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Univalence Criteria for General Integral Operators using the Struve and Bessel Functions

Breaz Daniel[1] and Tanase Carmen-Ioana[2*]

[1]*Daniel Breaz, University of Alba Iulia, N. Iorga Street, No. 11-13, 510009 Alba Iulia, Alba, Romania*
[2]*Tanase Carmen-Ioana, University of Pitesti, Targul din Vale Str., No. 1, 110040 Pitesti, Arges, Romania*

Abstract

In this paper, we consider the class of Bessel functions and the class of Struve functions. We obtain some univalence criteria for two general integral operators.

Keywords: Analytic functions; Struve functions; Bessel functions; Integral operators

Introduction and Preliminaries

Let consider U the unit disc. Let H(U) be the set of holomorphic functions in the unit disc U [1-9].

Consider A={f ∈ H(U): f(z)=z+$a_2z^{2+}a_3z^3$+,z ∈ U} be the class of analytic functions in U and S={f ∈ A: f is univalent in U}.

Theorem 1.1

If the function f is regular in unit disc U, f(z)=z+a_2z^{2+}... and

$$\left(1-|z|^2\right).\left|\frac{zf''(z)}{f'''(z)}\right|<1 \tag{1}$$

for all z ∈ U, then the function is univalent in U [1].

Theorem 1.2

If the function g is regular in U and |g(z)|<1 in U, then for all ξ ∈ U and z ∈ U the following inequalities hold [4]

$$\left|\frac{g(\xi)-g(z)}{1-g(z).g(\xi)}\right|\le\left|\frac{\xi-z}{1-\bar{z}.\xi}\right| \tag{2}$$

and

$$\left|g'(z)\right|\le\frac{1-\left|g'(z)\right|^2}{1-z^2} \tag{3}$$

the equalities hold in case $g(z)=\varepsilon\frac{z+u}{1+\bar{u}z}$ where |ε|=1 and |u|<1.

Remark 1.1

For z=0 from inequality (2) we obtain for every ξ ∈ U [2]

$$\left|\frac{g(\xi)-g(0)}{1-g(\bar{0}).g(\xi)}\right|\le|\xi| \tag{4}$$

and hence

$$\left|g'(\xi)\right|\le\frac{|\xi|+g(0)|}{1+|g(0)||\xi|} \tag{5}$$

Considering g(0)=a and ξ=z, then

$$|g(z)|<\frac{|z|+|a|}{1+|a||z|} \tag{6}$$

for all z ∈ U.

Let us consider the second-order inhomogeneous differential equation (([10]), p.341)

$$z^2w''(z)+zw'(z)+\left(z^2-v^2\right)w(z)=\frac{4\left(\frac{z}{2}\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} \tag{7}$$

whose homogeneous part is Bessel's equation, where v is an unrestricted real (or complex) number. The function H_v, which is called the Struve function of order v, is defined as a particular solution of (7). This function has the form

$$H_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{3}{2}\right).\Gamma\left(v+n+\frac{3}{2}\right)}.\left(\frac{z}{2}\right)^{2n+v+1}\quad \text{for all } z\in\mathbb{C} \tag{8}$$

We consider the transformation

$$g_v=2^v\sqrt{\pi}\Gamma\left(v+\frac{3}{2}\right).z^{\frac{-v-1}{2}}H_v\left(\sqrt{z}\right)) \tag{9}$$

After some calculus we obtain

$$g_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}{4^n.\Gamma\left(n+\frac{3}{2}\right).\Gamma\left(v+n+\frac{3}{2}\right)}.z^n \tag{10}$$

Using Theorem 2.1 ([5]) for our case with $b=c=1,\ k=v+\frac{3}{2}$ we obtain that:

Theorem 1.3 [5],[3], if v>$\frac{\sqrt{3}-7}{8}$ then the function g_v is univalent in U.

