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Dissipative Nonlinear Schrödinger Equations with Singular Data

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Abstract

We consider the long time asymptotics for dissipative nonlinear Schrödinger equations of order $1<p<3$ with singular data including the Dirac delta function in one space dimension. 2000 Mathematics Subject Classification: 35Q55, 35B40.

Keywords: Dissipative NLS equations; Large initial data; Large time asymptotics

Introduction

We consider the initial value problem for the following nonlinear Schrödinger equations in one space dimension

$$i\partial_t u+\frac{1}{2}\partial_x^2 u=\lambda|u|^{p-1}u, u(0,x)=\phi(x), \tag{1}$$

where $x\in\mathbb{R}$, $t\in\mathbb{R}$, $\lambda=\lambda_1+i\lambda_2$, $\lambda_j\in\mathbb{R}$, $j=1,2$, $\lambda_2<0, |\lambda_2|>\frac{p-1}{2\sqrt{p}}|\lambda_1|$, $1<p<3$, $u(t,x)$ is an unknown complex-valued function. There are some works concerning the physical applications of (1) [1,2]. We note that $\lambda_2<0$ implies the dissipation of $|u(t,x)|$ by nonlinear Ohm's law [1]. We are interested in the initial data involving the Dirac delta function. Therefore the data are not necessarily in $\mathbf{L}^2$. Related work can be seen in [3] in which homogeneous weighted $\mathbf{L}^2$ space was considered. Let $\mathbf{L}^\infty$ denote the usual Lebesgue space with the norm $\|\phi\|_{\mathbf{L}^\infty}=\operatorname{ess.sup}_{x\in\mathbb{R}}|\phi(x)|$. For $m,s\in\mathbb{R}$, weighted homogeneous Sobolev space $\mathring{\mathbf{H}}^m$ is defined by $\mathring{\mathbf{H}}^m=\left\{f;\|f\|_{\mathring{\mathbf{H}}^m}=\left\|(-\Delta)^{m/2}f\right\|_{\mathbf{L}^2}<\infty\right\}$, where $\|f\|_{\mathbf{L}^2}^2=\int_{\mathbb{R}}|f(x)|^2\,dx$. Let us introduce some notations. We define the dilation operator by $(D_t\phi)(x)=(it)^{-\frac{1}{2}}\phi(x/t)$ for $t\neq 0$ and define $M=e^{\frac{i}{2t}x^2}$ for $t\neq 0$, $E=e^{-\frac{i}{2}t\xi^2}$. Evolution operator $U(t)$ is written as $U(t)=MD_t\mathcal{F}M$, where $\mathcal{F}$ denotes the Fourier transform. We also have $U(-t)=M^{-1}\mathcal{F}^{-1}D_t^{-1}M^{-1}$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform. We denote by the same letter C various positive constants. The standard generator of Galilei transformations is given by $J(t)=U(t)xU(-t)=x+it\partial_x$. We also have commutation relations with J and $L=i\partial_t+\frac{1}{2}\partial_x^2$ such that $[L,J]=0$. To prove our main result, we introduce the function space:

$$X_{m,T}=\{y;\mathcal{F}U(-t)y\in C([0,T);\mathbf{L}^\infty\cap\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}\cap\mathring{\mathbf{H}}^m),\|y\|_{X_{m,T}}<\infty\},$$

where $\varepsilon>0$ is small enough, $\|y\|_{X_{m,T}}=\sup_{0\le t<T}\|\mathcal{F}U(-t)y(t)\|_{\mathbf{Y}_m}$ with $\mathbf{Y}_m=\mathbf{L}^\infty\cap\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}\cap\mathring{\mathbf{H}}^m$ and $m=1,2.$, our main result is

Theorem 1: We assume that $\mathcal{F}\phi\in\mathbf{Y}_1$. Then the Cauchy problem (1) with $\frac{5+\sqrt{33}}{4}<p<3$ has a unique global solution $u\in X_{1,\infty}$ satisfying the time decay estimate

$$\|u(t)\|_{\mathbf{L}^\infty}\le Ct^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}$$

for any $t>0$.

