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Self-generated vortex flows in a tokamak magnetic island with a background flow

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Abstract

We present a gyrokinetic theory of self-generated E × B vortex flows in a magnetic island in a collisionless tokamak plasma with a background vortex flow. We find that the long-term evolution of the self-generated vortex flows can be classified into two regimes by the background vortex flow potential Φ, with an asymptotic criterion given by $e\Phi_{\rm cr}/T = \epsilon w/r$, where T is temperature, ϵ is the inverse aspect ratio and r is the radial coordinate. We find that the background vortex flow above the criterion significantly weakens the toroidal precession-induced long-term damping and structure change of the self-generated vortex flows. That is, the finite background vortex flow is beneficial to maintain the self-generated vortex flows, favorable to an internal transport barrier formation. Our result indicates that the island boundary region is a prominent location for triggering the transition to an enhanced confinement state of the magnetic island.

Keywords: gyrokinetics, tokamak, magnetic island, vortex flow, turbulence, self-generation, transport barrier

1. Introduction

In magnetically confined fusion plasmas, it is widely accepted that the self-generation of the E × B shear flow (streaming on the magnetic surfaces) from the microturbulence [1–4] with equivalent instantaneous microturbulence reduction, is a trigger of the transition to an enhanced confinement regime accompanied by transport barrier formation [5–8]. Experiments [9–11] and reduced models [12, 13] of the H-mode transition [14] have shown that after the triggering by the turbulence-induced E × B zonal flow, the contribution from the profile-induced E × B shear flow (via radial force balance) continues to increase with the profile gradient as a result of the E × B shear suppression of turbulent transport [15, 16]. It finally replaces the role of the self-generated zonal flow, terminating the transport barrier formation. This competition between the macroscale E × B shear flow and the mesoscale zonal flow in turbulence reduction has been interpreted as a feature of the two predators-one prey system [17].

The mechanisms of the E × B flow shear suppression of turbulence have been applied to interpret non-trivial transport behaviors in the vicinity of a magnetic island observed in experiments [18–20], such as internal transport barrier formation [21, 22] and transition between the low and high-accessibility states [23]. Experimental measurements [24, 25] have found that often the E × B flow in a magnetic island circulates on the island contours (helical magnetic surfaces), and is therefore called the vortex flow. Meanwhile, nonlinear fluid and gyrokinetic simulations [26, 27–31] have shown that the vortex flow can be self-generated from microturbulence, just like the zonal flows. At this point, we would like to mention that so far, the interpretations suggested from the experimental studies have considered only the profile-induced E × B flow in the candidate mechanisms, while the simulation works have focused on the turbulence-induced E × B flow only. In this paper, we present a gyrokinetic theory showing

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the effect of the background vortex flow on the evolution of the self-generated vortex flow, which thereby reveals the relationship between the two E × B vortex flows.

Gyrokinetics has been extensively used to study self-generated flows in toroidal fusion devices, as a pioneering theory work by Rosenbluth and Hinton [32] using gyrokinetics has revealed that a proper treatment of the polarization density, which originates from the finite Larmor radius (FLR) effect [33], is essential to capture residual zonal flows after fast collisionless damping. Extending theories of the residual zonal flows in axisymmetric tokamaks [32–34] and 3D stellarators [35–37], we have developed gyrokinetic theories of self-generated E × B flows in magnetic configurations with broken symmetries [38, 39]. In a previous work [39] on a self-generated vortex flow in a tokamak magnetic island, we have shown that the toroidal precession of the vortex flow-carrying particles breaks the quasi-helical symmetry of the self-generated vortex flow, resulting in a long-term flow damping and significant deviation of the streamlines from the island contours (we call this 'finite surface deviation (FSD)') forming a zonal-vortex flow mixture. This toroidicity-induced vortex flow damping makes turbulence self-regulation via vortex flow self-generation hard. In the present work, we generalize the previous work by including a background E × B vortex flow to the system. As a result, we show that a large enough background E × B vortex flow above a criterion $e\Phi/T > \epsilon w/r$, where T is temperature, ϵ is the inverse aspect ratio and r is the radial position, could significantly reduce the toroidicity-induced FSD, long-term damping and structure change of the self-generated vortex flow. That is, a finite background E × B flow could lead to a favorable condition for the transition to an enhanced confinement state of a magnetic island triggered by the vortex flow self-generation.

This paper is organized as follows. In section 2, we present a gyrokinetic description of the self-generated vortex flow in a stationary tokamak magnetic island in the presence of a background vortex flow. In section 3, we explicitly calculate the residual vortex flow after fast collisionless damping in a short term. In section 4, we present general properties of the motion of a bounce/transit orbit center and obtain approximate explicit expressions of the orbit center motion by defining three asymptotic regimes. In section 5, we obtain the general solution of the long-term flow level and explicitly calculate its expression with weak and strong background vortex flows to show that a large background flow gives a positive synergism with the self-generated flow. In section 6, we discuss indications of our results and future works and close the paper with a summary.

2. Gyrokinetic description of self-generated vortex flow

We use the nonlinear gyrokinetic equations [40–42] for a precise description of the response of a collisionless magnetized fusion plasma to a source much slower than the gyrofrequency ($\partial_t \sim \omega \ll \Omega_c$). In our study, the total magnetic field is given by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{1}$$

where

$$\mathbf{B}_0 = I\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\psi_0, \tag{2}$$

is the tokamak magnetic field and $\mathbf{B}_1$ is a magnetic field perturbation associated with the magnetic island. For simplicity, we consider a low-β circular-concentric high-aspect-ratio tokamak plasma neglecting Shafranov shift, shaping effects, and parallel magnetic compression. Then, the equilibrium magnetic field strength becomes

$$B_0 = |\mathbf{B}_0| \simeq B_a(1 - \epsilon\cos\theta), \tag{3}$$

and the island-perturbed magnetic field is

$$\mathbf{B}_1 = \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\delta\psi, \tag{4}$$

where we consider a widely used simple model for the magnetic island perturbation [43],

$$\delta\psi = \tilde{\psi}\cos(ky). \tag{5}$$

Here, the amplitude of the perturbed poloidal magnetic flux $\tilde{\psi}$ is approximated to be a constant considering a slowly-time-varying magnetic island satisfying constant-ψ approximation [43]. In this study, we consider the magnetic island geometry in which we use the local Cartesian coordinates (x, y, z), where $x = r - r_s$ and $y = r_s(\theta - \zeta/q_s)$ denotes the radial distance from the mode rational surface and the distance in helical angle direction, respectively. Here, r_s and q_s are the radial position and safety factor at the mode rational surface, and θ and ζ are poloidal and toroidal angles, respectively. The third component $z = q_s R\theta$ represents the direction of the unperturbed magnetic field at the mode rational surface. Accordingly, the total magnetic field is decomposed as, after an expansion in $x/r_s \sim w/r \ll 1$ with respect to the mode rational surface,

$$\mathbf{B} \simeq \mathbf{B}_z + \mathbf{B}_x + \mathbf{B}_y, \tag{6}$$

where

$$\frac{B_x}{B_z} = \frac{w^2}{4L_s}k\sin(ky), \quad \frac{B_y}{B_z} = -\frac{x}{L_s}. \tag{7}$$

