Journal of
Applied & Computational Mathematics

Huang et al., J Appl Computat Math 2016, 5:1
DOI: 10.4172/2168-9679.1000291

Research Article Open Access

Mathematical Modelling and Computer Simulation

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Abstract

This article adopts and analyzes a stochastic collocation method to approximate the solution of four order elliptic partial differential equations with random coefficients and forcing terms, which are applied for some mathematical-biology model. The method is composed of a Galerkin finite approximation in space and a collocation in the zeros of suitable tensor product orthogonal polynomials (Gauss points) in the probability space, and natural brings on the solution of uncoupled deterministic problems. The well-posedness of the elliptic partial differential equations is investigated as well under some regular assumptions. Strong error estimates for the fully discrete solution using L2 norms are obtained in this work.

Keywords: Collocation techniques; Galerkin finite element methods; Stochastic PDEs; Smolyak approximation

Introduction

Mathematical modeling and computer simulation are nowadays widely used tools to predict the behavior of biological research problems. To illustrate the idea, we consider nonlocal effects and long range diffusion mathematical biology model [1]. The classical approach to diffusion is the following form

$$\frac{\partial u(x,t)}{\partial t} = \nabla.\alpha_1(x,u)\nabla u + f(u,x,t) \tag{1}$$

where $u(x, t)$ is the concentration of the species and α_1 is the diffusion coefficient or diffusivity of $u(x, t)$. Situations where $_1$ is space-dependent are arising in more and more modeling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey matter in the growth and spread of brain tumors. The source term or forcing term f in an ecological context, for example, could represent the birth-death process. However, the equation (1) is strictly only applicable to dilute systems, that is the diffusion is a local or short range effect. In many biological areas, such as embryological development, the densities of cells involved are not small and a local or short range diffusive flux proportional to the gradient is not sufficiently accurate. When we discuss the mechanical theory of biological pattern formation in certain circumstances, it is intuitively reasonable, perhaps necessary, to include long range effects. In 1969, Othmer derived the following formulation (1).

$$\frac{\partial u(x,t)}{\partial t} = \nabla.\alpha_1(x,u)\nabla u - \Delta(\alpha_2\Delta u) + f(u,x,t) \tag{2}$$

where $\alpha_1 > 0$ and α_2 are constants. α_2 is a measure of the long range effects and in general is smaller in magnitude than α_1. The biharmonic term is stabilizing if $\alpha_2 > 0$, or destabilizing if $\alpha_2 < 0$. In this form, the first term represents an average of nearest neighbors and the second biharmonic term is a contribution from the average of nearest averages.

We then consider the stationary Dirichelet boundary value problem of the equation (2)

$$\begin{aligned} -\nabla\cdot(\alpha_1(x,u)\nabla u) + \Delta(\alpha_2\Delta u) &= f(u,x) \\ u(x) &= 0, on\,\partial D, \\ \frac{\partial u(x)}{\partial n} &= 0, on\,\partial D \end{aligned} \tag{3}$$

where D represents the species living area, which can be considered bounded and the Dirichelet boundary condition can be interpreted as the number of the species is zero on the boundary of the domain D. Yet many biological applications are affected by a relatively large amount of uncertainty in the input data, such as model coefficients, source term/forcing term, boundary conditions, and geometry. In the case, to obtain a reliable numerical prediction, one has to include uncertainty quantification due to uncertainty in the input data. In this paper we focus on problem (3) with a probabilistic description of the uncertainty in the input data. Let D be a convex bounded polygonal domain in $\mathbb{N}^d$, (d = 1, 2, 3) and $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events, and $P:\mathcal{F} \to [0,1]$ is probability measure. Consider the stochastic linear fourth-order elliptic boundary value problem: find a random function $u(\omega;x):\Omega\times\bar{D} \to \mathbb{R}$ such that P-almost everywhere (a.e.) in Ω, or in other words, almost surely (a. s.) the following equations hold

$$\begin{aligned} -\nabla\cdot\alpha_1(\omega,\cdot)\nabla u + \Delta(\alpha_2(\omega,\cdot)\Delta u(\omega,\cdot)) &= f(\omega,\cdot), on\, D, \\ u(\omega,\cdot) &= 0,\ on\ \partial D, \\ \frac{\partial u(\omega,\cdot)}{\partial n} &= 0,\ on\ \partial D. \end{aligned} \tag{4}$$

We make the following assumptions:

(A_1) $\alpha_1(\omega,\cdot)$, $\alpha_2(\omega,\cdot)$ is uniformly bounded, i.e. there exist a_{min}, $a_{max} > 0$ such that

$$P\left(\omega\in\Omega : \alpha_i(x) \in [a_{\min}, a_{\max}], \forall x \in \bar{D}, i = 1,2\right) = 1\cdot$$

where $\bar{D}$ is the closure of D.

(A_2) $f(\omega,\cdot)$ is square integrable with respect to P, i.e. $\int_D E[f^2]dx < \infty$ this article will establish a stochastic collocation method for problem

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Received February 15, 2016; **Accepted** February 24, 2016; **Published** March 01, 2016

Citation: Huang Y, Zhang H, Laibin G (2016) Mathematical Modelling and Computer Simulation. J Appl Computat Math 5: 291. doi:10.4172/2168-9679.1000291

(1). To describe the method, we first introduce used spaces as follows

$$H = L^2(D) = \left\{ v : \int_D |v|^2 \, dx < \infty \right\}$$

and

$$L^\infty(D) = \left\{ v : \int_D ess \sup_{x\in D} |v(x)| < +\infty \right\}.$$

Denote $\alpha = (\alpha_1,\cdots,\alpha_n)$, α_i is nonnegative integers, $i = 1,\cdots,n$, the length of α is given by $|\alpha| = \sum_{i=1}^n \alpha_i$. Suppose that the weak derivatives $D^{|\alpha|}$ exist for all $|\alpha| \le k$, where k is nonnegative integer, and define the Sobolev space.

$$H^k(D) = v\left\{ \upsilon : D^\alpha v \in L^2(D), |\alpha| \le k \right\}.$$

The space $H^k(D)$ is endowed with the norm associated to the inner product

$$(v,w)k = \sum_{|\alpha|\le k} \left(D^\alpha v, D^\alpha w \right),\ \forall v,w \in H^k(D),$$

and the corresponding norm

$$\|v\|_k = \left((v,v)_k \right)^{1/2}.$$

Define Semi-norm in the Sobolev space $H^k(D)$

$$|v|_k^2 = \sum_{|\alpha|=k} \left(D^\alpha v, D^\alpha \right),\ v \in H^k(D).$$

Denote by $H_0^k(D)$ the closure of $C_0^\infty(D)$ with the norm $\|v\|_k$ in $H^k(D)$. Let v is strongly measurable and Define

$$L_P^2\left(\Omega; H_0^2(D)\right) = \left\{ v : \Omega \to H_0^2(D) \mid \int_\Omega \|v\|_{H_0^2}^2 \, dP(\omega) < \infty \right\},$$

and

$$L_P^\infty\left(\Omega; H_0^2(D)\right) = \left\{ v : \Omega \to H_0^2(D) \mid P - ess \sup_{x\in\omega} \|v\|_{H_0^2}^2 < \infty \right\}.$$

Next, using the Green formula, problem (4) can be written in the following weak form: find $u \in L_P^2\left(\Omega; H_0^2(D)\right)$ such that

$$\int_D E\left([\alpha_1 \nabla u \cdot \nabla v] + [\alpha_2 \Delta u \cdot \Delta v] \right) dx = \int_D E[f \cdot v] dx, \forall v \in L_P^2\left(\Omega; H_0^2(D)\right). \quad (5)$$

finally, we discuss the well-posedness of problem (5). To the end, we quote the following Poincare's inequality

$$\|v\|_k^2 \le C_P |v|_k^2,\ \forall v \in H_0^k(D),$$

with $C_P = C_P(D, n, k) > 0$, [2].

