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An Efficient Solver of Eigenmodes for a Class of Complex Optical Waveguides

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Abstract

In this paper, for a class of complex optical waveguide, the high-precision computation of the propagation constants β are studied. The corresponding Sturm-Liouville (S-L) problem is represented as $\varphi_{zz}+\tilde{\alpha}(x,z)\varphi_z+\tilde{\beta}(x,z)\varphi=\beta^2\varphi$ in an open domain (open on one side), where *x* is a given value. Firstly, a perfectly matched layer is used to terminate the open domain. Secondly, both the equation and the complex coordinate stretching transformations are constructed. Thirdly, the S-L problem is turned to a simplified form such as $\hat{\phi}_{\hat{z}\hat{z}}+s(\hat{z})\phi=\beta^2\phi$ in a bounded domain. Finally, the coefficient function $s(\hat{z})$ is approximated by a piecewise polynomial of degree two. Since the simplified equation in each layer can be solved analytically by the Kummer functions, the approximate dispersion equation is established to the TE case. When the coefficient function is continuous, the approximate solutions converge fast to the exact ones, as the maximum value of the subinterval sizes tends to zero. Numerical simulations show that high-precision eigenmodes may be obtained by the Müller's method with suitable initial values.

Keywords: Eigenmode • Helmholtz equation • Optical waveguide • Numerical method • Kummer functions

Introduction

It is known that the eigenmodes play an important role in optimizing the designs of the microwave engineering for the integrated circuitry [1], microstrip substrates [2], and the integrated optical devices [3,4]. When we compute the wave propagation by use of the eigenmode expansion method, the high-precision propagation constants are needed [5-17]. For the complex and open waveguide with a curved interface, firstly we can construct both the local orthogonal coordinate and the equation transformations [18], and then the original propagation model (Helmholtz equation) is turned to a linear second-order partial differential equation with a flatted interface. The corresponding Sturm-Liouville (S-L) problem is represented as $\varphi_{zz}+\tilde{\alpha}(x,z)\varphi_z+\tilde{\beta}(x,z)\varphi=\beta^2\varphi,$ where the variable x is a given value and β is a propagation constant. That is, β^2 is known as the eigenvalue λ. Secondly, the open domain is truncated by the perfectly matched layer (PML) [19], or by a coordinate stretching transformation [20,21]. Then, the corresponding S-L problem is changed as $\hat{\varphi}_{\hat{z}\hat{z}}+\tilde{\alpha}(x,z(\hat{z}))\hat{\varphi}_{\hat{z}}+\tilde{\beta}(x,z(\hat{z}))\hat{\varphi}=\hat{\beta}^2\hat{\varphi}$ with the interface and the boundary conditions (the domain is bounded), where $\hat{\beta}$ is an approximate value of β. When the transverse variable $\hat{z}$ is discretized, there is a discrete set of eigenmodes, which is composed of a finite number of perturbed propagating modes, an infinite sequence of perturbed leaky modes and an infinite number of Berenger modes. Some numerical methods, such as the finite element method (FEM) [22-25], the finite difference method (FDM) [26], the multidomain pseudospectral method [27-29], and the B-spline modal method [30], approximately turn the original S-L problem to a matrix eigenvalue problem. However, we still hardly obtain high-precision eigenvalues, because these methods will produce large and complex matrices causing the difficulties in numerical implementation. For this reason, a different kind of treatment has been proposed, that is, turning the S-L problem to a root-finding problem of a nonlinear dispersion equation, where the roots of the equation are the eigenvalues or the propagation constants. There are some efficient methods to treat slab waveguides [31-33] and rib waveguides [34]. For the varying refractive index's waveguides terminated by the PMLs, the corresponding S-L problem is expressed as $\varphi_{zz}+\tilde{\beta}(x,z)\varphi=\beta^2\varphi,$ and there are some approximate dispersion equations, which are established by the Wentzel-Kramers-Brillouin (WKB) method [35] and the differential transfer matrix method [36-40]. Yet, these methods cannot guarantee the approximation accuracy of the propagation constants. Although an exact dispersion equation for the waveguide with varying refractive-index profile has been constructed [41], it involves the derivatives of the refractive-index function and highly oscillatory integrals leading to the difficulties in numerical computation. Recently, we give efficient approximations of dispersion relations for the varying refractive-index waveguides with the two flat interfaces, where the waveguides are open on both sides and terminated by two PMLs, and whose S-L problem is the simple form: $\hat{\varphi}_{\hat{z}\hat{z}}+\tilde{\beta}(x,z(\hat{z}))\hat{\varphi}=\hat{\beta}^2\hat{\varphi}$ with the interface and the boundary conditions [42]. In this paper, we extend our previous works [42] to the ones for a class of complex waveguides. Namely, the S-L problems are extended to $\varphi_{zz}+\tilde{\alpha}(x,z)\varphi_z+\tilde{\beta}(x,z)\varphi=\beta^2\varphi$ with the interface and the boundary conditions, where the domain is open on one side. For simplicity, we only develop a solver of computing the transverse electric (TE) eigenmodes for the waveguides, which are terminated by the PML along one transverse direction. The rest of this paper is organized as follows. In Section 2, a modified S-L problem (dispersion equation) is introduced when a PML is used, and a solver for the dispersion equations for the TE case is constructed, where the coefficient functions are approximated by the piecewise polynomials of degree two, and the approximate eigenfunctions are expressed analytically by the Kummer functions. In Section 3, the numerical simulations for the TE case are given. Finally, the conclusions are presented in Section 4.

