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Solution of Ordinary Differential Equation with Variable Coefficient Using Shehu Transforms

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Abstract

Shehu transform is a new integral transform type used to solve differential equations as other integral transforms. In this study, we will discuss the Shehu transform method to solve ordinary differential equation of variable coefficient. In order to solve, first we discussed the relationship between this new integral transform with Laplace transform.

Keywords: Shehu transform • Laplace transform • Variable coefficient • Ordinary differential equation

Introduction

Many problems in engineering and science can be formulated in terms of differential equations. The ordinary differential equations arise in many areas of Mathematics, as well as in Sciences and Engineering. In the literature there are numerous integral transforms [1] and widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral transform were extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Shehu Transform, Mellin, and Hankel, Fourier Transform, Sumudu Transform, Elzaki Transform and Aboodh Transform. Aboodh Transform [2,3] was introduced by Khalid Aboodh in 2013, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform [4-6]. New integral transform, named as ZZ Transformation [7-10] introduce by Zain Ul Abadin Zafar [2016] , ZZ transform was successfully applied to integral equations, ordinary differential equations . The main objective of this article is to construct (introduce) relationship between Shehu transform and Laplace Transform which helps us to use Shehu transform simply to solve ordinary differential equation of variable coefficient. The plane of the paper is as follows: In section 2, we introduce the basic idea of Shehu transform, in section 3, we introduce Laplace Transform, Application in 4 and conclusion in 5, respectively.

Shehu transform

Definition: A new transform called the Shehu transform of the function (t) belonging to a class A, where:

$|t|$

$A = \{ (t): \exists N, \eta_1, \eta_2 > 0, |f(t)| < Ne^{\eta}i, if t \in (-1)^i \times [0\ \infty) \}$

Where (t) defined by $\mathbb{S}[f(t)]$ and is given by:

$$\mathbb{S}\{f(t)\} = W(s,u) = \int_0^\infty e^{\left(\frac{-st}{u}\right)}(t)dt \tag{1.1}$$

Or $\mathbb{S}\{f(t)\} = W(s,u) = u\int_0^\infty f(ut)e^{-st}dt$ (1.2)

Laplace transform

Definition: The Laplace transform of the function (t), $f$or all$t \geq 0$ is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1.3}$$

Connection between Shehu transform and Laplace transform

In this section, we present connections between Shehu transform and Laplace Transform

Theorem 1: If $\{f(t)\} = W(s, u)$ and $\mathcal{L}\{f(t)\} = F(s)$ then

s

$\{f(t)\} = W(s, u) = F\ ( )$ (1.4)

u

Proof: Since, from (1.2) we have:

$$\mathbb{S}\{f(t)\} = W(s,u) = u\int_0^\infty f(ut)e^{-st}dt$$

Put $w = ut \Rightarrow \frac{dw}{} = dt$ in the above equation, we have

u

∞

$\Rightarrow \mathbb{S}\{f(t)\} = W(s, u) = u \quad e^{\left(\frac{-s}{u}\right)wdw} \int_0 f(w)$

u

$\Rightarrow \mathbb{S}u^w dw = F\ (^s)\ \{f(t)\} = W(s,u) = \int_0 f(w)e$

u

$\Rightarrow W(s, u) = F\ (^s)$

u

Hence the proof is completed.

Theorem: If $[f(t)] = W(s, u)$and $\mathcal{L}\{f(t)\} = F(s)$ then

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$$[tf(t)] = [-u\frac{d}{ds}\mathbb{S}[f(t)]] \tag{1.5}$$

Proof: We know that from (1.3), $(s)=\int_0^\infty f(t)e^{-st}dt$

∞

$s_t \quad -\left(\frac{s}{u}\right)$

$\Rightarrow F\,( ) = \int f(t)edt$

$u \qquad 0$

∞

$d \quad s \qquad d_t -\left(\frac{s}{u}\right)$

$\Rightarrow \frac{}{}F\,( ) = \frac{}{}[\int f(t)e$

$ds \quad u \qquad ds$

0

$$=\int_0^\infty f(t)\frac{\partial}{\partial s}\left(e^{-\left(\frac{s}{u}\right)t}\right)dt$$

$$=\int_0^\infty f(t)e^{-\left(\frac{s}{u}\right)t}\left(\frac{-t}{u}\right)dt$$

$$=-\frac{1}{u}\int_0^\infty f(t)e^{-\left(\frac{s}{u}\right)t}(t)dt$$

∞

$d \qquad s_t -\left(\frac{s}{u}\right)$

$\Rightarrow -u\frac{}{}F\,( ) = \int tf(t)e$

$ds \quad u$

0

But, from (1.4) we have that $\{f(t)\} = W(s,u) = F\,(^s$)