Lemma 3.1: Under assumptions (A_1) and (A_2), problem (2) admits a unique solution $u \in L_P^2\left(\Omega; H_0^2(D)\right)$ which satisfies the following estimate

$$\|u\|_{L_P^2\left(\Omega; H_0^2(D)\right)} \le \frac{C_P}{a_{\min}} \left(\int_D E|f|^2 \, dx \right)^{1/2}.$$

Proof: Define a bilinear form on $L_P^2\left(\Omega; H_0^2(D)\right)$

$$a(u,v) = \int_D E\left([\alpha_1 \nabla u \cdot \nabla v] + [\alpha_2 \Delta u \cdot \Delta v] \right) dx.$$

We will prove $a(u, v)$ is continuous and coercive.

$$|a(u,v)| \le \left| \int_D E[\alpha_1 \nabla u \cdot \nabla v] dx \right| + \left| \int_D E[\alpha_2 \Delta u \cdot \Delta v] dx \right| \le a_{\max} \|u\|_{L_P^2\left(\Omega; H_0^2(D)\right)} \|v\|_{L_P^2\left(\Omega; H_0^2(D)\right)}$$

By the Poincare' inequality, we have

$$a(u,u) = \int_D E\left([\alpha_1 \nabla u \cdot \nabla u] + [\alpha_2 \Delta u \cdot \Delta u] \right) dx \ge \frac{a_{\min}}{C_P} \|u\|_{L_P^2\left(\Omega; H_0^2(D)\right)}^2,$$

Where $C_P = \min\left\{ C_P(D,n,1), C_P(D,n,2) \right\}$. Thus, problem (5) admits a unique solution $u \in L_P^2\left(\Omega; H_0^2(D)\right)$ by the Lax-Milgram theorem, moreover, the following estimate holds

$$\|u\|_{L_P^2\left(\Omega; H_0^2(D)\right)} \le \frac{C_P}{a_{\min}} \left(\int_D E|f|^2 \, dx \right)^{1/2}.$$

The proof is now complete.

Finite-Dimensional Noise Assumption

In many problems the source of randomness can be approximated using just a small number of uncorrelated, sometimes independent, random variables, for example, the case of a truncated Karhunen-Loeve expansion [3]. This motivates us to make the following assumption.

Assumption: The coefficients and forcing terms used in the computations have the forms

$$a(\omega,x) = a\left(Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega), x \right),\ on\ \Omega \times \bar{D}.$$

and

$$f(\omega,x) = f\left(Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega), x \right),\ on\ \Omega \times \bar{D}.$$

where N ∈ N_+ and $\{Y_n\}_{n=1}^N$ are real-value random variables with mean value zero and unit variance. Denote with $\Gamma_n \equiv Y_n(\omega)$, the image of $Y_n, \Gamma^N = \Pi \sum_{n=1}^N \Gamma_n$, where we assume Yn(ω), to be bounded, without loss of generality we can assume $\Gamma_n = [-1,1]$, and we suppose that the random variables $Y = [Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega)]$ have a joint probability density function $\rho : \Gamma^N \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma^N)$.

After making Assumption, the solution u of the stochastic fourth-order elliptic boundary value problem (5) can be described by just a finite number of random variables, i.e. $u(\omega,x) = u\left(Y_1(w), Y_2(w), \cdots, Y_N(w), x \right)$. Then, the goal is to approximate the $u(y, x)$, where $y \in \Gamma^N$ and $x \in \bar{D}$. Observe that the stochastic variational formulation (5) has a deterministic equivalent which is the following: find $u \in L_\rho^2\left(\Gamma^N; H_0^2(D) \right)$ such that

$$\int_{\Gamma^N} \rho\left((\alpha_1 \nabla u, \nabla u)_{L^2(D)} + (\alpha_2 \Delta u, \Delta v)_{L^2(D)} \right) dy = \int_{\Gamma^N} \rho(f,v)_{L^2(D)} dy\ \forall v \in L_\rho^2\left(\Gamma^N; H_0^2(D) \right) \quad (6)$$

where

$$L_\rho^2\left(\Gamma^N; H_0^2(D) \right) = \left\{ v : \Omega \to H_0^2(D) \mid \int_{\Gamma^N} \rho \|v\|_{H_0^2(D)}^2 \, dy < \infty \right\}.$$

Since the solution of (1.6) is unique and is also a solution of (5), it follows that the solution has the form $u(\omega,x) = u\left(Y_1(\omega), Y_2(\omega), \cdots, Y_N(\omega), x \right)$. The stochastic boundary value problem (4) now becomes a deterministic boundary value problem (6) for a fourth-order elliptic PDE with an N-dimensional parameter. For convenience, we consider the solution u as a function $u : \Gamma^N \to H_0^2(D)$ and use the notation $u(y)$ whenever we want to highlight the dependence on the parameter y. We use similar notation for the coefficient α_1, α_2, and the forcing term f. Given $u \in L_\rho^2(\Gamma^N) \otimes H_0^2(D)$, it can be shown that problem(4) is equivalent to

$$\int_{\Gamma^N} \rho\left((\alpha_1 \nabla u, \nabla v)_{L^2(D)} + (\alpha_2 \Delta u, \Delta v)_{L^2(D)} \right) dy = \int_{\Gamma^N} \rho(f,v)_{L^2(D)} dy\ \forall v \in_\rho^2 (\Gamma^N) \otimes H_0^2(D). \quad (7)$$

Thus, we turn the original stochastic fourth-order elliptic equation into a deterministic parametric fourth-order elliptic equation and we will adopt finite element technique to approximate the solution of the resulting deterministic problem.

Regularity Assumption

The convergence properties of the collocation techniques that will be developed in the next section depend on the regularity that the solution u has with respect to y. Denote $\Gamma_n^* = \Pi_{j=1,j\neq n}^N \Gamma_j$, and let y_n^* be an arbitrary element of Γ_n^*. Here we require the solution of problem (4) to satisfy the following assumption. To make this assumption, we introduce the functional space

$$C^0\left(\Gamma^N; H_0^2\right) = \left\{v : \Gamma^N \to H_0^2(D) \mid \max_{y\in\Gamma^N} \left\|v(y,\cdot)\right\|_{H_0^2(D)} < +\infty\right\},$$

where v is continuous in y.

Assumption: For each $y_n \in \Gamma_n$, there exists $\tau_n > 0$ such that the function $u(y_n, y_n^*, x)$ as a function of $y_n, u : \Gamma_n \to C^0\left(\Gamma_n^*; H_0^2(D)\right)$, admits an analytic extension $u(z, y_n^*, x)$, $z \in \mathcal{C}$, in the region of the complex plane

$$\sum\left(\Gamma_n;\tau_n\right) \equiv \left\{z\in\mathcal{C}, dist\left(z,\Gamma_n\right) \le \tau_n\right\}.$$

Moreover, $\forall z \in \sum\left(\Gamma_n;\tau_n\right)$

$$\left\|u(z)\right\| C^0\left(\Gamma_n^*; H_0^2(D)\right) \le \lambda ,$$

with λ a constant independent of n.

The following lemma will verify that this assumption is sound.

Lemma 5.1: Under the assumption that there exists $\gamma n < \infty$ for every $y = (y_n, y_n^*) \in \Gamma^N$ such that

$$\left\|\frac{\partial_{yn}^k \alpha(y,\cdot)}{\alpha(y,\cdot)}\right\| L\infty(D) \le \gamma_n^k k! \; and \; \frac{\left\|\partial_{y_n}^k f(y,\cdot)\right\| L^2(D)}{1+\left\|f(y,\cdot)\right\| L^2(D)} \le \gamma_n^k k! \,.$$

Then, the solution $u(y_n, y_n^*, x)$ of problem (7) as a function of $y_n, u : \Gamma_n \to C^0\left(\Gamma_n^*; H_0^2(D)\right)$ admits an analytic extension $u(z, y_n^*, x)$, $z\in\mathcal{C}$, in the region of the complex plane

$$\sum\left(\Gamma_n;\tau_n\right) \equiv \left\{z\in\mathcal{C}, dist\left(z,\Gamma_n\right) \le \tau_n\right\},$$

with $0 < \tau_n < 1/(2\gamma n)$, moreover, for all $z \in \sum\left(\Gamma_n;\tau_n\right)$

$$\left\|u(z)\right\| C^0\left(\Gamma_n^*; H_0^2(D)\right) \le \frac{C_0}{1-2\gamma_n\tau_n}\left(2\left\|f\right\| C^0\left(\Gamma^N; L^2(D)\right)+1\right),$$

where C_0 is a constant depending on a_{min}, a_{max} and Poincare's constant C_P

Proof: For simplicity, we first study the following problem: there exists $u \in L_\rho^2\left(\Gamma^N\right) \otimes H_0^2(D)$ such that

$$\int_{\Gamma^N} \rho\left(\alpha\Delta u, \Delta v\right)_{L^2(D)} dy = \int_{\Gamma^N} \rho\left(f, v\right)_{L^2(D)} dy \;\; \forall v \in_\rho^2\left(\Gamma^N\right) \otimes H_0^2(D).$$