Mathematical Modelling and Implementation of Eigenmodes

Now, we start from the following equation, which can be obtained by transforming the Helmholtz equation as mentioned above:

$$V_{zz}+\tilde{\alpha}(x,z)V_z+\tilde{\beta}(x,z)V=0,\quad \lim_{z\to\infty}V=0. \tag{1}$$

When the eigenmode expansion method is used to solve the Eq. (1), the S-L problem of the operator $L=\partial_z^2+\tilde{\alpha}(x,z)\partial_z+\tilde{\beta}(x,z)$ must be

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considered [43], where x is a given value.

For a dual-layer planar waveguide with the transverse variable z shown in Figures 1 & 2, we consider the S-L problem of the operator **L** (dispersion equation) as follows:

$$\begin{cases} \mathrm{L}\,\varphi = \varphi_{zz} + \tilde{\alpha}(x,z)\varphi_z + \tilde{\beta}(x,z)\varphi = \lambda\varphi, \quad z > 0; \\ \varphi(0) = 0, \quad \lim\limits_{z\to+\infty} \varphi(z) = 0; \end{cases} \tag{2}$$

where $\tilde{\alpha}(x,z)$ and $\tilde{\beta}(x,z)$ can be regarded as the functions only related with variable z since x is a given value, λ is an eigenvalue of the operator L, φ is an eigenfunction corresponding the eigenvalue λ, $\lambda = \beta^2$, and β is called the propagation constant.

By the feature of the optical waveguides, we assume that $\tilde{\alpha}(x,z)$ and $\tilde{\beta}(x,z)$ are constants as the variable z is large enough. Namely, there is a d making $\tilde{\alpha}(x,z) = \tilde{\alpha}_1$ and $\tilde{\beta}(x,z) = \tilde{\beta}_1$ as $z > d$, where $\tilde{\alpha}_1$ and $\tilde{\beta}_1$ are two constants. So, when $z > d$, the dispersion equation can be simply written as follows:

$$\varphi_{zz} + \tilde{\alpha}_1\varphi_z + \tilde{\beta}_1\varphi = \lambda\varphi, \quad z > d. \tag{3}$$

In order to numerically solve Eq.(2), we must truncate the open domain to the finite one. Thus, we choose a PML to truncate the open domain, that

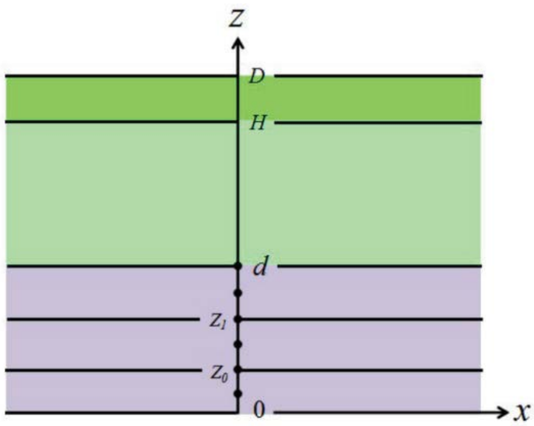

Figure 1. Sketch of the optical waveguide terminated by a PML (marked by green).

