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TOPOLOGICAL
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Limits in the category Seg of Segal topological algebras

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Abstract. In this paper we find several sufficient conditions for a family of Segal topological algebras to have a limit in the category **Seg** of Segal topological algebras.

Keywords: Segal topological algebras, category theory, inverse systems, limits.

1. INTRODUCTION

Throughout the paper, $\mathbb{K}$ stands either for the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A *topological algebra* is a topological linear space over $\mathbb{K}$, which is also an algebra over $\mathbb{K}$ with separately continuous multiplication.

We start the paper by recalling some definitions given in [1].

Let I be any set of indices, $\{(A_i,\tau_i)\}_{i\in I}$ a collection of topological algebras and

$$\prod_{i\in I} A_i = \{(a_i)_{i\in I} : a_i \in A_i \text{ for every } i\in I\}$$

the *direct product* of the collection $\{(A_i,\tau_i)\}_{i\in I}$. Defining algebraic operations of $\prod\limits_{i\in I} A_i$ pointwise, we turn the direct product also into an algebra. On this algebra we consider the product topology, the base of which is the collection

$$\mathscr{B} = \Big\{\prod_{i\in I} U_i : \text{ there are } n\in\mathbb{Z}^+, i_1,\ldots,i_n\in I \text{ such that}$$

$$U_i\in\tau_i \text{ for every } i\in\{i_1,\ldots,i_n\}, \text{ and } U_i = A_i, \text{ otherwise}\Big\},$$

where $\prod\limits_{i\in I} U_i = \{(u_i)_{i\in I} : u_i\in U_i \text{ for every } i\in I\}$, then the direct product becomes a topological algebra (for the details, see [1], pp. 26–27) which is called the *topological direct product* of the topological algebras $\{(A_i,\tau_i)\}_{i\in I}$.

Let $(I,\preceq)$ be a partially ordered set and $(A_i)_{i\in I}$ a collection of algebras. Let us recall that an inverse system of algebras is the ordered pair $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$, where A_i is an algebra for each $i\in I$, $\psi_j^k : A_k \to A_j$

is an algebra homomorphism for all $j,k \in I$ with $j \preceq k$, $\psi_i^i = 1_{A_i}$ for each $i \in I$, and $\psi_i^j \circ \psi_j^k = \psi_i^k$ for all $i,j,k \in I$ with $i \preceq j \preceq k$.

It is said that a partially ordered set $(I, \preceq)$ is a *partially ordered set with the greatest element* i_g, if there exists $i_g \in I$ such that $i \preceq i_g$ for each $i \in I$. Notice, that such i_g is unique.

Example 1. *Some of the examples of partially ordered sets with the greatest element $i_g \in I$ are:*

a) $I = \mathbb{Z}^+$ *with $a \preceq b$ if and only if $b \leqslant a$ and $i_g = 1$;*

b) $I = (0,1]$ *with $a \preceq b$ if and only if $a \leqslant b$ and $i_g = 1$;*

c) $I = P(X)$*, the power set of any nonempty set X, with $A \preceq B$ if and only if $A \subseteq B$ and $i_g = X$.*

We finish this section by recalling some notions from [3].

Let $(I, \preceq)$ be a partially ordered set and $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$ an inverse system of algebras.

A *thread* of the inverse system $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$ is an element $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$ such that $\psi_j^k(a_k) = a_j$ for all $j,k \in I$ with $j \preceq k$.

Notice, that the collection

$$C = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i : \psi_j^k(a_k) = a_j \text{ for all } j \preceq k\}$$

of all threads of the inverse system $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$ becomes also an algebra, when we consider on it the pointwise defined algebraic operations. Indeed, for any $(a_i)_{i\in I}, (\overline{a_i})_{i\in I} \in C$ and $\lambda \in \mathbb{K}$ we have

$$\psi_j^k(a_k + \overline{a_k}) = \psi_j^k(a_k) + \psi_j^k(\overline{a_k}) = a_j + \overline{a_j},\ \psi_j^k(a_k\overline{a_k}) = \psi_j^k(a_k)\psi_j^k(\overline{a_k}) = a_j\overline{a_j}$$

and

$$\psi_j^k(\lambda a_k) = \lambda\, \psi_j^k(a_k) = \lambda a_j,$$

which means that

$$(a_i)_{i\in I} + (\overline{a_i})_{i\in I} = (a_i + \overline{a_i})_{i\in I} \in C,\ (a_i)_{i\in I}(\overline{a_i})_{i\in I} = (a_i\overline{a_i})_{i\in I} \in C \text{ and } \lambda(a_i)_{i\in I} = (\lambda a_i)_{i\in I} \in C.$$

When we consider on C the subspace topology, induced by the product topology on the direct product $\prod\limits_{i\in I} A_i$, then C becomes a topological algebra. In what follows, the algebra C as above, will be referred as the *inverse limit algebra* of the inverse system $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$.

