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TOPOLOGICAL ALGEBRAS

Coproducts in the category Seg of Segal topological algebras

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Abstract. In this paper we find a sufficient condition for a family of Segal topological algebras to have a coproduct in the category **Seg**.

Key words: Segal topological algebras, category, tensor product algebra, free product, coproduct.

1. INTRODUCTION

Let $\mathbb{K}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. By a *topological algebra* we will always mean a topological linear space over $\mathbb{K}$, where the separately continuous multiplication has been defined.

Recall that a topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f: A \to B$, if

(1) $\mathrm{cl}_B(f(A)) = B$;

(2) f is continuous;

(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In short, we will denote Segal topological algebra by a triple (A, f, B).

Let us briefly recall the definition of the category **Seg** of Segal topological algebras. Its objects are all left (right or two-sided) Segal topological algebras. For any $(A,f,B),(C,g,D) \in \mathrm{Ob}(\mathbf{Seg})$, the set $\mathrm{Mor}((A,f,B),(C,g,D))$ of morphisms from (A,f,B) to (C,g,D) consists of all such pairs (α,β) of continuous algebra homomorphisms $\alpha: A \to C$ and $\beta: B \to D$, for which $g \circ \alpha = \beta \circ f$, i.e. we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ C & \xrightarrow{g} & D \end{array}.$$

The composition of morphisms of **Seg** is defined componentwise as follows:
for any $(A,f,B),(C,g,D),(E,h,F) \in \mathrm{Ob}(\mathbf{Seg})$ and any morphisms $(\alpha,\beta): (A,f,B) \to (C,g,D)$, $(\gamma,\delta): (C,g,D) \to (E,h,F)$, the composition of (γ,δ) and (α,β) is $(\gamma,\delta)\circ(\alpha,\beta) = (\gamma\circ\alpha, \delta\circ\beta)$.

In [1], pp. 2–4, it was shown that this composition of morphisms is correctly defined and associative. Moreover, it was demonstrated that the identity morphism for an object (A, f, B) of **Seg** is a pair $(1_A, 1_B)$ of identity maps.

First categorical properties of the category **Seg** were studied in [3] and [4]. The paper [3] also provides some historical overview of Segal topological algebras.

The aim of this research is to study whether there exists a coproduct of a family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ of Segal topological algebras in the category Seg.

2. TENSOR PRODUCT ALGEBRA

Let Λ be an index set (which can be finite or infinite) and let $(A_\lambda, \tau_\lambda)_{\lambda\in\Lambda}$ be a family of topological algebras. Equip the direct product $\prod\limits_{\lambda\in\Lambda} A_\lambda$ with the box topology $\tau_{\prod\limits_{\lambda\in\Lambda} A_\lambda}$, the base of which consists of sets in the form $\{\prod\limits_{\lambda\in\Lambda} U_\lambda : U_\lambda \in \tau_\lambda\}$.

Then we can consider the topological tensor product algebra $(\bigotimes\limits_{\lambda\in\Lambda} A_\lambda, \tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda})$, where the topology $\tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda}$ is the topology in which the map $l : \prod\limits_{\lambda\in\Lambda} A_\lambda \to \bigotimes\limits_{\lambda\in\Lambda} A_\lambda$, defined by $l(\prod\limits_{\lambda\in\Lambda} a_\lambda) = \bigotimes\limits_{\lambda\in\Lambda} a_\lambda$ for each $\prod\limits_{\lambda\in\Lambda} a_\lambda \in \prod\limits_{\lambda\in\Lambda} A_\lambda$, is continuous. This means that $\tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda} = \{l(W) : W \in \tau_{\prod\limits_{\lambda\in\Lambda} A_\lambda}\}$. In this topology on the tensor product, for each neighbourhood O of zero in $\bigotimes\limits_{\lambda\in\Lambda} A_\lambda$, there exist neighbourhoods $(O_\lambda)_{\lambda\in\Lambda}$ of zero in algebras $(A_\lambda)_{\lambda\in\Lambda}$, such that $\bigotimes\limits_{\lambda\in\Lambda} O_\lambda \subseteq O$. The topology $\tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda}$ is called the *tensor product topology* on $\bigotimes\limits_{\lambda\in\Lambda} A_\lambda, \tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda}$.

Notice that the general form of an element a of $\bigotimes\limits_{\lambda\in\Lambda} A_\Lambda$ is $a = \sum\limits_{i=1}^{k} \bigotimes\limits_{\lambda\in\Lambda} a_{(\lambda,i)}$, where $k \in \mathbb{Z}^+$, i.e. every element of the tensor product is a finite sum of simple tensors $\bigotimes\limits_{\lambda\in\Lambda} a_\lambda$.

We start this paper with a result about the density of images of maps between tensor products.

Lemma 1. *Let Λ be an index set, $(A_\lambda, \tau_\lambda)_{\lambda\in\Lambda}$, $(B_\lambda, \sigma_\lambda)_{\lambda\in\Lambda}$ two families of topological algebras and $(f_\lambda : A_\lambda \to B_\lambda)_{\lambda\in\Lambda}$ a family of maps. Let $(\bigotimes\limits_{\lambda\in\Lambda} A_\lambda, \tau_{\bigotimes\limits_{\lambda\in\Lambda} A_\lambda}), (\bigotimes\limits_{\lambda\in\Lambda} B_\lambda, \tau_{\bigotimes\limits_{\lambda\in\Lambda} B_\lambda})$ be the respective topological tensor product algebras and $f : \bigotimes\limits_{\lambda\in\Lambda} A_\lambda \to \bigotimes\limits_{\lambda\in\Lambda} B_\lambda$ be a map, which is given by*

$$f\left(\sum_{i=1}^{k} \bigotimes_{\lambda\in\Lambda} a_{(\lambda,i)}\right) = \sum_{i=1}^{k} \bigotimes_{\lambda\in\Lambda} f_\lambda(a_{(\lambda,i)}) \text{ for each } \sum_{i=1}^{k} \bigotimes_{\lambda\in\Lambda} a_{(\lambda,i)} \in \bigotimes_{\lambda\in\Lambda} A_\lambda.$$

If $f_\lambda(A_\lambda)$ is dense in B_λ for each $\lambda \in \Lambda$, then the set $f(\bigotimes\limits_{\lambda\in\Lambda} A_\lambda)$ is dense in $\bigotimes\limits_{\lambda\in\Lambda} B_\lambda$.

Proof. Take any $b \in \bigotimes\limits_{\lambda\in\Lambda} B_\lambda$. Then there exist $k \in \mathbb{Z}^+$, and for each $\lambda \in \Lambda$, elements $b_{(\lambda,1)}, \dots, b_{(\lambda,k)}$ such that $b = \sum\limits_{i=1}^{k} \bigotimes\limits_{\lambda\in\Lambda} b_{(\lambda,i)}$. Set $K = \{(\lambda, i) : \lambda \in \Lambda, i \in \{1, \dots, k\}\}$ and let U be any neighbourhood of b in $\bigotimes\limits_{\lambda\in\Lambda} B_\lambda$. Then there exists a neighbourhood O of zero in $\bigotimes\limits_{\lambda\in\Lambda} B_\lambda$ such that $b + O \subseteq U$. As the addition is continuous in $\bigotimes\limits_{\lambda\in\Lambda} B_\lambda$, then there exists a neighbourhood V of zero in $\bigotimes\limits_{\lambda\in\Lambda} B_\lambda$ such that $\underbrace{V + \cdots + V}_{k \text{ times}} \subseteq O$.