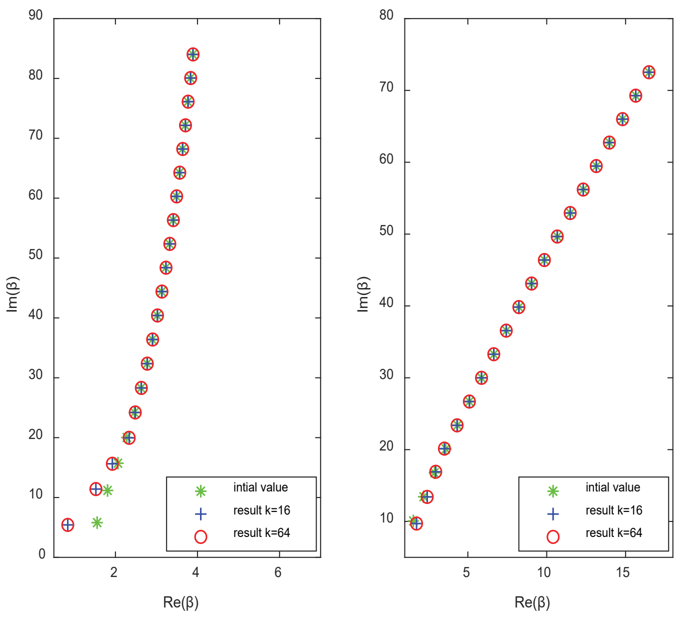

Figure 2. The propagation constants from Example. Left: leaky modes. Right: Berenger modes. '*' stands for obtained from the asymptotic solutions for the slab waveguide, '+'stands for the results of with k=16, and 'o' stands for the \results of with k=64.

is, making a complex coordinate stretching transform:

$$\hat{z} = z + \mathrm{i}\int_0^z \sigma(\tau)d\tau, \tag{4}$$

where $\sigma(\tau) = 0$ for $0 \le z \le H$, otherwise $\sigma(\tau) = C\cdot\tau^3/(1+\tau^2)$; here $\tau = \frac{z-H}{D-H}$, $\mathrm{i} = \sqrt{-1}$ and $0 < d < H < D$.

Then, Eq.(2) is approximately turned as the following form:

$$\begin{cases} \hat{\varphi}_{zz} + \tilde{\alpha}(x,z)\varphi_z + \tilde{\beta}(x,z)\varphi = \hat{\lambda}\varphi, \quad 0 < z \le H; \\ \frac{1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}z}\right) + \tilde{\alpha}_1\frac{1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}\varphi}{\mathrm{d}z} + \tilde{\beta}_1\hat{\varphi} = \hat{\lambda}\varphi, \\ H < z < D; \\ \hat{\varphi}(0) = 0, \quad \varphi(D) = 0; \end{cases} \tag{5}$$

where $\hat{\lambda} = \beta^2$.

Case 1: when $0 \le z \le H$, we make the transformation $\hat{\varphi}(z) = \exp\left(-\int_0^z \frac{\tilde{\alpha}(x,t)}{2}\mathrm{d}t\right)\cdot\phi(z)$. Then, the equation

$$\hat{\varphi}_{zz} + \tilde{\alpha}(x,z)\varphi_z + \tilde{\beta}(x,z)\varphi = \hat{\lambda}\varphi, \quad 0 < z \le H \tag{6}$$

can be transformed to the following form:

$$\phi_{zz} + s_0(z)\phi = \hat{\lambda}\phi, \quad 0 < z \le H; \tag{7}$$

where $s_0(z) = \tilde{\beta}(x,z) - \frac{\tilde{\alpha}^2(x,z)}{4} - \frac{\tilde{\alpha}_z(x,z)}{2}$.

Case 2: when $H < z \le D$, we make the other transformation $\hat{\varphi}(\hat{z}) = \mathrm{e}^{-\frac{\tilde{\alpha}_1}{2}\hat{z}}\cdot\phi(z)$. Then, the equation becomes

$$\frac{1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}z}\right) + \frac{\tilde{\alpha}_1}{1+\mathrm{i}\sigma(z)}\frac{\mathrm{d}\varphi}{\mathrm{d}z} + \tilde{\beta}_1\hat{\varphi} = \hat{\lambda}\varphi, \quad H < z < D; \tag{8}$$

that is,

$$\hat{\varphi}_{\hat{z}\hat{z}} + \tilde{\alpha}_1\varphi_z + \tilde{\beta}_1\varphi = \hat{\lambda}\varphi. \tag{9}$$

It can be transformed to the following form:

$$\phi_{\hat{z}\hat{z}} + s_1\phi = \hat{\lambda}\phi, \tag{10}$$

where $s_1 = \tilde{\beta}_1 - \frac{\tilde{\alpha}_1^2}{4}$ is a constant.