2. THE CATEGORY SEG

In this section we recall some notions and facts from [2].

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

1) $\mathrm{cl}_B(f(A)) = B$, i.e., $f(A)$ is dense in B;

2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;

3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a Segal topological algebra will be denoted shortly by a triple (A, f, B).

The set $\mathrm{Ob}(\mathbf{Seg})$ of all objects of the category **Seg** consists of all left (right or two-sided) Segal topological algebras. For any $(A, f, B), (C, g, D) \in \mathrm{Ob}(\mathbf{Seg})$, the set $\mathrm{Mor}((A, f, B), (C, g, D))$ of morphisms from

(A,f,B) to (C,g,D) consists of all such pairs (α,β) of continuous algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to D$, for which $g \circ \alpha = \beta \circ f$, i.e., there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ C & \xrightarrow{g} & D \end{array}.$$

The composition of morphisms of **Seg** is defined componentwise as follows:
for any objects $(A,f,B),(C,g,D),(E,h,F) \in \mathrm{Ob}(\mathbf{Seg})$ and any morphisms $(\alpha,\beta) \in \mathrm{Mor}((A,f,B),(C,g,D))$, $(\gamma,\delta) \in \mathrm{Mor}((C,g,D),(E,h,F))$, the composition of (γ,δ) and (α,β) is $(\gamma,\delta)\circ(\alpha,\beta) = (\gamma\circ\alpha,\delta\circ\beta)$.

In [2], pp. 2–4, it was proved that this composition of morphisms is well defined and associative. Moreover, it was showed that the identity morphism for an object (A,f,B) of **Seg** is a pair $(1_A,1_B)$ of identity maps.

3. INVERSE SYSTEMS OF SEGAL TOPOLOGICAL ALGEBRAS

In this section we prove some results about inverse systems of Segal topological algebras.

Definition 1. *An* **inverse system** *of Segal topological algebras in the category* **Seg** *is the collection* $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$, *where* $(f_i : A_i \to B_i)_{i\in I}$ *is the collection of continuous algebra homomorphisms such that* $f_i(A_i)$ *is a dense ideal of* B_i *and* $(\psi_j^k,\phi_j^k) \in \mathrm{Mor}((A_k,f_k,B_k),(A_j,f_j,B_j))$ *for all* $j,k \in I$ *with* $j \preceq k$, *such that* $\psi_i^j \circ \psi_j^k = \psi_i^k$, $\phi_i^j \circ \phi_j^k = \phi_i^k$ *for all* $i,j,k \in I$ *with* $i \preceq j \preceq k$, $\psi_i^i = 1_{A_i}$ *and* $\phi_i^i = 1_{B_i}$ *for all* $i \in I$.

It is easy to see that each inverse system $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ of Segal topological algebras defines inverse systems $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$ and $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$ of topological algebras.

We state the next result for algebras, adding the topology only in claim c), where it is needed.

Proposition 1. *Let* I *be a partially ordered set,* $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$ *inverse systems of algebras,* C *and* D *their respective inverse limit algebras and* $(f_i : A_i \to B_i)_{i\in I}$ *a collection of maps such that* $f_j \circ \psi_j^k = \phi_j^k \circ f_k$ *for all* $j,k \in I$ *with* $j \preceq k$. *Then the following claim holds.*

a) *The map* $f : \prod\limits_{i\in I} A_i \to \prod\limits_{i\in I} B_i$, *defined by* $f((a_i)_{i\in I}) = (f_i(a_i))_{i\in I}$ *for all* $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$, *maps a thread of the inverse system* $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$ *to a thread of the inverse system* $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$, *i.e.,* $f(C) \subseteq D$.

Moreover, if the partially ordered set $(I,\preceq)$ *has the greatest element* i_g, *then the following claims also hold.*

b) *If* $f(A_{i_g})$ *is a left (right or two-sided) ideal of* B_{i_g}, *then* $f(C)$ *is a right (respectively, left or two-sided) ideal of* D.

c) *If the algebras* $(A_i)_{i\in I},(B_i)_{i\in I}$ *are topological algebras, the algebras* C *and* D *are equipped with the subspace topology, inherited from the topological direct products* $\prod\limits_{i\in I} A_i$ *and* $\prod\limits_{i\in I} B_i$, *the maps* $(\phi_j^k)_{j\preceq k}$ *are continuous and* $f_{i_g}(A_{i_g})$ *is dense in* B_{i_g}, *then* $f(C)$ *is dense in* D.

Proof. Take any $(a_i)_{i\in I} \in C$. Then $\psi_j^k(a_k) = a_j$ for all $j,k \in I$ with $j \preceq k$. Set $b_i = f_i(a_i)$ for each $i \in I$. Then $f((a_i)_{i\in I}) = (b_i)_{i\in I}$.