Now, for each $\lambda \in \Lambda$, there exists a neighbourhood V_λ of zero in B_λ such that $\bigotimes\limits_{\lambda\in\Lambda} V_\lambda \subseteq V$, and for every $(\lambda, i) \in K$, $b_{(\lambda,i)} + V_\lambda \in b_{(\lambda,i)} + \bigotimes\limits_{\lambda\in\Lambda} V_\lambda$. As the general element of a tensor product is some finite sum of simple tensors, then it is clear that, for each $i \in \{1, \dots, k\}$, we have

$$\bigotimes_{\lambda\in\Lambda} (b_{(\lambda,i)} + V_\lambda) \subseteq \bigotimes_{\lambda\in\Lambda} (b_{(\lambda,i)} + \bigotimes_{\lambda\in\Lambda} V_\lambda) \subseteq \bigotimes_{\lambda\in\Lambda} b_{(\lambda,i)} + \bigotimes_{\lambda\in\Lambda} V_\lambda.$$

For each $(\lambda, i) \in K$, set $U_{(\lambda,i)} = b_{(\lambda,i)} + V_\lambda$. Then, for each $(\lambda, i) \in K$, $U_{(\lambda,i)}$ is a neighbourhood of $b_{(\lambda,i)}$ and

$$\sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} U_{(\lambda,i)} \subseteq \sum_{i=1}^{k} \left(\underset{\lambda\in\Lambda}{\otimes} b_{(\lambda,i)} + \underset{\lambda\in\Lambda}{\otimes} V_\lambda \right) = \sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} b_{(\lambda,i)} + \sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} V_\lambda \subseteq b + \sum_{i=1}^{k} V \subseteq b + O \subseteq U.$$

Since $f_\lambda(A_\lambda)$ is dense in B_λ for each $\lambda \in \Lambda$, then there exist partially ordered sets $(I_\lambda, \succ_\lambda)_{\lambda\in\Lambda}$, and for each $(\lambda, i) \in K$, the family $(a_{\zeta_{(\lambda,i)}})_{\zeta_{(\lambda,i)}\in I_\lambda}$ of elements of A_λ such that $(f(a_{\zeta_{(\lambda,i)}}))_{\zeta_{(\lambda,i)}\in I_\lambda}$ converges to $b_{(\lambda,i)}$. This means that, for every $(\lambda, i) \in K$, there exists an element $\eta_{(\lambda,i)} \in I_\lambda$ such that from $\zeta_{(\lambda,i)} \succ_\lambda \eta_{(\lambda,i)}$ it follows that $f_\lambda(a_{\zeta_{(\lambda,i)}}) \in U_{(\lambda,i)}$.

Define the multi-index set $\prod\limits_{\lambda\in\Lambda} I_\lambda$ and consider on it the partial order $\succ$ defined by $(\phi_{(\lambda,i)})_{\lambda\in\lambda} \succ (\psi_{(\lambda,i)})_{\lambda\in\Lambda}$ if and only if $\phi_{(\lambda,i)} \succ_\lambda \psi_{(\lambda,i)}$ for each $\lambda \in \Lambda$. Then $(\prod\limits_{\lambda\in\Lambda} I_\lambda, \succ)$ becomes a partially ordered set of multi-indices.

Take any $(a_{\zeta_{(\lambda,i)}})_{\lambda\in\Lambda} \in \underset{\lambda\in\Lambda}{\otimes} A_\lambda$ with $(\zeta_{(\lambda,i)})_{\lambda\in\Lambda} \succ (\eta_{(\lambda,i)})_{\lambda\in\Lambda}$ and $i \in \{1, \ldots, k\}$ fixed. Then $\zeta_{(\lambda,i)} \succ_\lambda \eta_{(\lambda,i)}$ for each $\lambda \in \Lambda$ and we have that $f_\lambda(a_{\zeta_{(\lambda,i)}}) \in U_{(\lambda,i)}$. This means that

$$f\left(\sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} a_{\zeta_{(\lambda,i)}} \right) = \sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} f_\lambda(a_{\zeta_{(\lambda,i)}}) \in \sum_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} U_{(\lambda,i)} \subseteq U$$

for all $(\zeta_{(\lambda,i)})_{\lambda\in\Lambda} \in \prod\limits_{\lambda\in\Lambda} I_\lambda$ with $(\zeta_{(\lambda,i)})_{\lambda\in\Lambda} \succ (\eta_{(\lambda,i)})_{\lambda\in\Lambda}$. Hence, the family $(f(\sum\limits_{i=1}^{k} \underset{\lambda\in\Lambda}{\otimes} a_{\zeta_{(\lambda,i)}}))_{(\zeta_{(\lambda,i)})_{\lambda\in\Lambda}\in \prod\limits_{\lambda\in\Lambda} I_\lambda}$ converges to b.

As b is an arbitrary element of $\underset{\lambda\in\Lambda}{\otimes} B_\lambda$, then the set $f(\underset{\lambda\in\Lambda}{\otimes} A_\lambda)$ is dense in $\underset{\lambda\in\Lambda}{\otimes} B_\lambda$. □

Remark 1. Notice that Lemma 1 is also true in case we have families $(A_\lambda, \tau_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda, \sigma_\lambda)_{\lambda\in\Lambda}$ of topological linear spaces instead of topological algebras. Moreover, the map f, given in Lemma 1, is continuous, and if all the maps $(f_\lambda)_{\lambda\in\Lambda}$ are algebra homomorphisms, then the map f is also an algebra homomorphism.

3. SOME PROPERTIES OF THE FREE PRODUCT OF ALGEBRAS

Remember (see [2], p. 203) that for a collection $(A_\lambda)_{\lambda\in\Lambda}$ of algebras, their tensor algebra is an algebra

$$T = \left(\bigoplus_{\lambda\in\Lambda} A_\lambda \right) \oplus \left(\bigoplus_{\lambda,\mu\in\Lambda} (A_\lambda \otimes A_\mu) \right) \oplus \left(\bigoplus_{\lambda,\mu,\nu\in\Lambda} (A_\lambda \otimes A_\mu \otimes A_\nu) \right) \oplus \ldots$$

and every element $t \in T$ is in the form

$$t = \bigoplus_{l=1}^{k} \left(\bigoplus_{m=1}^{p_l} \left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l} \right) \right)$$

for some $k, p_l, r_{m,l} \in \mathbb{Z}^+$ and $t_{q,m,1}, \ldots, t_{q,m,i_l} \in \bigcup_{\lambda\in\Lambda} A_\lambda$.

In [2], pp. 203–205, we defined the algebraic operations in T as follows. If $\rho \in \mathbb{K}$,

$$t = \bigoplus_{l=1}^{k_t} \left(\bigoplus_{m=1}^{p_l} \left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l} \right) \right) \in T$$

and

$$s=\bigoplus_{f=1}^{k_s}\left(\bigoplus_{g=1}^{u_f}\left(\sum_{h=1}^{v_{g,f}} s_{h,g,1}\otimes\ldots\otimes s_{h,g,j_f}\right)\right)\in T,$$

then

$$\rho t=\bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}(\rho t_{q,m,1})\otimes\ldots\otimes t_{q,m,i_l}\right)\right),$$

$$t+s=\bigoplus_{l=1}^{k_t+k_s}\left(\bigoplus_{m=1}^{w_l}\left(\sum_{q=1}^{x_{m,l}} z_{q,m,1}\otimes\ldots\otimes z_{q,m,L_l}\right)\right),$$

where

$$L_l=\begin{cases} i_l, & \text{if } 1\leqslant l\leqslant k_t\\ j_{l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases},\quad w_l=\begin{cases} p_l, & \text{if } 1\leqslant l\leqslant k_t\\ u_{l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}, \tag{3.1}$$

$$x_{m,l}=\begin{cases} r_{m,l}, & \text{if } 1\leqslant l\leqslant k_t\\ v_{m,l-k_t}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}\quad\text{and}\quad z_{q,m,d}=\begin{cases} t_{q,m,d}, & \text{if } 1\leqslant l\leqslant k_t\\ s_{q,m,d}, & \text{if } k_t<l\leqslant k_t+k_s\end{cases}. \tag{3.2}$$

The multiplication of elements had to satisfy the rule

$$t\cdot s=\bigoplus_{\varepsilon=1}^{k_tk_s}\bigoplus_{\delta=1}^{p_{X_1}u_{X_2}}\sum_{y=1}^{r_{X_3,X_1}v_{X_4,X_2}}\left(\bigotimes_{u=1}^{i_{X_1}} t_{X_5,X_3,u}\otimes\bigotimes_{d=1}^{j_{X_2}} s_{X_6,X_4,d}\right),$$

where

$$X_1=\left\lfloor\frac{\varepsilon-1}{k_s}\right\rfloor+1,\;\; X_2=\varepsilon-X_1k_s=\varepsilon-\left\lfloor\frac{\varepsilon-1}{k_s}\right\rfloor k_s,$$

$$X_3=\left\lfloor\frac{\delta-1}{p_{X_1}}\right\rfloor+1=\left\lfloor\frac{\delta-1}{p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1}}\right\rfloor+1,\;\; X_4=\delta-X_3p_{X_1}=\delta-\left\lfloor\frac{\delta-1}{p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1}}\right\rfloor p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1},$$

$$X_5=\left\lfloor\frac{y-1}{v_{X_4,X_2}}\right\rfloor+1=\left\lfloor\frac{y-1}{v_{\delta-\left\lfloor\frac{\delta-1}{p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1}}\right\rfloor p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1},\,\varepsilon-\lfloor\frac{\varepsilon-1}{k_s}\rfloor k_s}}\right\rfloor+1$$

and

$$X_6=y-(X_5-1)v_{X_4,X_2}=y-\left\lfloor\frac{y-1}{v_{X_4,X_2}}\right\rfloor+1$$

$$=y-\left\lfloor\frac{y-1}{v_{\delta-\left\lfloor\frac{\delta-1}{p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1}}\right\rfloor p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1},\,\varepsilon-\lfloor\frac{\varepsilon-1}{k_s}\rfloor k_s}}\right\rfloor v_{\delta-\left\lfloor\frac{\delta-1}{p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1}}\right\rfloor p_{\lfloor\frac{\varepsilon-1}{k_s}\rfloor+1},\,\varepsilon-\lfloor\frac{\varepsilon-1}{k_s}\rfloor k_s}.$$

