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On P-Laplacian Problem with Decaying Cylindrical Potential and Critical Exponent

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Abstract

In this paper we prove the existence and multiplicity of solutions for p-laplacian problem with decaying cylindrical potential and critical exponent by using Palais-Smale condition and by splitting the Nehari manifold N in two disjoint subsets N^+ and N^-, thus considering the minimization problems on N^+ and N^- respectively.

Keywords: Cylindrical potential; Palais-Smale condition; Nehari manifold; Critical exponent

Introduction

In this paper we consider the following problem

$$\begin{cases} L_{a,\mu}u \\ u\in\mathcal{D}_1^p(\mathbb{R}^N), \end{cases} = \quad h|y|^{-p_*b}|u|^{p_*-2}u+\lambda g \, in\, \mathbb{R}^N, y\neq 0 \qquad (1.1)$$

Where $L_{a,\mu}w=-div\left(|y|^{-pa}|\nabla w|^{p-2}\nabla w\right)-\mu|y|^{-p(a+1)}|w|^{p-2}w, 1<p<k$ with k and N are integers such that $N\geq p+1$ and k belongs to $\{3,.,N-1\}$ and where each point x in $\mathbb{R}^N$ is written as a pair

$$(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k} -\infty<a<(k-p)/p, a\leq b<a+1, p^*=$$

$$pN/(N-p+p(b-a)) -\infty<\mu<\bar{\mu}\%_{a,k,p}=((k-p(a+1))/p)$$

h is a bounded positive function on $\mathbb{R}^k$ and λ is real parameter. $\mathcal{H}_\mu'$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_1^p(\mathbb{R}^N)$ will be defined later.

Some results are already available for (1.1) in the case k=N and p=2, Example [1,2] and the references therein. Wang and Zhou [1,2] proved that there exist at least two solutions for (1.1) with $a=0, 0<\mu\leq\bar{\mu}_{0,N}=((N-2)/2)^2$ and $h\equiv 1$, under certain conditions on g. Bouchekif and Matallah [3] showed the existence of two solutions of (1.1) under certain conditions on functions g and h; when $0<\mu\leq\bar{\mu}_{0,N}\$,\$\lambda\in(0,\Lambda_*)$, $-\infty<a<(N-2)/2$ and $a\leq b<a+1$ with Λ_* a positive constant. Concerning existence results in the case k<N and p=2, [4,5]. Musina [5] considered (1.1) with -a/2 instead of a and $\lambda=0$, also (1.1) with a=0, b=0, $\lambda=0$, with $h\equiv 1$ and $a\neq 2-k$. She established the existence of a ground state solution when $2<k\leq N$ and $0<\mu<\bar{\mu}_{a,k}=((k-2+a)/2)^2$ for (1.1) with -a/2 instead of a and $\lambda=0$. She also showed that (1.1) with a=0, b=0, $\lambda=0$ does not admit ground state solutions. Badiale et al. [6] studied (1.1) with a=0, b=0, $\lambda=0c$ and $h\equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution u, satisfying $u(y,z)=u(|y|,z)$ when $2\leq k<N$ and $\mu<0$. Bouchekif and El Mokhtar [7] proved that (1.1) admits two distinct solutions when $2<k\leq N\$,\$b=N-p(N-2)/2$ with $p\in(2,2^*]$, $\mu<\bar{\mu}_{0,k}$, and $\lambda\in(0,\Lambda_*)$ where Λ_* is a positive constant. Terracini [8] proved that there is no positive solutions of (1.1) with b=0, $\lambda=0$ when $a\neq 0$, $h\equiv 1$ and $\mu<0$. The regular problem corresponding to $a=b=\mu=0$ and $h\equiv 1$ has been considered on a regular bounded domain by Tarantello [9]. She proved that, for $g\in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions. For instance, Xuan studied the multiple weak solutions for p-Laplace equation with singularity and cylindrical symmetry in bounded domains [10]. However, they only considered the equation with sole critical Hardy-Sobolev term.