Let $i_g \in I$ be the greatest element in I, i.e., $i \preceq i_g$ for each $i \in I$, and consider the algebra homomorphisms $\psi_i^{i_g} : A_{i_g} \to A_i, \phi_i^{i_g} : B_{i_g} \to B_i$ for all $i \in I$.

a) Take any $j,k \in I$ with $j \preceq k$. Then

$$\phi_j^k(b_k) = \phi_j^k(f_k(a_k)) = (\phi_j^k \circ f_k)(a_k) = (f_j \circ \psi_j^k)(a_k) = f_j(\psi_j^k(a_k)) = f_j(a_j) = b_j.$$

Hence, $f((a_i)_{i\in I}) = (b_i)_{i\in I} \in D$ for each $(a_i)_{i\in I} \in C$. Thus, $f(C) \subseteq D$.

b) We will prove it for left ideals. The cases for right and two-sided ideals are similar. Take any $(d_i)_{i\in I} \in D \subseteq \prod_{i\in I} B_i$. Then $\phi_j^k(d_k) = d_j$ for all $j,k \in I$ with $j \preceq k$. Now,

$$(d_i)_{i\in I} f((a_i)_{i\in I}) = (d_i)_{i\in I}(f_i(a_i))_{i\in I} = (d_i f_i(a_i))_{i\in I}.$$

As $f_{i_g}(A_{i_g})$ is a left ideal of B_{i_g} and $d_{i_g} \in B_{i_g}$, there exists $\overline{a_{i_g}} \in A_{i_g}$ such that $f_{i_g}(\overline{a_{i_g}}) = d_{i_g} f_{i_g}(a_{i_g})$. Define $a = (\overline{a_i})_{i\in I}$, where $\overline{a_i} = \psi_i^{i_g}(\overline{a_{i_g}})$ for each $i \in I$. Then $a \in \prod_{i\in I} A_i$.

Take any $j,k \in I$ with $j \preceq k$. Then

$$\overline{a_j} = \psi_j^{i_g}(\overline{a_{i_g}}) = (\psi_j^k \circ \psi_k^{i_g})(\overline{a_{i_g}}) = \psi_j^k(\psi_k^{i_g}(\overline{a_{i_g}})) = \psi_j^k(\overline{a_k}),$$

$$\phi_j^k(f_k(\overline{a_k})) = (\phi_j^k \circ f_k)(\overline{a_k}) = (f_j \circ \psi_j^k)(\overline{a_k}) = f_j(\psi_j^k(\overline{a_k})) = f_j(\overline{a_j})$$

and

$$d_j f_j(a_j) = \phi_j^{i_g}(d_{i_g}) f_j(\psi_j^{i_g}(a_{i_g})) = \phi_j^{i_g}(d_{i_g})(f_j \circ \psi_j^{i_g})(a_{i_g}) = \phi_j^{i_g}(d_{i_g})(\phi_j^{i_g} \circ f_{i_g})(a_{i_g}) = \phi_j^{i_g}(d_{i_g})\phi_j^{i_g}(f_{i_g}(a_{i_g}))$$

$$= \phi_j^{i_g}(d_{i_g} f_{i_g}(a_{i_g})) = \phi_j^{i_g}(f_{i_g}(\overline{a_{i_g}})) = (\phi_j^{i_g} \circ f_{i_g})(\overline{a_{i_g}}) = (f_j \circ \psi_j^{i_g})(\overline{a_{i_g}}) = f_j(\psi_j^{i_g}(\overline{a_{i_g}})) = f_j(\overline{a_j}).$$

Hence, $a \in C, f(a) \in D$ and $(d_i)_{i\in I}(f_i(a_i))_{i\in I} = f(a) \in f(C)$. This means that $f(C)$ is a left ideal of D.

c) Take any neighbourhood O of an element $(d_i)_{i\in I} \in D$. Then there exist neighbourhoods $\{O_i \subseteq B_i\}_{i\in I}$ of elements $\{d_i\}_{i\in I}$, respectively, such that there exists a positive number n and indices $i_1,\dots,i_n \in I$ so that $O_i = B_i$ for all $i \notin \{i_1,\dots,i_n\}$ and $(d_i)_{i\in I} \in D \cap \prod_{i\in I} O_i \subseteq O$.

Notice that from $\phi_i^{i_g}(d_{i_g}) = d_i \in O_i$ it follows that $d_{i_g} \in (\phi_i^{i_g})^{-1}(O_i) \subseteq B_{i_g}$ for each $i \in I$. As $\phi_{i_1}^{i_g},\dots,\phi_{i_n}^{i_g}$ are continuous, then $(\phi_{i_1}^{i_g})^{-1}(O_{i_1}),\dots,(\phi_{i_n}^{i_g})^{-1}(O_{i_n})$ are neighbourhoods of d_{i_g} in B_{i_g}.