Suppose that we have two collections of algebras, $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$, indexed by the same set Λ. We can consider the algebras $(A_\lambda)_{\lambda\in\Lambda}$ disjoint by setting $a=(a,\lambda)$ for every $a\in A_\lambda$. Similarily, we can consider the algebras $(B_\lambda)_{\lambda\in\Lambda}$ disjoint. We need the disjointness of these families of algebras in order to be able to choose for every $a\in\bigcup\limits_{\lambda\in\Lambda}A_\lambda$ and every $b\in\bigcup\limits_{\lambda\in\Lambda}B_\lambda$ unique indices $\lambda_a\in\Lambda$ and $\lambda_b\in\Lambda$ such that $a\in A_{\lambda_a}$ and $b\in B_{\lambda_b}$. Thus, in what follows, for $\lambda,\mu\in\Lambda$ with $\lambda\neq\mu$ we have $A_\lambda\cap A_\mu=\emptyset=B_\lambda\cap B_\mu$. Morever, for any $a \in \bigcup_{\lambda\in\Lambda} A_\lambda$ and every $b \in \bigcup_{\lambda\in\Lambda} B_\lambda$ we will denote by λ_a the unique index from Λ such that $a \in A_{\lambda_a}$ and by λ_b the unique index from Λ such that $b \in B_{\lambda_b}$. Notice that in some places we need to write μ_a instead of λ_a and μ_b instead of λ_b.

Let T be the tensor algebra of algebras $(A_\lambda)_{\lambda\in\Lambda}$ and S the tensor algebra of algebras $(B_\lambda)_{\lambda\in\Lambda}$.

Suppose that there are also algebra homomorphisms $f_\lambda : A_\lambda \to B_\lambda$ for all $\lambda \in \Lambda$. Define a map $\widetilde{h_T} : \bigcup_{\lambda\in\Lambda} A_\lambda \to \bigcup_{\lambda\in\Lambda} B_\lambda$ by $\widetilde{h_T}(a) = f_{\lambda_a}(a)$. Now, define a map $h_T : T \to S$ by setting

$$h_T(t) = \bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}} \widetilde{h_T}(t_{q,m,1}) \otimes \cdots \otimes \widetilde{h_T}(t_{q,m,i_l})$$

for every element

$$t = \bigoplus_{l=1}^{k}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right)$$

of T. Modifying the ideas of [2], pp. 208–209, we can show that h_T is an algebra homomorphism. Indeed, using the symbols given in (3.1)−(3.2), we obtain that for $\rho \in \mathbb{K}$,

$$t = \bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right) \in T$$

and

$$s = \bigoplus_{f=1}^{k_s}\left(\bigoplus_{g=1}^{u_f}\left(\sum_{h=1}^{v_{g,f}} s_{h,g,1} \otimes \ldots \otimes s_{h,g,j_f}\right)\right) \in T,$$

and we have

$$\begin{aligned} h_T(t) + h_T(s) &= \bigoplus_{l=1}^{k_t}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}}\bigotimes_{u=1}^{i_l}\widetilde{h_T}(t_{q,m,u}) + \bigoplus_{f=1}^{k_s}\bigoplus_{g=1}^{u_f}\sum_{h=1}^{v_{g,f}}\bigotimes_{v=1}^{j_f}\widetilde{h_T}(s_{h,g,v}) \\ &= \bigoplus_{l=1}^{k_t+k_s}\bigoplus_{m=1}^{w_l}\sum_{q=1}^{x_{m,l}}\bigotimes_{d=1}^{L_l}\widetilde{h_T}(z_{q,m,d}) \\ &= h_T\left(\bigoplus_{l=1}^{k_t+k_s}\left(\bigoplus_{m=1}^{w_l}\left(\sum_{q=1}^{x_{m,l}} z_{q,m,1} \otimes \ldots \otimes z_{q,m,L_l}\right)\right)\right) = h_T(t+s), \end{aligned}$$

$$h_T(\rho t) = \bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}} \widetilde{h_T}(\rho t_{q,m,1}) \otimes \ldots \otimes \widetilde{h_T}(t_{q,m,i_l})$$

$$= \bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}} (\rho\widetilde{h_T}(t_{q,m,1})) \otimes \ldots \otimes \widetilde{h_T}(t_{q,m,i_l}) = \rho\left(\bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}}\bigotimes_{u=1}^{i_l}\widetilde{h_T}(t_{q,m,u})\right) = \rho h_T(t)$$

and

$$h_T(t \cdot s) = h_T\left(\bigoplus_{l=1}^{k_t}\bigoplus_{f=1}^{k_s}\left(\bigoplus_{m=1}^{p_l}\bigoplus_{g=1}^{u_f}\sum_{y=1}^{r_{m,l}v_{g,f}}\bigotimes_{u=1}^{i_l} t_{\left\lfloor\frac{y-1}{v_{g,f}}\right\rfloor+1,m,u} \otimes \bigotimes_{d=1}^{j_f} s_{y-\left\lfloor\frac{y-1}{v_{g,f}}\right\rfloor v_{g,f},g,d}\right)\right)$$

$$= \bigoplus_{l=1}^{k_t}\bigoplus_{f=1}^{k_s}\left(\bigoplus_{m=1}^{p_l}\bigoplus_{g=1}^{u_f}\sum_{y=1}^{r_{m,l}v_{g,f}}\bigotimes_{u=1}^{i_l}\widetilde{h_T}\left(t_{\left\lfloor\frac{y-1}{v_{g,f}}\right\rfloor+1,m,u}\right) \otimes \bigotimes_{d=1}^{j_f}\widetilde{h_T}\left(s_{y-\left\lfloor\frac{y-1}{v_{g,f}}\right\rfloor v_{g,f},g,d}\right)\right)$$

$$= \left(\bigoplus_{l=1}^{k_t}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}}\bigotimes_{u=1}^{i_l}\widetilde{h_T}(t_{q,m,u})\right)\cdot\left(\bigoplus_{f=1}^{k_s}\bigoplus_{g=1}^{u_f}\sum_{h=1}^{v_{g,f}}\bigotimes_{d=1}^{j_f}\widetilde{h_T}(s_{h,g,d})\right) = h_T(t)\cdot h_T(s).$$

Hence, h_T is indeed an algebra homomorphism.

Suppose that, for every $\lambda \in \Lambda$, $f_\lambda(A_\lambda)$ is a left (right or two-sided) ideal of B_λ. It is natural to ask whether it is then true that $h_T(T)$ is a left (right or two-sided) ideal of S. Actually, we will show that the answer to the question "Whether $h_T(T)$ is a left (right or two-sided) ideal of S" does not depend on the fact whether $f_\lambda(A_\lambda)$ is or is not a left (right or two-sided) ideal of B_λ for every $\lambda \in \Lambda$.

As h is an algebra homomorphism, then $\rho h(t) = h(\rho t) \in h(T)$ and $h(t)+h(s) = h(t+s) \in h(T)$ for every $t,s \in T$ and every $\rho \in \mathbb{K}$. What concerns the multiplication of elements of $h_T(T)$ with elements of S, then it is not always true that $v \cdot h_T(t), h_T(t)\cdot v \in h_T(T)$ for arbitrary $t \in T$ and $v \in S$.