Before formulating our results, we give some definitions and notation. We denote by $\mathcal{D}_1^p=\mathcal{D}_1^p(\mathbb{R}^k\setminus\{0\}\times\mathbb{R}^{N-k})$ and $\mathcal{H}_\mu=\mathcal{H}_\mu(\mathbb{R}^k\setminus\{0\}\times\mathbb{R}^{N-k})$, the closure of $C_0^\infty(\mathbb{R}^k\setminus\{0\}\times\mathbb{R}^{N-k})$ with respect to the norms.

$$\|u\|_{a,p,0}=\left(\int_{\mathbb{R}^N}|y|^{-pa}|\nabla u|^p\,dx\right)^{1/p}$$

and

$$\|u\|_{a,p,\mu}=\left(\int_{\mathbb{R}^N}\left(|y|^{-pa}|\nabla u|^p-\mu|y|^{-p(a+1)}|u|^p\right)dx\right)^{1/p},$$

respectively, with $\mu<\bar{\mu}_{a,k,p}=((k-p(a+1))/p)^p$ for $k\neq p(a+1)$.

From the Hardy-Sobolev-Mazfiya inequality, it is easy to see that the norm $\|u\|_{a,p,\mu}$ is equivalent to $\|u\|_{a,p,0}$.

Since our approach is variational, we define the functional $I_{a,b,\lambda,\mu}$ on $\mathcal{H}_\mu$ by $I(u)=I_{a,b,\lambda,\mu}(u):=(1/p)\|u\|^p_{a,p,\mu}-(1/p_*)\int_{\mathbb{R}^N}h|y|^{-p_*b}|u|^{p_*}dx-\lambda\int_{\mathbb{R}^N}gudx.$ We say that $u\in\mathcal{H}_\mu$ is a weak solution of the problem $(\mathcal{P})$ if it satisfies $\langle I'(u),v\rangle=\int_{\mathbb{R}^N}\left(|y|^{-pa}\nabla u\nabla v-\mu|y|^{-p(a+1)}uv-h|y|^{-p_*b}|u|^{p_*-2}uv-\lambda gv\right)dx=0$, for $v\in\mathcal{H}_\mu$.

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}_\mu', \mathcal{H}_\mu$.

Throughout this work, we consider the following assumptions:

(G) There exist $\nu_0>0$ and $\delta_0>0$ such that $g(x)\geq\nu_0$, for all x in $B(0,2\delta_0)$.

(H) $\lim_{|y|\to 0}h(y)=\lim_{|y|\to\infty}h(y)=h_0>0, h(y)\geq h_0, y\in\mathbb{R}^k$.

Here, B(a,r) denotes the ball centered at a with radius r.

Under some sufficient conditions on coefficients of equation of (1.1), we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$ respectively.

Remark 1: Note that all solutions of our problem (1.1) are nontrivial.

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We shall state our main results:

Theorem 1: Assume that $3 \le k \le N-1, -1 < a < (k-p)/p, \%0 \le \mu < \bar{\mu}_{a,k,p}$ and (G) holds, then there exists $\Lambda_1 > 0$ such that the problem (1.1), has at least one nontrivial solution on $\mathcal{H}_\mu$ for all $\lambda \in (0, \Lambda_1)$.

Theorem 2: In addition to the assumptions of the Theorem 1, if (H) holds, then there exists $\Lambda_2 > 0$ such that the problem (1.1), has at least two nontrivial solutions on $\mathcal{H}_\mu$ for all $0 < \lambda < \Lambda_3 = \min(\Lambda_1, \Lambda_2)$.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2.