Here, B_z is the unperturbed magnetic field strength at the mode rational surface, w is the magnetic island half-width defined as $w^2 = 4L_s\tilde{\psi}/RB$, $L_s = q_s R/\hat{s}$, $\hat{s} = (r/q)\mathrm{d}q/\mathrm{d}r$ is the magnetic shear, and $k = m/r_s$ is the wavenumber of the island perturbation $\mathbf{B}_1$. Note that $B_z(z)$, $B_x \propto \tilde{\psi}$, and $B_y \propto \hat{s}x$ capture the toroidicity, the island perturbation, and the magnetic shear effects, respectively. Then, the expression of the normalized helical magnetic flux, an appropriate magnetic surface label for the magnetic island geometry [44], becomes

$$X = \frac{2x^2}{w^2} - \cos(ky). \tag{8}$$

In this study, we consider a two-species collisionless tokamak plasma consisting of bulk ions with charge number $Z_i = 1$

(H, D, or T) and electrons. For every species, the gyrokinetic Vlasov equation for the vortex perturbation is, with the use of an eikonal representation and assuming a Maxwellian unperturbed particle distribution function F_0 [39],

$$\begin{aligned}\partial_t g + v_\parallel \mathbf{b}\cdot\nabla g + \mathbf{u}_{\mathrm{E0}}\cdot\nabla g + \mathbf{v}_{\mathrm{d}}\cdot\nabla g + i\omega_{\mathrm{D}} g \\ = \frac{e}{T}F_0 J_0(k_\perp\rho)\partial_t\phi + \frac{1}{B}\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''}[\mathbf{b}\cdot(\mathbf{k}'\times\mathbf{k}'')] \\ \times J_0(k'_\perp\rho)\phi_{\mathbf{k}'} g_{\mathbf{k}''},\end{aligned} \tag{9}$$

where g is the envelope of the gyroangle-independent part of the non-adiabatic response

$$\delta g = g(\mathbf{R},\mu,\mathcal{E},t)e^{iS(\mathbf{R})}. \tag{10}$$

Here, $\mathbf{R}=\mathbf{x}-\boldsymbol{\rho}$ is the gyrocenter position, $\mathbf{x}$ is the particle position, $\boldsymbol{\rho}$ is the gyroradius vector, $\mu = Mv_\perp^2/2B$ is the magnetic moment, $v_\perp$ is the perpendicular particle velocity, M is the particle mass, and $B=|\mathbf{B}|$ is the total magnetic field strength. δg appears as a result of a decomposition of the perturbed particle distribution function $\delta f(\mathbf{x},\mathbf{v},t)$ into the adiabatic and the non-adiabatic parts [40, 42],

$$\delta f = -\frac{e\delta\phi}{T}F_0 + \delta g e^{-\mathbf{k}\cdot\boldsymbol{\rho}}. \tag{11}$$

Here, F_0 is the unperturbed distribution function and $\mathbf{k}=\nabla S$ is the wavevector. Note that in our case of the vortex perturbation, we have $S(\mathbf{R})=S(X)$, where X is the normalized helical magnetic flux of which expression is presented in equation (8). Similarly, for the potential perturbation, we have considered

$$\delta\phi = \phi(\mathbf{R},t)e^{iS(X)}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{12}$$

so that

$$\langle\delta\phi\rangle = J_0(k_\perp\rho)\phi(\mathbf{R},t)e^{iS(X)} \tag{13}$$

results in the first term in the right-hand side (RHS) of equation (9) with a Bessel function $J_0(k_\perp\rho)$ which captures the FLR effect. Note that in equations (10) and (13), the eikonal factor $S(X)$ captures characteristics of the vortex perturbation, and slower variation of the envelopes g and ϕ, initially set to be uniform in space, becomes important in the long term evolution. In equation (9), the parallel velocity is expressed as $v_\parallel = \sigma\sqrt{2(\mathcal{E}-e\Phi(X)-\mu B)/M}$, where $\sigma=\pm1$ denotes the direction of the parallel streaming, $\mathcal{E}$ is the particle energy. $\mathbf{b}=\mathbf{B}/B$ is the unit vector parallel to the total magnetic field, $\mathbf{u}_{\mathrm{E0}}=\mathbf{b}\times\nabla\Phi(X)/B$ is the background E × B flow velocity (note that $\mathbf{k}\cdot\mathbf{u}_{\mathrm{E0}}=0$), $\mathbf{v}_d$ is the magnetic drift velocity and $\omega_{\mathrm{D}}=\mathbf{k}\cdot\mathbf{v}_d$ is the magnetic drift frequency. The last term in the RHS of equation (9) is the E × B nonlinearity [40] which is the nonlinear generation source of the vortex flow from the microturbulence [32, 35, 36]. Here, the subscripts $\mathbf{k}'$ and $\mathbf{k}''$ in $\phi_{\mathbf{k}'}$ and $g_{\mathbf{k}''}$ clarify that they are envelopes of turbulent perturbations with wavevectors $\mathbf{k}'$ and $\mathbf{k}''$, respectively. Hereafter, for the wavevector of the vortex perturbation, we use a notation $\mathbf{k}_X=\nabla S(X)$ instead of a general notation $\mathbf{k}$ to emphasize that we consider the vortex perturbation and not be confused with $k=m/r_s$ for the island perturbation.

The other part of the self-consistent gyrokinetic system is the gyrokinetic Poisson equation [42], which is equivalent to the quasi-neutrality condition $n_i=n_e$. Using equation (11), its perturbed part can be written as

$$-n_0\frac{e\delta\phi}{T_i} + \int \mathrm{d}^3v J_0\delta g_i = n_0\frac{e\delta\phi}{T_e} + \int \mathrm{d}^3v\delta g_e, \tag{14}$$

where we have neglected the electron FLR effect. Here, $n_0 = n_{i0}=n_{e0}$ is the equilibrium density. In the later part of this paper, we further neglect the electron finite-orbit-width (FOW) effect assuming $\Delta_{be}<\rho_i$, where $\Delta_b\sim q\rho/\sqrt{\epsilon}$ is the banana orbit width.

Following previous works [38, 39], we would like to obtain a bounce/transit averaged kinetic equation to study the long-term evolution of the vortex perturbation after fast collisionless damping of geodesic acoustic mode (GAM) oscillation by transit resonance [45]. In the magnetic island geometry, the particle streaming along the magnetic field line is mostly in the z-direction, and the projected streaming motions in the x- and y-directions are much slower. Based on this clear scale disparity, we define the bounce/transit average as [46, 47]

$$\overline{A} \equiv \oint_{\mathrm{b/t}}\frac{\mathrm{d}z}{v_\parallel}A \Big/ \oint_{\mathrm{b/t}}\frac{\mathrm{d}z}{v_\parallel}, \tag{15}$$

where b/t represents the lowest order bounce/transit trajectory by parallel streaming. In addition, the magnetic drift gives finite width of the bounce/transit orbit, and this effect is captured in the magnetic drift frequency as [37]

$$\omega_{\mathrm{D}} = v_\parallel\partial_z Q + \overline{\omega}_{\mathrm{D}}. \tag{16}$$

Here, $Q\sim k_X\Delta_b$ in the first term of the RHS represents the FOW effect and the second term represents the secular magnetic drift of the orbit center. Substituting equation (16) into equation (9), we obtain the lowest-order equation

$$v_\parallel\partial_z g + i\,(v_\parallel\partial_z Q)g = 0, \tag{17}$$

which yields $g = h\exp(-iQ)$ where $\partial_z h = 0$, i.e. $h=h(x,y)$. Here, we have used an ordering $\partial_t\sim\omega\ll v_\parallel\partial_z\sim v_\parallel/qR$ as we are interested in phenomena much slower than the bounce/transit motion. Multiplying the next order part of equation (9) by $\exp(iQ)$ and taking the bounce/transit average, we obtain the bounce/transit-kinetic equation as follows.