The Bessel function of the first kind is defined by

$$J_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+v+1)}.\left(\frac{z}{2}\right)^{2n+v}\quad . \tag{11}$$

We consider the transformation

***Corresponding author:** Tanase Carmen-Ioana, University of Pitesti, Targul din Vale Str., No. 1, 110040 Pitesti, Arges, Romania, Tel: +40 348 453 100; E-mail: carmen_tase@yahoo.com

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$$f_v(z)=2^v\Gamma(1+v)z^{-\frac{v}{2}}j_v(\sqrt{z}) \tag{12}$$

After some calculus we obtain

$$f_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+v)}{n!\Gamma(n+v+1).4^n}.z^n \tag{13}$$

Theorem 1.4

If $v>-2$ then $\operatorname{Re} f z\in U_1(0,4(v+2))$ and f_v is univalent in $U_1(0,4(v+2))$ [7,9,3].

Main Results

Theorem

Let f_{vi} Bessel functions, $z\in U, v\in(-2,-1), \alpha_i\in C$ where

$$f_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+v_i)}{n!\Gamma(n+v_i+1).4^n}.z^n \quad i\in(1,2,\ldots\ldots.,n)$$

If

$$\left|\frac{zf'_{vi}(z)-f_{vi}(z)}{zf_{vi}(z)}\right|<1\text{, for all } i\in\{1,2,\ldots,n\}, (\forall)z\in U \tag{14}$$

$$\frac{|\alpha_1|+|\alpha_2|+\ \ldots\ldots+|\alpha_n|}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}<1 \tag{15}$$

$$|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|<\frac{1}{\max\limits_{|z|<1}\left[\left(1-|z|^2.|z|.\frac{|z|+|c|}{1+|z||c|}\right)\right]} \tag{16}$$

where $|c|=\frac{1}{32}.\left|\frac{1}{(2+v_1)(1+v_1)}+\frac{1}{(2+v_2)(1+v_2)}+\ldots..+\frac{1}{(2+v_n)(1+v_n)}\right|$

then $G(z)=\int_0^z\left(\frac{f_{vi}(t)}{t}\right)^{\alpha 1}.\left(\frac{f_{v2}(t)}{t}\right)^{\alpha 2}\ldots\ldots\left(\frac{f_{v_n}(t)}{t}\right)^{\alpha n}dt\in S$

Proof.

We have $f_{vi}\in S$, $i\in\{1,2,\ldots,n\}$ and $\frac{f_{v_i}(z)}{z_{\alpha_2}}\neq 0$.

For z=0 we have $\left(\frac{f_{vi}(z)}{z}\right)^{\alpha 1}.\left(\frac{f_{v2}(z)}{z}\right)^{\alpha 2}\ldots\ldots\left(\frac{f_{v_n}(z)}{z}\right)^{\alpha_n}=1$.

Consider the function

$$h(z)=\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}.\frac{F''(z)}{F'(z)}$$

The function h has the form:

$$h(z)=\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}.\alpha_1.\frac{zf'_{vi}(z)-f_{vi}(z)}{zf_{vi}(z)}+\ldots\ldots.+\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}'.\alpha_2\frac{zf'_{vn}(z)-f_{vn}(z)}{zf_{vn}(z)}$$

We have:

$$h(0)=\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}.\alpha_1.\alpha^{\frac{1}{2}}+\ldots\ldots.+\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}.\alpha_n.\alpha_2^n$$

Where $a_2^1=\frac{1}{32(2+v_1)(1+v_1)}$

$$a_2^2=\frac{1}{32(2+v_2)(1+v_2)}$$

$$a_2^n=\frac{1}{32(2+v_n)(1+v_n)}$$

By using the relations (14) and (15) we obtain $|h(z)|<1$ and

$$h(0)=\frac{\left|\alpha_1.\alpha^{\frac{1}{2}}+\ldots..+\alpha_2.\alpha_2^n\right|}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}|c| \quad \text{where}$$

$$|c|=\frac{1}{32}.\left|\frac{1}{(2+v_1)(1+v_1)}+\frac{1}{(2+v_2)(1+v_2)}+\ldots..+\frac{1}{(2+v_n)(1+v_n)}\right|$$

Applying Remark 1.1 for the function h we obtain

$$\frac{1}{|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|}.\left|\frac{G''(z)}{G'(z)}\right|<\frac{|z|+|c|}{1+|c||z}$$

$$\left|(1-|z|^2).z.\frac{F''(z)}{F'(z)}\right|<|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|.(1-|z|^2).|z|.\frac{|z|+|c|}{1+|c||z}$$

for all $z\in U$

Let's consider the function H: [0,1] → R

$$H(x)=(1-x^2)x\frac{x+|c|}{1+|c|}.x=|z|$$

$H\left(\frac{1}{2}\right)=\frac{3}{8}.\frac{1+|c|}{2+|c|}.>0$ then $\max\limits_{x\in|0,1|}\max H(x)>0$.

