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The Nothing That Really Matters

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Synopsis

Zero has (a) special role(s) in mathematics. In the current century, we take negative numbers and zero for granted, but we should also be aware that their acceptance and their emergence in mathematics, and their ubiquity today, have not come to happen as rapidly as, for example, for natural numbers. Students can quickly become confused by the question: is zero a natural number? The answer is simple: a matter of definition. The history of zero and that of negative numbers are closely linked. It was in the calculations of debts that the negative numbers first appeared, where the state of balance between positive and negative assets was also given. We now take them for granted, but the history of science shows that it was far from a smooth process, and interesting mistakes about zero still occur in education today. I present a few examples below, and how they have been resolved.

Keywords: mathematics education, the number zero, history of mathematics, student errors.

The result was incorrect because of a zero,
the exam was a fail because of the error,
my mid-term score became a C because of the fail,
my mom got angry because of the bad grade,
I didn't get the gift because she was angry,
and all because of the zero, exactly!
(Márta Békés, translated by the author)

1. Introduction

Zero has two roles in today's mathematics: as a number and as a place-filling symbol. The role of zero in local-valued number systems, such as the decimal number system of today's Western culture, is actually that of an empty space indicator. For example, the 32, 320, 3200 and 302 signal lines are completely different [3]. All this may seem trivial, but it is not; just consider that the symbols that are concatenated to the results of counting do not contain zero (since zero is not a "plural" in the ordinary sense), but our place-value-based symbol system makes its use indispensable. It can be seen that the zero here is merely a sign, not necessary to refer to some abstract concept. Its appearance in this form is very early: according to the tablets, it was used in Babylonia as early as the second millennium BCE [1].

The Babylonians used a numerical system based on 60, which included decimal elements. They were familiar with and used the place-value notation, although zero as a sign was not known to them. The easiest and most obvious solution for them was to simply leave out the zero where it should have been, and not mark it with anything. As a result, the zeros at the end of the number did not appear, which could cause confusion when examining the numbers without context. Besides Babylonian mathematics, Greek mathematics was also unfriendly to zero. To them, zero was even more absurd as quantity. The reason for this is very simple: in their geometry-centred mathematics, where they worked with ratios and sections, it was absolutely inconceivable to them to have a section of length being either negative or zero. Even in advanced Chinese mathematics, zero did not appear, although they used a decimal number system. The surviving sources show that they used nine digits and the place value components that could be assigned to them. In addition, there were even numbers that had a numeric value described by their own digit, for example, the numbers ten, twenty, thirty, and forty also had independent signs.

These two roles of zero have appeared separately throughout history. It makes sense that the place-filling role may have appeared earlier than the number in cultures with a particular place value, because there was a need for it. True, if they did use some kind of space-filling signal, it was not the zero we know today, but rather either a signal based on convention, or simply the empty space. Zero, as an abstract number, appeared in India long after the seventh century BCE. These two roles are represented by the same symbol, well adapted to their relationship, and therefore practical, but at the same time difficult to distinguish conceptually, which creates didactic difficulties.

The first attempt to incorporate zero into arithmetic was made by the Indian mathematician and astronomer Brahmagupta. He describes how the basic operations work with zero. While in the case of addition and subtraction he is correct in the present sense, in the case of multiplication and division inaccuracies and uncertainties arise, for example the idea that if you divide zero by zero you get zero, or if you divide zero by either a positive or negative number, it is either zero or a fraction with zero in the denominator.

Two centuries later, Mahavira tries to correct Brahmagupta's inaccuracies, and partially succeeds. He fails in the division by zero; he claims that if a number is divided by zero, the number itself remains. When dividing, Bhaskara observes that the smaller the number, the higher the value. Thus he concludes that if you divide a number by zero, you get infinity. This is obviously a false result, because if it were true, then if I multiplied zero by infinity, I could get any number.

Since a given number system was most useful when it could cope with the operations, there was a given need for the two functions of zero to be linked sooner or later. In different cultures, this need has emerged in different ways and at different times in history. As a negative example, we can take the Roman numerals, which, compared to the Arabic numerals, handled the work with the basic operations less efficiently, at least via the algorithms we use today.