Preliminaries

We list here a few integral inequalities. The first one that we need is the Hardy inequality with cylindrical weights [5]. It states that

$$\bar{\mu}_{a,k,p} \int_{\mathbb{R}^N} |y|^{-p(a+1)} v^p dx \le \int_{\mathbb{R}^N} |y|^{-pa} |\nabla v|^p \, dx, \text{ for all } v \in \mathcal{H}_\mu,$$

The starting point for studying (1:1), is the Hardy-Sobolev-Mazfiya inequality that is particular to the cylindrical case k<N and that was proved by Mazfiya in [4]. It states that there exists positive constant C_{a,p_*} such that

$$C_{a,p_*} \left(\int_{\mathbb{R}^N} |y|^{-p_* b} |v|^{2_*} \, dx \right)^{p/p_*} \le \int_{\mathbb{R}^N} \left(|y|^{-pa} |\nabla v|^p - \mu |y|^{-p(a+1)} v^p \right) dx,$$

for any $v \in C_c^\infty\left(\left(\mathbb{R}^k \setminus \{0\}\right) \times \mathbb{R}^{N-k}\right)$.

Proposition 1: The value [4]

$$S_{\mu,p_*} = S_{\mu,p_*}(k, p_*) = \inf_{v \in \mathcal{H}_\mu \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left(|y|^{-pa} |\nabla v|^p - \mu |y|^{-p(a+1)} v^p \right) dx}{\left(\int_{\mathbb{R}^N} |y|^{-p_* b} |v|^{p_*} \, dx \right)^{p/p_*}}, \quad (2.1)$$

is achieved on $\mathcal{H}_\mu$, for $p \le k < N$ and $\mu \le \bar{\mu}\%_{a,k,p}$

Definition 1: Let $c \in \mathbb{R}, E$ a Banach space and $I \in C^1(E, \mathbb{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$.

(ii) We say that I satisfies the (PS)c condition if any (PS)c sequence in E for I has a convergent subsequence.

Nehari manifold

It is well known that I is of class C[1] in $\mathcal{H}_\mu$ and the solutions of (1.1) are the critical points of I which is not bounded below on $\mathcal{H}_\mu$. Consider the following Nehari manifold

$$\mathcal{N} = \{u \in \mathcal{H}_\mu \setminus \{0\} : \langle I'(u), u \rangle = 0\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{a,p,\mu}^p - \int_{\mathbb{R}^N} h|y|^{-p_* b} |u|^{p_*} \, dx - \lambda \int_{\mathbb{R}^N} gudx = 0. \quad (2.2)$$

Note that N contains every nontrivial solution of the problem (1.1) Moreover, we have the following results.

Lemma 1: The functional I is coercive and bounded from below on $\mathcal{N}$.

Proof: If $u \in \mathcal{N}$, then by (2.2) and the Holder inequality, we deduce that

$$\begin{aligned} I(u) &= \left((p_* - p)/p_* p\right) \|u\|_{a,p,\mu}^p - \lambda\left(1 - (1/p_*)\right) \int_{\mathbb{R}^N} gudx \\ &\ge \left((p_* - p)/p_* p\right) \|u\|_{a,p,\mu}^p - \lambda\left(1 - (1/p_*)\right) \|u\|_{a,p,\mu} \|g\| \mathcal{H}'_\mu \\ &\ge -\lambda^p C_0, \end{aligned} \quad (2.3)$$

where

$$C_0 := C_0\left(\|g\|_{\mathcal{H}'_\mu}\right) = \left[(p_* - 1)^p / p_* p (p_* - p)\right] \|g\|_{\mathcal{H}'_\mu}^p > 0.$$

Thus, I is coercive and bounded from below on N

Define

$$\Psi_\lambda(u) = \langle I'(u), u \rangle.$$

Then, for $u \in \mathcal{N}$

$$\begin{aligned} \langle \Psi'_\lambda(u), u \rangle &= p\|u\|_{a,p,\mu}^p - p_* \int_{\mathbb{R}^N} h|y|^{-p_* b} |u|^{p_*} \, dx - \lambda \int_{\mathbb{R}^N} gudx \\ &= \|u\|_{a,p,\mu}^p - (p_* - 1) \int_{\mathbb{R}^N} h|y|^{-p_* b} |u|^{p_*} \, dx \\ &= \lambda(p_* - 1) \int_{\mathbb{R}^N} gudx - (p_* - p)\|u\|_{a,p,\mu}^p. \end{aligned} \quad (2.4)$$

Now, we split $\mathcal{N}$ in three parts:

$\mathcal{N}^+ = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u \rangle > 0\}$, $\mathcal{N}^0 = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u \rangle = 0\}$

and $\mathcal{N}^- = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u \rangle < 0\}$.