$$\begin{aligned}\partial_t h + \overline{e^{iQ}v_\parallel\mathbf{b}_\perp\cdot\nabla(he^{-iQ})} + \overline{e^{iQ}\mathbf{u}_{\mathrm{E0}}\cdot\nabla(he^{-iQ})} \\ + \overline{e^{iQ}\mathbf{v}_{\mathrm{d}}\cdot\nabla(he^{-iQ})} + i\overline{\omega}_{\mathrm{D}}h = \frac{e}{T}F_0\overline{e^{iQ}J_0\partial_t\phi} + \overline{e^{iQ}NF_0},\end{aligned} \tag{18}$$

where $\mathbf{b}_\perp = (\mathbf{B}_x+\mathbf{B}_y)/B$ and NF_0 denotes the E × B nonlinearity [32, 35, 36]. In the long-wavelength limit, we simplify the left-hand-side (LHS) of equation (18) so that

$$\partial_t h + \left(\overline{v}_\parallel\mathbf{b}_\perp + \overline{\mathbf{u}}_{\mathrm{E0}} + \overline{\mathbf{v}}_{\mathrm{d}}\right)\cdot\nabla h + i\overline{\omega}_{\mathrm{D}}h = \frac{e}{T}F_0\overline{e^{iQ}J_0\partial_t\phi} + \overline{e^{iQ}NF_0}. \tag{19}$$

Here, note that we keep using the general expression of the RHS of equation (18) for a proper description of the polarization shielding.

3. Residual vortex flow

In the *short term* slower than ion bounce motion but faster than the secular drift motions, that is, $\omega_b^{-1} < t < \overline{\omega}_{\rm D}^{-1}$ where $\omega_{\rm b/t}$ is the bounce/transit frequency, equation (19) is approximated to

$$\partial_t h = \frac{e}{T} F_0 \overline{e^{iQ} J_0 \partial_t \phi} + \overline{e^{iQ} N} F_0, \tag{20}$$

neglecting spatial inhomogeneity of the envelope h guaranteed from the initial condition of the vortex flow perturbation. Note that equation (20) is the same with equation (6) of Rosenbluth and Hinton [32]. The solution of equation (20) is

$$h = \frac{e}{T} F_0 \overline{e^{iQ} J_0 \phi} + \overline{e^{iQ}} P F_0, \tag{21}$$

where $P = \int {\rm d}t N$. Substituting equation (21) into the quasi-neutrality condition, equation (14), we obtain

$$\frac{e}{T_{\rm i}} \phi \left\langle 1 - \int {\rm d}^3 v \frac{F_{0{\rm i}}}{n_0} J_0 e^{-iQ} \overline{e^{iQ} J_0} \right\rangle = \langle s \rangle , \tag{22}$$

where $\langle \cdots \rangle$ is the flux surface average [32, 39]. The source term s is given by

$$s = \int {\rm d}^3 v \frac{F_{0{\rm i}}}{n_0} J_0 e^{-iQ} \overline{e^{iQ} J_0 P_{\rm i}} - \int {\rm d}^3 v \frac{F_{0{\rm e}}}{n_0} \overline{P}_{\rm e}, \tag{23}$$

where $P_{\rm i,e} = \int {\rm d}t N_{\rm i,e}$ represents the amount of the vortex flow production from ions and electrons. We consider an initial kick as a source following previous works [32, 36, 39],

$$\begin{aligned} N F_0 = f_{\rm g}(t=0)\delta(t) &= -\langle n_{\rm pol}(t=0)\rangle \frac{F_0}{n_0}\delta(t) \\ &\simeq \frac{e}{T} F_0 \left\langle k_X^2 \rho_T^2 \right\rangle \phi(t=0)\delta(t), \end{aligned} \tag{24}$$

where $f_{\rm g}$ is the envelope of the perturbed gyrocenter distribution function $\delta f_{\rm g}$ and $n_{\rm pol}$ is that of the polarization density $\delta n_{\rm pol}$. Here, $n_{\rm pol}(0)$ and $\phi(0)$ denote the polarization density and the potential at the initial time $t = 0$, respectively. Substituting equation (24) into equation (22), we obtain

$$\frac{\phi}{\phi(t=0)} = \frac{\chi_{\rm cl}}{\chi_{\rm cl} + \chi_{\rm nc}}, \tag{25}$$

where

$$\chi_{\rm cl} = \left\langle k_X^2 \rho_{\rm Ti}^2 \right\rangle , \quad \chi_{\rm nc} = \left\langle \int {\rm d}^3 v \frac{F_{0{\rm i}}}{n_0} \left(\overline{Q_{\rm i}^2} - \overline{Q}_{\rm i}^2 \right) \right\rangle , \tag{26}$$

are long-wavelength expressions of the classical and neoclassical susceptibilities relevant to the polarization shielding [33, 34], which originate from the FLR and FOW effects, respectively. In equation (26), $\rho_{\rm Ti}$ is the ion thermal Larmor radius, $F_{0{\rm i}}$ is the unperturbed ion distribution function, and $Q_{\rm i}$ is the ion FOW factor formally defined in equation (16).

We then perform explicit calculations of the susceptibilities for the case of the circular concentric tokamak. Taking the bounce/transit average to the magnetic drift frequency $\omega_{\rm D}$, we obtain the explicit expression of the secular magnetic drift frequency

$$\overline{\omega}_{\rm D} = -\frac{S'}{R_a \Omega_{ca}} \frac{\mu B_a}{M} k \sin(ky) \left[2 \frac{E(\kappa)}{K(\kappa)} - 1 \right] \quad \text{(trapped)}, \tag{27}$$

$$= -\frac{S'}{R_a \Omega_{ca}} \frac{\mu B_a}{M} k \sin(ky) \left[2\kappa^2 \frac{E(\kappa^{-1})}{K(\kappa^{-1})} + 1 - 2\kappa^2 \right] \quad \text{(passing)}, \tag{28}$$

where $S' = {\rm d}S/{\rm d}X$, and R_a and Ω_{ca} are the major radius and the gyrofrequency at the magnetic axis. κ is the pitch angle parameter defined as

$$\kappa^2 \equiv \frac{\mathcal{E} - e\Phi - \mu B_a (1-\epsilon)}{2\epsilon \mu B_a}, \tag{29}$$

and $K(k)$ and $E(k)$ are the complete elliptic integral of the first and the second kinds, respectively. Substituting equations (27) and (28) into equation (16), we obtain

$$Q = Q_0 + Q_1, \tag{30}$$

where

$$Q_0 = \frac{B_a S'}{\Omega_{ca}} \frac{4x}{w^2} \frac{q R_a}{r} \frac{v_\parallel}{B}, \tag{31}$$

$$\begin{aligned} Q_1 = &-\sigma \frac{S'}{\Omega_{ca}} \frac{q}{\sqrt{\epsilon}} \sqrt{\frac{\mu B_a}{M}} 2k \sin(ky) \\ &\times \left[E(\vartheta, \kappa) - \frac{E(\kappa)}{K(\kappa)} F(\vartheta, \kappa) \right] \quad \text{(trapped)}, \end{aligned} \tag{32}$$

$$\times \kappa \left[E(\vartheta, \kappa^{-1}) - \frac{E(\kappa^{-1})}{K(\kappa^{-1})} F(\vartheta, \kappa^{-1}) \right] \quad \text{(passing)}. \tag{33}$$

Here, $\sigma = {\rm sgn}(v_\parallel)$, and ϑ is the bounce/transit angle given as [33]

$$\vartheta = \arcsin\left(\kappa^{-1} \sin(\theta/2) \right) \quad \text{(trapped)}, \tag{34}$$

$$= \theta/2 \quad \text{(passing)}, \tag{35}$$

and $F(x,k)$ and $E(x,k)$ are the incomplete elliptic integral of the first and the second kinds, respectively. Note that Q_0 originates from the unperturbed tokamak magnetic field $\mathbf{B}_0$, and Q_1 from the magnetic island perturbation $\mathbf{B}_1$.