u

∞

$-\left(\frac{s}{u}\right)d_t$

$\Rightarrow -u\frac{}{}\mathbb{S}\{f(t)\} = \int tf(t)e \quad dt$

$ds \qquad 0$

$= \mathbb{S}\{tf(t)\}$

$$\therefore \mathbb{S}\{tf(t)\} = -u\frac{d}{ds}\mathbb{S}\{f(t)\}$$

Generalization: $\{t^n f(t)\} = -u\frac{d^n}{ds^n}\mathbb{S}\{f(t)\}$

Example 1: Find $\{te^{at}\}$ Solution: We know that $\mathbb{S}\{tf(t)\} = -u\frac{d}{ds}\mathbb{S}\{f(t)\}$

Here $(t) = e^{at}$

$$\Rightarrow W(s,u) = \left[\frac{u}{s-au}\right]$$

$$\Rightarrow \mathbb{S}\{te^{at}\} = -u\frac{d}{ds}\frac{u}{s-au}[\,]$$

$$= -u\left[\frac{-u}{(s-au)^2}\right]$$

$$= \frac{u^2}{(s-au)^2}$$

Example 2: Let us consider the second-order differential equation with variable coefficient

$$ty'' + 2y' + ty = \cos t\,, (0) = 1, y'(0) = 1 \tag{1.6}$$

Solution: Taking Shehu Transform on both sides of Eq. 1.6, we get

$[ty'' + 2y' + ty = \cos t]$

$\Rightarrow -uds\frac{}{}{}^d[u^{s2}{}_2(s,u) - u\ {}^sy(0) - y'(0)] + 2\,[u\ {}^sY(s,u) - y(0)] - uds\frac{}{}{}^dY(s,u) = s\frac{}{}{}_2{}^{us} + u2$

Using the given Initial conditions (0) = 1, $y'(0) = 1$

$d \quad s^2 \qquad s \qquad s \qquad d \qquad us$

$\Rightarrow -uds\frac{}{}\ \frac{}{}[u_2(s,u) - u\frac{}{} - 1] + 2\,[u\frac{}{}Y(s,u) - 1] - uds\frac{}{}Y(s,u) = s\frac{}{}2 + u2$

$s^2 d \qquad 2s \qquad 1 \qquad s \qquad d \qquad us$

$\Rightarrow -u\,[u\frac{}{}\ \frac{}{}{}_2\,ds(s,u) + u\frac{}{}2\,Y(s,u) - u] + 2\,u\frac{}{}Y(s,u) - 2 - uds\frac{}{}Y(s,u) = s\frac{}{}2 + u2$

$s^2 d \qquad 2s \qquad s \qquad d \qquad us$

$\Rightarrow [-\frac{}{}\ \frac{}{}uds(s,u) - \frac{}{}uY(s,u) + 1 + 2\,u\frac{}{}Y(s,u) - 2 - uds\frac{}{}Y(s,u) = s\frac{}{}{}_2 + u2]$

$s^2 d \qquad d \qquad us$

$\Rightarrow -\frac{}{}\ \frac{}{}uds(s,u) - uds\frac{}{}Y(s,u) = s\frac{}{}{}_2 + u2 + 1$

$d \qquad s^2 + u^2 \qquad us$

$\Rightarrow -ds\frac{}{}\{y(t)\}\ [\ \frac{}{}\quad \frac{}{}u\ ] = s_2 + u2 + 1$

$d \qquad u^2 s \qquad u$

$\Rightarrow -ds\frac{}{}\{y(t)\} = (\frac{}{}\quad \frac{}{}s_2 + u2)2 + s2 + u2$

If we multiplying both sides by u, we get:

$d \qquad u^3 s \qquad u^2$

$\Rightarrow -uds\frac{}{}\{y(t)\} = (\frac{}{}\quad \frac{}{}s_2 + u2)2 + s2 +$

$u2$

But, from (1.5) we know that $[tf(t)] = -u\frac{d}{ds}\mathbb{S}[f(t)]$

$u3s \qquad u2$

$\Rightarrow [ty(t)] = (\frac{}{}\quad \frac{}{}s_2 + u2)2 + s2 + u2$

Apply Inverse Shehu Transformation on both sides, we get

$u3s \qquad u2$

$t(t) = \mathbb{S}{-1}\,((\frac{}{}s2 + u2)2) + \mathbb{S}{-1}\,(s\frac{}{}2 + u2)$

From Table 1, we have that $\mathbb{S}{-1}\,(s2u+2u2) = \sin t$ and $(\frac{}{}s2u+3us2)2 = -2udsd\,(s2u+2u2) = \mathbb{S}\,(2t\sin t)$

$$\Rightarrow t(t) = \frac{t}{}\sin t + \sin t2$$

$$\Rightarrow (t) = \frac{1}{2}\sin t + \frac{\sin t}{t}$$

Table 1. Shehu transform of some common functions.