We have the following results.

Lemma 2: Suppose that there exists a local minimizer u_0 for I on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then, $I'(u_0) = 0$ in $\mathcal{H}'_\mu$.

Proof: If u0 is a local minimizer for I on N, then there exists $\theta \in \mathbb{R}$ such that $\langle I'(u_0), \varphi \rangle = \theta \langle \Psi^{v}_\lambda(u_0), \varphi \rangle$ for any $\varphi \in \mathcal{H}_\mu$.

If $\theta = 0$, then the lemma is proved. If not, taking $\varphi \equiv u_0$ and using the assumption $u_0 \in \mathcal{N}$, we deduce $0 = \langle I'(u_0), u_0 \rangle = \theta \langle \Psi'_\lambda(u_0), u_0 \rangle$.

Thus,

$$\langle \Psi'_\lambda(u_0), u_0 \rangle = 0,$$

which contradicts the fact that $u_0 \notin \mathcal{N}^0$.

Let be

$$\Lambda_1 = (p_* - p)(p_* - 1)^{-(p_*-1)/(p_*-p)} \left[(h_0)^{-1} S_{\mu,p_*}\right]^{p_*/p(p_*-p)} \|g\|_{\mathcal{H}'_\mu}^{-1}. \quad (2.5)$$

Lemma 3: We have $\mathcal{N}^0 = \varnothing$ for all $\lambda \in (0, \Lambda_1)$.

Proof: Let us reason by contradiction.

Suppose $\mathcal{N}^0 \ne \varnothing$ for some $\lambda \in (0, \Lambda_1)$. Then, by (2.4) and for $u \in \mathcal{N}^0$, we have

$$\begin{aligned} \|u\|_{a,p,\mu}^p &= (p_* - 1) \int \mathbb{R}^N h|y|^{-p_* b} |u|^{p_*} \, dx \\ &= \lambda\left((p_* - 1)/(p_* - p)\right) \int_{\mathbb{R}^N} gudx. \end{aligned} \quad (2.6)$$

Moreover, by (G), the Holder inequality and the Sobolev embedding theorem, we obtain

$$\left[\left((h_0)^{-1} S_{\mu,p_*}\right)^{p_*/p} / (p_* - 1)\right]^{1/(p_*-p)} \le \|u\|_{a,p,\mu} \le [\lambda\left((p_* - 1)\|g\|_{\mathcal{H}'_\mu} / (p_* - p)\right)]. \quad (2.7)$$

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This implies that $\lambda \ge \Lambda_1$, which is a contradiction with the fact that $\lambda \in (0,\Lambda_1)$.

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $\lambda \in (0,\Lambda_1)$.

Define

$c = \inf_{u\in\mathcal{N}} I(u)$, $c^+ = \inf_{u\in\mathcal{N}^+} I(u)$ and $c^- = \inf_{u\in\mathcal{N}^-} I(u)$.

For the sequel, we need the following Lemma.

Lemma 4: (i) If $\lambda \in (0,\Lambda_1)$, then one has $c \le c^+ < 0$.