Thus, the original eigenvalue problem of the operator L is approximated by

$$\phi_{\hat{z}\hat{z}} + s(\hat{z})\phi = \hat{\lambda}\phi, \quad 0 < z < D, \tag{11}$$

where $\phi(0) = \phi(D) = 0$, $s(\hat{z}) = s_0(z)$ for $0 < z \le d$, and $s(\hat{z}) = s_1$ for $d < z < D$.

Remark 1: $\hat{z} = z$ as $0 \le z \le H$.

For the reason that it is a differential equation with variable coefficients in $[0,d]$, we first divide the interval $[0,d]$ into k subintervals with $h = d/k$. Then the subintervals are denoted by $I_j = [z_{j-1}, z_j]$ related to $z_j = jh \quad (j = 1, 2, \cdots, k)$, and the function $s_0(z)$ is interpolated by a polynomial of degree two, and $y_j(z)$ is the approximation of the $\phi(z)$ as $z \in I_j$. Thus, the form of approximated equation is

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}z^2} + (a_j z^2 + b_j z + c_j)y_j = 0, \tag{12}$$

where the coefficients of approximated equation are given by

$$\begin{cases} a_j = 2[s_0(t_0) - 2s_0(t_1) + s_0(t_2)]/h^2, \\ b_j = [(1-4j)s_0(t_0) + (8j-4)s_0(t_1) + (3-4j)s_0(t_2)]/h, \\ c_j = (2j^2 - j)s_0(t_0) + 4(j - j^2)s_0(t_1) + (2j^2 - 3j + 1)s_0(t_2) - \hat{\lambda}, \end{cases} \tag{13}$$

here $t_0 = (j-1)h$, $t_1 = (j-1/2)h$, and $t_2 = jh$ $(j = 1,2,\cdots,k)$.

Eq.(12) can be changed to the form of the confluent hypergeometric equation, and a pair of linearly independent solutions $\{m_j(z), n_j(z)\}$ as $z \in I_j$ are given by the Kummer functions. Then the approximated solution of ϕ as $z \in I_j$ can be expressed in the following form $y_j(z) = A_j m_j(z) + B_j n_j(z)$, $j = 1,2,\cdots,k$.

Suppose $\omega_j(z)$ is the approximation of $\hat{\varphi}(z)$ in $z \in I_j$, we have

$$\omega_j(z) = e^{-\int_0^z \frac{\tilde{\alpha}(x,t)}{2} \mathrm{d}t} [A_j m_j(z) + B_j n_j(z)], (j = 1,2,\cdots,k). \tag{14}$$

Since s_1 is constant for $d < z < D$, the results of field $\phi(\hat{z})$ satisfy $\phi(\hat{z}) = Ae^{\mathrm{i}\gamma\hat{z}} + Be^{-\mathrm{i}\gamma\hat{z}}$, $d < z < D$, where $\gamma = \sqrt{s_1 - \hat{\lambda}}$.

Set $z_0 = 0$ and $z_{k+1} = D$, then two linearly independent solutions are shown on $[z_k, z_{k+1}]$ as follows:

$$m_{k+1}(z) = e^{\mathrm{i}\gamma\hat{z}}, \quad n_{k+1}(z) = e^{-\mathrm{i}\gamma\hat{z}}, \tag{15}$$

with the boundary conditions $\phi(0) = 0$ and $\phi(\hat{D}) = 0$. Therefore, the approximated solutions of ϕ at $z_0 = 0$ and ϕ at $z_{k+1} = D$ should be defined as $y_1(0) = 0$ and $y_{k+1}(D) = 0$, respectively.

