}}$$
$$\times\left(\mu\rho\int_{h_2(t)}^{t}\mathrm{e}^{(\mu+\rho)(h_1(t)-g_1(s))}\,ds\times\int_{h_2(t)}^{t}\mathrm{e}^{(\mu+\rho)(h_2(t)-g_2(s))}\,ds\right)^{\frac{1}{2}}.$$

Therefore,

$$I(t)\geq\left(\mu\rho\int_{t-\frac{175}{100}}^{t}\mathrm{e}^{(\mu+\rho)(t-s-\frac{1}{4})}\,ds\times\int_{t-\frac{175}{100}}^{t}\mathrm{e}^{(\mu+\rho)(t-s)}\,ds\right)^{\frac{1}{2}}$$
$$\times\left(\mu\rho\int_{t-\frac{1}{4}}^{t}\mathrm{e}^{(\mu+\rho)(t-s-\frac{1}{4})}\,ds\times\int_{t-\frac{1}{4}}^{t}\mathrm{e}^{(\mu+\rho)(t-s)}\,ds\right)^{\frac{1}{2}}$$
$$=\frac{\mu\rho\mathrm{e}^{\frac{-1}{4}(\mu+\rho)}(\mathrm{e}^{\frac{7}{4}(\mu+\rho)}-1)(\mathrm{e}^{\frac{1}{4}(\mu+\rho)}-1)}{(\mu+\rho)^2}>\frac{1}{4}\quad\text{for }\mu\geq\frac{1}{2},\rho\geq\frac{56}{115}.$$

Consequently, Theorem 2.2 with $k=0$ implies that $D_3(x)\leq\sup_{t\geq3}\{w_2^{-4}(t)-t\}\leq 8$ for $\mu\geq\frac{1}{2}$, $\rho\geq\frac{56}{115}$.

Observe that none of the results in [11] apply to Eq. (18) when $0<\mu+\rho\leq\frac{4}{\mathrm{e}}$. The reason for this is that

$$\max_{1\leq j\leq2}g_j(t)=t-\frac{1}{4},$$

which leads to

$$\int_{\max_{1\leq j\leq2}g_j(t)}^{t}\bigl(a_1(s)+a_2(s)\bigr)\,ds\leq\frac{1}{4}(\mu+\rho)<\frac{1}{\mathrm{e}}\quad\text{for }\mu+\rho<\frac{4}{\mathrm{e}}.$$

Next, we move to the next example.

Example 3.2 Consider the differential equation

$$x'(t)+\frac{1}{2}x\left(t-\frac{11}{10}\right)+\frac{1}{2}x(t-1)+x(t-\epsilon)=0,\quad t\geq\frac{11}{10},\tag{19}$$

where $0<\epsilon<\frac{1}{2}$. This equation is of the form (1) with $a_1(t)=a_2(t)=\frac{1}{2}$, $a_3(t)=1$, $g_1(t)=t-\frac{11}{10}$, $g_2(t)=t-1$, and $g_3(t)=t-\epsilon$. Clearly,

$$h_2(t)=\max_{1\leq j\leq2}g_j(t)=t-1,\qquad w_2(t)=\min_{1\leq j\leq2}g_j(t)=t-\frac{11}{10},\qquad w_3(t)=\min_{1\leq j\leq3}g_j(t)=t-\frac{11}{10}$$

and

$$h_2^{-k}(t)=t+k,\qquad w_2^{-k}(t)=\max_{1\leq j\leq2}w_j^{-k}(t)=t+\frac{11}{10}k,\qquad w_3^{-k}(t)=\max_{1\leq j\leq3}w_j^{-k}(t)=t+\frac{11}{10}k.$$

Since

$$\sum_{i=1}^{1}\prod_{j=2}^{i}R^{k+1-j}(\eta_r)\int_{h_2(t)}^{t}\sum_{j=1}^{2}B_{j,2}^{i}\big(s,h_2(t)\big)\,ds=\int_{h_2(t)}^{t}\sum_{j=1}^{2}a_j(s)\,ds=1.$$

Then, according to Theorem 2.1 with $k=0$, Eq. (19) is oscillatory and $D_{\frac{11}{10}}(x)\leq\sup_{t\geq\frac{11}{10}}\{h_2^{-1}(w_2^{-1}(w_3^{-1}(t)))-t\}=\frac{16}{5}$.

Observe, however, that

$$\max_{1\leq j\leq 3}g_j(t)=t-\epsilon.$$

It is not difficult to show that all results of [11], [3, Theorem 3] and [3, Theorem 4] fail to apply to Eq. (19) for sufficiently small ϵ. Also, observe that

$$\int_{g_j(t)}^{t}a_j(s)\,ds<1\quad\text{for } j=1,2,3.$$

Therefore, [3, Theorem 2] cannot give an approximation to $D_{\frac{11}{10}}(x)$ for sufficiently small ϵ better than $\frac{16}{5}$.

4 Conclusion

In this paper, we studied the distribution of zeros of first-order delay differential equations. Also, we obtained upper bounds for the zeros of a first-order differential equation with several delays. Finally, some examples are demonstrated to prove the theoretical results.

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