For every point $y \in \Gamma^N$, the kth-derivative of u with respect to y_n is obtained by above equation, which satisfies as follows

$$\int_D \alpha(y)\cdot\partial_{y_n}^k \Delta u . \Delta\phi dx = -\sum_{l=1}^k C_k^l \int_D \partial_{y_n}^l \alpha(y)\cdot\partial_{y_n}^{k-1}\Delta u\cdot\Delta\phi dx + \int_D \partial_{y_n}^k f(y)\cdot\phi dx .$$

Taking $\phi = \partial_{y_n}^k u$, we obtain the following form

$$\begin{aligned}&\left\|\sqrt{\alpha(y)}\Delta\partial_{y_n}^k u\right\|_{L^2(D)}^2\\ &\le \sum_{l=1}^k C_k^l \left\|\frac{\partial_{y_n}^l \alpha(y)}{\alpha(y)}\right\|_{L\infty(D)} \cdot \left\|\sqrt{\alpha(y)}\partial_{y_n}^{k-1}\Delta u\right\|_{L^2(D)} \cdot \left\|\sqrt{\alpha(y)}\partial_{y_n}^k \Delta u\right\|_{L^2(D)}\\ &+\left\|\partial_{y_n}^k f(y)\right\|_{L^2(D)} \cdot \left\|\sqrt{\alpha(y)}\partial_{y_n}^k u\right\|_{L^2(D)}\end{aligned} \quad (8)$$

Using the Poincare' inequality, to obtain

$$\left\|\partial_{y_n}^k\right\|_{L^2(D)} \le \sqrt{C_P}\left\|\partial_{y_n}^k \Delta u\right\|_{L^2(D)} \le \frac{\sqrt{C_P}}{\sqrt{a_{\min}}}\left\|\sqrt{\alpha(y)}\partial_{y_n}^k \Delta u\right\|_{L^2(D)}. \quad (9)$$

Combination of (8), (9) yields

$$\left\|\sqrt{\alpha(y)}\Delta\partial_{y_n}^k\right\|_{L^2(D)} \le \sum_{l=1}^k C_k^l \left\|\frac{\partial_{l_n}^l \alpha(y)}{\alpha(y)}\right\|_{L^\infty(D)} \cdot \left\|\sqrt{\alpha(y)}\partial_{y_n}^{k-1}\Delta u\right\|_{L^2(D)} + \frac{\sqrt{C_P}}{\sqrt{a_{\min}}}\left\|\partial_{y_n}^k f\right\|_{L^2(D)}.$$

Setting $R_k = \frac{\left\|\sqrt{\alpha(y)}\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!}$ and using the bounds on the derivatives of α and f, we get the recursive inequality

$$R_k \le \sum_{l=1}^k \gamma_n^l R_{k-1} + \frac{\sqrt{C_P}}{\sqrt{a_{\min}}}\left(1+\left\|f\right\|_{L^2(D)}\right)\gamma_n^k \quad (10)$$

Next, we will prove the following form

$$R_k \le \tilde{C}_0\left(1+\left\|f\right\|_{L^2(D)}\right)\left(2\gamma_n\right)^k, \; \tilde{C}_0 = \frac{1}{2}\max\left\{\frac{C_P\sqrt{a_{\max}}}{a_{\min}}, \frac{\sqrt{C_P}}{\sqrt{a_{\min}}}\right\} \quad (11)$$

Denote $B = \frac{\sqrt{C_P}}{\sqrt{a_{\min}}}\left(1+\left\|f\right\|_{L^2(D)}\right)$, and obtain

$$R_1 \le \gamma_n R_0 + \gamma_n B = \gamma_n\left(R_0+B\right) = 2^{1-1}\gamma_n^1\left(R_0+B\right),$$

supposing

$$R_k \le 2^{k-1}\gamma_n^k\left(R_0+B\right).$$

arrive at

$$\begin{aligned}R_{k+1} &\le \sum_{l=1}^{k+1}\gamma_n^l R_{k+1-l} + B\gamma_n^{k+1}\\ &\le \sum_{l=1}^k \gamma_n^l\cdot 2^{k-1}\gamma_n^{k+1-l}\left(B+R_0\right)+\gamma_n^{k+1}R_0 + B\gamma_n^{k+1}\\ &= \gamma_n^{k+1}\left(B+R_0\right)\left(\sum_{l=1}^k 2^{k-1}+1\right)\\ &= 2^k\gamma_n^{k+1}\left(B+R_0\right)\end{aligned} \quad (12)$$

By Lemma 5.1, the following estimate holds

$$R_0 = \left\|\sqrt{\alpha(y)}\Delta u\right\|_{L^2(D)} \le \sqrt{a_{\max}}\left\|\Delta u\right\|_{L^2(D)} \le \frac{C_P\sqrt{a_{\max}}}{a_{\min}}\left\|f\right\|_{L^2(D)} \quad (13)$$

(11) follows from (12) and (13). Using that $R_k = \frac{\left\|\sqrt{\alpha(y)}\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!} \ge \frac{\sqrt{a_{\min}}\left\|\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!}$, we obtain

$$\frac{\left\|\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!} \le \frac{R_k}{\sqrt{a_{\min}}}. \quad (14)$$

Hence (11) and (14) imply

$$\frac{\left\|\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!} \le \frac{\tilde{C}_0}{\sqrt{a_{\min}}}\left(1+\|f\|_{L^2(D)}\right)(2\gamma_n)^k \cdot$$

For every $y_n \in \Gamma_n$, we get the final estimates on the growth of the derivatives of u.

$$\frac{\left\|\partial_{y_n}^k u\right\|_{L^2(D)}}{k!} \le \sqrt{C_P}\frac{\left\|\Delta\partial_{y_n}^k u\right\|_{L^2(D)}}{k!} \le C_0\left(1+\|f\|_{L^2(D)}\right)(2\gamma_n)^k \cdot \quad (15)$$

where $C_0 = \sqrt{\frac{C_P}{a_{\min}}}\tilde{C}_0$.

For every $(y_n, y_n^*, x) \in \Gamma^N \times D$, we consider $u(y_n, y_n^*, x) \in C^0\left(\Gamma^N; H_0^2(D)\right)$ as a function of $y_n, u : \Gamma_n \to C^0\left(\Gamma^N; H_0^2(D)\right)$. Besides, for every $y_n \in \Gamma_n$, define the power series $u : \mathbb{C} \to C^0\left(\Gamma_n^*; H_0^2(D)\right)$ as follows

$$\sum_{k=0}^{\infty} \partial_{y_n}^k u(y_n, \cdot, \cdot) \cdot \frac{(z-y_n)^k}{k!} \cdot \quad (16)$$

Then, we will prove the uniform convergence of the power series (16) with norm in $C^0\left(\Gamma_n^*; H_0^2(D)\right)$.

$$\frac{|z-y_n|^k}{k!} \cdot \left\|\partial_{y_n}^k u(y_n)\right\| C^0\left(\Gamma_n^*; H_0^2(D)\right)$$
$$\le C_0 \max_{y_n \in \Gamma_n}\left(1+2\|f\|C^0\left(\Gamma_n^*; L^2(D)\right)\right)|z-y_n|^k (2\gamma_n)^k$$
$$\le C_0\left(1+2\|f\|C^0\left(\Gamma^N; L^2(D)\right)\right)|z-y_n|^k (2\gamma_n)^k.$$

Setting $0 < \tau_n < \frac{1}{2\gamma_n}$, and in the ball $|z-y_n| \le \tau_n$, we have

$$\left(|z-y_n| 2\gamma_n\right)^k \le (2\gamma_n\tau_n)^k$$

Using $\sum_{k=0}^{\infty}(2\gamma_n\tau_n)^k = \frac{1}{1-2\gamma_n\tau_n}$, the power series $\sum_{k=0}^{\infty}\left(|z-y_n|2\gamma_n\right)^k, |z-y_n| \le \tau_n$, admits uniform convergence, in addition,

$$\sum_{k=0}^{\infty}\left(|z-y_n|2\gamma_n\right)^k \le \frac{1}{1-2\gamma_n\tau_n}.$$

Thus, we get $\sum_{k=0}^{\infty}\partial_{y_n}^k u(y_n,\cdot,\cdot).\frac{(z-y_n)^k}{k!}$ converges uniformly for every y_n. Let $\tilde{u}\left(y_n, y_n^*, x\right)$ satisfy

$$\tilde{u}\left(z, y_n^*, x\right) = \sum_{k=0}^{\infty}\partial_{y_n}^k u\left(y_n, y_n^*, x\right)\frac{(z-y_n)^k}{k!}, \ |z-y_n| \le \tau_n,$$

which implies $\tilde{u}\left(z, y_n^*, x\right)$ admits an analytic extension in the region $z \in \sum(\Gamma_n; \tau_n)$ and

$$\tilde{u}\left(z_n, y_n^*, x\right) = u\left(z_n, y_n^*, x\right) \ z_n = y_n, \ y_n \in \Gamma_n.$$

We get the following formula [4]

$$\tilde{u}\left(z, y_n^*, x\right) = u\left(z, y_n^*, x\right) = \sum_{k=0}^{\infty}\partial_{y_n}^k u\left(y_n, y_n^*, x\right)\frac{(z-y_n)^k}{k!}, \ |z-y_n| \le \tau_n.$$

Furthermore,

$$\|u(z)\| C^0\left(\Gamma_n^*; H_0^2(D)\right) \le \frac{C_0}{1-2\gamma_n\tau_n}\left(1\|f\|C^0\left(\Gamma^N; L^2(D)\right)+1\right).$$

Similarly, for the solution $u\left(y_n, y_n^*, x\right)$ of problem (7), the conclusion which is above drawn is correct. This finishes the proof.