Now, we explicitly calculate the classical and neoclassical susceptibilities $\chi_{\rm cl}$ and $\chi_{\rm nc}$. Note that the flux surface average in the magnetic island geometry can be written as

$$\langle A \rangle = \left\langle \langle A \rangle_z \right\rangle_y , \tag{36}$$

where

$$\langle A\rangle_z = \frac{1}{2\pi}\oint \mathrm{d}\theta A, \tag{37}$$

$$\langle A\rangle_y = \oint_X \frac{\mathrm{d}yA}{\pm\sqrt{X+\cos(ky)}} \Big/ \oint_X \frac{\mathrm{d}y}{\pm\sqrt{X+\cos(ky)}}. \tag{38}$$

In equation (38), the integration range is limited to $[-y_t, y_t]$, where $X+\cos(ky_t)=0$ at the turning points y_t and $-y_t$. The $+(-)$ sign in the integrand denotes the outer(inner) half of the integration curve along the island contour ($X=$ const). Using

$$\int \mathrm{d}^3v = \int_{\sigma>0} \frac{2\pi B\mathrm{d}\mu\mathrm{d}\mathcal{E}}{M|v_\parallel|} + \int_{\sigma<0} \frac{2\pi B\mathrm{d}\mu\mathrm{d}\mathcal{E}}{M|v_\parallel|}, \tag{39}$$

$\mathrm{d}\mu\mathrm{d}\mathcal{E} = 2\epsilon\mu B_a \mathrm{d}\mu\mathrm{d}(\kappa^2)$, and

$$\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{\kappa_0}^{1}\mathrm{d}(\kappa^2) = \int_0^1 \mathrm{d}(\kappa^2)\int_{-\theta_t}^{\theta_t}\mathrm{d}\theta, \tag{40}$$

where $\kappa_0=\sin(\theta/2)$, one can obtain from equation (26) the trapped and passing particle contributions to the neoclassical susceptibility

$$\chi_{\mathrm{nc}}^t = \left\langle 1.20\frac{x^2}{w^2} + 0.15\frac{1}{4}\left(\frac{kw}{2}\right)^2 \sin^2(ky)\right\rangle_y \frac{q^2}{\sqrt{\epsilon}}k_w^2\rho_{\mathrm{Ti}}^2, \tag{41}$$

$$\chi_{\mathrm{nc}}^p = \left\langle 0.43\frac{x^2}{w^2} + 0.09\frac{1}{4}\left(\frac{kw}{2}\right)^2 \sin^2(ky)\right\rangle_y \frac{q^2}{\sqrt{\epsilon}}k_w^2\rho_{\mathrm{Ti}}^2, \tag{42}$$

where $k_w \equiv 4S'/w$ is a characteristic vortex flow wavenumber. Note that from equation (26) the classical susceptibility is

$$\chi_{\mathrm{cl}} = \left\langle \frac{x^2}{w^2} + \frac{1}{4}\left(\frac{kw}{2}\right)^2 \sin^2(ky)\right\rangle_y k_w^2\rho_{\mathrm{Ti}}^2. \tag{43}$$

Using polar coordinates for the magnetic island geometry [20, 39]

$$\rho\cos\chi = \frac{x}{w}, \quad \rho\sin\chi = \sin\left(\frac{ky}{2}\right), \tag{44}$$

where $\rho^2=(X+1)/2$, the weighted y-average becomes

$$\langle A\rangle_y = \oint_\rho \frac{\mathrm{d}\chi\, A}{\sqrt{1-\rho^2\sin^2\chi}} \Big/ \oint_\rho \frac{\mathrm{d}\chi}{\sqrt{1-\rho^2\sin^2\chi}}. \tag{45}$$

Consequently, equations (41)–(43) yield [39]

$$\chi_{\mathrm{cl}} = [G_0(\rho)+G_1(\rho)]k_w^2\rho_{\mathrm{Ti}}^2, \tag{46}$$

$$\chi_{\mathrm{nc}} = [1.63G_0(\rho)+0.24G_1(\rho)]\frac{q^2}{\sqrt{\epsilon}}k_w^2\rho_{\mathrm{Ti}}^2, \tag{47}$$

where the geometrical factors G_0 and G_1 are given by

$$G_0(\rho) = \left\langle \frac{x^2}{w^2}\right\rangle_y = \frac{E(\rho)}{K(\rho)} - 1 + \rho^2, \tag{48}$$

$$G_1(\rho) = \left\langle \frac{1}{4}\left(\frac{kw}{2}\right)^2 \sin^2(ky)\right\rangle_y = \left(\frac{kw}{2}\right)^2 \frac{1}{3}\left[(2\rho^2-1)\frac{E(\rho)}{K(\rho)} + 1 - \rho^2\right]. \tag{49}$$

In equation (47), small coefficient 0.24 (which corrects 0.12 in Choi and Hahm [39]) for G_1 is because it originates from the helical angle y component of the magnetic drift $v_{dy}\propto\partial_r B\propto\cos\theta$ having even parity in θ. Consequently, the helical angle component mainly gives mean precession and has a minor contribution to the orbit width.

Substituting equations (46) and (47) into equation (25), we obtain an explicit expression of the residual vortex flow level in the long-wavelength limit [39],

$$\frac{\phi_{\mathrm{R}}(X)}{\phi(t=0)} = 1\Big/\left(1+\frac{1.63G_0(\rho)+0.24G_1(\rho)}{G_0(\rho)+G_1(\rho)}\frac{q^2}{\sqrt{\epsilon}}\right). \tag{50}$$

In the no-island limit $\tilde{\psi}\to 0$, we recover the famous Rosenbluth–Hinton residual level [32] as a result of vanishing G_1. (Recall that $G_1\sim w^2\sim\tilde{\psi}$ in equation (49).) In the presence of a magnetic island, due to the small coefficient for G_1 in χ_{nc}, the neoclassical enhancement of polarization shielding compared to the original classical one is weakened. Therefore, the residual vortex flow level presented in equation (50) is higher than Rosenbluth and Hinton. Note that $G_1\propto w^2$, so it could be interpreted as a finite island width effect which makes the higher residual level.

We would like to mention that in our study, we do not consider the contribution of the magnetic island perturbation $\mathbf{B}_1$ to the magnetic drift velocity $\mathbf{v}_{\mathrm{d}}$ through ∇B because of its negligible magnitude $\sim(w/r)^4\epsilon/q \ll 1$ compared to ∇B_0. That is, the magnetic island effect in equations (47) and (50) comes from $\mathbf{k}_X = S'\boldsymbol{\nabla}X$ in $\omega_{\mathrm{D}} = \mathbf{k}_X\cdot\mathbf{v}_{\mathrm{d}}$. It is a geometrical effect.