To obtain the constants of A_j and B_j $(j = 1,2,\cdots,k)$ we require the solutions $\omega_j(z)$ of field $\hat{\varphi}(z)$ and their first order derivatives are continuous at z_j. Thus, the interface conditions at $z_j (j = 1,2,\cdots,k)$ are

$$\omega_j(z_j) = \omega_{j+1}(z_j), \quad \omega'_j(z_j) = \omega'_{j+1}(z_j); \tag{16}$$

that is,

$$y_j(z_j) = y_{j+1}(z_j), \quad y'_j(z_j) = y'_{j+1}(z_j). \tag{17}$$

According to these conditions, a linear system of A_j and B_j $(j = 1,2,\cdots,k)$ can be obtained.

$$\begin{cases} A_1 m_1(0) + B_1 n_1(0) = 0, \\ A_j m_j(z_j) + B_j n_j(z_j) = A_{j+1} m_{j+1}(z_j) + B_{j+1} n_{j+1}(z_j), \\ A_j m'_j(z_j) + B_j n'_j(z_j) = A_{j+1} n'_{j+1}(z_j) + B_{j+1} n'_{j+1}(z_j), \\ A_{k+1} m_{k+1}(D) + B_{k+1} n_{k+1}(D) = 0, \end{cases} \tag{18}$$

here $(j = 1,2,\cdots,k)$. In order to simplify these equations, let $R_j = B_j / A_j$ $(j = 1,2,\cdots,k)$, the dispersion relations of $\hat{\beta}$ are

$$\begin{cases} R_1 = -\dfrac{m_1(0)}{n_1(0)}, \\ R_{j+1} = -\dfrac{m_{j+1}[m'_j + n'_j R_j] - m'_{j+1}[m_j + n_j R_j]}{n_{j+1}[m'_j + n'_j R_j] - n'_{j+1}[m_j + n_j R_j]}, \\ R_{k+1} = -\dfrac{m_{k+1}(D)}{n_{k+1}(D)} = -e^{2\mathrm{i}\gamma\hat{D}}. \end{cases} \tag{19}$$

Specially, when $j = k$,

$$R_{k+1} = -e^{2\mathrm{i}\gamma d} \cdot \frac{[m'_k + n'_k R_k] + \mathrm{i}\gamma[m_k + n_k R_k]}{[m'_k + n'_k R_k] - \mathrm{i}\gamma[m_k + n_k R_k]}. \tag{20}$$

The evaluation on R_{k+1} depends on R_1 recursively. Furthermore, R_{k+1} is determined by $\gamma = \sqrt{s_1 - \hat{\lambda}}$. Hence, the dispersion relation with respect to $\hat{\beta}$ is as follow:

$$f(\hat{\lambda}) = R_{k+1} + e^{2\mathrm{i}\gamma\hat{D}}. \tag{21}$$

Special Case: when $\tilde{\alpha}(x,z) = 0$, the S-L problem of the operator L becomes

$$\begin{cases} \hat{\varphi}_{zz} + \tilde{\beta}(x,z)\varphi = \hat{\lambda}\varphi, & 0 < z \le d; \\ \hat{\varphi}_{\hat{z}\hat{z}} + \tilde{\beta}_1\varphi = \hat{\lambda}\varphi, & d < z < D; \end{cases} \tag{22}$$

where $\tilde{\beta}(x,z)$ depends only on z, and $\tilde{\beta}_1$ is a constant. Moreover, the top and bottom boundary conditions are $\hat{\varphi}(0) = \varphi(D) = 0$. Applying a polynomial of degree two to interpolate the function $\tilde{\beta}(x,z)$ as $z \in I_j$, then we get the approximated equation

$$\frac{\mathrm{d}^2 y_j}{\mathrm{d}z^2} + (\tilde{a}_j z^2 + \tilde{b}_j z + \tilde{c}_j) y_j = 0, \tag{23}$$

where

$$\begin{cases} \tilde{a}_j = 2[\tilde{\beta}(x,t_0) - 2\tilde{\beta}(x,t_1) + \tilde{\beta}(x,t_2)]/h^2; \\ \tilde{b}_j = [(1-4j)\tilde{\beta}(x,t_0) + (8j-4)\tilde{\beta}(x,t_1) + (3-4j)\tilde{\beta}(x,t_2)]/h; \\ \tilde{c}_j = (2j^2 - j)\tilde{\beta}(x,t_0) + 4(j-j^2)\tilde{\beta}(x,t_1) + (2j^2 - 3j + 1)\tilde{\beta}(x,t_2) - \hat{\lambda}. \end{cases} \tag{24}$$