The Indian number system reached Europe through Arabic mediation. Fibonacci played a major role in its establishment. He was the first to recognise that zero could be a solution to second-degree equations, but he was alone in this recognition for a long time. The Roman numerals already mentioned have been in use for a long time in history, with all their drawbacks. One of the main reasons for this was their role in accounting. Unlike Arabic numerals, where you could just write a few zeros behind the numbers and get a completely different value, Roman numerals could not be simply faked. This was also the reason why traders used them, despite all the advantages of Arabic numerals. The final "victory" over Roman numerals for Arabic numerals came with the printing of books in the fifteenth century. To avoid misunderstandings, the numbers were also written down in words on financial transactions or even on banknotes. It is interesting that this idea had already been conceived in Florence a few centuries earlier.

In addition to zero, Nicolas Chuquet included operations with negative numbers in his set of rules. He was the first to use zero and negative numbers as exponent and is considered a pioneer of algebraic symbolism. He composed the words billion, trillion, quadrillion.

J.J. O'Connor and E.F. Robertson [2] in *A History of Zero* suggest that the difficulties associated with zero have not disappeared in education. Two such and similar difficulties are described below, which can be considered fascinating yet instructive.

We can see that zero has followed an interesting path throughout history. It still has an impact on our lives today. It is the cornerstone of mathematics — with its algebraic properties and its essential role in analysis (differential and integral calculus), which have given rise to sciences such as physics, finance and economics, and even computer science.

Zero is also given special attention in the teaching of mathematics. During the introduction of the four operations, the algebraic properties of zero are discussed separately: for example, in addition, zero is the neutral element (or identity element), and in division (or in fractions), if the divisor (denominator) is zero, it is said to be undefined, because zero has no multiplicative inverse. Teachers at different levels of education already try to teach the rules related with zero to the students:

- any number multiplied by zero equals zero,
- we do not divide by zero,
- zero is an integer,
- zero is neither positive nor negative,
- zero is an even number,
- zero divided by any integer is again zero,

- $0! = 1$,
- it is neither a prime number nor a composite number,
- any non-zero number to the zero power equals one...

... and what does equal zero to the power of zero?

In introductory calculus classes the answer to this last question is usually left undefined, because the expression cannot be interpreted in a way that assumes continuity. But depending on the context we are working, one answer seems to rule them all[1]. Most mathematicians, including Euler [4], agree, that in several cases it makes sense to assume $0^0 = 1$, and they have the following reasons for this:

(a) Libri [5] following Mascheroni, gave the equation below:

$$0^0 = (x - x)^{y-y} = \frac{(x-x)^y}{(x-x)^y} = 1.$$

(b) Libri [5] also invoked the binomial theorem. Using this idea, we get:

$$(x+1)^y = \sum_{k=0}^{y} \binom{n}{k} x^k 1^{y-k}.$$

Then for $x = 0$ we obtain:

$$1 = 0^0 \cdot 1.$$

So for the binomial theorem to hold, 0^0 should be 1.

(c) We can also look at it from a set-theoretic point of view. We can consider that x^y is actually the number of sets of elements y that can be chosen from the set of elements x. So if $x = y = 0$, then the previous statement implies that 0^0 is the number of sets of 0 elements, that can be chosen from a set of 0 elements, which equals to 1.

[1] Referring to J.R.R. Tolkien's "One ring to rule them all".

However, the following limits show a different picture from the previous ones:

- $\lim\limits_{x \to 0} x^0 = 1$,
- $\lim\limits_{x \to 0^+} 0^x = 0$,
- $\lim\limits_{x \to 0^-} 0^x = \text{undefined}$,
- $\lim\limits_{x \to 0^+} x^{\frac{1}{\ln x}} = e$.

In summary, it seems $0^0 = 1$ is the right decision in most cases. But a calculus teacher when teaching limits, for this kind of question would say, that there is no way to define it in a way that makes the function involved a continuous one.

So, despite the best efforts of the teachers, as we can see, interesting situations can occur during the teaching-learning process, that arise from the special properties of zero.