Example 1: Let us consider the case where the coefficient $\alpha(\omega,x)$ is expanded in a linear truncated Karhunen-Loeve expansion

$$\alpha(\omega,x) = b_0(x) + \sum_{n=1}^{N}\sqrt{\lambda_n b_n}(x)Y_n(\omega),$$

provided that such an expansion guarantees $\alpha(\omega) \ge a_{min}$ for almost every $\omega \in \Omega$ and $x \in D$ [5], in the case we have

$$\left\|\frac{\partial_{y_n}^k \alpha(y,\cdot)}{\alpha(y,\cdot)}\right\|_{L^\infty(\Gamma_N;D)} \le \begin{cases}\sqrt{\lambda_n}\|b_n\|_{L^\infty(D)/a_{\min}} & for \ \ k=1 \\ 0 & for \ \ k>1,\end{cases}$$

and we can take $\gamma_n = \sqrt{\lambda_n}\|b_n\|_{L^\infty(D)/a_{\min}}$.

If we consider, instead, a truncated exponential expansion

$$\alpha(\omega,x) = a_{\min} + \exp\left(b_0(x) + \sum_{n=1}^{N}\sqrt{\lambda_n}b_n(x)Y_n(\omega)\right),$$

we get

$$\left\|\frac{\partial_{y_n}^k \alpha(y,\cdot)}{\alpha(y,\cdot)}\right\|_{L^\infty(\Gamma_N;D)} \le \left(\sqrt{\lambda_n}\|b_n\|_{L^\infty(D)}\right)^k,$$

and we can take $\gamma_n = \sqrt{\lambda_n}\|b_n\|_{L^\infty(D)}$. Hence, both choice fulfill the assumption in lemma 5.1.

Example 2: Let us consider a forcing $f(\omega; x)$ of the form

$$f(\omega,x) = c_0(x) + \sum_{n=1}^{N}c_n(x)Y_n(\omega),$$

where the random variables $Y_n(\omega)$are mean value zero and unit variance, i.e. $E[Y_i]=0, E[Y_i Y_j] = \delta_{i,j}$, for $i,j \in \mathcal{N}_+$ where $\delta_{i,j}$ is the Kronecker symbol, besides, the functions $cn(x)$ are square integrable for any n = 1,2,⋯,N. Then the function f belongs to the space $C^0(\Gamma^N, L^2(D))$, we have

$$\frac{\left\|\partial_{y_n}^k f(y,\cdot)\right\|_{L^2(D)}}{1+\|f(y,\cdot)\|_{L^2(D)}} \le \begin{cases}\|c_n\|_{L^2(D)} & for \ \ k=1 \\ 0 & for \ \ k>1,\end{cases}$$

and we can take $\gamma_n = \|c_n\|_{L^2(D)}$ in Lemma 5.1, in the case, the solution u is linear with respect to the random variabhes Y_n.

Collocation Method

Collocation techniques

We seek a numerical approximation to the solution of (7) in a finite-dimensional subspace $V_{p,h}$ based on a tensor product $\mathcal{P}_p(\Gamma^N) \otimes H_h(\mathcal{D})$, where the following descriptions hold

$H_h(\mathcal{D}) \subset H_0^2(\mathcal{D})$ is a standard finite element space of dimension N_h, which contains continuous quintic piecewise polynomials defined on Argyris triangulations $\mathcal{T}_h$ that a maximum mesh spacing parameter $h > 0$.

$\mathcal{P}_p(\Gamma^N) \subset L_\rho^2(\Gamma^N)$ is the span of tensor product polynomials with degree at most $p = (p_1, p_2, \cdots, p_N)$, i.e. $\mathcal{P}_p = \otimes_{n=1}^N \mathcal{P}_{p_n}(\Gamma_n)$, with

$$\mathcal{P}_{p_n}(\Gamma_n) = span\left(y_n^m, m=0,\cdots,pn\right), \ \ n=1,\cdots,N.$$

Hence the dimension of $\mathcal{P}_p$ is $N_p = \Pi_{n=1}^N(p_n+1)$. We will establish spatial approximation and stochastic approximation, respectively

Spatial Galerkin finite element approximation

We first introduce the semidiscrete approximation $u_h : \Gamma^N \to H_h(\mathcal{D})$. Obtained by projecting (7) onto the subspace $L^2_\rho(\Gamma^N) \otimes H_h(\mathcal{D})$, as follows

$$\int_{\Gamma^N} \rho\Big((\alpha_1 \nabla_{uh}, \nabla_v)_{L^2(D)} + (\alpha_2 \Delta_{uh}, \Delta_v)_{L^2(D)} \Big) dy = \int_{\Gamma^N} \rho(f, v)_{L^2(D)} dy,\ \forall v \in^2_\rho (\Gamma^N) \otimes H_h(D) \tag{17}$$

In order to prove error estimates for stochastic partial differential equation, we need estimates for deterministic fourth order elliptic problem. Let us consider the stationary deterministic problem

$$-\nabla \cdot (\alpha_1(x) \nabla u) + \Delta(\alpha_2(x) \Delta u) = f(x),\ \ x \in D \qquad u \mid \partial D = 0,\ \frac{\partial u}{\partial n} \mid \partial D = 0. \tag{18}$$

we make the following assumptions:

(AA_1) there exist a_{min}; a_{max} > 0 such that

$\alpha_i(x) \in [a_{\min}, a_{\max}],\ \forall x \in \bar{D}, i = 1,2$.

(AA2) f(x) 2 L^2(D).

The variational form of problem (2.2) is to $u \in H^2_0(D)$ such that

$$a(u,\phi) = (\alpha_1(x)\nabla u, \nabla\phi) + (\alpha_2(x)\Delta u, \Delta\phi) = < f, \phi >,\ \ \forall \phi \in H^2_0(D), \tag{19}$$

where <·;· > represents the duality pairing.

Let $H_h(D) \subset H^2_0(D)$ be the finite element space. The discrete problem of (2.3) is to find $u_h \in H_h(D)$ such that

$$a(u_h, \chi) = (\alpha_1(x)\nabla u_h, \nabla\chi) + (\alpha_2(x)\Delta u_h, \Delta\chi) = < f, \chi >,\ \ \forall \chi \in H_h(D).$$

Then, we will estimate the error between u and uh. In order to get the estimate, we need the following two lemmas.

Lemma 6.1: Suppose the conditions (1) (H; (·;·)) is a Hilbert space, and V is (closed) subspace of H, (2) a(·;·) is a bilinear form on V , which is continuous and coercive on V , and that u solves, Given F ∈ V′ [6].

$$a(u,v) = F(v),\ \ \forall v \in V.$$

For the finite element variational problem: Given a finite-dimensional subspace $V_h \subset V$ and F ∈ V′, moreover, $u_h \in V_h$ satisfies

$$a(u_h, v) = F(v),\ \ \forall v \in V_h.$$

Then, the following inequality holds

$$\|u - u_h\| \le \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|,$$

where C, a are the continuity constant and the coercivity constant of a(·,·) on V, respectively.

Lemma 6.2: Let $\{\mathcal{T}^h\}$, 0 < h ≤ 1; be a non-degenerate family of subdivisions of a polyhedral domain $D \subset \mathbb{R}^d$. Let (*K*, $\mathcal{P}$, $\mathcal{N}$) be a reference element, satisfying the conditions [6].