Indeed, suppose that there exist $\lambda_0, \lambda_1 \in \Lambda$ such that A_{λ_0} is a proper subalgebra[1] of B_{λ_0}, f_{λ_0} is the identity map on A_{λ_0} (i.e. f_{λ_0} is an inclusion), $A_{\lambda_1} = B_{\lambda_1} = \mathbb{K}$, where B_{λ_0} is an algebra over the field $\mathbb{K}$ and f_{λ_1} is the identity map on $\mathbb{K}$.

As A_{λ_0} is a proper subalgebra of B_{λ_0}, then there exists $b \in B_{\lambda_0}$ such that $b \notin A_{\lambda_0}$. Now, take the unit element $e_\mathbb{K}$ of the field $\mathbb{K}$. Then $e_\mathbb{K} \in A_{\lambda_1} \subset T$. Hence, $f_{\lambda_1}(e_\mathbb{K}) = e_\mathbb{K} \in h_T(T)$ and $b \in B_{\lambda_0} \subset S$. Therefore, we can consider the product $b\cdot e_\mathbb{K} = b\otimes e_\mathbb{K} \in Sh_T(T) \subset S$. As the algebras $(B_\lambda)_{\lambda\in\Lambda}$ are considered pairwise disjoint, then we obtain $b \otimes e_\mathbb{K} \in B_{\lambda_0}\otimes B_{\lambda_1}$.

Suppose that $b\otimes e_\mathbb{K} \in h_T(T)$. Then $b\otimes e_\mathbb{K} \in f_{\lambda_0}(A_{\lambda_0})\otimes f_{\lambda_1}(A_{\lambda_1})$. Hence, there exist $m \in \mathbb{Z}^+$ and elements $b_1,\dots,b_m \in A_{\lambda_0}$, $k_1,\dots,k_m \in A_{\lambda_1} = \mathbb{K}$ such that $b\otimes e_\mathbb{K} = \sum_{i=1}^m b_i\otimes k_i$. Thus, for every bilinear map $g : B_{\lambda_0}\otimes B_{\lambda_1} \to B_{\lambda_0}$, we must have $g(b\otimes e_\mathbb{K}) = g(\sum_{i=1}^m b_i \otimes k_i)$.

Let $g : B_{\lambda_0}\otimes B_{\lambda_1} \to B_{\lambda_0}$ be a map, for which $g(\sum_{j=1}^n c_j\otimes l_j) = \sum_{j=1}^n l_j c_j$ for every $\sum_{j=1}^n c_j \otimes l_j \in B_{\lambda_0}\otimes B_{\lambda_1}$. Then it is easy to see that g is well defined and is a bilinear map. Moreover, $g(b\otimes e_\mathbb{K}) = b$ and $g(\sum_{i=1}^m b_i\otimes k_i) = \sum_{i=1}^m k_i b_i$. As A_{λ_0} is a subalgebra of B_{λ_0}, then $\sum_{i=1}^m k_i b_i \in A_{\lambda_0}$, while $b \notin A_{\lambda_0}$. Hence, $g(b\otimes e_\mathbb{K}) \neq g(\sum_{i=1}^m b_i\otimes k_i)$. This is a contradiction, which shows that $b\otimes e_\mathbb{K} \notin h(T)$. Therefore, $S\cdot h_T(T) \not\subset h_T(T)$.

Similarly, we can show that $h_T(T)\cdot S \not\subset h_T(T)$ in general. Thus, we have shown that $h_T(T)$ is not always a left (right or two-sided) ideal of S.

With that we have given a proof (in case of left ideals, the other cases are similar) of the following Lemma.

Lemma 2. *Let $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ be two collections of disjoint algebras indexed by the same set Λ. Let $(f_\lambda : A_\lambda \to B_\lambda)_{\lambda\in\Lambda}$ be a collection of algebra homomorphisms, T be the tensor algebra of algebras $(A_\lambda)_{\lambda\in\Lambda}$ and S the tensor algebra of algebras $(B_\lambda)_{\lambda\in\Lambda}$. Let $\widetilde{h_T} : \bigcup_{\lambda\in\Lambda} A_\lambda \to \bigcup_{\lambda\in\Lambda} B_\lambda$ be the map, defined by $\widetilde{h_T}(a) = f_{\lambda_a}(a)$, where $\lambda_a \in \Lambda$ is the unique index such that $a \in A_{\lambda_a}$. Let $h_T : T \to S$ be the map, defined by*

$$h_T(t) = \bigoplus_{l=1}^{k}\bigoplus_{m=1}^{p_l}\sum_{q=1}^{r_{m,l}}\widetilde{h_T}(t_{q,m,1})\otimes\cdots\otimes\widetilde{h_T}(t_{q,m,i_l})$$

for every element

$$t = \bigoplus_{l=1}^{k}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1}\otimes\ldots\otimes t_{q,m,i_l}\right)\right)$$

of T. Then $h_T(T)$ is a left (right or two-sided) ideal of S if and only if $S\cdot h_T(T) \subseteq h_T(T)$ (respectively, $h_T(T)\cdot S \subseteq h_T(T)$ or $S\cdot h_T(T)\cdot S \subseteq h_T(T)$).

[1] This situation is possible, for example, when B_{λ_0} is a topological algebra, which has a maximal ideal A_{λ_0} that is not closed in the topology of B_{λ_0}.

Consider the two-sided ideals I of T and J of S, generated by the sets

$$\{x \otimes y - xy : x, y \in A_\lambda, \lambda \in \Lambda\} \quad \text{and} \quad \{z \otimes w - zw : z, w \in B_\lambda, \lambda \in \Lambda\},$$

respectively. As h_T is an algebra homomorphism, then, for every fixed $\lambda \in \Lambda$ and $x, y \in A_\lambda$, we have

$$\begin{aligned} h_T(x \otimes y - xy) &= h_T(x \otimes y) - h_T(xy) = \widetilde{h_T}(x) \otimes \widetilde{h_T}(y) - \widetilde{h_T}(xy) \\ &= f_\lambda(x) \otimes f_\lambda(y) - f_\lambda(xy) = f_\lambda(x) \otimes f_\lambda(y) - f_\lambda(x) f_\lambda(y) \in J, \end{aligned}$$

which means that $h_T(I) \subseteq J$.

Consider the free product T/I of algebras $(A_\lambda)_{\lambda\in\Lambda}$ and the free product S/J of algebras $(B_\lambda)_{\lambda\in\Lambda}$.

Let

$$\kappa_I : T \to T/I, \quad \kappa_J : S \to S/J$$

be the respective quotient maps. Define a map $h : T/I \to S/J$ by $h(\kappa_I(t)) = \kappa_J(h_T(t))$ for every $t \in T$. This map is well defined because $h_T(I) \subseteq J$. Moreover, h is an algebra homomorphism because the maps h_T, κ_I and κ_J are algebra homomorphisms.

Lemma 3. *Let $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ be two collections of disjoint algebras indexed by the same set, $(f_\lambda : A_\lambda \to B_\lambda)_{\lambda\in\Lambda}$ a collection of algebra homomorphisms, T the tensor algebra of algebras $(A_\lambda)_{\lambda\in\Lambda}$ and S the tensor algebra of algebras $(B_\lambda)_{\lambda\in\Lambda}$. Consider the two-sided ideals I of T and J of S, generated by the sets*

$$\{x \otimes y - xy : x, y \in A_\lambda, \lambda \in \Lambda\} \quad \textit{and} \quad \{z \otimes w - zw : z, w \in B_\lambda, \lambda \in \Lambda\},$$

respectively, the free product T/I of algebras $(A_\lambda)_{\lambda\in\Lambda}$ and the free product S/J of algebras $(B_\lambda)_{\lambda\in\Lambda}$. Define a map $h : T/I \to S/J$ by $h(\kappa_I(t)) = \kappa_J(h_T(t))$ for every $t \in T$, where h_T is defined as in Lemma 2. If $S \cdot h_T(T) \subseteq h_T(T)$ ($h_T(T) \cdot S \subseteq h_T(T)$ or $S \cdot h_T(T) \cdot S \subseteq h_T(T)$), then $h(T/I)$ is a left (respectively, right or two-sided) ideal of S/J.

Proof. We will prove the claim for left ideals. The other cases are similar.

As h is an algebra homomorphism and T/I is an algebra, then $h(T/I) + h(T/I) \in h(T/I)$ and $\lambda h(T/I) \subseteq h(T/I)$ for every $\lambda \in \mathbb{K}$.