(ii) If $0 < \lambda < \Lambda_3 = \min(\Lambda_1,\Lambda_2)$ then $c^- > C_1$, where

$$C_1 = C_1(\lambda, S_{\mu,p_*}\|g\|_{\mathcal{H}'_\mu}) = ((p_* - p)/p_* p)(p_* - 1)^{p/(p_*-p)}(S_{\mu,p_*})^{p_*/(p_*-p)} + -\lambda(1-(1/p_*))(p_*-1)^{p/(p_*-p)}\|g\|_{\mathcal{H}'_\mu}.$$

Proof: (i) Let $u \in \mathcal{N}^+$. By (2.4), we have

$$[1/(p_*-1)]\|u\|^p_{a,p,\mu} > \int \mathbb{R}^N h|y|^{-p_*b}|u|^{p_*}dx$$

and so

$$I(u) = (-1/p)\|u\|^p_{a,p,\mu} + (1-(1/p_*))\int \mathbb{R}^N h|y|^{-p_*b}|u|^{p_*}dx$$
$$< [(-1/p)+(1-(1/p_*))(1/(p_*-1))]\|u\|^p_{a,p,\mu} =$$
$$-((p_*-p)/p_*p)\|u\|^p_{a,p,\mu},$$

we conclude that $c \le c^+ < 0$.

(ii) Let $u \in \mathcal{N}^-$. By (2.4), we get

$$[1/(p_*-1)]\|u\|^p_{a,p,\mu} < \int_{\mathbb{R}^N} h|y|^{-p_*b}|u|^{p_*}dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\mathbb{R}^N} h|y|^{-p_*b}|u|^{p_*}dx \le (S_{\mu,p_*})^{-p_*/p}\|u\|^{p_*}_{a,\mu}.$$

This implies

$$\|u\|_{a,p,\mu} > [(p_*-1)]^{-1/(p_*-p)}(S_{\mu,p_*})^{p_*/p(p_*-p)}, \text{ for all } u \in \mathcal{N}^-.$$

By (2.3), we get

$$I(u) \ge ((p_*-p)/p_*p)\|u\|^p_{a,p,\mu} - \lambda(1-(1/p_*))\|u\|_{a,p,\mu}\|g\|_{\mathcal{H}'_\mu}.$$

Thus, for all

$$0 < \lambda < \Lambda_3 = \min(\Lambda_1,\Lambda_2), \tag{2.8}$$

with

$$\Lambda_2 = ((p_*-p)/p_*p)\left[\frac{p-1}{(p_*-1)h_0}\right]^{p/(p_*-p)}\left[\frac{p_*}{(p_*-1)\|g\|_{\mathcal{H}'_\mu}}\right](S_{\mu,p_*})^{1/p_*},$$

we have $I(u) \ge C_1$.

For each $u \in \mathcal{H}_\mu$, we write

$$t_m = t_{\max}(u) = \left[\frac{\|u\|_{a,p,\mu}}{(p_*-1)\int_{\mathbb{R}^N} h|y|^{-p_*b}|u|^{p_*}dx}\right]^{1/(p_*-p)} > 0.$$

Lemma 5: Let $\lambda \in (0,\Lambda_1)$. For each $u \in \mathcal{H}_\mu$, one has the following:

(i) If $\int_{\mathbb{R}^N} g(x)udx \le 0$, then there exists a unique $t^- > t_m$ such that $t^-u \in \mathcal{N}^-$ and $I(t^-u) = \sup_{t\ge0} I(tu)$.

(ii) If $\int_{\mathbb{R}^N} g(x)udx > 0$, then there exist unique t^+ and t^- such that $0<t^+ < t_m < t^-, t^+u \in \mathcal{N}^+, t^-u \in \mathcal{N}^-$,

$I(t^+u) = \inf_{0\le t\le t_m} I(tu)$ and $I(t^-u) = \sup_{t\ge0} I(tu)$.

Proof: With minor modifications, we refer to [11].

Proof of theorem 1

For the proof we get, firstly, the following results:

Proposition 2:

(i) If $\lambda \in (0,\Lambda_1)$, [11] then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}$ such that

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1) \text{ in } \mathcal{H}'_\mu, \tag{3.1}$$

where $o_n(1)$ tends to 0 as n tends to ∞.