Referring to the Kummer functions, eventually, we derive the dispersion relations as follows:

$$\begin{cases} R_1 = -\dfrac{m_1(0)}{n_1(0)}, \\ R_{j+1} = -\dfrac{m_{j+1}[m'_j + n'_j R_j] - m'_{j+1}[m_j + n_j R_j]}{n_{j+1}[m'_j + n'_j R_j] - n'_{j+1}[m_j + n_j R_j]}, \\ R_{k+1} = -\dfrac{m_{k+1}(D)}{n_{k+1}(D)} = -e^{2\mathrm{i}\sqrt{\tilde{\beta}_1 - \hat{\lambda}} \cdot \hat{D}}. \end{cases} \tag{25}$$

Asymptotic solutions for the TE case: for the Berenger modes in a dual-layer optical waveguide terminated by a PML, $\hat{s}_0$ is the average value of $s_0(z)$ at the 17 points distributing equally within the interval $[0, d]$. We can obtain the following formula [44]:

$$\hat{\lambda}_\eta \approx s_1 - \left(\frac{W_1}{a} - \frac{aa_2}{W_1^2} - \frac{a^2 a_3}{W_1^3} - \frac{a^3 a_4}{W_1^4} \right)^2, \tag{26}$$

where $\mathrm{Im}(W_1) > 0$, η is an integer, and

$a = \mathrm{i}(\hat{D} - d)$; $a_2 = \delta_1/4$; $a_3 = -\delta_1/4a$;

$a_4 = \dfrac{\delta_1}{4a^2} - \dfrac{1}{16}\delta_1^2$; $\delta_1 = \hat{s}_0 - s_1$; $W_1 = \mathrm{Lambert\,W}(\eta, \pm\frac{\mathrm{i}}{2} \cdot (\hat{D} - d)\sqrt{\delta_1})$;

$\eta \ge 0$ except $\mathrm{LambertW}(0, -\frac{\mathrm{i}}{2}(\hat{D} - d)\sqrt{\delta_1})$.

And asymptotic solutions of leaky modes are also obtained as follow [44]:

$$\gamma_0 \approx \frac{W_\iota}{A} - \frac{AA_2}{W_\iota^2} - \frac{A^2 A_3}{W_\iota^3} - \frac{A^3 A_4}{W_\iota^4}, \tag{27}$$

where

$\gamma_0 = \sqrt{\hat{s}_0 - \hat{\lambda}}$; $a = -\mathrm{i}\,d$; $A_2 = -\dfrac{\delta_1}{4}$; $A_3 = \dfrac{\mathrm{i}\delta_1}{4d}$; $A_4 = -\dfrac{3\delta_1^2}{32} + \dfrac{\delta_1}{4d^2}$;

$\delta_1 = \hat{s}_0 - s_1$; $W_\iota = \mathrm{LambertW}(p, \frac{(-1)^{q+1} d}{2} \cdot \sqrt{\delta_1})$; $\iota = -2(p+1) + q$; for $q = 0,1$; $p = -1,-2,\cdots$.

At last, we find the roots of the equation $f(\hat{\lambda}) = 0$ by the Müller's method with suitable initial values, which are taken by the asymptotic solutions.

Numerical Examples

The theory of this paper has been implemented and tested on a number of examples. For simplicity, we still denote β as $\hat{\beta}$ in the following statements. For example, let $C = 16$, $d = 0.8$, $H = 1.6$, $D = 1.7$, $s_1 = 1.45^2$, $\tilde{\alpha}(x,z) = 15.6(0.4 - z)^3$, and $\tilde{\beta}(x,z) = 60.84 \times \mathrm{sech}(z - 0.4)$.

Therefore, $s_0(z) = 60.84[\mathrm{sech}(z-0.4)-(0.4-z)^6+3(0.4-z)^2/7.8]$ for $0 < z \le d$.

The propagation constants β of propagating modes satisfy $\min(s_0(z)) < \beta^2 < \max(s_0(z))$ for $0 < z \le d$. Thus, the propagation constants of propagating modes, leaky modes and Berenger modes are shown in Tables 1, 2 and 3, respectively. In Table 1, β_1, β_2 and β_3 are taken as the initial values. In Tables 2 and 3, β_1, β_2 and β_3 are represented as the iterative values as k = 16, 32 and 64, respectively.

Table 1: Propagation constants of propagating modes.

β_1	β_2	β_3	Approximate Solution β
7.740	7.745	7.750	7.02573-0.00000i
7.755	7.760	7.765	7.02573-0.00000i
7.770	7.775	7.780	7.02573-0.00000i
7.785	7.790	7.795	7.02573-0.00000i

Table 2: Propagation constants of leaky modes for the TE case.