Take any $a \in h(T/I)$ and any $b \in S/J$. Then $a \in h(\kappa_I(T)) = \kappa_J(h_T(T))$ and $b \in \kappa_J(S)$. As $S \cdot h_T(T) \subseteq h_T(T)$ and κ_J is an algebra homomorphism, then

$$b \cdot a \in \kappa_J(S) \cdot \kappa_J(h_T(T)) \subseteq \kappa_J(S \cdot h_T(T)) \subseteq \kappa_J(h_T(T)) = h(\kappa_I(T)) = h(T/I).$$

With that we have proved that $S/J \cdot h(T/I) \subseteq h(T/I)$, i.e. that $h(T/I)$ is a left ideal of S/J. □

Open question 1. Is the condition $S \cdot h_T(T) \subseteq h_T(T)$ ($h_T(T) \cdot S \subseteq h_T(T)$ or $S \cdot h_T(T) \cdot S \subseteq h_T(T)$) necessary for $h(T/I)$ to be a left (respectively, right or two-sided) ideal of S/J?

4. SOME PROPERTIES OF TENSOR ALGEBRA OF TOPOLOGICAL ALGEBRAS

Let $(i_\mu : A_\mu \to T)_{\mu\in\Lambda}$ be a family of inclusion maps sending elements of A_μ into the direct summand A_μ of T, respectively, i.e. $i_\mu(a) = a \in A_\mu \subset T$ for every $a \in A_\mu$ and every $\mu \in \Lambda$. Then the map i_μ is an algebra homomorphism for every $\mu \in \Lambda$. Moreover, the quotient map κ_I is an algebra homomorphism. Hence, all maps of the family $(\alpha_\mu = \kappa_I \circ i_\mu : A_\mu \to T/I)_{\mu\in\Lambda}$ are algebra homomorphisms.

Similarly, let $(j_\mu : B_\mu \to S)_{\mu\in\Lambda}$ be a family of inclusion maps, which are also algebra homomorphisms, and $(\beta_\mu = \kappa_J \circ j_\mu : B_\mu \to S/J)_{\mu\in\Lambda}$ be respective algebra homomorphisms. Notice that $h \circ \alpha_\lambda = \beta_\lambda \circ f_\lambda$ for each $\lambda \in \Lambda$. Indeed, fix any $\lambda \in \Lambda$ and take $a \in A_\lambda$. Then

$$(h\circ\alpha_\lambda)(a) = h(\kappa_I(i_\lambda(a))) = h(\kappa_I(a)) = \kappa_J(h_T(a)) = \kappa_J(f_\lambda(a))$$
$$= \kappa_J(j_\lambda(f_\lambda(a))) = ((\kappa_J\circ j_\lambda)\circ f_\lambda)(a) = (\beta_\lambda\circ f_\lambda)(a).$$

If all algebras $(A_\lambda)_{\lambda\in\Lambda}$ are topological algebras, set

$$F = \Big\{ v : T/I \to C : C \text{ is a topological algebra}, v \text{ is an algebra homomorphism such that } v\circ\alpha_\mu \text{ is continuous for each } \mu\in\Lambda \Big\}.$$

On the tensor algebra T, consider the direct sum topology

$$\tau_T = \{O \subseteq \underset{i\in\mathbb{Z}^+}{\oplus} X_i : f_i^{-1}(O)\in\tau_i \text{ for each } i\in\mathbb{Z}^+\},$$

where

$$X_i = \underset{\lambda_1,\ldots,\lambda_i\in\Lambda}{\oplus}(A_{\lambda_1}\otimes\cdots\otimes A_{\lambda_i})$$

and τ_i is the tensor product topology on X_i. It is known that the topology τ_T is the final topology defined by the inclusion maps $f_i : X_i \to T$. Hence, all inclusion maps are continuous in the topology τ_T. The topology τ_T on tensor algebra T is also called the *tensor algebra topology*.

Equip T/I with the topology $\tau_{\underset{\lambda\in\Lambda}{\sqcup} A_\lambda}$, in which all maps $v\in F$ are continuous. Then $(T/I, \tau_{\underset{\lambda\in\Lambda}{\sqcup} A_\lambda})$ is a topological algebra (see [2], pp. 210–212).

If all algebras $(B_\lambda)_{\lambda\in\Lambda}$ are topological algebras, we consider on S the tensor algebra topology τ_S and take the quotient topology

$$\tau_{S/J} = \{U\subseteq S/J : \{s\in S, \kappa_J(s)\in U\}\in\tau_S\}$$

on S/J. Then the quotient algebra $(S/J, \tau_{S/J})$ is a topological algebra and $\kappa_J : S\to S/J$ is a continuous map. Since the inclusion map j_μ is continuous with respect to the topology τ_S, then $\beta_\mu = \kappa_J\circ f_\mu$ is also continuous for each $\mu\in\Lambda$.

Suppose now that the maps $(f_\lambda)_{\lambda\in\Lambda}$ are also continuous. With respect to topologies $\tau_{\sqcup_{\lambda\in\Lambda}A_\lambda}$ and $\tau_{S/J}$, the map h becomes continuous, because from the fact that $h\circ\alpha_\lambda = \kappa_J\circ f_\lambda$ is a continuous map for each $\lambda\in\Lambda$, it follows that $h\in F$.

Using the symbols defined above, we obtain another result.

Proposition 1. *Let T and S be tensor algebras of two collections of topological algebras, $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$, indexed by the same set Λ, respectively, and let I and J be the two-sided ideals of T and S, generated by the sets*

$$\{x\otimes y - xy : x,y\in A_\lambda, \lambda\in\Lambda\} \quad \textit{and} \quad \{z\otimes w - zw : z,w\in B_\lambda, \lambda\in\Lambda\},$$

respectively. Suppose that there are also maps $f_\lambda : A_\lambda\to B_\lambda$ for all $\lambda\in\Lambda$ such that $f_\lambda(A_\lambda)$ is dense in B_λ for all $\lambda\in\Lambda$. Then $h(T/I)$ is also dense in S/J.

Proof. Take any $w\in S/J$ and any neighbourhood W of w in S/J. Then there exist some element

$$v = \bigoplus_{l=1}^{k_v}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} v_{q,m,1}\otimes\ldots\otimes v_{q,m,i_l}\right)\right)\in S$$

and a neighbourhood V of v in S such that $w=\kappa_J(v)$ and $\kappa_J(V)\subseteq W$. Let

$$K = \{\mu = (\kappa,\nu,\rho) : l\in\{1,\ldots,k_v\}, \nu\in\{1,\ldots,p_l\}, \kappa\in\{1,r_{\nu,l}\}, \rho\in\{1,\ldots,i_l\}\}.$$

Notice that the set K is a finite set. Now, for every $\mu \in K$, there exists unique $\lambda_\mu = \lambda_{\nu_\mu} \in \Lambda$ such that $v_\mu := v_{\kappa,\nu,\rho} \in B_{\lambda_\mu}$. Similarly to the proof of Lemma 1, we can find for each $\mu \in K$ a neighbourhood V_{λ_μ} of v_μ in B_{λ_μ} such that

$$\bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} V_{\lambda_{(q,m,1)}} \otimes \ldots \otimes V_{\lambda_{(q,m,i_l)}}\Big)\Big) \subseteq V.$$

Since $f_\lambda(A_\lambda)$ is dense in B_λ for every $\lambda \in \Lambda$, then there exist partially ordered sets $(I_\lambda, \succ_\lambda)_{\lambda\in\Lambda}$ and for each $\mu \in K$ a family $(a_{\zeta_\mu})_{\zeta_\mu \in I_{\lambda_\mu}}$ of elements of A_{λ_μ} such that $(f_{\lambda_\mu}(a_{\zeta_\mu}))_{\zeta_\mu\in I_{\lambda_\mu}}$ converges to v_μ. This means that, for every $\mu \in K$, there exists an element $\eta_\mu \in I_\mu$ such that from $\zeta_\mu \succ_{\lambda_\mu} \eta_\mu$ it follows that $f_{\lambda_\mu}(a_{\zeta_\mu}) \in V_{\lambda_\mu}$.

Define the multi-index set $\prod\limits_{\mu\in K} I_{\lambda_\mu}$ and consider on it the partial order $\succ$, defined by $(\phi_\mu)_{\mu\in K} \succ (\psi_\mu)_{\mu\in K}$ if and only if $\phi_\mu \succ_{\lambda_\mu} \psi_\mu$ for each $\mu \in K$. Then $(\prod\limits_{\mu\in K} I_{\lambda_\mu}, \succ)$ becomes a partially ordered set of multi-indices.