(ii) If $0 < \lambda < \Lambda_3 = \min(\Lambda_1,\Lambda_2)$, then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}^-$ such that

$I(u_n) = c^- + o_n(1)$ and $I(u_n) = c^- + o_n(1)$ and $I'(u_n) = o_n(1)$ in $\mathcal{H}'_\mu$.

Now, taking as a starting point the work of Tarantello [9], we establish the existence of a local minimum for I on $\mathcal{N}^+$.

Proposition 3: If $\lambda \in (0,\Lambda_1)$, then I has a minimizer $u_1 \in \mathcal{N}^+$ and it satisfies

(i) $I(u_1) = c = c^+ < 0$,

(ii) u_1 is a nonnegative solution of (1.1)

Proof: (i) By Lemma 1, I is coercive and bounded below on $\mathcal{N}$. We can assume that there exists $u_1 \in \mathcal{H}_\mu$ such that

$$u_n \rightharpoonup u_1 \text{ weakly in } \mathcal{H}_\mu, \tag{3.2}$$

$$u_n \rightharpoonup u_1 \text{ weakly in } L^{p_*}(\mathbb{R}^N, |y|^{-p_*b}),$$

$$u_n \to u_1 \text{ a.e in } \mathbb{R}^N.$$

Thus, by (3.1) and (3.2), u_1 is a weak solution of (1.1) since c<0 and $I(0) = 0$. Now, we show that u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Suppose otherwise. Then $\|u_1\|_{a,p,\mu} < \liminf_{n\to\infty}\|u_n\|_{a,p,\mu}$ and we obtain

$$c \le I(u_1) = ((p_*-p)/p_*p)\|u_1\|^p_{a,p,\mu} - \lambda(1-(1/p_*))\int \mathbb{R}^N gu_1dx$$
$$< \liminf_{n\to\infty} I(u_n) = c.$$

We get a contradiction. Therefore, u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Moreover, we have $u_1 \in \mathcal{N}^+$. If not, then by Lemma 5, there are two numbers t_0^+ and t_0^-, uniquely defined so that $t_0^+u_1 \in \mathcal{N}^+$ and $t_0^-u_1 \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since $\frac{d}{dt}I(tu_1)_{|t=t_0^+} = 0$ and $\frac{d^2}{dt^2}I(tu_1)_{|t=t_0^+} > 0$, there exists $t_0^+ < t^- \le t_0^-$ such that $I(t_0^+u_1) < I(t^-u_1)$. By Lemma 5, $I(t_0^+u_1) < I(t^-u_1) < I(t_0^-u_1) = I(u_1)$, which is a contradiction.

Proof of theorem 2

In this section, we establish the existence of a second nonnegative solution of (1.1). For this, we require the following Lemmas with C_0 is given in (2.3).

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Lemma 6: Assume that (G) holds and let $(u_n)_n \subset \mathcal{H}_\mu$ be a $(PS)_c$ sequence for I for some $c \in \mathbb{R}$ with $u_n \rightharpoonup u$ in $\mathcal{H}_\mu$. Then, $I'(u)=0$ and $I(u) \geq -C_0\lambda^p$.

Proof: It is easy to prove that $I'(u)=0$, which implies that $\langle I'(u),u\rangle = 0$, and $\int_{\mathbb{R}^N} h|y|^{-p_*b}|u|^{2_*}dx = \|u\|^p_{a,p,\mu} - \lambda\int_{\mathbb{R}^N} gudx.$

Therefore, we get

$$I(u) = ((p_* - p)/p_*p)\|u\|^p_{a,p,\mu} - \lambda(1-(1/p_*))\int_{\mathbb{R}^N} gudx.$$

Using (2.3), we obtain that $I(u) \geq -C_0\lambda^p$.

Lemma 7: Assume that (G) holds and for any (PS)c sequence with c is a real number such that $c < c_\lambda^*$. Then, there exists a subsequence which converges strongly.