We obtain

$$\left|(1-|z|^2).z.\frac{F''(z)}{F'(z)}\right|<|\alpha_1.\alpha_2\ldots\ldots.\alpha_n|.\max_{|x|<1}(1-|z|^2).|z|.\frac{|z|+|c|}{1+|c||z}$$

Applying the condition (16) we obtain:

$$\left|(1-|z|^2).z.\frac{F''(z)}{F'(z)}\right|<1(\forall)z\in U$$

and from Theorem 1.1 then $F\in S$.

For $\alpha_1=\alpha_2=\ldots=\alpha_n$ in Theorem 2.1 we obtain the next corollary:

Corollary 2.1

Let f_{vi} Bessel functions, $z\in U, v\in(-2,-1), \alpha_i\in C$ where

$$f_{vi}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+v_i)}{n!\Gamma(n+v_i+1).4^n}.z^n \quad i\in(1,2,\ldots\ldots.,n)$$

If

$$\left|\frac{zf'_{vi}(z)-f_{vi}(z)}{zf_{vi}(z)}\right|<1\text{, for all } i\in\{1,2,\ldots,n\}, (\forall)z\in U(17)$$

$$\max_{|x|<1}(1-|z|^2).|z|.\frac{|z|+|c|}{1+|c||z}<1 \tag{18}$$

where $|c|=\frac{1}{32}.\left|\frac{1}{(2+v_1)(1+v_1)}+\frac{1}{(2+v_2)(1+v_2)}+\ldots..+\frac{1}{(2+v_n)(1+v_n)}\right|$ then

$$F(z)=\int_0^z\left(\frac{f_{vi}(t)}{t}\right)^{\alpha 1}.\left(\frac{f_{v2}(t)}{t}\right)^{\alpha 2}\ldots\ldots\left(\frac{f_{v_n}(t)}{t}\right)^{\alpha n}dt\in S$$

Theorem 2.2

Let g_{vi} Struve functions, $z\in U, v\in(-2,-1), \alpha_i\in C$ where

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$$g_{vi}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\frac{3}{2}\right)\Gamma\left(v+\frac{3}{2}\right)}{4^n\Gamma\left(n+\frac{3}{2}\right).\Gamma\left(v+n+\frac{3}{2}\right)}.z^n \quad i\in(1,2,......,n)$$

If

$$\left|\frac{zg'(z)-g_{vi}(z)}{zg_{vi}(\xi)}\right|\le 1 \text{, for all i} \in \{1,2,...,n\}, (\forall)z \in U \tag{19}$$

$$\frac{|\alpha_1|+|\alpha_2|+ ......+|\alpha_n|}{|\alpha_1.\alpha_2........\alpha_n|}<1 \tag{20}$$

$$|\alpha_1.\alpha_2........\alpha_n| < \frac{1}{\max\limits_{|z|<1}\left[\left(1-|z|^2.|z|.\frac{|z|+|c|}{1+|z||c|}\right)\right]} \tag{21}$$

where $|c|=\frac{1}{15}.\left|\frac{1}{(2v_i+3)(2v_i+5)}+\frac{1}{(2v_1+3)(2v_2+5)}+.....+\frac{1}{(2v_n+3)(2v_n+5)}\right|$ then

$$G(z)=\int_0^z\left(\frac{g_{vi}(t)}{t}\right)^{\alpha 1}.\left(\frac{g_{v2}(t)}{t}\right)^{\alpha 2}.......\left(\frac{g_{v_n}(t)}{t}\right)^{\alpha n}dt\in S$$

Proof.

We have $g_{vi} \in S, i \in \{1,2,...,n\}$ and $\frac{g_{v_i}(z)}{z}\neq 0$.