Since the solution of the linear problem with the Dirac delta δ is given by $U(t)\delta=ct^{-\frac{1}{2}}e^{\frac{ix^2}{2t}-i\frac{\pi}{4}}, c=(2\pi)^{-1/2}$, we look for the solution of (1) with the Dirac delta δ as the initial function in the form

$$u(t,x)=t^{-\frac{1}{2}}e^{\frac{ix^2}{2t}-i\frac{\pi}{4}}C(t), C(0)=c. \tag{2}$$

We have the ordinary differential equation $idC(t)/dt=\lambda t^{-\frac{p-1}{2}}|C(t)|^{p-1}C(t)$. We change $C(t)=w(t)e^{i\arg C(t)}$, then

$$\frac{d}{dt}w(t)-\lambda_2 t^{-\frac{p-1}{2}}|w(t)|^{p-1}w(t)=0, w(0)=c, \arg C(0)=0,$$

which can be solved explicitly as $w(t)=c\left(1-\lambda_2\frac{2(p-1)}{3-p}c^{p-1}t^{\frac{3-p}{2}}\right)^{-\frac{1}{p-1}}$ and we also have $d\arg C(t)/dt=-\lambda_1 t^{-\frac{p-1}{2}}|w(t)|^{p-1}$ from which it follows that

$$\arg C(t)=-\lambda_1\int_0^t\tau^{-\frac{p-1}{2}}|w(\tau)|^{p-1}d\tau=\frac{\lambda_1}{\lambda_2(p-1)}\log\left(1-\lambda_2\frac{2(p-1)}{3-p}c^{p-1}t^{\frac{3-p}{2}}\right).$$

Thus the solution has the form (2) with $C(t)=w(t)e^{i\arg C(t)}$. It is expected that the solution of (1) with data involving the Dirac delta function behaves like (2). Our result says that the upper bound for solutions is the same as given in (2).

Local Existence

In this section we prove the local existence of solutions in $X_{1,T}$. We denote the remainder terms

$$R_1=|\mathcal{F}M\mathcal{F}^{-1}\varphi|^{p-1}\mathcal{F}M^{-1}\varphi-|\varphi|^{p-1}\varphi,$$

$$R_2=\mathcal{F}(M^{-1}-1)\mathcal{F}^{-1}|\mathcal{F}M\mathcal{F}^{-1}\varphi|^{p-1}\mathcal{F}M\mathcal{F}^{-1}\varphi,$$

where we denote $\varphi=\mathcal{F}U(-t)v$. We need

Lemma 1: The estimate is true

$$\|R_j\|_{\mathbf{L}^\infty}\le Ct^{-\frac{\varepsilon}{2}}\left(t^{-\frac{\mu}{2}(p-1)}\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\mu}}^{p-1}+\|\varphi\|_{\mathbf{L}^\infty}^{p-1}\right)\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}},$$

For $0<\varepsilon$, $\mu\le\frac{3}{2}$, $j=1,2$, provided the right-hand side is finite.

Proof: By the Sobolev inequality

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$$\left\|\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi\right\|_{\mathbf{L}^\infty} \le C\left\|\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi\right\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}^{\frac{1}{1+2\varepsilon}}\left\|\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi\right\|_{\mathbf{L}^2}^{\frac{2\varepsilon}{1+2\varepsilon}}$$
$$\le Ct^{-\frac{\varepsilon}{2}}\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}} \tag{3}$$

for $0<\varepsilon\le\frac{3}{2}$. Also we write $\mathcal{F}M\mathcal{F}^{-1}\varphi=\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi+\varphi$. Hence the first term is estimated as

$$\|R_1\|_{\mathbf{L}^\infty} \le C\left(\left\|\mathcal{F}M\mathcal{F}^{-1}\varphi\right\|_{\mathbf{L}^\infty}^{p-1}+\|\varphi\|_{\mathbf{L}^\infty}^{p-1}\right)\left\|\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi\right\|_{\mathbf{L}^\infty}$$
$$\le Ct^{-\frac{\varepsilon}{2}}\left(t^{-\frac{\mu}{2}(p-1)}\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\mu}}^{p-1}+\|\varphi\|_{\mathbf{L}^\infty}^{p-1}\right)\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}.$$