Take any $(a_{\zeta_\mu})_{\mu\in K} \in \bigotimes\limits_{\mu\in K} A_{\lambda_\mu}$ with $(\zeta_\mu)_{\mu\in K} \succ (\eta_\mu)_{\mu\in K}$. Then $\zeta_\mu \succ_{\lambda_\mu} \eta_\mu$ for each $\mu \in K$ and we have $f_{\lambda_\mu}(a_{\zeta_\mu}) \in V_\mu$. As $h(\kappa_I(t)) = \kappa_J(h_T(t))$ for each $t \in T$, then this means that

$$h\Bigg(\kappa_I\Bigg(\bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} a_{\zeta_{(q,m,1)}} \otimes \ldots \otimes a_{\zeta_{(q,m,i_l)}}\Big)\Big)\Bigg)\Bigg)$$

$$= \kappa_J\Bigg(\bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} f_{\lambda_{(q,m,1)}}(a_{\zeta_{(q,m,1)}}) \otimes \ldots \otimes f_{\lambda_{(q,m,i_l)}}(a_{\zeta_{(q,m,i_l)}})\Big)\Big)\Bigg)$$

$$\in \kappa_J\Bigg(\bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} V_{\lambda_{(q,m,1)}} \otimes \ldots \otimes V_{\lambda_{(q,m,i_l)}}\Big)\Big)\Bigg) \subseteq \kappa_J(V) \subseteq W$$

for every

$$t = \bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} a_{\zeta_{(q,m,1)}} \otimes \ldots \otimes a_{\zeta_{(q,m,i_l)}}\Big)\Big) \in T$$

with $(\zeta_\mu)_{\mu\in K} \succ (\eta_\mu)_{\mu\in K}$. Hence, the family

$$\big(t_{(\zeta_\mu)_{\mu\in K}}\big)_{(\zeta_\mu)_{\mu\in K}\in \prod\limits_{\mu\in K} I_{\lambda_\mu}} = \Bigg(h\Bigg(\kappa_I\Bigg(\bigoplus_{l=1}^{k_\nu}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} a_{\zeta_{(q,m,1)}} \otimes \ldots \otimes a_{\zeta_{(q,m,i_l)}}\Big)\Big)\Bigg)\Bigg)\Bigg)_{(\zeta_\mu)_{\mu\in K}\in \prod\limits_{\mu\in K} I_{\lambda_\mu}}$$

of elements of $h(T/I)$ converges to w.

As w is an arbitrary element of S/J, then the set $h(T/I)$ is dense in S/J. □

Corollary 1. *Let $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\lambda)_{\lambda\in\Lambda}$ be two sets of disjoint topological algebras, indexed by the same set Λ. For every $\lambda \in \Lambda$, let $f_\lambda : A_\lambda \to B_\lambda$ be a continuous algebra homomorphism such that $f_\lambda(A_\lambda)$ is dense in B_λ. Define a map $h : T/I \to S/J$ by $h(\kappa_I(t)) = \kappa_J(h_T(t))$ for every $t \in T$, where h_T is defined as in Lemma 2. If $S \cdot h_T(T) \subseteq h_T(T)$ ($h_T(T) \cdot S \subseteq h_T(T)$ or $S \cdot h_T(T) \cdot S \subseteq h_T(T)$), then $h(T/I)$ is a dense left (respectively, right or two-sided) ideal of S/J.*

Proof. The claim follows from Lemma 3 and Proposition 1. □

Corollary 2. *Let $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ be a family of Segal topological algebras, T the tensor algebra of algebras $(A_\lambda)_{\lambda\in\Lambda}$, S the tensor algebra of algebras $(B_\lambda)_{\lambda\in\Lambda}$, I and J two-sided ideals of T and S, generated by the sets*

$$\{x \otimes y - xy : x, y \in A_\lambda, \lambda \in \Lambda\} \quad \textit{and} \quad \{z \otimes w - zw : z, w \in B_\lambda, \lambda \in \Lambda\},$$

respectively, and $h : T/I \to S/I$ a map, defined in Lemma 3. If $S \cdot h_T(T) \subseteq h_T(T)$ ($h_T(T) \cdot S \subseteq h_T(T)$ or $S \cdot h_T(T) \cdot S \subseteq h_T(T)$), then $(T/I, h, S/I)$ is a left (respectively, right or two-sided) Segal topological algebra.

Remark 2. Notice that the result in Corollary 2 does not depend on whether some particular Segal topological algebra $(A_{\lambda_0}, f_{\lambda_0}, B_{\lambda_0})$ from the family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ is left, right or two-sided Segal topological algebra.

5. COPRODUCTS IN THE CATEGORY SEG

Definition 1. The coproduct *of the family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ of Segal topological algebras in the category* **Seg** *is an ordered pair $((\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda), ((\alpha_\mu, \beta_\mu))_{\mu\in\Lambda})$, consisting of a Segal topological algebra $(\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda)$ and a family $((\alpha_\mu, \beta_\mu) : (A_\mu, f_\mu, B_\mu) \to (\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda))_{\mu\in\Lambda}$ of morphisms in* **Seg** *such that for any object (C, g, D) of* **Seg** *and every family $((\gamma_\mu, \delta_\mu) : (A_\mu, f_\mu, B_\mu) \to (C, g, D))_{\mu\in\Lambda}$ of morphisms in* **Seg***, there exists a unique morphism $(\theta, \omega) : (\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda) \to (C, g, D)$ in* **Seg** *such that the diagram*

$$
\begin{tikzcd}
D & & & & B_\nu \arrow[llll, "\delta_\nu"'] \arrow[dlll, "\beta_\nu"] & \\
& \underset{\lambda\in\Lambda}{\sqcup} B_\lambda \arrow[ul, dashed, "\omega"] & B_\mu \arrow[l, "\beta_\mu"] \arrow[ull, "\delta_\mu"'] & & & \\
& \underset{\lambda\in\Lambda}{\sqcup} A_\lambda \arrow[u, "h"] \arrow[dl, dashed, "\theta"'] & A_\mu \arrow[l, "\alpha_\mu"] \arrow[u, "f_\mu"] \arrow[dll, "\gamma_\mu"] & \cdots & & \cdots \\
C \arrow[uuu, "g"] & & & & A_\nu \arrow[llll, "\gamma_\nu"] \arrow[ulll, "\alpha_\nu"'] \arrow[uuu, "f_\nu"'] &
\end{tikzcd}
$$

commutes.

Thus, to have a coproduct $((\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda), ((\alpha_\mu, \beta_\mu))_{\mu\in\Lambda})$ in **Seg**, it is equivalent to having the following conditions fulfilled:

(1) there exists $(\underset{\lambda\in\Lambda}{\sqcup} A_\lambda, h, \underset{\lambda\in\Lambda}{\sqcup} B_\lambda) \in \mathrm{Ob}(\mathbf{Seg})$;

(2) there exist two families $(\alpha_\mu : A_\mu \to \underset{\lambda\in\Lambda}{\sqcup} A_\lambda)_{\mu\in\Lambda}$ and $(\beta_\mu : B_\mu \to \underset{\lambda\in\Lambda}{\sqcup} B_\lambda)_{\mu\in\Lambda}$ of continuous algebra homomorphisms such that $h \circ \alpha_\mu = \beta_\mu \circ f_\mu$ for each $\mu \in \Lambda$;

(3) for any $(C, g, D) \in \mathrm{Ob}(\mathbf{Seg})$ and families $(\gamma_\mu : A_\mu \to C)_{\mu\in\Lambda}$, $(\delta_\mu : B_\mu \to D)_{\mu\in\Lambda}$ of continuous algebra homomorphisms such that $g \circ \gamma_\mu = \delta_\mu \circ f_\mu$ for each $\mu \in \Lambda$, there exist continuous algebra homomorphisms $\theta : \underset{\lambda\in\Lambda}{\sqcup} A_\lambda \to C$ and $\omega : \underset{\lambda\in\Lambda}{\sqcup} B_\lambda \to D$ such that

(3a) $\theta \circ \alpha_\mu = \gamma_\mu$ for each $\mu \in \Lambda$;

(3b) $\omega \circ \beta_\mu = \delta_\mu$ for each $\mu \in \Lambda$;

(3c) $g \circ \theta = \omega \circ h$;

(3d) if $\overline{\theta} : \underset{\lambda\in\Lambda}{\sqcup} A_\lambda \to C$ and $\overline{\omega} : \underset{\lambda\in\Lambda}{\sqcup} B_\lambda \to D$ are continuous algebra homomorphisms such that $g \circ \overline{\theta} = \overline{\omega} \circ h$, $\gamma_\mu = \overline{\theta} \circ \alpha_\mu$ and $\delta_\mu = \overline{\omega} \circ \beta_\mu$ for each $\mu \in \Lambda$, then $\overline{\theta} = \theta$ and $\overline{\omega} = \omega$.