Set $W_{i_g} = O_{i_g} \cap (\phi_{i_1}^{i_g})^{-1}(O_{i_1}) \cap \dots \cap (\phi_{i_n}^{i_g})^{-1}(O_{i_n})$ and $W_i = \phi_i^{i_g}(W_{i_g})$ for each $i \in I$. Then W_{i_g} is a neighbourhood of d_{i_g} and

$$d_i = \phi_i^{i_g}(d_{i_g}) \in \phi_i^{i_g}(W_{i_g}) = W_i \subseteq O_i$$

for each $i \in I$. Hence, $D \cap \prod_{i\in I} W_i \subseteq D \cap \prod_{i\in I} O_i \subseteq O$ is a neighbourhood of $(d_i)_{i\in I}$ in the topology of D.

As $f_{i_g}(A_{i_g})$ is dense in B_{i_g}, there exist $b_{i_g} \in W_{i_g}$ and $\overline{a_{i_g}} \in A_{i_g}$ such that $b_{i_g} = f_{i_g}(\overline{a_{i_g}})$. Take $\overline{a_i} = \psi_i^{i_g}(\overline{a_{i_g}})$ for each $i \in I$. As we showed in part b) of the proof, $a = (\overline{a_i})_{i\in I} \in C$. Set $b_i = f_i(\overline{a_i})$ for every $i \in I$ and $b = (b_i)_{i\in I}$. For each $i \in I$, we have

$$b_i = f_i(\overline{a_i}) = f_i(\psi_i^{i_g}(\overline{a_{i_g}})) = (f_i \circ \psi_i^{i_g})(\overline{a_{i_g}}) = (\phi_i^{i_g} \circ f_{i_g})(\overline{a_{i_g}}) = \phi_i^{i_g}(f_{i_g}(\overline{a_{i_g}})) = \phi_i^{i_g}(b_{i_g}) \in \phi_i^{i_g}(W_{i_g}) = W_i,$$

which means that $f(a) = b \in \prod_{i\in I} W_i$.

Take any $j,k \in I$ with $j \preceq k$. Then

$$\phi_j^k(b_k) = \phi_j^k(f_k(\overline{a_k})) = (\phi_j^k \circ f_k)(\overline{a_k}) = (f_j \circ \psi_j^k)(\overline{a_k}) = f_j(\psi_j^k(\overline{a_k})) = f_j(\overline{a_j}) = b_j,$$

which means that $b \in D$.

Hence, $f(a) = b = (b_i)_{i\in I} \in D \cap \prod_{i\in I} W_i \subseteq O$ and $f(C)$ is dense in D. $\square$

Now, let us state and prove a simple corollary of the Proposition 1.

Corollary 1. *If $(I,\preceq)$ is a partially ordered set with the greatest element, $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ is the inverse system of left (right or two-sided) Segal topological algebras and let C and D be the inverse limit algebras for the inverse systems $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$, respectively. Equip C and D with the subspace topology, inherited from the topological direct products $\prod\limits_{i\in I}A_i$ and $\prod\limits_{i\in I}B_i$, respectively, and the map $f:\prod\limits_{i\in I}A_i\to\prod\limits_{i\in I}B_i$ is defined by $f((a_i)_{i\in I})=(f_i(a_i))_{i\in I}$ for all $(a_i)_{i\in I}\in\prod\limits_{i\in I}A_i$, then (C,g,D), where g is the restriction of f to C, is a left (respectively, right or two-sided) Segal topological algebra.*

Proof. Notice that all the assumptions of Proposition 1 are fulfilled. Therefore, $g(C)=f(C)$ is a dense left (right or two-sided, respectively) ideal of D, which means that (C,g,D) is a Segal topological algebra. □

Remark 1. *Notice, that in the proof of part b) of the Proposition 1 we used one and the same element $\overline{a_{i_g}}\in A_{i_g}$ in order to define elements $\overline{a_i}$ for all $i\in I$. Similarly, in part c) of the same proof, we represented all elements d_i through the images of the unique element $d_{i_g}\in B_{i_g}$.*

These ideas would not work in the case of an upward directed set $(I,\preceq)$ (i.e., a partially ordered set $(I,\preceq)$, where for each pair $i,j\in I$ there exists $k\in I$ such that $i\preceq k$ and $j\preceq k$), because we can not define the thread $a=(\overline{a_i})_{i\in I}$ (for each pair $i,j\in I$, the elements $\overline{a_i},\overline{a_j}$ could be still using $\overline{a_k}$ but for an infinite family $(i_n)_{n\in\mathbb{Z}^+}$ of elements of I, one can not find such k_0 that $i_n\preceq k_0$ for all $n\in\mathbb{Z}^+$). The similar reason holds for the part c) of the proof, where the thread $a=(\overline{a_i})_{i\in I}$ is defined exactly in the same way.