4. Bounce/transit center motion and drift surface

In this section, we analyze the bounce/transit center orbit to find the drift surface and the orbit frequency. We consider a closed bounce/transit center trajectory and relevant drift action-angle pair (Ω,φ) [48]. That is, from equation (19), we have

$$\left(\bar{v}_\parallel\mathbf{b}_\perp + \bar{\mathbf{u}}_{\mathrm{E}0} + \bar{\mathbf{v}}_{\mathrm{d}}\right)\cdot\boldsymbol{\nabla} = \omega_\varphi \left.\frac{\partial}{\partial\varphi}\right|_{\Omega,\mathcal{E},\mu}, \tag{51}$$

where ω_φ is the drift orbit frequency. Then,

$$0 = \left(\bar{v}_\parallel\mathbf{b}_\perp + \bar{\mathbf{u}}_{\mathrm{E}0} + \bar{\mathbf{v}}_{\mathrm{d}}\right)\cdot\boldsymbol{\nabla}\Omega = \left(\bar{v}_\parallel - \frac{R_a\Phi}{\tilde{\psi}}\right)\frac{B_x}{B_z}\frac{\partial\Omega}{\partial x} + \left\{\left(\bar{v}_\parallel - \frac{R_a\Phi}{\tilde{\psi}}\right)\frac{B_y}{B_z} + \bar{v}_{\mathrm{d}}\right\}\frac{\partial\Omega}{\partial y}. \tag{52}$$

On a drift surface $\Omega = \text{const}$, we have

$$0 = \mathrm{d}\Omega = \frac{\partial \Omega}{\partial x}\mathrm{d}x + \frac{\partial \Omega}{\partial y}\mathrm{d}y. \tag{53}$$

Recall that $\mathcal{E} = \text{const}$ and $\mu = \text{const}$ for a single particle. Substituting equation (52) into equation (53), we obtain

$$-\left(\bar{v}_{\parallel}\frac{B_y}{B_z} + \bar{v}_{\mathrm{d}}\right)\mathrm{d}x + \frac{B_x}{B_z}\mathrm{d}y = 0. \tag{54}$$

Here, the explicit expression of the precession velocity $\bar{v}_{\mathrm{d}}$ is

$$\bar{v}_{\mathrm{d}} = -\frac{1}{R\Omega_c}\frac{\mu B_a}{M}\left\{2\frac{E(\kappa)}{K(\kappa)} - 1\right\} \quad \text{(trapped)}, \tag{55}$$

$$= -\frac{1}{R\Omega_c}\frac{\mu B_a}{M}\left\{2\kappa^2\frac{E(\kappa^{-1})}{K(\kappa^{-1})} + 1 - 2\kappa^2\right\} \quad \text{(passing)}, \tag{56}$$

and the bounce/transit-averaged parallel velocity is

$$\bar{v}_{\parallel} = 0 \quad \text{(trapped)}, \tag{57}$$

$$= \mathrm{sgn}(v_{\parallel})\sqrt{\epsilon}\sqrt{\frac{\mu B_a}{M}}\pi\frac{\kappa}{K(\kappa^{-1})} \quad \text{(passing)}. \tag{58}$$

To the lowest order in w/r, one may neglect spatial variations of $\bar{v}_{\mathrm{d}}$ and $\bar{v}_{\parallel}$ in equation (54) and obtain

$$\left(\bar{v}_{\parallel} - \frac{R_a}{\tilde{\psi}}\Phi'\right)\mathrm{d}X - \bar{v}_{\mathrm{d}}\frac{4L_s}{w^2}\mathrm{d}x = 0. \tag{59}$$

Note that equation (59) yields a general drift surface label

$$\Omega = X - \frac{R_a}{\tilde{\psi}}\frac{\Phi}{\bar{v}_{\parallel}} - \frac{4L_s}{w^2}\frac{\bar{v}_{\mathrm{d}}}{\bar{v}_{\parallel}}x, \tag{60}$$

consistent with previous drift-kinetic calculations [49, 50]. Note that the ratio of the two terms in equation (59) are, from equations (56) and (58),

$$\frac{(\text{first term})}{(\text{third term})} \sim \frac{w}{L_s}\frac{\sqrt{\epsilon}v_{\perp}}{v_{\perp}^2/R\Omega_c} \sim \frac{\sqrt{\epsilon}\hat{s}w}{q\rho_T} = \frac{w}{w_{\mathrm{cr}}}, \tag{61}$$

where $v_{\perp}$ is the perpendicular particle velocity, and the critical magnetic island half-width is

$$w_{\mathrm{cr}} = \frac{q\rho_T}{\sqrt{\epsilon}\hat{s}} \sim \frac{\Delta_b}{\hat{s}}. \tag{62}$$

Next, we estimate ordering of the second term in equation (59) compared to the third,

$$\frac{(\text{second term})}{(\text{third term})} \sim \frac{R_a\Phi/\tilde{\psi}}{4L_s\bar{v}_d/w} \sim \frac{e\Phi/T}{\epsilon w/r} = \frac{\Phi}{\Phi_{\mathrm{cr}}}, \tag{63}$$

where the critical amplitude of the background vortex flow is

$$\frac{e\Phi_{\mathrm{cr}}}{T} = \epsilon\frac{w}{r}. \tag{64}$$

Note that the critical background flow level is proportional to the inverse aspect ratio ϵ and the relative island width w/r. In this study, we define three asymptotic regimes depending on the magnetic island half-width w and the background vortex potential Φ as follows.

1. *Toroidal regime* : $w < w_{\mathrm{cr}}$ and $\Phi < \Phi_{\mathrm{cr}}$,
2. *Shifted vortex regime* : $\Phi > \max\{\Phi_{\mathrm{cr}}, \Phi_{\mathrm{cr}}w/w_{\mathrm{cr}}\}$,
3. *Shifted island regime* : otherwise.

Note that while the relative importance of the terms in equation (59) depends on the velocity pitch κ, we simplify our arguments by considering representative values only.

4.1. Toroidal regime

In the toroidal regime, the toroidal precession, the third term of the LHS of equation (59) is dominant, and we can simply approximate $\Omega \approx \psi$ ($\leftrightarrow x$) and $\varphi \approx \zeta$ as in our previous study [39] by keeping only the toroidal precession, the last term in the LHS of equation (59). Here, we have

$$\omega_{\varphi} = n\omega_{\mathrm{pr}} \simeq k\bar{v}_{\mathrm{d}}. \tag{65}$$

4.2. Shifted vortex regime

Meanwhile, in the shifted vortex regime, large background vortex flow dominates over the averaged streaming and the toroidal precession, that is, the second term in equation (59) is much larger than the first and the third terms. As a result, we have

$$\Omega = \frac{2(x-d)^2}{w^2} - \cos(ky), \tag{66}$$

where the radial shift

$$d = -\frac{L_s\bar{v}_d}{R_a\Phi'/\tilde{\psi}} = -\frac{\bar{v}_{\mathrm{d}}}{4\Phi'/Bw}w \sim \frac{\bar{v}_{\mathrm{d}}}{u_{\mathrm{E}}}w, \tag{67}$$

which satisfies $|d| \ll w$, characterizes the relative importance of the magnetic precession along the tokamak magnetic surface compared to the background vortex flow along the island magnetic surface. It is obvious that equation (66) is described by the polar coordinates

$$\eta\cos\varphi = \frac{x-d}{w}, \quad \eta\sin\varphi = \sin\left(\frac{ky}{2}\right), \quad \eta^2 = \frac{\Omega+1}{2}, \tag{68}$$

where φ is the drift angle. Note that the general expression of the drift orbit frequency ω_{φ} can be obtained from equation (51) as follows.