Theorem 1. *Let $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ be a family of left (right or two-sided) Segal topological algebras, T the tensor algebra of algebras $(A_\lambda)_{\lambda\in\Lambda}$, S the tensor algebra of algebras $(B_\lambda)_{\lambda\in\Lambda}$, I and J two-sided ideals of T and S, generated by the sets*

$$\{x\otimes y - xy : x,y\in A_\lambda, \lambda\in\Lambda\} \quad \text{and} \quad \{z\otimes w - zw : z,w\in B_\lambda, \lambda\in\Lambda\},$$

respectively, and $h: T/I \to S/I$ a map, defined in Lemma 3. If $S\cdot h_T(T)\subseteq h_T(T)$ (respectively, $h_T(T)\cdot S\subseteq h_T(T)$ or $S\cdot h_T(T)\cdot S\subseteq h_T(T)$), then the coproduct of the family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda\in\Lambda}$ exists and is in the form $((\bigsqcup_{\lambda\in\Lambda} A_\lambda, h, \bigsqcup_{\lambda\in\Lambda} B_\lambda), ((\alpha_\mu,\beta_\mu))_{\mu\in\Lambda})$, where $\bigsqcup_{\lambda\in\Lambda} A_\lambda = T/I$, $\bigsqcup_{\lambda\in\Lambda} B_\lambda = S/J$, $\alpha_\mu = \kappa_I\circ i_\mu$ and $\beta_\mu = \kappa_J\circ j_\mu$ for each $\mu\in\Lambda$.

Proof. We follow the steps (1)–(3d), as described after the definition of a coproduct in **Seg**, in order to prove the present theorem.

(1) By Corollary 2, we know that $(\bigsqcup_{\lambda\in\Lambda} A_\lambda, h, \bigsqcup_{\lambda\in\Lambda} B_\lambda)\in \mathrm{Ob}(\mathbf{Seg})$.

(2) In the beginning of Section 4 we already checked that $h\circ\alpha_\mu = \beta_\mu\circ f_\mu$ for every $\mu\in\Lambda$.

(3) Take any $(C,g,D)\in\mathrm{Ob}(\mathbf{Seg})$ and families $(\gamma_\mu : A_\mu\to C)_{\mu\in\Lambda}$, $(\delta_\mu : B_\mu\to D)_{\mu\in\Lambda}$ of continuous algebra homomorphisms such that $g\circ\gamma_\mu = \delta_\mu\circ f_\mu$ for each $\mu\in\Lambda$.

Remember that $\bigsqcup_{\lambda\in\Lambda} A_\lambda = T/I$ and $\bigsqcup_{\lambda\in\Lambda} B_\lambda = S/J$, which means that every element of $\bigsqcup_{\lambda\in\Lambda} A_\lambda$ is of the form $\kappa_I(t)$ for some

$$t = \bigoplus_{l=1}^{k_t}\Big(\bigoplus_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}} t_{q,m,1}\otimes\ldots\otimes t_{q,m,i_l}\Big)\Big)\in T$$

and every element of $\bigsqcup_{\lambda\in\Lambda} B_\lambda$ is of the form $\kappa_J(v)$ for some

$$v = \bigoplus_{o=1}^{k_v}\Big(\bigoplus_{p=1}^{u_o}\Big(\sum_{n=1}^{w_{p,o}} v_{n,p,1}\otimes\ldots\otimes v_{n,p,i_o}\Big)\Big)\in S.$$

Define maps $\theta: \bigsqcup_{\lambda\in\Lambda} A_\lambda\to C$ and $\omega: \bigsqcup_{\lambda\in\Lambda} B_\lambda\to D$ as follows:

$$\theta(\kappa_I(t)) = \sum_{l=1}^{k_t}\Big(\sum_{m=1}^{p_l}\Big(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}\tilde{\gamma}(t_{q,m,d})\Big)\Big),$$

where $\tilde{\gamma}(t_{q,m,d}) = \gamma_\mu(t_{q,m,d})$ for $t_{q,m,d}\in A_\mu$ (here $\mu = \lambda_{t_{q,m,d}}$) and

$$\omega(\kappa_J(v)) = \sum_{o=1}^{k_v}\Big(\sum_{p=1}^{u_o}\Big(\sum_{n=1}^{w_{p,o}}\prod_{d=1}^{i_o}\tilde{\delta}(v_{n,p,d})\Big)\Big),$$

where $\tilde{\delta}(v_{n,p,d}) = \delta_\mu(v_{n,p,d})$ for $v_{n,p,d}\in B_\mu$ (here $\mu = \lambda_{v_{n,p,d}}$).

Take any $u\in T$ such that $\kappa_I(u) = \kappa_I(t)$. Then $s = u - t\in I$, which means that s has the form

$$s = \bigoplus_{f=1}^{k_s}\Big(\bigoplus_{g=1}^{u_f}\Big(\sum_{h=1}^{v_{g,f}} s_{h,g,1}\otimes\ldots\otimes s_{h,g,j_f}\Big)\Big),$$

where, for all possible values of q,m,d, we have $s_{q,m,d} = x_{s_{q,m,d}}\otimes y_{s_{q,m,d}} - x_{s_{q,m,d}}y_{s_{q,m,d}}$ for some $x_{s_{q,m,d}}, y_{s_{q,m,d}}\in A_{\lambda_{s_{q,m,d}}}$ and $u = t+s$ has the form

$$u = \bigoplus_{l=1}^{k_t+k_s}\Big(\bigoplus_{m=1}^{w_l}\Big(\sum_{q=1}^{x_{m,l}} z_{q,m,1}\otimes\ldots\otimes z_{q,m,L_l}\Big)\Big),$$

where $L_l, w_l, x_{m,l}$ and $z_{q,m,d}$ are defined as in (3.1)–(3.2). Notice that, for all possible values of q,m,d, we have

$$\theta(\kappa_I(s_{q,m,d})) = \theta(\kappa_I(x_{s_{q,m,d}} \otimes y_{s_{q,m,d}} - x_{s_{q,m,d}} y_{s_{q,m,d}})) = \tilde{\gamma}(x_{s_{q,m,d}})\tilde{\gamma}(y_{s_{q,m,d}}) - \tilde{\gamma}(x_{s_{q,m,d}} y_{s_{q,m,d}})$$
$$= \gamma_{\lambda_{s_{q,m,d}}}(x_{s_{q,m,d}})\gamma_{\lambda_{s_{q,m,d}}}(y_{s_{q,m,d}}) - \gamma_{\lambda_{s_{q,m,d}}}(x_{s_{q,m,d}} y_{s_{q,m,d}}) = \theta_C,$$

because $\gamma_{\lambda_{s_{q,m,d}}}$ is an algebra homomorphism.

This means that $\theta(\kappa_I(s)) = \theta_C$ and $\theta(\kappa_I(u)) = \theta(\kappa_I(s+t)) = \theta(\kappa_I(s)) + \theta(\kappa_I(t)) = \theta(\kappa_I(t))$. Hence, θ is correctly defined. Similarly, we can also check that ω is correctly defined, i.e. if $\kappa_J(v_1) = \kappa_J(v_2)$, then also $\omega(\kappa_J(v_1)) = \omega(\kappa_J(v_2))$.

As the maps $(\gamma_\mu : A_\mu \to C)_{\mu\in\Lambda}$, $(\delta_\mu : B_\mu \to D)_{\mu\in\Lambda}$ were continuous algebra homomorphisms, then the maps θ and ω are also continuous algebra homomorphisms.

(3a) Fix any $\mu \in \Lambda$ and any $a \in A_\mu$. Then $\alpha_\mu(a) = (\kappa_I \circ i_\mu)(a) = \kappa_I(i_\mu(a)) = \kappa_I(a)$. Hence, $(\theta \circ \alpha_\mu)(a) = \theta(\kappa_I(a)) = \gamma_\mu(a)$. Thus, $\theta \circ \alpha_\mu = \gamma_\mu$ for each $\mu \in \Lambda$.