In order to use upward directed set, the proofs of parts b) and c) have to be done differently. Unfortunately, the author does not have any idea for that yet.

To avoid the condition that the index set has to have the greatest element, we can also state and prove the following lemma.

Lemma 1. *Let $(I,\preceq)$ be a partially ordered set, $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ be the inverse system of left (right or two-sided) Segal topological algebras and let C and D be the inverse limit algebras for the inverse systems $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$, respectively. Equip C and D with the subspace topology, inherited from the topological direct products $\prod\limits_{i\in I}A_i$ and $\prod\limits_{i\in I}B_i$, respectively, let the map $f:\prod\limits_{i\in I}A_i\to\prod\limits_{i\in I}B_i$ be defined by $f((a_i)_{i\in I})=(f_i(a_i))_{i\in I}$ for all $(a_i)_{i\in I}\in\prod\limits_{i\in I}A_i$, and let g be the restriction of f to C. If $g(C)$ is a dense left (respectively, right or two-sided) ideal of D, then (C,g,D) is a left (respectively, right or two-sided) Segal topological algebra.*

Proof. As $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ is the inverse system of left (right or two-sided) Segal topological algebras, then $f_j\circ\psi_j^k=\phi_j^k\circ f_k$ for all $j,k\in I$ with $j\preceq k$, all the maps $(f_i)_{i\in I},(\psi_j^k)_{j\preceq k}$, $(\phi_j^k)_{j\preceq k}$ are continuous algebra homomorphisms and $f_i(A_i)$ is a dense left (right or two-sided) ideal of B_i for every $i\in I$. Moreover, f and g are continuous algebra homomorphisms, because all maps $(f_i)_{i\in I}$ were continuous algebra homomorphisms.

As $g(C)$ is a dense left (respectively, right or two-sided) ideal of D, then (C,g,D) is a left (respectively, right or two-sided) Segal topological algebra. □

4. LIMIT OF AN INVERSE SYSTEM OF SEGAL TOPOLOGICAL ALGEBRAS

Now we are ready to describe the limit of an inverse system of Segal topological algebras. For that, we first need the following definition.

Definition 2. *The* **limit of an inverse system** $(((A_i,f_i,B_i))_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ *in* **Seg** *is the pair $((C,g,D),(\alpha_i,\beta_i)_{i\in I})$, where $(C,g,D)\in\mathrm{Ob}(\mathbf{Seg})$ and $(\alpha_j,\beta_j)\in\mathrm{Mor}((C,g,D),(A_j,f_j,B_j))$ for each $j\in I$ such that*

1) $(\psi_j^k,\phi_j^k)\circ(\alpha_k,\beta_k)=(\alpha_j,\beta_j)$, *whenever* $j\preceq k$;

2) *for any* $(X,h,Y) \in \mathrm{Ob}(\mathbf{Seg})$ *and* $\{(\gamma_i,\delta_i) \in \mathrm{Mor}((X,h,Y),(A_i,f_i,B_i))\}_{i\in I}$ *which satisfy the condition* $(\psi_j^k,\phi_j^k)\circ(\gamma_k,\delta_k) = (\gamma_j,\delta_j)$, *whenever* $j \preceq k$, *there exists a unique morphism* $(\theta,\omega) : (X,h,Y) \to (C,g,D)$ *making the diagrams*

$$\begin{tikzcd}
C \arrow[dr, "\alpha_j"] \arrow[ddd, "g"'] & & X \arrow[ll, dashed, "\theta"'] \arrow[dl, "\gamma_j"'] \arrow[ddd, "h"] \\
& A_j \arrow[d, "f_j"] & \\
& B_j & \\
D \arrow[ur, "\beta_j"] & & Y \arrow[ll, dashed, "\omega"] \arrow[ul, "\delta_j"']
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
C \arrow[d, "\alpha_j"] \arrow[ddd, "g"'] & X \arrow[l, dashed, "\theta"'] \arrow[d, "\gamma_k"'] \arrow[ddd, "h"] \\
A_j \arrow[d, "f_j"] & A_k \arrow[l, "\psi_j^k"] \arrow[d, "f_k"] \\
B_j & B_k \arrow[l, "\phi_j^k"'] \\
D \arrow[u, "\beta_j"'] & Y \arrow[l, dashed, "\omega"] \arrow[u, "\delta_k"]
\end{tikzcd}$$

Diagram 2

commutative for all $j,k \in I$ *with* $j \preceq k$.

Now we are ready to state and prove the main result of this paper.