***f*(*t*)**	**$\mathbb{S}\{f(t)\} = W(s, u)$**
1	$\frac{u}{s}$
t	$\frac{u2}{s2}$
$t2$	$\frac{2!\, u^3}{s3}$
tn	$\frac{n!\, un+1}{sn+1}$
eat	$\frac{u}{s - au}$
$\cos(at)$	$\frac{us}{s2 + \alpha 2u2}$
$\sin(at)$	$\frac{\alpha u^2}{s2 + \alpha 2u2}$

Example 3: Consider the ordinary differential equation with variable coefficients (Table 2)

$ty'' + (1 - 2t)' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$ (1.7)

Solution: Applying the Shehu transform of both sides of Eq. 1.7, we get

$[ty'' + y' - 2ty' - 2y = 0\,]$ (1.8)

Using the differential property of Shehu transform Eq.1.8 can be written as:

(s, u)

$-y(0) - y$

$-uds^{d}[u^{s^2}2 \qquad u^{s} \qquad '(0)] + [u^{s}Y(s, u) - y(0)] + 2uds^{d}[u^{s}Y(s, u) - y(0)]$

$- 2(s, u) = 0$

Using the given Initial conditions $(0) = 1$, $y'(0) = 2$

$$\Rightarrow -uds\frac{d}{\quad}\frac{s^2}{\quad}[u_2(s, u) - u\frac{s}{\quad} - 2] + [u\frac{s}{\quad}Y(s, u) - 1] + 2uds\frac{d}{\quad}[u\frac{s}{\quad}Y(s, u) - 1] - 2Y(s, u) = 0$$

Table 2. Laplace transform of some common functions.

***f*(*t*)**	**$\mathcal{L}\{f(t)\} = F(s)$**
1	$\frac{1}{s}$
t	$\frac{1}{s2}$
$t2$	$\frac{2!}{s3}$
tn	$\frac{n!}{sn+1}$
eat	$\frac{1}{s - a}$
$\cos(at)$	$\frac{s3}{s2 + \alpha 2}$
$\sin(at)$	$\frac{\alpha s^2}{s2 + \alpha 2}$

$$\Rightarrow -u\left[u\frac{s^2}{\quad}\frac{d}{\quad}{}_2 ds(s, u) + u\frac{2s}{\quad}2\, Y(s, u) - u\right] + \left[u\frac{1}{\quad}Y(s, u) - 1\right] + 2u\left[u\frac{s}{\quad}\frac{sd}{\quad}dsY(s, u) + u\frac{1}{\quad}Y(s, u)\right]$$

$- 2(s, u) = 0$

$$\Rightarrow \left[-\frac{s^2}{uds}\frac{d}{\quad}(s, u) - \frac{2s}{u}Y(s, u) + 1\right] + \left[\frac{s}{u}Y(s, u) - 1\right] + \left[2s\frac{d}{ds}Y(s, u) + 2Y(s, u)\right]$$

$- 2(s, u) = 0$

$$\Rightarrow -\frac{s^2}{uds}\frac{d}{\quad}(s, u) - \frac{s}{u}Y(s, u) + 2s\frac{d}{ds}Y(s, u) = 0$$

Rearranging the terms, we have

$ds^2 s \Rightarrow \underline{\quad}(s, u)\,[2s - \underline{\quad}] = Y(s, u)\,[\;]$

$\Rightarrow \underline{\quad}Y(s, u) = Y(s, u)\,[- \underline{\qquad}]^{1}_{-2}$

$\Rightarrow \underline{\quad}Y(s, u) = Y(s, u)\,[\underline{\qquad}]^{1}_{-2}$

Separating variables, we have

$$\frac{d(s, y)}{(s, u)} \quad \frac{-1}{s - 2u} = ds$$

and an integration yield

$\ln|(s, u)| = -\ln|s - 2u| + c_1$

Or

$$\mathbb{S}[y(t)] = \frac{c}{s - 2u}$$

By using Shehu inverse transform we obtain the solution in the following way,

$(t) = ce^{2t}$ (1.9)

We now determine value of c by using given initial value $(0) = 1$, implies $c = 1$

Therefore, from equation (1.9),

$(t) = e^{2t}$

Conclusion

In this paper, we have successfully conducted a relationship between Shehu transform and Laplace transform which help as Shehu Transform is applied directly to obtain the solution of ordinary differential equation with variable coefficient. It may be concluded that Shehu Transform is very powerful and efficient in finding the solution for a wide class of ordinary differential equation with variable coefficient.

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