$$\omega_{\varphi} = \left(\bar{v}_{\parallel}\mathbf{b}_{\perp} + \bar{\mathbf{u}}_{\mathrm{E0}} + \bar{\mathbf{v}}_{\mathrm{d}}\right)\cdot\boldsymbol{\nabla}\varphi. \tag{69}$$

Then, the drift orbit frequency can be obtained from equations (7), (68) and (69) as follows.

$$\omega_{\varphi} \simeq \left(\bar{\mathbf{u}}_{\mathrm{E0}} + \bar{\mathbf{v}}_{\mathrm{d}}\right)\cdot\boldsymbol{\nabla}\varphi = \frac{2k\Phi'}{Bw}\cos\left(\frac{ky}{2}\right)\left(1 - (\partial_x d)\sin^2\varphi\right). \tag{70}$$

4.3. Shifted island regime

In the shifted island regime, the averaged streaming is dominant. As we are interested in the deviation of the drift surface from the magnetic surface by toroidal precession [39], we neglect the background vortex flow while keeping the toroidal precession. Then, the approximate drift surface label becomes

$$\Omega = \frac{2(x-d)^2}{w^2} - \cos(ky), \tag{71}$$

where

$$d = L_s \bar{v}_\mathrm{d}/\bar{v}_\parallel \tag{72}$$

is the radial shift of the drift surface from the magnetic surface X due to toroidal precession. Again, $|d| \ll w$. The drift orbit frequency is approximated to, using equations (7), (68) and (69),

$$\begin{aligned}\omega_\varphi &\simeq \left(\bar{v}_\parallel \mathbf{b}_\perp + \bar{\mathbf{v}}_\mathrm{d}\right)\cdot\boldsymbol{\nabla}\varphi \\ &= -\frac{kw}{2L_s}\bar{v}_\parallel \cos\left(\frac{ky}{2}\right)\left(1-(\partial_x d)\sin^2\varphi\right).\end{aligned} \tag{73}$$

5. Long-term evolution of residual vortex flow

In the *long term* $\overline{\omega}_\mathrm{D}^{-1} < t$, the secular drift $\overline{\omega}_\mathrm{D}$ participates in the vortex flow evolution. Extending our previous work [39], we quantify the effect of deviation of the drift surface from the magnetic surface by defining the general FSD factor

$$\Lambda \equiv \int_\Omega \mathrm{d}\varphi \frac{\overline{\omega}_\mathrm{D}}{\omega_\varphi}. \tag{74}$$

Note that the secular drift frequency is

$$\overline{\omega}_\mathrm{D} = S'\bar{v}_d k \sin(ky). \tag{75}$$

Therefore, in the toroidal regime, we have [39]

$$\Lambda \simeq -S'\cos(ky), \tag{76}$$

since $\omega_\varphi \simeq k\bar{v}_\mathrm{d}$. In the shifted vortex regime and the shifted island regime, using equations (68), (70) and (73) and the ordering $|\partial_x d| \ll 1$, we obtain

$$\Lambda \simeq 4S'\frac{d}{w}\eta\cos\varphi = 4S'\frac{d(x-d)}{w^2}. \tag{77}$$

From equations (76) and (77), we realize that $\Lambda \sim \mathcal{O}(1)$ in the toroidal regime with $S' \sim \mathcal{O}(1)$, while $\Lambda \sim \mathcal{O}(d/w) \ll 1$ in the shifted vortex and the shifted island regimes. That is, the background vortex flow or the parallel streaming could significantly reduce the deviation of the drift surface from the magnetic surface.

Using the drift action-angle pair and the FSD factor, equations (51) and (74), we rewrite equation (19), the bounce/transit-averaged kinetic equation, as follows,

$$\partial_t h + \omega_\varphi \left.\frac{\partial h}{\partial\varphi}\right|_\Omega + i\omega_\varphi \left.\frac{\partial\Lambda}{\partial\varphi}\right|_\Omega h = \frac{e}{T}F_0\overline{e^{iQ}J_0\partial_t\phi} + \overline{e^{iQ}NF_0}. \tag{78}$$

Multiplying equation (78) by $\exp(i\Lambda)$ and taking the drift average [48]

$$[\cdots] \equiv \oint_\Omega \frac{\mathrm{d}\varphi}{\omega_\varphi}(\cdots)\Big/\oint_\Omega \frac{\mathrm{d}\varphi}{\omega_\varphi}, \tag{79}$$

we obtain the drift-averaged kinetic equation

$$\partial_t H = \partial_t \frac{e}{T}\left[e^{i\Lambda}\overline{F_0 e^{iQ}J_0\phi}\right] + \left[e^{i\Lambda}\overline{e^{iQ}NF_0}\right], \tag{80}$$

where $h = He^{-i\Lambda}$, which yields a solution

$$g = e^{-iQ}e^{-i\Lambda}\left[e^{i\Lambda}\overline{F_0 e^{iQ}J_0\frac{e}{T}\phi}\right] + e^{-iQ}e^{-i\Lambda}\left[e^{i\Lambda}\overline{e^{iQ}f_\mathrm{g}(0)}\right]. \tag{81}$$

Substituting equation (81) into the quasi-neutrality condition, we obtain the general solution for the long-term potential, which can be written in the following compact form [36].

$$\mathcal{D}\delta\phi = \mathcal{I}, \tag{82}$$

where the dielectricity is

$$\begin{aligned}\mathcal{D}\delta\phi &= \sum_{j=e,i}\mathcal{D}_j\delta\phi \\ &= \sum_{j=e,i}\frac{e}{T_j}\left\langle\int \mathrm{d}^3v\frac{F_{0j}}{n_0}\left\{\delta\phi - J_0 e^{-iQ}e^{-i\Lambda}\left[e^{i\Lambda}\overline{e^{iQ}J_0\phi}\right]e^{iS}\right\}\right\rangle_z,\end{aligned} \tag{83}$$

and the source is

$$\mathcal{I} = \frac{e}{T_\mathrm{i}}\left\langle\int \mathrm{d}^3v\frac{F_{0\mathrm{i}}}{n_0}J_0 e^{-iQ}e^{-i\Lambda}\left[e^{i\Lambda}\overline{e^{iQ}}\chi_\mathrm{cl}\phi(0)\right]e^{iS}\right\rangle_z. \tag{84}$$

Note that ϕ is outside of the bounce average in equations (83) and (84), provided that it is independent of z in the long term as poloidal angle-dependent GAM sidebands were already damped by transit resonance [45]. Now, we obtain explicit expressions of the long-term potential by approximating equations (83) and (84) for *narrow* ($w_\mathrm{cr,e} \ll w \ll w_\mathrm{cr,i}$) and *thick* ($w_\mathrm{cr,i} \ll w$) magnetic islands.