Theorem 1. *Let* $(I,\preceq)$ *be a partially ordered set,* $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ *an inverse system in the category* **Seg** *and let* C *and* D *be the inverse limit algebras for the inverse systems* $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$, *respectively. Equip* C *and* D *with the subspace topology, inherited from the topological direct products* $\prod\limits_{i\in I} A_i$ *and* $\prod\limits_{i\in I} B_i$, *respectively, and let* g *be the restriction of the map* $f : \prod\limits_{i\in I} A_i \to \prod\limits_{i\in I} B_i$, *defined by* $f((a_i)_{i\in I}) = (f_i(a_i))_{i\in I}$ *for all* $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$, *to* C, *and* $(\alpha_i : C \to A_i, \beta_i : D \to B_i)_{i\in I}$ *be a family of maps, defined by* $\alpha_j((a_i)_{i\in I}) = a_j$, $\beta_j((b_i)_{i\in I}) = b_j$ *for all* $j \in I$ *and all* $(a_i)_{i\in I} \in C$, $(b_i)_{i\in I} \in D$. *If* $(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$, *then the limit of the inverse system* $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ *in the category* **Seg** *exists and has the form* $((C,g,D),(\alpha_i,\beta_i)_{i\in I})$.

Proof. Take any $j,k \in I$ with $j \preceq k$. Then

$$(\psi_j^k \circ \alpha_k)((a_i)_{i\in I}) = \psi_j^k(\alpha_k((a_i)_{i\in I})) = \psi_j^k(a_k) = a_j = \alpha_j((a_i)_{i\in I})$$

and

$$(\phi_j^k \circ \beta_k)((b_i)_{i\in I}) = \phi_j^k(\beta_k((b_i)_{i\in I})) = \phi_j^k(b_k) = b_j = \beta_j((b_i)_{i\in I})$$

for every $(a_i)_{i\in I} \in C$ and every $(b_i)_{i\in I} \in D$. Hence,

$$(\psi_j^k,\phi_j^k)\circ(\alpha_k,\beta_k) = (\alpha_j,\beta_j)$$

for all $j,k \in I$ with $j \preceq k$. Thus, 1) of Definition 2 is fulfilled.

Take any $(X,h,Y) \in \mathrm{Ob}(\mathbf{Seg})$ and a family $((\gamma_i,\delta_i) : (X,h,Y) \to (A_i,f_i,B_i))_{i\in I}$ of morphisms, which satisfy the condition $(\psi_j^k,\phi_j^k)\circ(\gamma_k,\delta_k) = (\gamma_j,\delta_j)$ for all $j,k \in I$ with $j \preceq k$. Then $f_i \circ \gamma_i = \delta_i \circ h$ for all $i \in I$ and $\gamma_j = \psi_j^k \circ \gamma_k, \delta_j = \phi_j^k \circ \delta_k$ for all $j,k \in I$ with $j \preceq k$.

Define the maps $\theta : X \to \prod\limits_{i\in I} A_i$ and $\omega : Y \to \prod\limits_{i\in I} B_i$ by $\theta(x) = (\gamma_i(x))_{i\in I}$ for each $x \in X$ and $\omega(y) = (\delta_i(y))_{i\in I}$ for each $y \in Y$. Then θ and ω are continuous algebra homomorphisms, since the morphisms $\gamma_i,\delta_i, i \in I$ are continuous algebra homomorphisms.

Take any $(a_i)_{i\in I} \in \theta(X)$ and any $(b_i)_{i\in I} \in \omega(Y)$. Then $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$, $(b_i)_{i\in I} \in \prod\limits_{i\in I} B_i$ and there exist $x \in X$ and $y \in Y$ such that $\gamma_i(x) = a_i$ and $\delta_i(y) = b_i$ for each $i \in I$. Notice that

$$\psi_j^k(a_k) = \psi_j^k(\gamma_k(x)) = (\psi_j^k \circ \gamma_k)(x) = \gamma_j(x) = a_j$$

and

$$\phi_j^k(b_k) = \phi_j^k(\delta_k(y)) = (\phi_j^k \circ \delta_k)(y) = \delta_j(y) = b_j$$

for all $j,k \in I$ with $j \preceq k$. Hence, $(a_i)_{i\in I} \in C$ and $(b_i)_{i\in I} \in D$, which means that $\theta(X) \subseteq C$ and $\omega(D) \subseteq D$. Thus, θ and ω are actually maps in the form $\theta : X \to C$ and $\omega : Y \to D$.