2. The Classroom Observations

While we are on the subject of Nicolas Chuquet, I can relate an interesting didactic observation about the zero in the exponent. I noticed an interesting fact in a university student's work. The topic was integration, and we were looking for a primitive function of a sum. The algorithm of integration was correctly applied by the student to the example, except for one step. The linear term without coefficients was incorrectly integrated. To correct the error, I tried to lead to the correct solution by asking a question about how to integrate functions of type x^n when $n \neq -1$. The answer given by the student was correct, so I asked him to correct the incorrect solution as he had just described. He tried, but the result was the same incorrect solution:

$$\int x\,dx = \frac{x^1}{1} + c.$$

I asked him to verbalise his own solution. The first sentence revealed the "error". Literally in the middle of nowhere. In the exponent of the linear member, we do not write the one, so the student reflexively, one might say instinctively, treated it as zero. When I asked for the value of x^0, I got a clear answer: one. My question to give the value of x^1 was also answered correctly, and this was the first time that the student seemed to recognise what and where he could have gone wrong. When I asked him to tell me what he was thinking/considering when he first worked on the problem, he said: "There was nothing in the exponent, and nothing equals to zero."

An interesting "fallacy" also arose in the field of probability. Students were asked to write down the binomial coefficient

$$\binom{5}{0}$$

in fractional form. One of the students wrote the usual formula in his notebook:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and then, after the substitution, described the following:[2]

$$\binom{5!}{5! \cdot 0!} = 0.$$

Then, seconds later, he corrected the next zero after the equal sign by one:

$$\binom{5!}{5! \cdot 0!} = \cancel{0} = 1.$$

After witnessing this, I asked him to tell me what he did and why. He replied, "I substituted in the formula, then, after simplification, I multiplied the members in the denominator in my head, and finally I did the division. For some reason, however, a bad result came out, as common sense dictates that the solution should be one and not zero, so I corrected the zero by one." Reflecting on this, I asked my first question about dividing by zero. The student reflexively replied that dividing by zero was not understood. Here, in fact, he received confirmation from himself that he could not have come

[2] In the student's note, it was written in this form, without the fraction bar and in brackets.

up with zero, so he had made a mistake somewhere — which he had already been aware of, since he had corrected zero to one. In the following, I asked him to enter the value of 0!, if he knows. There was no response to this question. I could detect a puzzlement, the student felt more and more the “disturbance in the Force”. I then began to write the following lines in his notebook:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3!$$
$$3! = 3 \cdot 2 \cdot 1 = 3 \cdot 2!$$
$$2! = 2 \cdot 1 = 2 \cdot 1!$$

“Can you see the pattern yet?,” I asked. And then his eyes lit up. I knew that he had discovered it, and that he would never forget this realisation, because he hadn’t learned it, he had “discovered” it himself. I asked him to write down the few remaining steps himself:

$$n! = n \cdot (n-1)!$$

substituting $n = 1$, we get that

$$1! = 1 \cdot (1-1)! = 1 \cdot 0!$$

so

$$0! = \left(\frac{1!}{1}\right) = \left(\frac{1}{1}\right) = 1$$

and from this, we can conclude that $0! = 1$.

This brief justification was enough for the student not only to understand but also to apply $0! = 1$ correctly from now on.

3. Discussion

Zero is the sign of “nothing”, and it is impossible to ignore, because we owe a lot to it. Yet in Europe, the concept of zero has been very difficult to come to terms with. Not so in South America, where the Maya, thousands of years ago, had a number like zero, which they marked in their own script. Today, “nothing” is a great power, and zeros in a crowd are truly frightening. A million seconds takes less than two weeks, while a trillion seconds takes more than 30,000 years. But the difference between a million and a trillion is only a few zeros!

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[3] Uwe Springfeld, *Die Geschichte der Null*, Campus Verlag, Frankfurt am Main, 2000.

[4] Leonhard Euler, *Introduction to analysis of the infinite, Book 1.* Translated by Blanton, J. D., Springer, New York, 1988.

[5] Guillaume Libri, "Mémoire sur les fonctions discontinues." *Journal für die reine und angewandte Mathematik*, Volume **10** (1833), pages 303–316.