Here $c_\lambda^* = ((p_* - p)/p_*p)(h_0)^{-p/(p_*-p)}(S_{\mu,p_*})^{p_*/(p_*-p)} - C_0\lambda^p$.

Proof: Using standard arguments, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Thus, there exist a subsequence of $(u_n)_n$ which we still denote by $(u_n)_n$ and $u \in \mathcal{H}_\mu$ such that

$u_n \rightharpoonup u$ weakly in $\mathcal{H}_\mu$,

$u_n \rightharpoonup u$ weakly in $L^{p_*}(\mathbb{R}^N, |y|^{-p_*b})$.

$u_n \to u$ a.e in $\mathbb{R}^N$.

Then, u is a weak solution of (1.1). Let $v_n = u_n - u$, then by Brezis-Lieb [12], we obtain

$$\|v_n\|^p_{a,p,\mu} = \|u_n\|^p_{a,p,\mu} - \|u\|^p_{a,p,\mu} + o_n(1) \qquad (4.1)$$

and

$$\int_{\mathbb{R}^N} h|y|^{-p_*b}|v_n|^{p_*}dx = \int_{\mathbb{R}^N} h|y|^{-p_*b}|u_n|^{p_*}dx - \int_{\mathbb{R}^N} h|y|^{-p_*b}|u|^{p_*}dx + o_n(1). \qquad (4.2)$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|y|^{-p_*b}|v_n|^{p_*}dx = h_0 \lim_{n\to\infty}\int_{\mathbb{R}^N} |y|^{-p_*b}|v_n|^{p_*}dx. \qquad (4.3)$$

Since $I(u_n) = c + o_n(1), I'(u_n) = o_n(1)$ and by (4.1), (4.2), and (4.3) we can deduce that

$$(1/p)\|v_n\|^p_{a,p,\mu} - (1/p_*)\int_{\mathbb{R}^N} h|y|^{-p_*b}|v_n|^{p_*}dx = c - I(u) + o_n(1), \qquad (4.4)$$

$$\|v_n\|^p_{a,p,\mu} - \int_{\mathbb{R}^N} h|y|^{-p_*b}|v_n|^{p_*}dx = o_n(1).$$

Hence, we may assume that

$$\|v_n\|^p_{a,p,\mu} \to l, \int_{\mathbb{R}^N} h|y|^{-p_*b}|v_n|^{p_*}dx \to l. \qquad (4.5)$$

Sobolev inequality gives $\|v_n\|^p_{a,p,\mu} \geq (S_{\mu,p_*})\int_{\mathbb{R}^N} h|y|^{-p_*b}|v_n|^{p_*}dx$. Combining this inequality with (4.5), we get $l \geq S_{\mu,p_*}(l^{-1}h_0)^{-p/p_*}$.

Either l=0 or $l \geq (h_0)^{-p/(p_*-p)}(S_{\mu,p_*})^{p_*/(p_*-p)}$. Suppose that $l \geq (h_0)^{-p/(p_*-p)}(S_{\mu,p_*})^{p_*/(p_*-p)}$.

Then, from (4.4), (4.5) and Lemma 6, we get

$$c \geq ((p_* - p)/p_*p)l + I(u) \geq c_\lambda^*,$$

which is a contradiction. Therefore, l = 0 and we conclude that u_n converges to u strongly in $\mathcal{H}_\mu$.

Lemma 8: Assume that (G) and (H) hold. Then, there exist $v \in \mathcal{H}_\mu$ and $\Lambda_* > 0$ such that for $\lambda \in (0,\Lambda_*)$, one has

$$\sup_{t\geq 0} I(tv) < c_\lambda^*,$$

where C0 is the positive constant given in (2.3). In particular,

$c^- < c_\lambda^*$, for all $\lambda \in (0,\Lambda_*)$.