It follows from the definitions of θ and ω immediately that

$$(\alpha_i, \beta_i) \circ (\theta, \omega) = (\gamma_i, \delta_i)$$

for each $i \in I$. Now, as $(\alpha_j, \beta_j) \in \mathrm{Mor}((C,g,D),(A_j, f_j, B_j))$ and $(\gamma_j, \delta_j) \in \mathrm{Mor}((X,h,Y),(A_j, f_j, B_j))$ for each $j \in I$, then $\beta_j \circ g = f_j \circ \alpha_j$ and $f_j \circ \gamma_j = \delta_j \circ h$ for each $j \in I$. Therefore,

$$\beta_j \circ g \circ \theta = (\beta_j \circ g) \circ \theta = (f_j \circ \alpha_j) \circ \theta = f_j \circ (\alpha_j \circ \theta) = f_j \circ \gamma_j = \delta_j \circ h \tag{4.1}$$

for each $j \in I$. Moreover, by the definitions of θ and ω, we also obtain that

$$(\omega \circ h)(x) = \omega(h(x)) = (\delta_i(h(x)))_{i\in I} = ((\delta_i \circ h)(x))_{i\in I} = ((f_i \circ \gamma_i)(x))_{i\in I}$$

$$= (f_i(\gamma_i(x)))_{i\in I} = g((\gamma_i(x))_{i\in I}) = g(\theta(x)) = (g \circ \theta)(x)$$

for each $x \in X$. Thus, $\omega \circ h = g \circ \theta$ and the Diagram 1 is commutative.

Notice that $\alpha_j \circ \theta = \gamma_j = \psi_j^k \circ \gamma_k$ and $\beta_j \circ \omega = \delta_j = \phi_j^k \circ \delta_k$ for all $j,k \in I$ with $j \preceq k$. Moreover, using (4.1), we get

$$\beta_j \circ g \circ \theta = \delta_j \circ h = (\phi_j^k \circ \delta_k) \circ h = \phi_j^k \circ \delta_k \circ h$$

for each $j,k \in I$ with $j \in I$. Thus, the Diagram 2 is commutative.

With that we have demonstrated that there exists $(\theta, \omega) \in \mathrm{Mor}((X,h,Y),(C,g,D))$, which makes these two diagrams commute.

Let $(\overline{\theta}, \overline{\omega}) \in \mathrm{Mor}((X,h,Y),(C,g,D))$ be such that the respective two diagrams are commutative. Take any $x \in X, y \in Y$ and set $(c_i)_{i\in I} = \overline{\theta}(x)$, $(d_i)_{i\in I} = \overline{\omega}(y)$. From the commutativity of the first diagram, it follows that

$$\gamma_j(x) = (\alpha_j \circ \overline{\theta})(x) = \alpha_j(\overline{\theta}(x)) = \alpha_j((c_i)_{i\in I}) = c_j$$

and that

$$\delta_j(y) = (\beta_j \circ \overline{\omega})(y) = \beta_j(\overline{\omega}(y)) = \beta_j((d_i)_{i\in I}) = d_j.$$

Hence,

$$\overline{\theta}(x) = (c_i)_{i\in I} = (\gamma_i(x))_{i\in I} = \theta(x) \text{ and } \overline{\omega}(y) = (d_i)_{i\in I} = (\delta_i(y))_{i\in I} = \omega(y).$$

As it happens so for every $x \in X$ and for every $y \in Y$, then $\overline{\omega} = \omega$ and $\overline{\theta} = \theta$. Therefore, $(\theta, \omega) \in \mathrm{Mor}((X,h,Y),(C,g,D))$ is the unique morphism in **Seg**, which makes these two diagrams commutative, which completes the proof that (C,g,D) is the limit of the inverse system $(((A_i, f_i, B_i))_{i\in I}, (\psi_j^k, \phi_j^k)_{j<k})$. □

The next two corollaries are consequences of Theorem 1.

Corollary 2. *Let $(I, \preceq)$ be a partially ordered set, $((A_i, f_i, B_i)_{i\in I}, (\psi_j^k, \phi_j^k)_{j\preceq k})$ an inverse system in the category* **Seg** *and let C and D be the inverse limit algebras for the inverse systems $((A_i)_{i\in I}, (\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I}, (\phi_j^k)_{j\preceq k})$, respectively. Equip C and D with the subspace topology, inherited from the topological direct products $\prod_{i\in I} A_i$ and $\prod_{i\in I} B_i$, respectively, let g be the restriction of the map $f : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$, defined by $f((a_i)_{i\in I}) = (f_i(a_i))_{i\in I}$ for all $(a_i)_{i\in I} \in \prod_{i\in I} A_i$, to C, and $(\alpha_i : C \to A_i, \beta_i : D \to B_i)_{i\in I}$ be the family of maps, defined by $\alpha_j((a_i)_{i\in I}) = a_j$, $\beta_j((b_i)_{i\in I}) = b_j$ for all $j \in I$ and all $(a_i)_{i\in I} \in C, (b_i)_{i\in I} \in D$. If $g(C)$ is a dense ideal of D, then the limit of the inverse system $((A_i, f_i, B_i)_{i\in I}, (\psi_j^k, \phi_j^k)_{j\preceq k})$ in the category* **Seg** *exists and has the form $((C,g,D),(\alpha_i, \beta_i)_{i\in I})$.*