ι	β_1 for $k=16$	β_2 for $k=32$	β_3 for $k=64$
1	0.83852+ 5.42434i	0.83852+ 5.42434i	0.83852+ 5.42434i
2	1.52393+11.41193i	1.52393+11.41192i	1.52393+11.41192i
3	1.92469+15.63788i	1.92468+15.63788i	1.92468+15.63788i
4	2.33529+19.97154i	2.33529+19.97154i	2.33529+19.97154i
5	2.48289+24.20780i	2.48289+24.20780i	2.48289+24.20780i
6	2.63137+28.30444i	2.63137+28.30444i	2.63137+28.30444i
7	2.77607+32.36134i	2.77607+32.36133i	2.77607+32.36133i
8	2.90682+36.39285i	2.90682+36.39285i	2.90682+36.39285i
9	3.02505+40.40465i	3.02504+40.40464i	3.02504+40.40464i
10	3.13316+44.40159i	3.13316+44.40159i	3.13316+44.40159i
11	3.23285+48.38733i	3.23284+48.38733i	3.23284+48.38733i
12	3.32531+52.36441i	3.32531+52.36440i	3.32531+52.36440i
13	3.41152+56.33462i	3.41151+56.33462i	3.41151+56.33462i
14	3.49226+60.29931i	3.49224+60.29931i	3.49224+60.29931i
15	3.56816+64.25948i	3.56815+64.25949i	3.56815+64.25949i
16	3.63978+68.21591i	3.63977+68.21592i	3.63977+68.21592i
17	3.70756+72.16920i	3.70755+72.16921i	3.70755+72.16921i
18	3.77190+76.11984i	3.77190+76.11984i	3.77190+76.11985i
19	3.83312+80.06821i	3.83312+80.06821i	3.83312+80.06821i
20	3.89152+84.01462i	3.89152+84.01462i	3.89152+84.01462i

Table 3: Propagation constants of Berenger modes for the TE case.

η	β_1 for $k=16$	β_2 for $k=32$	β_3 for $k=64$
1	1.74842+ 9.71020i	1.74842+9.710198i	1.74842+9.710198i
1	2.42335+13.41599i	2.42335+13.41599i	2.42335+13.41599i
2	2.96306+16.91230i	2.96306+16.91230i	2.96306+16.91230i
2	3.51505+20.14151i	3.51505+20.14151i	3.51505+20.14151i
3	4.31788+23.37743i	4.31788+23.37743i	4.31788+23.37743i
3	5.09828+26.69102i	5.09828+26.69102i	5.09828+26.69102i
4	5.86661+29.99240i	5.86661+29.99241i	5.86661+29.99241i
4	6.64466+33.27954i	6.64466+33.27954i	6.64466+33.27954i
5	7.43504+36.56089i	7.43504+36.56089i	7.43504+36.56089i
5	8.23428+39.83971i	8.23429+39.83971i	8.23429+39.83971i
6	9.04015+43.11623i	9.04016+43.11623i	9.04016+43.11623i
6	9.85166+46.39052i	9.85167+46.39052i	9.85167+46.39052i
7	10.66815+49.66286i	10.66815+49.66286i	10.66816+49.66286i
7	11.48902+52.93352i	11.48902+52.93352i	11.48902+52.93352i
8	12.31375+56.20274i	12.31376+56.20274i	12.31376+56.20274i
8	13.14192+59.47070i	13.14193+59.47069i	13.14193+59.47070i
9	13.97316+62.73752i	13.97317+62.73752i	13.97317+62.73752i
9	14.80716+66.00335i	14.80717+66.00335i	14.80717+66.00335i
10	15.64367+69.26828i	15.64367+69.26827i	15.64367+69.26827i
10	16.48245+72.53239i	16.48245+72.53239i	16.48245+72.53239i

Conclusions

To compute the eigenmodes of the complex optical waveguide which is open on one side, based on our previous methods, the second-order linear differential equation in standard form of the corresponding S-L problem has been reduced to the equation without the first derivative term. Then, a PML is used to terminate the open waveguide. Also, the coefficient function of the simplified S-L problem is approximated by a piecewise polynomial of degree two. Since the solutions of the approximated equation in each layer are analytically expressed by the Kummer functions, the approximate dispersion equation is established to the TE case. Apparently, the approximate solutions will converge to the exact ones as the number k of subintervals tends to infinity, or equivalently the step size tends to zero. In the numerical example, we find out the roots of the dispersion equation by the Müller's method, where three different asymptotic solutions of slab waveguides play the roles of initial values. Numerical simulations show that the iteration converges fast and high-precision values for propagation constants may be obtained only if a suitable root-finding method is adopted and some good initial values are given. Thus, further research will be performed in future.