5.1. Narrow magnetic island: $w_\mathrm{cr,e} \ll w \ll w_\mathrm{cr,i}$

5.1.1. Negligible background flow: $\Phi \ll \Phi_\mathrm{cr}$. For a narrow magnetic island with a weak background vortex flow, ions and trapped electrons are in the toroidal regime while passing electrons are in the drift island regime. Then, we have a large FSD factor $\Lambda \sim \mathcal{O}(1)$ for ions, and therefore the ion FLR and FOW effects and the electron FSD effect, much smaller than unity, become negligible. As a result, as emphasized in the previous work [39], the ion toroidal precession homogenizes the flow potential envelope along the tokamak magnetic surface ψ. Then, taking the flux surface average to equation (82) over the unperturbed magnetic surface, we finally obtain

$$\frac{\phi_\mathrm{L}(\psi)}{\phi(t=0)} = k_w^2\rho_\mathrm{Ti}^2\frac{J_0^2(S')}{1-J_0^2(S')}\left[\frac{4x^2}{w^2} + \left(\frac{kw}{2}\right)^2\frac{J_0(S')+J_2(S')}{2J_0(S')}\right], \tag{85}$$

for the long-term flow envelope $\phi_{\rm L}$. Equation (85) indicates that we have a *zonal-vortex flow mixture* [39] as a result of the long-term evolution of the self-generated vortex flow,

$$\delta\phi = \phi_{\rm L}(\psi)e^{iS(X)}, \tag{86}$$

which is a combination of the zonal-like envelope $\phi_{\rm L}(\psi)$ and the vortex-like eikonal part $\exp[iS(X)]$. Note that equation (85) indicates that the final mixture flow level is small due to the factor $k_w^2\rho_{\rm Ti}^2 \ll 1$ in our ordering for the long-wavelength vortex flow.

5.1.2. Finite background flow: $\Phi_{\rm cr} \ll \Phi$. Meanwhile, in the presence of a large enough background vortex flow, ions and trapped electrons are in the shifted vortex regime, with a reduced FSD factor $\Lambda \sim d/w \ll 1$ from equations (67) and (77). Since the maximal ordering for the background vortex flow is $e\Phi/T \sim \mathcal{O}(1)$, with an ordering $e\Phi_{\rm cr}/T \gg w_{\rm cr,e}/w$ the passing electrons are in the shifted island regime having an even smaller FSD factor compared to ions, leading to only a minor correction to the plasma dielectricity, equation (83). Note that in the shifted vortex regime,

$$\langle d\rangle_v \equiv \left\langle \int {\rm d}^3v \frac{F_0}{n_0} d \right\rangle_z = 0, \tag{87}$$

that is, there is no averaged deviation of the trajectories of flow-carrying particles from the magnetic surface X. This is due to an exact cancellation of the contributions from the toroidal precession of trapped and passing particles. Therefore, the potential surface (streamline) is maintained to be the same as the magnetic surface X in a long term up to the linear order of $\mathcal{O}(d/w)$. In other words, the flow structure is maintained as the concentric vortex in a long term.

Then, we readily obtain the long-term vortex flow level as follows, from equations (83) and (84) with the flux surface average.

$$\frac{\phi_{\rm L}(X)}{\phi(t=0)} = \frac{\chi_{\rm cl}}{\chi_{\rm cl}+\chi_{\rm nc}+\chi_{\rm d}}, \tag{88}$$

where we have defined

$$\chi_{\rm d} = \left\langle \langle [\Lambda^2]\rangle_v - \langle [\Lambda]^2\rangle_v \right\rangle, \tag{89}$$

the *drift susceptibility* which represents the long-term enhancement of the polarization shielding due to the magnetic precession, reduced by the background vortex flow. Note that minor electron contributions have been neglected for a simple estimation. Then, the explicit expression of the drift susceptibility is

$$\chi_{\rm d} = k_w^2 D_i^2 G_0(\rho), \tag{90}$$

where

$$D \equiv \sqrt{\langle d^2\rangle_v} \simeq 1.97\frac{V_D}{V_E}\epsilon^{1/4}w, \tag{91}$$

captures the effect of the FSD of the orbit center trajectory from the magnetic surface to the long-term vortex flow level. Here, $V_D \equiv T/eBR$ and $V_E \equiv 4\Phi'/Bw$ characterize the magnitude of the toroidal precession and the mean E × B flow, respectively. Note that $D \propto 1/\Phi'$ from equation (91), so that the secular drift-induced enhancement of polarization shielding $\chi_{\rm d} \propto D^2$ decreases with the background vortex flow amplitude. It clearly shows that *a background vortex flow is beneficial to maintain self-generated vortex flows*. In the large-flow limit $\Phi' \to \infty$, we have a vanishing drift susceptibility $\chi_{\rm d} \to 0$ and accordingly recover $\phi_{\rm L} \to \phi_{\rm R}$. That is, there is no further damping (shielding) of the residual vortex flow in the presence of a very large background vortex flow.

5.2. Thick magnetic island: $w \gg w_{\rm cr,i}$

In a thick magnetic island, we still find similar aspects of the long-term evolution of the self-generated vortex flow as those in a narrow magnetic island in a collisionless plasma. We have a toroidal precession-induced deviation of the streamlines from the magnetic island contours, which can be significantly reduced by a strong background vortex flow. Notable differences are dominant trapped particle contribution over passing particles and the equal role of electrons with ions in the presence of weak or moderate background vortex flow.

5.2.1. Negligible background flow: $\Phi \ll \Phi_{\rm cr}$. We have trapped particles in the toroidal regime, and passing particles in the shifted island regime in a thick magnetic island with a weak background vortex flow. From equations (72), (76) and (77), we then have much smaller FSD $\Lambda \sim d/w$ for the passing particles than that for the trapped particles $\Lambda \sim \mathcal{O}(1)$. We thus have a dominant trapped particle contribution to the plasma dielectricity, resulting in

$$\frac{\phi_{\rm L}(\psi)}{\phi(t=0)} = \frac{k_w^2\rho_{\rm Ti}^2}{1+T_{\rm i}/T_{\rm e}}\frac{J_0^2(S')}{1-J_0^2(S')} \times \left[\frac{4x^2}{w^2} + \left(\frac{kw}{2}\right)^2 \frac{J_0(S')+J_2(S')}{2J_0(S')}\right]. \tag{92}$$

That is, in the absence of a strong background vortex flow and a collisional relaxation, toroidal precession again leads to the formation of the zonal-vortex flow mixture in a thick magnetic island. Note that electrons also contribute to the dielectricity in addition to ions as shown in the factor $1+T_{\rm i}/T_{\rm e}$, which gives a lower mixture flow level compared to the case of a narrow magnetic island.

5.2.2. Moderate background flow: $\Phi_{\rm cr} \ll \Phi \ll \Phi_{\rm cr}w/w_{\rm cr,i}$. With a moderate background vortex flow in a thick magnetic island, we have trapped particles in the shifted vortex regime by the strong flow, while passing particles are in the shifted island regime due to a stronger effect from the averaged streaming. Therefore, the FSD factor for the passing particles is smaller than that for the trapped particles overall. As a result, we have a dominant trapped particle contribution to the long-term evolution of a self-generated vortex flow, which leads

to equation (88), but with a different expression of the drift susceptibility

$$\chi_{\rm d} = k_w^2(D_{\rm i}^2 + D_{\rm e}^2)G_0(\rho), \tag{93}$$

where

$$D = 1.41\frac{V_D}{V_E}\epsilon^{1/4}w, \tag{94}$$

contains only trapped particle contribution to the FSD.

5.2.3. Strong background flow: $\Phi \gg \Phi_{\rm cr}w/w_{\rm cr,i}$. In the presence of a very strong background vortex flow, ions and trapped electrons are in the shifted vortex regime, and only passing electrons are in the shifted island regime. Then, we have the same result with equation (88) in section 5.1.2 for a narrow magnetic island with a strong flow.