Proof. The statement follows directly from Lemma 1 and Theorem 1. □

Corollary 3. *If $(I,\preceq)$ is a partially ordered set, which has the greatest element, then the limit of the inverse system $((A_i,f_i,B_i)_{i\in I},(\psi_j^k,\phi_j^k)_{j\preceq k})$ in the category* **Seg** *exists and has the form $((C,g,D),(\alpha_i,\beta_i)_{i\in I})$, where C and D are the inverse limit algebras for the inverse systems $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, $((B_i)_{i\in I},(\phi_j^k)_{j\preceq k})$, equipped with the subspace topology, inherited from the topological direct products $\prod\limits_{i\in I}A_i$ and $\prod\limits_{i\in I}B_i$, respectively, g is the restriction of the map $f:\prod\limits_{i\in I}A_i\to\prod\limits_{i\in I}B_i$, defined by $f((a_i)_{i\in I})=(f_i(a_i))_{i\in I}$ for all $(a_i)_{i\in I}\in\prod\limits_{i\in I}A_i$, to C, and $(\alpha_i:C\to A_i,\beta_i:D\to B_i)_{i\in I}$ is the family of maps, defined by $\alpha_j((a_i)_{i\in I})=a_j$, $\beta_j((b_i)_{i\in I})=b_j$ for all $j\in I$, $(a_i)_{i\in I}\in C$ and $(b_i)_{i\in I}\in D$.*

Proof. By Corollary 1, we know that $(C,g,D)\in\mathrm{Ob}(\mathbf{Seg})$. The claim follows now from Theorem 1. □

5. APPLICATION OF COROLLARY 3 TO THE CATEGORY $\mathscr{S}(B)$

In [3] we studied the existence of limits in the category $\mathscr{S}(B)$ of Segal topological algebras. The results of the present paper enable us to have also some new results about the existence of limits in the category $\mathscr{S}(B)$. Let us first shortly remind the definition of the category $\mathscr{S}(B)$.

The set $\mathrm{Ob}(\mathscr{S}(B))$ of objects of the category $\mathscr{S}(B)$ consisted of all Segal topological algebras in a fixed topological algebra B, i.e., all Segal algebras in the form of triples $(A,f,B),(C,g,B),\ldots$.

The set $\mathrm{Mor}((A,f,B),(C,g,B))$ of morphisms between Segal topological algebras (A,f,B) and (C,g,B) consisted of all continuous algebra homomorphisms $\alpha:A\to C$, satisfying $g(\alpha(a))=(1_B\circ f)(a)=f(a)$ for every $a\in A$, i.e., making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle 1_B} \\ C & \xrightarrow{g} & B \end{array}$$

commutative. The composition of morphisms was defined componentwise.

In [3] we showed that all at most countable limits in $\mathscr{S}(B)$ existed in case when the algebra B is a Baire-like algebra for descending ideals (for the definition of this term, see [3], pp. 6–7). The results of the present paper allow us to prove the existence of some limits in $\mathscr{S}(B)$ also in case of non-countable families of objects of $\mathscr{S}(B)$. We will finish this paper by stating a result about the limits in $\mathscr{S}(B)$.

Corollary 4. *If $(I,\preceq)$ is a partially ordered set, which has the greatest element $i_g\in I$, then the limit of the inverse system $((A_i,f_i,B)_{i\in I},(\psi_j^k)_{j\preceq k})$ in the category $\mathscr{S}(B)$ exists and has the form $((C,g,B),(\alpha_i)_{i\in I})$, where C is the inverse limit algebra for the inverse system $((A_i)_{i\in I},(\psi_j^k)_{j\preceq k})$, equipped with the subspace topology, inherited from the topological direct product $\prod\limits_{i\in I}A_i$, g is the restriction of the map $f:\prod\limits_{i\in I}A_i\to B$, defined by $f((a_i)_{i\in I})=f_{i_g}(a_{i_g})$ for all $(a_i)_{i\in I}\in\prod\limits_{i\in I}A_i$, to C, and $(\alpha_i:C\to A_i)_{i\in I}$ is the family of maps, defined by $\alpha_j((a_i)_{i\in I})=a_j$ for all $j\in I$ and all $(a_i)_{i\in I}\in C$.*

Proof. The result is a special case of Corollary 3, where $B_i=B$ for each $i\in I$, $\phi_j^k=1_B$ for all $j,k\in I$ with $j\preceq k$, $D=B$ and $\beta_i=1_B$ for all $i\in I$. □

6. CONCLUSIONS

In this paper we showed that if the inverse system of Segal topological algebras is indexed by a partially ordered set, which has the greatest element, then the limit of this system exists both in the category **Seg** and in the category $\mathscr{S}(B)$. The explicit constructions of the limits in these categories are also provided.

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Piirid Segali topoloogiliste algebrate kategoorias Seg

Mart Abel

Käesolevas artiklis leitakse piisavad tingimused piiride leidumiseks Segali topoloogiliste algebrate kategooriates **Seg** ja $\mathscr{S}(B)$.