For z=0 we have $\left(\frac{g_{vi}(z)}{z}\right)^{\alpha 1}.\left(\frac{g_{v2}(z)}{z}\right)^{\alpha 2}......\left(\frac{g_{v_n}(z)}{z}\right)^{\alpha_n}=1$

Consider the function

$$h(z)=\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}.\frac{G''(z)}{G''(z)}$$

The function h has the form:

$$h(z)=\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}.\alpha_1.\frac{zg'_{vi}(z)-g'_{vi}(z)}{zg_{vi}(z)}+.......+\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}'.\alpha_2\frac{zg'_{v_n}(z)-g_{v_n}(z)}{zg_{v_n}(z)}$$

We have:

$$h(0)=\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}.\alpha_1,\mathrm{b}^{\frac{1}{2}}+.....+\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}.\alpha_n.b_2^n$$

Where

$$b_2^1=\frac{1}{15(2v_i+3)(2v_i+5)}$$

$$b_2^2=\frac{1}{15(2v_2+3)(2v_2+5)}\backslash$$

$$b_2^n=\frac{1}{15(2v_n+3)(2v_n+5)}$$

By using the relations (19) and (20) we obtain |h(z)|<1 and

$$h(0)=\frac{\left|\alpha_1.\mathrm{b}^{\frac{1}{2}}+.....+\alpha_2.\mathrm{b}_2^n\right|}{|\alpha_1.\alpha_2........\alpha_n|}=|c| \quad \text{where}$$

$$|c|=\frac{1}{15}.\left|\frac{1}{(2v_i+3)(2v_i+5)}+\frac{1}{(2v_1+3)(2v_2+5)}+.....+\frac{1}{(2v_n+3)(2v_n+5)}\right|.$$

Applying Remark 1.1 for the function h we obtain

$$\frac{1}{|\alpha_1.\alpha_2........\alpha_n|}.\left|\frac{G''(z)}{G'(z)}\right|<\frac{|z|+|c|}{1+|c||z}$$

$$\Leftrightarrow\left|(1-|z|^2).z.\left|\frac{G''(z)}{G'(z)}\right|\right|<|\alpha_1.\alpha_2........\alpha_n|.\max_{|x|<1}(1-|z|^2).|z|.\frac{|z|+|c|}{1+|c||z}$$

for all z∈U

Let's consider the function H : [0,1] → R

$$H(x)=(1-x^2)x\frac{x+|c|}{1+|c|};x=|z|$$

$H\left(\frac{1}{2}\right)=\frac{3}{8}.\frac{1+|c|}{2+|c|}.>0$ then $\max\limits_{x\in|0,1|}$ H(x) > 0.

We obtain

$$\Leftrightarrow\left|(1-|z|^2).z.\left|\frac{G''(z)}{G'(z)}\right|\right|<|\alpha_1.\alpha_2........\alpha_n|.\max_{|x|<1}(1-|z|^2).|z|.\frac{|z|+|c|}{1+|c||z}<1.$$

Applying the condition (21) we obtain:

$$\left|(1-|z|^2).z.\left|\frac{G''(z)}{G'(z)}\right|\right|<1$$

and from Theorem 1.1 then G ∈ S.

In Theorem 2.2 we consider $\alpha_{1=}\alpha_{2=}...=\alpha_{n=}1$ and obtain the next corollary:

Corollary 2.2 Let g_{vi} Bessel functions, $z \in U_1, v \in (-2,-1), \alpha_i \in C$ where

$$g_{vi}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+v_i)}{\Gamma(n+v_i+1).4^n}.z^n \quad i\in(1,2,......,n)$$

If

$$\left|\frac{zg'(z)-g_{vi}(z)}{zg_{vi}(\xi)}\right|\le 1 \text{, for all i} \in \{1,2,...,n\}, (\forall)z \in U \tag{22}$$

$$\max_{|x|<1}(1-|z|^2).|z|.\frac{|z|+|c|}{1+|z||c}<1 \tag{23}$$

where $|c|=\frac{1}{15}.\left|\frac{1}{(2v_i+3)(2v_i+5)}+\frac{1}{(2v_1+3)(2v_2+5)}+.....+\frac{1}{(2v_n+3)(2v_n+5)}\right|$
then

$$G(z)=\int_0^z\left(\frac{g_{vi}(t)}{t}\right)^{\alpha 1}.\left(\frac{g_{v2}(t)}{t}\right)^{\alpha 2}.......\left(\frac{g_{v_n}(t)}{t}\right)^{\alpha n}dt\in S$$

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