In the same manner, we estimate the second term

$$\|R_2\|_{\mathbf{L}^\infty} \le Ct^{-\frac{\varepsilon}{2}}\left\|\mathcal{F}M\mathcal{F}^{-1}\varphi\right\|_{\mathbf{L}^\infty}^{p-1}\left\|\mathcal{F}M\mathcal{F}^{-1}\varphi\right\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}$$
$$\le Ct^{-\frac{\varepsilon}{2}}\left(t^{-\frac{\mu}{2}(p-1)}\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\mu}}^{p-1}+\|\varphi\|_{\mathbf{L}^\infty}^{p-1}\right)\|\varphi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}.$$

This completes the proof of the lemma.

Let us consider the linearized version of (1)

$$i\partial_t u+\frac{1}{2}\partial_x^2 u=\lambda|v|^{p-1}v,\quad u(0,x)=\phi(x), \tag{4}$$

where $v\in X_{1,T}, \|v\|_{X_{1,T}}\le\rho$ and $\rho>0$. By the integral equation associated with (4), we have

$$\|\psi(t)\|_{\mathring{\mathbf{H}}^1}\le\|\mathcal{F}\phi\|_{\mathring{\mathbf{H}}^1}+C\int_0^T\|v(t)\|_{\mathbf{L}^\infty}^{p-1}\|\varphi(t)\|_{\mathring{\mathbf{H}}^1}dt,$$

where $\psi(t)=\mathcal{F}U(-t)u(t)$, $\varphi(t)=\mathcal{F}U(-t)v(t)$. Using the representation $v(t)=MD_t\varphi(t)+MD_t\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi(t)$, in view of (3), we have

$$\|v\|_{\mathbf{L}^\infty}\le t^{-\frac{1}{2}}\|\varphi(t)\|_{\mathbf{L}^\infty}+t^{-\frac{1}{2}}\|\mathcal{F}(M-1)\mathcal{F}^{-1}\varphi(t)\|_{\mathbf{L}^\infty}$$
$$\le t^{-\frac{1}{2}}\|\varphi(t)\|_{\mathbf{L}^\infty}+t^{-\frac{1+\varepsilon}{2}}\|\varphi(t)\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}\le Ct^{-\frac{1}{2}}\left(1+t^{-\frac{\varepsilon}{2}}\right)\rho. \tag{5}$$

Hence

$$\|\psi(t)\|_{\mathring{\mathbf{H}}^1}\le\|\mathcal{F}\phi\|_{\mathring{\mathbf{H}}^1}+C\rho^p\int_0^T t^{\frac{1-p}{2}}\left(1+t^{-\frac{\varepsilon}{2}(p-1)}\right)dt$$
$$\le\|\mathcal{F}\phi\|_{\mathring{\mathbf{H}}^1}+C\rho^pT^{\frac{3-p}{2}-\frac{\varepsilon}{2}(p-1)} \tag{6}$$

if $3-\frac{2\varepsilon}{1+\varepsilon}>p$ and $0<T<1$. In the same way as in the proof of (6), we have

$$\|\psi(t)\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}\le\|\mathcal{F}\phi\|_{\mathring{\mathbf{H}}^{\frac{1}{2}+\varepsilon}}+C\rho^pT^{\frac{3-p}{2}-\frac{\varepsilon}{2}(p-1)}, \tag{7}$$

if $3-\frac{2\varepsilon}{1+\varepsilon}>p$ and $0<T<1$. We multiply both sides of the equation (4) by $\mathcal{F}U(-t)$, to get

$$i\partial_t\psi(t)=\lambda t^{-\frac{p-1}{2}}|\varphi(t)|^{p-1}\varphi(t)+\lambda t^{-\frac{p-1}{2}}(R_1+R_2). \tag{8}$$