(i) K is star-shaped with respect to some ball,

(ii) $\mathcal{P}_{m-1} \subseteq \mathcal{P} \subseteq H^m_\infty(K)$,

(iii) $\mathcal{N} \subseteq (C^l(K))'$.

Suppose 1 ≤ p ≤ ∞ and either $m - l - n/p > 0$ when p > 1 or $m - l - n \ge 0$ when $p = 1$, for some *l*, *m* and *p*. For all $T \in \mathcal{T}^h$, 0 < *h* ≤ 1, let $(T, \mathcal{P}_T, \mathcal{N}_T)$ be the affine-equivalent element. $\mathcal{I}^h \in C^l(\bar{\Omega}) \to L^1(\Omega)$ is the global interpolation operator defined by

$$\mathcal{I}^h u \mid_{T:} = \mathcal{I}^h_T u,\ \ for\ T \in \mathcal{T}^h,\ \ h \in (0,1].$$

where $\mathcal{I}^h_T$ is the interpolation operator for the affine-equivalent element $(T, \mathcal{P}_T, \mathcal{N}_T)$. Then there exists a positive constant $\bar{C}$ depending on the reference element, *n*, *m*, *p* and the number ρ (regular constant) such that for $0 \le s \le m$,

$$\left(\sum_{T \in \mathcal{T}^h} \|v - I^h v\|^p_{H^s_p(D)} \right)^{1/p} \le \bar{C} h^{m-s} |v|_{H^m_p(D)},$$

for all $v \in H^m_p(D)$.

Theorem 6.1: Let *D* be a convex bounded polygonal domain in $\mathbb{R}^d$(d = 123), and the solutions $u \in H^4(D) \cap H^2_0(D)$ and $u_h \in H_h(D)$ satisfy (17) and (18), then there exists a positive constant $\breve{C} > 0$ and $\tilde{C} > 0$, the following estimates hold

$$\|u - u_h\|_{H^2_0(D)} \le \breve{C} h^2 |u|_4, \tag{20}$$

$$\|u - u_h\|_{L^2(D)} \le \breve{C} h^4 |u|_4. \tag{21}$$

Proof: By Lemma 6.1, Continuity and coercivity of the bilinear form *a(u; v)* and Lemma 6.2, successively, we have

$$\begin{aligned}
\|u - u_h\|_{H^2_0(D)} &\le \frac{C}{\alpha} \inf_{v \in H_h(D)} \|u - v\|_{H^2_0(D)} \\
&= \frac{C_P}{a_{\max} a_{\min}} \inf_{v \in H_h(D)} \|u - v\|_{H^2_0(D)} \\
&\le \frac{C_P}{a_{\max} a_{\min}} \left(\sum_{T \in \mathcal{T}^h} \|u - I^h u\|^2_{H^2(D)} \right)^{1/2} \\
&\le \breve{C} h^2 |u|_4,
\end{aligned}$$

Where $\breve{C} = \frac{CC_p}{a_{\max} a_{\min}}$. For error estimates for $u\text{-}u_h$ in the $L^2(D)$ norm, we proceed by a duality argument. Let φ be arbitrary in L^2, take $\varnothing \in H^4(D) \cap H^2_0(D)$ as the solution of the following equation

$$-\nabla \cdot (\alpha_1(x)\nabla\varnothing) + \Delta(\alpha_2(x)\Delta\varnothing) = \varphi,\ \ \ x \in D,\ \varnothing \mid_{\partial D} = 0,\ \ \Delta\varnothing \mid \partial D = 0.$$

The variational formulation of this problem is to seek $\varnothing \in H^2_0(D)$ such that

$$(\alpha_1(x)\nabla\varnothing, \nabla v) + (\alpha_2(x)\Delta\varnothing, \Delta v) = < \varphi, v >,\ \ \ \forall v \in H^2_0(D).$$

Since $u - u_h \in H^2_0(D)$, let $\varphi = u - u_h$, the solution exists uniquely. Using the Poincaré inequality, we have

$$|\varnothing|_4 \le a_{\max} C_P |\Delta\alpha(x)\Delta\varnothing|_4 = a_{\max} C_P \|\varphi\|_{L^2(D)} = a_{\max} C_P \|u - u_h\|_{L^2(D)},$$

By (2.5) and Lemma 2.2, together with $a(u - u_h, \mathcal{I}^h\psi) = 0, \mathcal{I}^h\psi \in H^2_0(D)$, we get

$$\begin{aligned}
\|u - u_h\|^2_{L^2(D)} &= (u - u_h, u - u_h) \\
&= a(u - u_h, \varnothing) \\
&= a(u - u_h, \varnothing - I^h\varnothing) \\
&\le a_{\max} \|u - u_h\|_{H^2_0(D)} \|\varnothing - I^h\varnothing\|_{H^2_0(D)} \\
&\le \breve{C} h^2 |u|_4 \hat{C} h^2 |\varnothing|_4 \\
&= \tilde{C} h^4 |u|_4 \|u - u_h\|_{L^2(D)},
\end{aligned}$$

where $\tilde{C} = a_{\max} C_P \breve{C}\overline{C}$. Therefore,

$$\|u - u_h\|_{L^2(D)} \le \tilde{C}h^4 |u|_4$$

Stochastic sparse grid collocation approximation

We then introduce the multidimensional Lagarange interpolant operator $\mathcal{I}_p : C^0(\Gamma^N; H_0^2(D)) \to \mathcal{P}_p(\Gamma^N) \otimes \otimes H_0^2(D)$. For each dimension n=1,2,…,N, let $y_n k_n, 1 \le k_n \le p_n+1$ and denote by y_k the point yk=($y_1 k_1, y_1 k_2, \ldots, y_n k_n$) $\in \Gamma^N$. For each n = 1, 2,…,N, the Lagrange basis $\{l_{n,j}\}$ of the space $\mathcal{P}_{pn}$ satisfies $l_{n,j} \in \mathcal{P}_{pn}(\Gamma_n)$, $l_{n,j}(y_{n,k}) = \delta_{j,k}$, $j,k = 1,2,\cdots,pn+1$, and we set $l_k(y) = \Pi_{n=1}^N ln,kn(y_n)$. Then, the final approximation is given by

$$u_{h,p}(y;x) = \sum_{n=1}^{N_p} u_h(y_k;x) l_k(y),$$

where $u_h(y_k, x)$ is the solution of problem (17) for y = y_k, N_p is the number of nodes. Equivalently, we introduce such that

$$I_p v(y) = \sum_{n=1}^{N_p} v(y_k) l_k(y) \quad \forall v \in C^0\left(\Gamma^N; H_0^2(D)\right).$$

Full tensor product interpolation: In this section we briey recall interpolation based on Lagarange polynomials. We first introduce an index $i \in \mathbb{N}_+, i \ge 1$. Then, for each value of *i*, let $\{y_1^i, \ldots, y_{m_i}^i\} \subset \Gamma^1$ be a sequence of abscissas for Lagarange interpolation on Γ^1. For $u \in C^0(\Gamma^1; \mathrm{H}_0^2(\mathrm{D}))$, we introduce a sequence of one-dimensional Lagarange

interpolation $\mathscr{U}^i : C^0(\Gamma^1; \mathrm{H}_0^2(D)) \to V_{m_i}(\Gamma^1; \mathrm{H}_0^2(D))$

$$\mathscr{U}^i(u)(y) = \sum_{k=1}^{m_i} u(y_k^i) l_k^i(y), \quad \forall u \in C^0\left(\Gamma^1; H_0^2(D)\right), \tag{22}$$

where $l_k^i \in \mathcal{P}_{m_i - 1}(\Gamma^1)$are the Lagarange polynomials of degree m_i - 1, i.e. $l_k^i(y) = \Pi_{j=1, j\ne k}^{m_i} \frac{(y - y^i)}{y_k^i - y_j^i}$. In fact, formula (22) reproduces exactly all polynomials of degree less than m_i. In the multivariate case $N > 1$, for each $u \in C^0(\Gamma^N; \mathrm{H}_0^2(D))$ and the multi-index $\boldsymbol{i}$ = ($i_1, \ldots, i_N$) $\in \mathbb{N}_+^N$ we define the full tensor product interpolation formulas

$$I_i^N u(y) = \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_N}\right)(u)(y) = \sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_N=1}^{m_{i_N}} u\left(y_{k_1}^{i_1}, \cdots, y_{k_N}^{i_N}\right)\left(l_{k_1}^{i_1} \otimes \cdots \otimes l_{k_N}^{i_N}\right). \tag{23}$$

Clearly, the above product needs $\prod_{n=1}^N \mathrm{m}_{i_n}$ function evaluations with $I_i^N u(y) \in P_p(\Gamma^N) \otimes H_0^2(D) \subset L_\rho^2(\Gamma^N) \otimes H_0^2(D)$. Thus, for $u_h \in C^0(\Gamma^N; H^h(D))$, the following form holds

$$I_i^N u_h(y) = \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_N}\right)(u_h)(y) = \sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_N=1}^{m_{i_N}} u_h\left(y_{k_1}^{i_1}, \cdots, y_{k_N}^{i_N}\right)\left(l_{k_1}^{i_1} \otimes \cdots \otimes l_{k_N}^{i_N}\right) \tag{24}$$

with $I_i^N u_h(y) \in P_p\left(\Gamma^N\right) \otimes H_h(D)$.