(3b) Fix any $\mu \in \Lambda$ and any $b \in B_\mu$. Then $\beta_\mu(b) = (\kappa_J \circ j_\mu)(b) = \kappa_J(j_\mu(b)) = \kappa_J(b)$. Hence, $(\omega \circ \beta_\mu)(b) = \omega(\kappa_J(b)) = \delta_\mu(b)$. Thus, $\omega \circ \beta_\mu = \delta_\mu$ for each $\mu \in \Lambda$.

(3c) Take any $x \in \bigsqcup_{\lambda\in\Lambda} A_\lambda$. Then there exists

$$t = \bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right) \in T$$

such that $x = \kappa_I(t)$.

Notice that, for any $a \in \bigcup_{\lambda\in\Lambda} A_\lambda$, we have

$$(g \circ \tilde{\gamma})(a) = (g \circ \gamma_{\mu_a})(a) = (\delta_{\mu_a} \circ f_{\mu_a})(a) = (\tilde{\delta} \circ \tilde{f})(a),$$

where $\tilde{f} : \bigcup_{\lambda\in\Lambda} A_\lambda \to \bigcup_{\lambda\in\Lambda} B_\lambda$ is defined as $\tilde{f}(a) = f_{\mu_a}(a)$ for each $a \in \bigcup_{\lambda\in\Lambda} A_\lambda$ and $\tilde{\delta} : \bigcup_{\lambda\in\Lambda} B_\lambda \to D$ is defined as $\tilde{\delta}(b) = \delta_{\mu_b}(b)$ for each $b \in \bigcup_{\lambda\in\Lambda} B_\lambda$. Hence, $g \circ \tilde{\gamma} = \tilde{\delta} \circ \tilde{f}$.

Define maps $\alpha : \bigcup_{\lambda\in\Lambda} A_\lambda \to \bigsqcup_{\lambda\in\Lambda} A_\lambda$ and $\beta : \bigcup_{\lambda\in\Lambda} B_\lambda \to \bigsqcup_{\lambda\in\Lambda} B_\lambda$ by $\alpha(a) = \alpha_{\mu_a}(a)$ and $\beta(b) = \beta_{\mu_b}(b)$, respectively. Then $\tilde{\delta} = \omega \circ \beta$, $\tilde{\gamma} = \theta \circ \alpha$ and $\beta \circ \tilde{f} = h \circ \alpha$.

Notice that, for every $\mu \in \Lambda$ and every $a \in A_\mu$, we have $(h \circ \alpha_\mu)(a) = (h \circ \kappa_I)(a)$.

Now, because of the definitions of addition and multiplication via direct sums and tensor products in T,

$$(g \circ \theta)(x) = g(\theta(\kappa_I(t))) = g\left(\sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}\tilde{\gamma}(t_{q,m,d})\right)\right)\right)$$
$$= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}(g \circ \tilde{\gamma})(t_{q,m,d})\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}(\tilde{\delta} \circ \tilde{f})(t_{q,m,d})\right)\right)$$
$$= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}((\omega \circ \beta) \circ \tilde{f})(t_{q,m,d})\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}(\omega \circ (h \circ \alpha))(t_{q,m,d})\right)\right)$$
$$= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\prod_{d=1}^{i_l}(\omega \circ (h \circ \kappa_I))(t_{q,m,d})\right)\right) = (\omega \circ h)\sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}\kappa_I\left(\prod_{d=1}^{i_l}t_{q,m,d}\right)\right)\right)$$
$$= (\omega \circ h)\left(\kappa_I\left(\sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}}(t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l})\right)\right)\right)\right)$$

$$= (\omega \circ h)\left(\kappa_I\left(\bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right)\right)\right) = (\omega \circ h)(\kappa_I(t)) = (\omega \circ h)(x)$$

for each $x \in \bigsqcup_{\lambda \in \Lambda} A_\lambda$. Hence, $g \circ \theta = \omega \circ h$.

(3d) Suppose that $\overline{\theta} : \bigsqcup_{\lambda \in \Lambda} A_\lambda \to C$ and $\overline{\omega} : \bigsqcup_{\lambda \in \Lambda} B_\lambda \to D$ are continuous algebra homomorphisms such that $g \circ \overline{\theta} = \overline{\omega} \circ h$, $\gamma_\mu = \overline{\theta} \circ \alpha_\mu$ and $\delta_\mu = \overline{\omega} \circ \beta_\mu$ for each $\mu \in \Lambda$. Take any $x \in \bigsqcup_{\lambda \in \Lambda} A_\lambda$. Then there exists

$$t = \bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right) \in T$$

such that $x = \kappa_I(t)$.

Now, because of the definitions of addition and multiplication via direct sums and tensor products in T and since $\overline{\theta}, \kappa_I, \theta$ are algebra homomorphisms, we obtain

$$\begin{aligned}
\overline{\theta}(x) &= (\overline{\theta} \circ \kappa_I)\left(\bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right)\right) \\
&= (\overline{\theta} \circ \kappa_I)\left(\sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} t_{q,m,d}\right)\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\overline{\theta} \circ \kappa_I)(t_{q,m,d})\right)\right) \\
&= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\overline{\theta} \circ \kappa_I)(i_{\mu_{t_{q,m,d}}}(t_{q,m,d}))\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\overline{\theta}(\kappa_I \circ i_{\mu_{t_{q,m,d}}}))(t_{q,m,d})\right)\right) \\
&= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\overline{\theta} \circ \alpha_{\mu_{t_{q,m,d}}})(t_{q,m,d})\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\gamma_{\mu_{t_{q,m,d}}})(t_{q,m,d}))\right)\right) \\
&= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\theta \circ \alpha_{\mu_{t_{q,m,d}}})(t_{q,m,d})\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\theta(\kappa_I \circ i_{\mu_{t_{q,m,d}}}))(t_{q,m,d})\right)\right) \\
&= \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\theta \circ \kappa_I)(i_{\mu_{t_{q,m,d}}}(t_{q,m,d}))\right)\right) = \sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} (\theta \circ \kappa_I)(t_{q,m,d})\right)\right) \\
&= (\theta \circ \kappa_I)\left(\sum_{l=1}^{k_t}\left(\sum_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} \prod_{d=1}^{i_l} t_{q,m,d}\right)\right)\right) = (\theta \circ \kappa_I)\left(\bigoplus_{l=1}^{k_t}\left(\bigoplus_{m=1}^{p_l}\left(\sum_{q=1}^{r_{m,l}} t_{q,m,1} \otimes \ldots \otimes t_{q,m,i_l}\right)\right)\right) = \theta(x)
\end{aligned}$$

for each $x \in \bigsqcup_{\lambda \in \Lambda} A_\lambda$. Using similar arguments for $\tilde{\omega}, \omega, \kappa_J$ and the definitions of addition and multiplication in S, we can show that $\tilde{\omega}(y) = \omega(y)$ for each $y \in \bigsqcup_{\lambda \in \Lambda} B_\lambda$. As it holds for each $x \in \bigsqcup_{\lambda \in \Lambda} A_\lambda$ and each $y \in \bigsqcup_{\lambda \in \Lambda} B_\lambda$, then we have $\tilde{\theta} = \theta$ and $\tilde{\omega} = \omega$.

With this we have proved our claim that $(\bigsqcup_{\lambda \in \Lambda} A_\lambda, h, \bigsqcup_{\lambda \in \Lambda} B_\lambda)$ is the coproduct of the family $(A_\lambda, f_\lambda, B_\lambda)_{\lambda \in \Lambda}$ of Segal topological algebras. Hence, the coproduct exists in the category **Seg**. □

Open question 2. Is the condition $S \cdot h_T(T) \subseteq h_T(T)$ ($h_T(T) \cdot S \subseteq h_T(T)$ or $S \cdot h_T(T) \cdot S \subseteq h_T(T)$) necessary for the existence of a coproduct?

6. CONCLUSIONS

In the present research we have found a sufficient condition for the existence of coproducts in the category **Seg** and stated some open problems.

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Kokorrutised Segali topoloogiliste algebrate kategoorias Seg

Mart Abel

On leitud piisav tingimus kokorrutiste leidumiseks kategoorias **Seg** ja sõnastatud mõned lahtised probleemid.