We have by (8) and Lemma 1 with $\mu=\varepsilon$

$$\|\psi(t)\|_{\mathbf{L}^\infty}\le\|\mathcal{F}\phi\|_{\mathbf{L}^\infty}+C\rho^p\int_0^T t^{\frac{1-p}{2}-\frac{\varepsilon}{2}p}dt$$
$$\le\|\mathcal{F}\phi\|_{\mathbf{L}^\infty}+C\rho^pT^{\frac{3-p}{2}-\frac{\varepsilon}{2}p} \tag{9}$$

if $3-\frac{3\varepsilon}{1+\varepsilon}>p$ and $0<T<1$. Hence by (6), (7) and (9), we have

$$\|\psi(t)\|_{\mathbf{Y}_1}\le\frac{\rho}{2}+C\rho^pT^{\frac{3-p}{2}-\frac{\varepsilon}{2}p} \tag{10}$$

which implies there exists a $T=T(\rho)>0$ such that $\|u\|_{X_{1,T}}\le\rho$. In the same way, we find that there exists a $T=T(\rho)>0$ such that

$$\|u_1-u_2\|_{X_{1,T}}\le\frac{1}{2}\|v_1-v_2\|_{X_{1,T}} \tag{11}$$

for the difference of two solutions, where $i\partial_t u_j+\frac{1}{2}\partial_x^2u_j=\lambda|v_j|^{p-1}v_j$, $u_j(0,x)=\phi(x)$, $j=1,2$. By (10) and (11), we find that there exists a time T such that the transformation $u=Sv$ is a contraction mapping from $X_{1,T}$ into itself. This implies the local existence of solutions in $X_{1,T}$.

Proof of Theorem 1

Under the assumptions $\lambda_2<0, |\lambda_2|>\frac{p-1}{2\sqrt{p}}|\lambda_1|$, we have a dissipation property of solutions to (1) such that $\|Ju(t)\|\le\|x\phi\|$ [2,4,5]. From the local existence theorem, it is enough to prove the a-priori estimate of $\mathcal{F}U(-t)u(t)$. We represent the solution of (8) in the form $\psi=re^{iw}$, $r=|\psi|$, $w=\arg\psi$, then we find

$$i\partial_t r-r\partial_t w=\lambda t^{-\frac{p-1}{2}}r^p+\lambda t^{-\frac{p-1}{2}}e^{-iw}(R_1+R_2)$$

from which it follows that

$$\begin{cases}\partial_t r=\lambda_2 t^{-\frac{p-1}{2}}r^p+\operatorname{Im}\left(\lambda t^{-\frac{p-1}{2}}e^{-iw}(R_1+R_2)\right),\\ \partial_t w=-\lambda_1 t^{-\frac{p-1}{2}}r^{p-1}-r^{-1}\operatorname{Re}\left(\lambda t^{-\frac{p-1}{2}}e^{-iw}(R_1+R_2)\right).\end{cases}$$

Then we have

$$\partial_t r-\lambda_2 t^{-\frac{p-1}{2}}r^p\le C\rho^pt^{\frac{1-p}{2}-\frac{1}{4}},$$

if $\mu(p-1)<\frac{1}{2}$, since we have $\|R_j\|_{\mathbf{L}^\infty}\le C\left(1+t^{-\frac{\mu}{2}(p-1)}\right)\langle t\rangle^{-\frac{1}{4}}\rho^p$ for $j=1,2$ by Lemma 1. Define

$$F(t)=\frac{r(0)}{\left(1-\frac{2\lambda_2(p-1)}{3-p}t^{\frac{3-p}{2}}r(0)^{p-1}\right)^{\frac{1}{p-1}}}.$$

By a direct calculation we get $dF/dt-\lambda_2 t^{-\frac{p-1}{2}}F^p=0, F(0)=r(0)$. Multiplying both sides of the above inequality by F^{-p}, we obtain

$$\frac{d}{dt}\left(F^{-p}r\right)+\lambda_2 t^{-\frac{p-1}{2}}\left(pF^{-1}r-F^{-p}r^p\right)\le CF^{-p}\rho^pt^{\frac{1-p}{2}}\left(1+t^{-\frac{\mu}{2}(p-1)}\right)\langle t\rangle^{-\frac{1}{4}}.$$