Denote $Y_{i_n} = \left\{y_1^{i_n}, \cdots, y_{m_{in}}^{i_n}\right\}, n = 1, \cdots, N$ and $\hat{Y} = Y_{i_1} \times \cdots \times Y_{i_N}$ Moreover, let $y_k = \left(y_{i_{k_1}}^{i_1} \cdots y_{k_N}^{i_N}\right) \in \hat{Y}$ and $l_k(y) = l_{i_{k_1}}^{i_1} \otimes \cdots \otimes l_{i_{k_N}}^{i_N}, k = 1, \cdots, \hat{N}$, where $\hat{N}$ represents the number of grid points of the set $\hat{Y}$. Let $\hat{Y}\{\phi_1, \ldots, \phi_{N_h}\}$ be a basis of . To obtain $(y_{k_1}^{i_1}, \ldots, y_{k_N}^{i_N})$ in (24), setting *v(y;x) = lk(y)* ϕ_j *(x)* in (17), *j* = 1,…,*Nh* and using Numerical integration, which is based on the set $\hat{Y}$ with corresponding weighs with corresponding weighs, we obtain

$$\sum_{i=1}^{\hat{N}} \omega_i \left\{\left(a_1(y_i;x)\nabla\left(\sum_{m=1}^{\hat{N}} u_h(y_m,\cdot) l_m(y_i)\right), \nabla\left(l_k(y_i)\phi_j(x)\right)\right) + \left(a_2(y_i;x)\Delta\left(\sum_{m=1}^{\hat{N}} u_h(y_m,\cdot) l_m(y_i)\right), \Delta\left(l_k(y_i)\phi_j(x)\right)\right)\right\}$$
$$= \sum_{i=1}^{\hat{N}} \omega_i\left(f(y_i;x), \left(l_k(y_i)\phi_j(x)\right)\right), \quad j = 1, \cdots, N_h$$

For $k = 1, \ldots, \hat{N}$, using $l_k(y_i) = \delta_{k,i}$, we get u_h(*yk; x*) satisfies

$$\left(a_1(y_k;x)\nabla u_h(y_k;x), \nabla\phi_j(x)\right) + \left(a_2(y_k;x)\Delta\left(u_h(y_k;x)\right), \phi_j(x)\right) = \left(f(y_k;x), \phi_j(x)\right), \quad j = 1, \cdots, N_h.$$

Smolyak approximation

Here we follow closely the work [7] and describe the Somlyak isotropic $\mathscr{A}(w, N)$. The Smolyak formulas are just linear combinations of product formula (23) with the following key properties: only products with a relatively a small number of points are used. With $\mathscr{U}^0 = 0$ and for $i \in \mathbb{N}_+$ define

$$\Delta^i := \mathscr{U}^i - \mathscr{U}^{i-1}.$$

Given an integer $w \in \mathbb{N}_+$, hereafter called the level, we define the sets

$$X(w,N) := \left\{ i \in \mathbb{N}_+^N, i \ge 1 : \sum_{n=1}^N (i_n - 1) \le w \right\}, \tag{25}$$

$$\tilde{X}(w,N) := \left\{ i \in \mathbb{N}_+^N, i \ge 1 : \sum_{n=1}^N (i_n - 1) = w \right\}, \tag{26}$$

$$Y(w,N) := \left\{ i \in \mathbb{N}_+^N, i \ge 1 : w - N + 1 \le \sum_{n=1}^N (i_n - 1) \le w \right\}, \tag{27}$$

and for $\mathrm{i} \in \mathbb{N}_+^N$ we set $|\mathrm{i}| = i_1 + \ldots + i_N$. Then the isotropic Smolyak formula is given by

$$\mathscr{A}(w,N) = \sum_{i \in X(w,N)} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N}\right). \tag{28}$$

Equivalently, formula (28) can be written as [8]

$$\mathscr{A}(w,N) = \sum_{i \in Y(w,N)} (-1)^{w+N-|i|} C_{N-1}^{w+N-|i|} \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_N}\right) \tag{29}$$

Obviously, A (w; 1) = U w. To compute A (w;N)(u), one only needs to know function values on the "spare grid"

$$\mathscr{H}(w,N) = \bigcup_{i \in Y(w,N)} \left(\vartheta^{i_1} \times \cdots \times \vartheta^{i_N}\right) \subset \Gamma^N \tag{30}$$

where $\vartheta^i = \left\{y_1^i, \cdots, y_{m_i}^i\right\} \subset \Gamma^1$ denotes the set of abscissas used by $\mathscr{U}^i$.

Choice of collocation nodes

In this section, we will determine how to select the collocation nodes. To the end, we introduce a conclusion.

Lemma 6.3: Let $\{y_k\}_{k=1}^{p=1}$ be the $p+1$ roots of the p + 1 degree ρ-orthogonal polynomial q_{p+1} on the interval Γ. Then, for every function $v \in C^0(\Gamma; \mathrm{H}_0^2)$ the interpolation error satisfies [9].

$$\|v - I_p v\|_{L_\rho^2}\left(\Gamma; H_0^2(D)\right) \le C_1 \inf_{w \in P_p(\Gamma) \otimes H_0^2(D)} \|v - w\|_{C^0(\Gamma; H_0^2)(D)}, \tag{31}$$

where the constant C_1 is independent of *p*.

This lemma relates the approximation error $v - I_p v$ in the L_ρ^2 -norm with the best approximation, hence we propose to use Gaussian abscissas, i.e. the zeros of the orthogonal polynomials with respect to some positive weight. The natural choice of the weight should be the probability density function ρ of the random variables $Y_i(\omega)$ for all i. However, in the general multivariate case, if the random variables $Y_i(\omega)$ are not independent, the joint density ρ does not factorize, i.e. $\rho(y_1, y_2, \cdots, y_N) \neq \Pi_{n=1}^{N} \rho_n(y_n)$ Now, we introduce an auxiliary probability density function $\hat{\rho}: \Gamma^N \to \mathbb{R}^+$ that can be seen as the joint probability of N independent random variables, i.e., it factorizes as

$$\hat{\rho}(y_1, y_2, \cdots, y_n) = \prod_{n=1}^{N} \hat{\rho}_n(y_N), \forall y = (y_1, y_2, \cdots, y_n) \in \Gamma^N, \quad (32)$$

and is such that $\left\| \frac{\rho}{\hat{\rho}} \right\|_{L\infty} (\Gamma^N) < \infty$ The auxiliary density $\hat{\rho}$ should be chosen as close as possible to the true density ρ so as to have the quotient $\frac{\rho}{\hat{\rho}}$ not too large. Indeed, such quotient will appear in final error estimate. For each dimension $n = 1, \cdots, N$, let the m_i Gaussian abscissas be the roots of the m_i degree polynomial that is $\hat{\rho}_n$-orthogonal to all polynomials of degree less than m_i on the interval Γ_n, i.e.

$$\int_{\Gamma_n} q_{m_i}(y) v(y) \hat{\rho}_n(y) dy = 0 \quad for\, all \quad v \in \mathcal{P}_{m_i - 1}(\Gamma_n).$$