6. Discussions

In the previous sections, we have studied the effects of the background vortex flow on the evolution of the self-generated vortex flow in the short-term $\omega_{bi}^{-1} < t < \overline{\omega}_{\rm D}^{-1}$ and the long-term $\overline{\omega}_{\rm D}^{-1} < t$. In the short term, we have found that the residual vortex flow level in a magnetic island is unchanged by the background flow. It is largely different from the case of the residual zonal flow in tokamak geometry, which is enhanced by the equilibrium radial electric field [51, 52] as a result of the reduction of the neoclassical polarization shielding due to orbit squeezing [53]. In tokamak geometry, it is the poloidal direction that determines the lowest-order bounce motion and thus we have a significant contribution from the equilibrium E × B flow to the projected bounce motion which can be comparable to the projected parallel streaming $\sim v_\parallel B_\theta/B$. Meanwhile, in magnetic island geometry, the lowest-order bounce motion is determined in the reference helical magnetic field direction, exactly orthogonal to the background vortex flow. This is the reason for the absence of the mean E × B flow effect on the residual vortex flow level.

In the long term, our theory predicts that the toroidicity-induced breaking of the helical symmetry induces further collisionless flow damping toward a zonal-vortex flow mixture with $\phi_{\rm L}/\phi(t=0) \sim \chi_{\rm cl}$. This symmetry breaking-induced damping makes it harder for the turbulence-induced modulational growth of the vortex flows [54] to overcome the flow damping, resulting in an increase of the bifurcation threshold. However, in the presence of a large enough background vortex flow with $e\Phi/T \gg \epsilon w/r$, the toroidicity-induced flow damping is significantly reduced so that $\phi_{\rm L}/\phi(t=0) \sim \chi_{\rm cl}/(\chi_{\rm cl} + \chi_{\rm nc} + \chi_{\rm d})$. The suppression of the long-term damping of the self-generated vortex flow lowers the transition threshold. The finite background vortex flow also prevents the structure deformation of the self-generated vortex flow from the concentric vortex, which therefore prevents a parallel collisional relaxation. Then, the dominant collision effect would be a slower collisional flow damping by the neoclassical friction between trapped and passing particles [55].

The above findings indicate that *the finite background vortex flow makes a positive synergism with the self-generated vortex flow leading to a more favorable condition for the transport barrier formation.* An important point is that the critical background flow level for the prevention of the toroidal precession-induced long-term damping of the self-generated vortex flow is much lower than that required for the background E × B flow shear-induced turbulence suppression, $\omega_{\rm E\times B} = \Delta\omega_{\rm T}$. Here, $\omega_{\rm E\times B}$ is the E × B shearing rate [20] and $\Delta\omega_{\rm T}$ is the turbulence decorrelation rate [15, 16]. For a simple estimation, let us consider orderings $L_{\rm E} \sim w$, $\Delta\omega_{\rm T} \sim \omega_{*e0} \sim (k_\perp/r)T/eB$ and $k_\perp\rho_i \sim 1$, where $L_{\rm E}$ is length scale of the background vortex flow potential, ω_{*0} is an unperturbed diamagnetic frequency, and $k_\perp$ is the perpendicular wavenumber of the microturbulence. Then, the E × B shear suppression criterion yields, roughly, $e\Phi/T \sim L_{\rm E}^2k_\perp/r \sim (w/\rho_i)w/r$, which is much larger than $e\Phi_{\rm cr}/T = \epsilon w/r$ for the prevention of the long-term damping of the self-generated vortex flow. Therefore, the condition $\Phi > \Phi_{\rm cr}$ addressed in this work could be considered as a preliminary condition for transition to an enhanced confinement state of a magnetic island. It is worth noting that experimental and simulation studies [22, 25, 29] indicate that this condition can be satisfied more easily near the island separatrix compared to the central region of the island due to a larger profile-induced background electric field. Mechanisms of confinement enhancement in the island region suggested in previous studies rely on turbulence suppression by the large background E × B flow shear around the island boundary [23, 56]. Our work indicates that in addition to that, the magnetic island boundary region is also a prime location for triggering the transition (bifurcation) to an enhanced confinement state of the magnetic island accompanied by transport barrier formation.

Considering the experimentally observed feature that only a thick enough magnetic island has a significant background electric field [24], a thick magnetic island is preferred over a narrow one for the transport barrier formation (either by the easier triggering through the vortex flow self-generation, or solely by the stronger background flow [13]). However, even with the internal transport barrier and local confinement enhancement, the global tokamak confinement is likely degraded with the thick magnetic island due to profile flattening [57] over a wide radial range in the island [18, 22]. A possible strategy to overcome this demerit of a thick magnetic island could be mitigating or suppressing the magnetic island using electron cyclotron current drive [58–60] or externally imposed magnetic perturbation [61–63] after the internal transport barrier formation. Thanks to the hysteresis of transport barrier dynamics [64–68], we could minimize the demerit of a thick magnetic island while maintaining the benefit.

We would like to mention several theory issues for a more thorough understanding of the vortex flow dynamics in the magnetic island region and its impact on confinement. First, the effect of the toroidal precession (or the magnetic drift),

with and without the background vortex flow, should also be investigated in the generation part. Recall that theoretical studies have shown a negative effect of the equilibrium E × B flow shear to the modulation growth of the tokamak zonal flow [12, 69], and that synergism of the toroidal precession and the E × B flow was analytically studied in the context of the shear suppression of turbulence [70, 71]. Second, a theoretical extension of the present work in a tokamak to the self-generated vortex flows in a stellarator magnetic island would be interesting where the secular radial drift of the orbit centers enters to the long-term flow evolution. Recall that there have been extensive experimental E × B shear flow and transport studies in stellarators including LHD, TJ-II, and W7-X. Third, note that while we have used an eikonal representation, the scale of spatial inhomogeneity of the flow potential envelope ϕ becomes comparable to that of the eikonal factor S as an initially uniform vortex potential envelope evolves. That is, we are touching the validity limit of the eikonal representation. For a more precise description of the dehomogenization of the envelope, one may need to consider variations of the envelope ϕ and the eikonal factor S on the same footing. Fourth, we have assumed the Maxwellian unperturbed distribution function F_0 in this work, and therefore we have not captured the contribution from the equilibrium parallel current via F_0. Since the equilibrium parallel current is an essential element of magnetic island physics, we will address its effect on the vortex flow evolution in the near future. Finally, an extension to the burning plasmas with abundant energetic particles should be pursued in the near future for a precise prediction of confinement and potential new operation scenarios in future fusion machines. The energetic particles are expected to amplify the toroidicity-induced helical symmetry breaking and long-term flow damping due to the proportionality of the toroidal precession to the particle energy. To be rigorous, we have to extend our calculation to a shorter wavelength vortex flows $k_X\rho_\theta > 1$, where ρ_θ is the poloidal gyroradius [33, 34], to fully consider effects of the energetic particles having large gyroradii.

In summary, we have shown by analytic gyrokinetic calculations that in the short term, the residual level of a self-generated vortex flow after fast collisionless damping in a stationary tokamak magnetic island, higher than the Rosenbluth–Hinton level due to a finite island width, is unaffected by a background vortex flow. In the long term, the residual vortex flow evolves to a zonal-vortex flow mixture with further damping by a toroidicity-induced breaking of the helical symmetry. However, in the presence of a finite background vortex flow with $e\Phi/T > \epsilon w/r$, the long-term flow damping and the structure deformation are significantly reduced. Since the deviation of the streamlines of the self-generated vortex flow from the island magnetic surfaces is suppressed by the finite background vortex flow, its parallel collisional relaxation is also prevented. As the self-generated E × B flows play a crucial role in the triggering of the transport barrier formation, the positive synergism of the background and the self-generated vortex flows leads to a more favorable condition for the transition to an enhanced confinement state of a magnetic island. The synergism that we have found could cooperate with previously suggested mechanisms relying on the background shear flow, supporting the argument that the island boundary region is a prominent location for the transition.

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