Appendix

When $a \ne 0$, let's begin to consider the second order differential equation as follow

$$y''(z) + (az^2 + bz + c)y(z) = 0, \tag{28}$$

by the change of variables $t = z + (b/2a)$, we have

$$y''(t) + (at^2 + C)y(t) = 0, \tag{29}$$

where $C = c - [b^2/(4a)]$.

Further change of variables

$$x = \sqrt{-a}t^2, \quad \exp(-x/2)w(x) = y(t) \tag{30}$$

leads to

$$x\frac{d^2w}{dx^2} + \left(\frac{1}{2} - x\right)\frac{dw}{dx} - \left(\frac{1}{4} - \frac{C}{4\sqrt{-a}}\right)w(x) = 0, \tag{31}$$

which is in the form of the confluent hypergeometric equation

$$x\frac{d^2w}{dx^2} + (B - x)\frac{dw}{dx} - Aw(x) = 0, \tag{32}$$

with $A = \frac{1}{4} - \frac{C}{4\sqrt{-a}}$ and $B = 1/2$. Two linearly independent solutions are given by the Kummer functions $M(A,B,x)$ and $U(A,B,x)$. However, $U(A,B,x)$ has a branch point $x = 0$, Hence, two linearly independent solutions of Eq. (28) are given by

$$\begin{cases} m(z) = \exp(-x(z)/2)\cdot M\left(A, \frac{1}{2}, x(z)\right), \\ n(z) = \exp(-x(z)/2)\cdot [x(z)]^{\frac{1}{2}} M\left(A + \frac{1}{2}, \frac{3}{2}, x(z)\right), \end{cases} \tag{33}$$

where

$$x(z) = \sqrt{-a}\left[z + b/(2a)\right]^2, \quad A = \frac{1}{4} - \frac{C}{4\sqrt{-a}} = \frac{1}{4} + \frac{b^2 - 4ac}{16a\sqrt{-a}}.$$

Recall the following differential formulas for the Kummer function:

$$\frac{d}{dx}M(a,b,x) = \frac{a}{b}M(a+1,b+1,x). \tag{34}$$

The derivatives of $m(z)$ and $n(z)$ can be expressed as

$$\begin{cases} m'(z) = -\frac{1}{2}x'(z)m(z) + 2Ax'(z)\exp(-x(z)/2) \\ \times M\left(A+1, \frac{3}{2}, x(z)\right), \\ n'(z) = -\frac{1-x(z)}{2x(z)}x'(z)n(z) + \frac{2A+1}{3}\sqrt{x(z)}x'(z) \\ \times \exp(-x(z)/2)M\left(A+\frac{3}{2}, \frac{5}{2}, x(z)\right). \end{cases} \quad (35)$$

Special Case: when a=0, and $b \neq 0$, Eq.(28) becomes

$$y''(z) + (bz + c)y(z) = 0 \quad (36)$$

and two linearly independent solutions are given by the Airy functions

$$m(z) = \mathrm{Ai}(w(z)), \quad n(x) = \mathrm{Bi}(w(z)), \quad (37)$$

where $w(z) = -\frac{bz+c}{b^{2/3}}$. Therefore,

$$m'(z) = -b^{1/3}\,\mathrm{Ai}'(w(z)), \quad n'(x) = -b^{1/3}\,\mathrm{Bi}'(w(z)). \quad (38)$$

The Airy functions and their derivatives can be evaluated in Matlab as

$$\mathrm{airy}(x) = \mathrm{Ai}(x),\ \mathrm{airy}(1,x) = \mathrm{Ai}'(x),$$
$$\mathrm{airy}(2,x) = \mathrm{Bi}(x),\ \mathrm{airy}(3,x) = \mathrm{Bi}'(x). \quad (39)$$

When $a = b = 0$, Eq.(28) becomes

$$y''(z) + cy(z) = 0, \quad (40)$$

which has two linearly indecent solutions

$$m(z) = e^{-\mathrm{i}\sqrt{c}z}, \quad n(z) = e^{\mathrm{i}\sqrt{c}z}. \quad (41)$$

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