In addition, let the number of abscissas mi in each level to grow according to the following formula

$$m_i = 2i - 1 \quad (33)$$

Error Analysis

In this section we show error estimates that will help us understand the sparse grid stochastic collocation method in this situation is efficient. Collocation methods can be used to approximate the solution using $u_h \in C^o(\Gamma^N; H_h(D))$ finitely many function values, each of them is computed by Galerkin finite elements. Besides, u_h admits an analytic extension by assumption. Let the fully discrete numerical approximation be $\mathscr{A}(w, \mathrm{N}) u_h \in P_p(\Gamma^N) \otimes H_h(\mathrm{D})$. Our aim is to give a priori estimates for the total error $e = u - \mathscr{A}(w, \mathrm{N}) u_h$, where the operator $\mathscr{A}(w, \mathrm{N})$ is described by (29). We will investigate the error

$$\left\| u - \mathscr{A}(w, N) u_h \right\| \leq \left\| u - u_h \right\| + \left\| u_h - \mathscr{A}(w, N) u_h \right\| = I + II \quad (34)$$

evaluated in the norm $L_\rho^2(\Gamma^N; H_0^2(D))$ This yields also control of the error in the expected value of $u \|E[e]\|_{H_0^2(D)} \leq E\left[\|e\|_{H_0^2(D)}\right] \leq \|e\|_{H_0^2(\Gamma^N; H_0^2(D))}$. The term I controls the convergence with respect to h, i.e. the finite element error, which will be dictated by approximability properties of the finite element space

$H_h(\mathrm{D})$, given by (21) in theorem

$$I = \left\| u - u_h \right\| L_\rho^2(\Gamma^N; H_0^2(D)) = \int_{\Gamma^N} \rho(y) \left\| u - u_h \right\|_{H_0^2(D)} dy)^{1/2}$$
$$\leq h^2 \left(\int_{\Gamma^N} \rho(y) |u|_4^2 \, dy \right)^{1/2}$$

Thus, we will only concern ourselves with the convergence results when implementing the Smolyak approximation formula, namely, our primary concern will be to analyze the Smolyak approximation error

$$II = \left\| u_h - \mathscr{A}(w, N) u_h \right\| L_\rho^2(\Gamma^N; H_0^2(D)),$$

for Gaussian versions of the Smolyak formula. In this work the technique to develop error bounds for multidimensional Smolyak approximation is based on one dimensional result. Therefore, we first address the case $N = 1$. Let us recall the best approximation error for a function $u: \Gamma^1 \to H_0^2(D)$ which admits an analytic extension in the region $\sum(\Gamma^1; \tau) = \{ z \in \mathrm{C}, dist(z, \Gamma^1) < \tau \}$ of the complex plane, for some $\tau > 0$, in this case, τ is smaller than the distance between $\Gamma^1 \subset \mathbb{R}$ and the nearest singularity of $u(Z)$ in the complex plane. For the case $N=1$, we quote the following results.

Lemma 7.1: Given a function $u \in C^0(\Gamma^1; H_0^2(D))$ which admits an analytic extension in the region of the complex plane $\sum(\Gamma^1; \tau) = \{ z \in \mathrm{C}, dist(z, \Gamma^1) < \tau \} 8$, for some $\tau > 0$. There holds [10]

$$E_{m_i - 1} = \min_{v \in V_{m_i}} \left\| u - v \right\|_{C^0(\Gamma^1; H_0^2(D))} \leq \frac{2}{e^\sigma - 1} e^{-\sigma(m_i - 1)} \max_{z \in \sum(\Gamma^1; \tau)} \left\| u(z) \right\|_{H_0^2(D)}. \quad (35)$$

where $0 < \sigma = \log\left(\frac{2\tau}{|\Gamma^1|} + \sqrt{1 + \frac{4\Gamma^2}{|\Gamma^1|^2}} \right)$ and

$$V_m = \left\{ v \in C^0(\Gamma^N; H_0^2(D)) : v(y; x) = \sum_{k=1}^{m} \tilde{v}_k(x) l_k(y), \{\tilde{v}_k\}_{k=1}^{m} \in H_0^2(D) \right\} \quad (36)$$

Setting

$$\hat{C} = \frac{2}{e^{\sigma - 1}} \max_{z \in \sum(\Gamma^1; \tau)} \left\| v(z) \right\|_{H_0^2(D)},$$

we have

$$E_{m_i} \leq \hat{C} e^{-\sigma m_i} \quad (37)$$

In what follows we will use shorthand notions $\|\cdot\|_{\infty, N}$ $for \|\cdot\|_{L\infty(\Gamma^N; H_0^2(D))}$ and $\|\cdot\|_{\rho, N}$ $for \|\cdot\|_{L_\rho^2(\Gamma^N; H_0^2(D))}$.

Lemma 7.2: For every function $u \in C^0(\Gamma^1; H_0^2(D))$ the interpolation error with Lagrange polynomials based on Gaussian abscissas satisfies [11].

$$\begin{aligned} \left\| u - \mathscr{U}^i(u) \right\|_{\hat{\rho}, 1} &\leq \sqrt{6} \sqrt{\int_{\Gamma^1} \hat{\rho}(y) dy} \inf_{v \in V_{m_i}} \left\| u - v \right\|_{\infty, 1} \\ &= \sqrt{6} \sqrt{\int_{\Gamma^1} \hat{\rho}(y) dy} E_{m_i - 1} \end{aligned} \quad (38)$$

The combination of (2.19) with (3.4), (3.5) yields

$$\begin{aligned} \left\| (I_1 - \mathscr{U}^i)(u) \right\|_{\hat{\rho}, 1} &= \left\| u - \mathscr{U}^i(u) \right\|_{\hat{\rho}, 1} \\ &\leq \sqrt{6} \sqrt{\int_{\Gamma^1} \hat{\rho}(y) dy \hat{C} e^{-\sigma(2i-2)}} \\ &\leq \sqrt{6} \sqrt{\int_{\Gamma^1} \hat{\rho}(y) dy \hat{C} e^{2\sigma} e^{-\sigma(2i)}} \\ &\leq C e^{-\sigma(2i)}, \end{aligned} \quad (39)$$

and

$$\begin{aligned} \left\| \Delta^i(u) \right\|_{\hat{\rho}, 1} &= \left\| \mathscr{U}^i(u) - \mathscr{U}^{i-1}(u) \right\|_{\hat{\rho}, 1} \\ &\leq \left\| (I_1 - \mathscr{U}^i)(u) \right\|_{\hat{\rho}, 1} + \left\| (I_1 - \mathscr{U}^{i-1})(u) \right\|_{\hat{\rho}, 1}, \\ &\leq 2C e^{-2\sigma(i-1)} \end{aligned} \quad (40)$$

for all $i \in N_+$ with positive constants $C = \sqrt{6} \sqrt{\int_{\Gamma^1} \hat{\rho}(y) dy} \hat{C} e^{2\sigma}$ and σ depending on $u$but not on i. Moreover, the Gaussian abscissas defined in Section 2 are constructed for the auxiliary density $\hat{\rho}(y_1, y_2, \cdots, y_n) = \Pi_{n=1}^{N} \hat{\rho}_n(y_n), \forall y = (y_1, y_2, \cdots, y_n) \in \Gamma^N$, still yielding control of the desired norm [12].

$$\left\| v \right\|_{L_\rho^2(\Gamma^N; H_0^2(D))} \leq \left\| \frac{\rho}{\hat{\rho}} \right\|_{L\infty(\Gamma^N)} \left\| v \right\|_{L_{\hat{\rho}}^2(\Gamma^N; H_0^2(D))},$$

for all $v \in C^0\left(\Gamma^N; H_0^2(D)\right)$, therefor we mainly investigate $\left\| I^N - \mathscr{A}(\omega, N)(u) \right\|_{L^2_\rho\left(\Gamma^N, H_0^2(D)\right)}$.