By the Young inequality $pF^{-1}r\le F^{-p}r^p+(p-1)$. If $\lambda_2<0$, then

$$\frac{d}{dt}\left(F^{-p}r\right)\le-\lambda_2(p-1)t^{-\frac{p-1}{2}}+CF^{-p}\rho^pt^{\frac{1-p}{2}}\left(1+t^{-\frac{\mu}{2}(p-1)}\right)\langle t\rangle^{-\frac{1}{4}}.$$

Integrating in time, we get

$$r(t)\le(F(t))^p\left(r(0)^{1-p}-\frac{2\lambda_2(p-1)}{3-p}t^{\frac{3-p}{2}}\right)$$
$$+C\rho^p(F(t))^p\int_0^t(F(\tau))^{-p}\tau^{\frac{1-p}{2}}\left(1+\tau^{-\frac{\mu}{2}(p-1)}\right)\langle\tau\rangle^{-\frac{1}{4}}d\tau$$
$$=F(t)+C\rho^p(F(t))^p\int_0^t(F(\tau))^{-p}\tau^{\frac{1-p}{2}}\left(1+\tau^{-\frac{\mu}{2}(p-1)}\right)\langle\tau\rangle^{-\frac{1}{4}}d\tau, \tag{12}$$

since by the definition $(F(t))^{1-p}=r(0)^{1-p}-2\lambda_2t^{\frac{3-p}{2}}(p-1)/(3-p)$. Let us consider the second term of the right-hand side of (12). We find $(F(\tau))^{-p}=r(0)^{-p}\left(1-\frac{2\lambda_2(p-1)}{3-p}\tau^{\frac{3-p}{2}}r(0)^{p-1}\right)^{\frac{p}{p-1}}$, therefore

$$(F(t))^p\int_0^t(F(\tau))^{-p}\tau^{\frac{1-p}{2}}\left(1+\tau^{-\frac{\mu}{2}(p-1)}\right)\langle\tau\rangle^{-\frac{1}{4}}d\tau$$
$$\le C\langle t\rangle^{-\frac{p(3-p)}{2(p-1)}}\int_0^1\tau^{-\frac{p-1}{2}-\frac{\mu}{2}(p-1)}d\tau+C\langle t\rangle^{-\frac{p}{2}\left(\frac{3-p}{p-1}\right)}\int_1^t\tau^{\frac{p}{2}\left(\frac{3-p}{p-1}\right)-\frac{p-1}{2}-\frac{1}{4}}d\tau$$
$$\le C\langle t\rangle^{-\frac{p}{2}+\frac{5}{4}}\le C\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}$$

for $-\frac{p-1}{2}-\frac{\mu}{2}(p-1)+1>0$ and $-\frac{p}{2}+\frac{5}{4}<\frac{1}{2}-\frac{1}{p-1}$ which is satisfied if $p>\frac{5+\sqrt{33}}{4}$. Hence

$$r(t)\le F(t)+C\rho^{p}\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}\le C\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}.$$

Since $u(t)=MD_t\psi+MD_t\mathcal{F}(M-1)\mathcal{F}^{-1}\psi$, we have

$$\|u(t)\|_{\mathbf{L}^\infty}\le Ct^{-\frac{1}{2}}\|r(t)\|_{\mathbf{L}^\infty}+Ct^{-\frac{1}{2}}\|\mathcal{F}(M-1)\mathcal{F}^{-1}\psi\|_{\mathbf{L}^\infty}\le Ct^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}.$$

This completes the proof of the theorem.

Finally we make a remark. In the same way as in the proof of Theorem 1.2 from [5], we obtain the following result.

Theorem 2: Assume that $\mathcal{F}\phi\in\mathbf{Y}_2$. Then the Cauchy problem (1) with $\frac{19+\sqrt{145}}{12}<p\le\frac{5+\sqrt{33}}{4}$ has a unique global solution $u\in X_{2,\infty}$ satisfying the time decay estimate

$$\|u(t)\|L\infty\le Ct^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}-\frac{1}{p-1}}$$

for all $t>0$.

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