Lemma 7.3: For function $u \in C^0\left(\Gamma^1; H_0^2(D)\right)$, satisfying the assumptions of Lemma 7.1 and Lemma 7.2. The isotropic Smolyak formula (28) based on Gaussian abscissas satisfies

$$\left\| \left(I^N - \mathscr{A}(w, N)\right)(u) \right\|_{L^2_\rho\left(\Gamma^N, H_0^2(D)\right)} \le \sqrt{\left\| \frac{\rho}{\hat{\rho}} \right\|_{L\infty\left(\Gamma^N\right)}} \sum_{d=1}^{N} R(w,d) \tag{41}$$

with

$$R(w,d) = \frac{(2C)^d}{2} \sum_{i \in \tilde{X}(w,d)} e^{-2\sigma(w+1)} \tag{42}$$

Proof: We start providing and equivalent representation of the isotropic Smolyak formula:

$$\begin{aligned}\mathscr{A}(w,N) &= \sum_{i \in X(w,N)} \bigotimes_{n=1}^{N} \Delta^{i_n} \\ &= \sum_{i \in X(w,N-1)} \bigotimes_{n=1}^{N-1} \Delta^{i_n} \otimes \sum_{j=1}^{1+w-\sum_{n=1}^{N-1}(i_n-1)} \Delta^j \\ &= \sum_{i \in X(w,N-1)} \bigotimes_{n=1}^{N-1} \Delta^{i_n} \otimes \mathscr{U}^{1+w-\sum_{n=1}^{N-1}(i_n-1)}.\end{aligned}$$

Defining the one-dimensional identity operator $I_1^{(n)} : \Gamma_n \to \Gamma_n$ for $n = 1, 2, \cdots, N$, the error estimate can be computed using the previous representation, namely

$$\begin{aligned}I^N - \mathscr{A}(w,N) &= I^N - \sum_{i \in X(w,N-1)} \bigotimes_{n=1}^{N-1} \Delta^{i_n} \otimes \left(\mathscr{U}^{1+w-\sum_{n=1}^{N-1}(i_n-1)} - I_1^{(N)} \right) \\ &- \sum_{i \in X(w,N-1)} \bigotimes_{n=1}^{N-1} \Delta^{i_n} \otimes I_1^{(N)} \\ &= \sum_{i \in X(w,N-1)} \bigotimes_{n=1}^{N-1} \Delta^{i_n} \otimes \left(I_1^{(N)} - \mathscr{U}^{1+w-\sum_{n=1}^{N-1}(i_n-1)} \right) \\ &+ \left(I_{N-1} - \mathscr{A}(w, N-1) \right) \otimes | \, I_1^{(N)} \\ &= \sum_{d=2}^{N} \left[\tilde{R}(w,d) \bigotimes_{n=d+1}^{N} I_1^{(n)} \right] + \left(I_1^{(1)} - \mathscr{A}(w,1) \right) \bigotimes_{n=2}^{N} I_1^{(n)}\end{aligned}$$

where, for a general dimension *d*,

$$\tilde{R}(w,d) = \sum_{i \in X(w,d-1)} \bigotimes_{n=1}^{d-1} \Delta^{i_n} \otimes \left(I_1^{(d)} - \mathscr{U}^{\hat{i}_d} \right).$$

For any $(i_1, \cdots, i_{d-1}) \in X(w, d-1)$, denote $\hat{i}_d = 1 + w - \sum_{n-1}^{d-1}(i_n - 1)$. Noting that the d-dimensional vector $j = \left(i_1, \cdots, i_{d-1}, \hat{i}_d\right)$ belongs to the set $\tilde{X}(w,d)$ with this definition, together with (39) and (40), the term $\tilde{R}(w,d)$ can now be bounded as follows

$$\begin{aligned}\left\| \tilde{R}(w,d)(u) \right\|_{\hat{\rho},d} &= \left\| \sum_{i \in X(w,d-1)} \bigotimes_{n=1}^{d-1} \Delta^{i_n} \otimes \left(I_1^{(d)} - \mathscr{U}^{\hat{i}_d} \right) \right\|_{\hat{\rho},d} \\ &\le \sum_{i \in X(w,d-1)} \prod_{n=1}^{d-1} \left\| \Delta^{i_n} \right\|_{\hat{\rho},1} \cdot \left\| I_1^{(d)} - \mathscr{U}^{\hat{i}_d} \right\|_{\hat{\rho},1} \\ &\le \sum_{i \in X(w,d-1)} \prod_{n=1}^{d-1} 2C e^{-2\sigma(i_n - 1)} \cdot C e^{-2\sigma \hat{i}_d} \\ &= \frac{(2C)^d}{2} \sum_{j \in \tilde{X}(w,d)} e^{-\sigma \sum_{n=1}^{d-1} 2(i_n-1) + 2\hat{i}_d} \\ &= \frac{(2C)^d}{2} \sum_{j \in \tilde{X}(w,d)} e^{-2\sigma(w+1)} \\ &= R(w,d)\end{aligned}$$

Observing the fact that the set $\tilde{X}(w,1)$ contains only the point $i_1 = 1 + w$ s $A(w,d) = \mathscr{U}^{1+w}$, the following form holds

$$\begin{aligned}\left\| \left(I_1^{(1)} - \mathscr{A}(w,1) \right) \bigotimes_{n=2}^{N} I_1^{(n)}(u) \right\|_{\hat{\rho},d} &= \left\| \left(I_1^{(1)} - \mathscr{A}(w,1)(u) \right) \right\|_{\hat{\rho},1} \\ &= \left\| \left(I_1^{(1)} - \mathscr{U}^{1+w} \right)(u) \right\|_{\hat{\rho},1} \\ &\le C e^{-2\sigma(w+1)} = R(w,1)\end{aligned}$$

Theorem 7.4: For function $u \in C^0\left(\Gamma^1; H_0^2(D)\right)$ under the assumptions of Lemma 7.3, the following bound holds for term $R(w,d), d = 1, \cdots, N$

$$R(w,d) = \frac{(2C)^d}{2} \cdot (1+w)^{d-1} \cdot e^{-2\sigma(w+1)} \tag{43}$$

moreover,

$$\left\| \left(I^N - \mathscr{A}(w,N)\right)(u) \right\|_{L^2_\rho\left(\Gamma^N, H_0^2(D)\right)} \le \sqrt{\left\| \frac{\rho}{\hat{\rho}} \right\|_{L\infty\left(\Gamma^N\right)}} \cdot \frac{1}{2} \frac{2C(1-2C)^N}{1-2C} \cdot (1+w)^{N-1} \cdot e^{-2\sigma(w+1)} \tag{44}$$

Proof: We first quote a result: $\sum_{n=d}^{k} \binom{n}{d} = \binom{k+1}{d+1}$ and we collocate the number of points $N_w^d = \# \tilde{X}(w,d)$, as follows

$$N_w^d = \binom{d+w-1}{d-1}$$

Obviously when *d* = 1, we have

$$N_w^1 = 1 = \binom{1+w-1}{1-1} = \binom{w}{0} = 1$$

suppose that the following equation holds

$$N_w^{d-1} = \binom{(d-1)+w-1}{(d-1)-1}$$

By induction, we obtain

$$N_w^d = \sum_{m=0}^{w} N_{w-m}^{d-1} = \sum_{m=0}^{w} N_m^{d-1} = \sum_{m=0}^{w} \binom{d-1+m-1}{d-1-1} = \binom{d+w-1}{d-1}$$

next, we present another useful Euler formula: Suppose $n, \left(n \in \mathbf{N}_+\right)$ to be Sufficiently large

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + c, \quad c = 0.57721 \le \cdots$$

where c is irrational number and represents Euler constant. Since the following form holds

$$\begin{aligned}\binom{d+w}{d} &= \frac{(d+w)\cdots(w+1)}{d\cdots 1} \\ &= \prod_{s=1}^{d} \left(1 + \frac{w}{s}\right) \le \left(\frac{1}{d} \sum_{s=1}^{d} \left(1 + \frac{w}{s}\right) \right)^d \\ &= \left(1 + \frac{w \sum_{s=1}^{d} \frac{1}{s}}{d} \right)^d \\ &\le \left(1 + w \frac{1 + \ln d}{d} \right)^d \\ &\le (1+w)^d\end{aligned}$$

we obtain

Citation: Huang Y, Zhang H, Laibin G (2016) Mathematical Modelling and Computer Simulation. J Appl Computat Math 5: 291. doi:10.4172/2168-9679.1000291

$$R(w,d) = \frac{(2C)^d}{2} \sum_{j \in \tilde{X}(w,d)} e^{-2\sigma(w+1)}$$
$$\leq \frac{(2C)^d}{2} \binom{d+w-1}{d-1} e^{-2\sigma(w+1)}$$
$$= \frac{(2C)^d}{2} \cdot (1+w)^{d-1} \cdot e^{-2\sigma(w+1)},$$

Therefore, we get the estimate as follows,

$$\sum_{d=1}^{N} R(w,d) \leq \frac{1}{2} \sum_{d=11}^{N} (2C)^d \cdot (1+w)^{d-1} \cdot e^{-2\sigma(w+1)}$$
$$= \frac{1}{2} \cdot e^{-2\sigma(w+1)} \sum_{d=1}^{N} (2C)^d \cdot (1+w)^{d-1}$$
$$\leq \frac{1}{2} \cdot e^{-2\sigma(w+1)} \cdot (1+w)^{N-1} \sum_{d=1}^{N} (2C)^d$$
$$= \frac{1}{2} \frac{2C\left(1-(2C)^N\right)}{1-2C} \cdot (1+w)^{N-1} \cdot e^{-2\sigma(w+1)}.$$

This finishes the proof.

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