Proof: Let φ_ε be such that

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \\ \omega_\varepsilon(x - x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \mathbb{R}^N \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \mathbb{R}^N \end{cases}$$

where ω_ε verifies (2.1). Then, we claim that there exists $\varepsilon_0 > 0$ such that

$$\lambda\int_{\mathbb{R}^N} g(x)\varphi_\varepsilon(x)dx > 0 \text{ for any } \varepsilon \in (0,\varepsilon_0). \qquad (4.6)$$

In fact, if $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, (4.6) obviously holds. If there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, then by the continuity of $g(x)$, there exists $\eta > 0$ such that $g(x) > 0$ for all $x \in B(x_0,\eta)$. Then by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that $\lambda\int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x - x_0)dx > 0$, for any $\varepsilon \in (0,\varepsilon_0)$.

Now, we consider the following functions

$$f(t) = I(t\varphi_\varepsilon) \text{ and } \tilde{f}(t) = (t^p/p)\|\varphi_\varepsilon\|^p_{a,p,\mu} - (t^{p_*}/p_*)\int_{\mathbb{R}^N} h|y|^{-p_*b}|\varphi_\varepsilon|^{p_*}dx.$$

Then, we get for all $\lambda \in (0,\Lambda_1)$

$$f(0) = 0 < c_\lambda^*.$$

By the continuity of f, there exists $t_0>0$ small enough such that $f(t) < c_\lambda^*$, for all $t \in (0,t_0)$.

On the other hand, we have

$$\max_{t\geq 0}\tilde{f}(t) = ((p_* - p)/p_*p)(h_0)^{-p/(p_*-p)}(S_{\mu,p_*})^{p_*/(p_*-p)}.$$

Then, we obtain

$$\sup_{t\geq 0} I(t\varphi_\varepsilon) < ((p_* - p)/p_*p)(h_0)^{-p/(p_*-p)}(S_{\mu,2_*})^{p_*/(p_*-p)} - \lambda t_0\int_{\mathbb{R}^N} g\varphi_\varepsilon dx.$$

Now, taking $\lambda > 0$ such that

$$-\lambda t_0\int_{\mathbb{R}^N} g\varphi_\varepsilon dx < -C_0\lambda^p,$$

and by (4.6), we get

$$0 < \lambda < \left[(t_0/C_0)\left(\int_{\mathbb{R}} g\varphi\right)\right]^{1/(\ \ 1)}, \text{for } \varepsilon << \varepsilon_0.$$

Set

$$\Lambda_* = \min\left\{\Lambda_1, \left[(t_0/C_0)\left(\int_{\mathbb{R}^N} g\varphi_\varepsilon\right)\right]^{1/(p-1)}\right\}.$$

We deduce that

$$\sup_{t\geq 0} I(t\varphi_\varepsilon) < c_\lambda, \text{ for all } \lambda \in (0,\Lambda_*).$$

Now, we prove that

$$c^- < c_\lambda^*, \text{ for all } \lambda \in (0,\Lambda_*).$$

By (G) and the existence of ψ_n satisfying (2.1), we have

$$\lambda\int_{\mathbb{R}^N} g\psi_n dx > 0.$$

Combining this with Lemma 5 and from the definition of c^- and (4.7), we obtain that there exists t_n>0 such that $t_n\psi_n \in \mathcal{N}^-$ and for all $\lambda \in (0, \Lambda_*)$, $c^- \le I(t_n\psi_n) \le \sup_{t\ge 0} I(t\psi_n) < c_\lambda^*$.

Now we establish the existence of a local minimum of I on $\mathcal{N}^-$.

Proposition 4: There exists $\Lambda_4 > 0$ such that for $\lambda \in (0, \Lambda_4)$, the functional I has

a minimizer u_2 in $\mathcal{N}^-$ and satisfies.

(i) $I(u_2) = c^-$,

(ii) u_2 is a solution of (1.1) in $\mathcal{H}_\mu$, where $\Lambda_4 = \min\{\Lambda_3, \Lambda_*\}$ with Λ_3 defined as in (2.8) and Λ_* defined as in the proof of Lemma 8.

Proof: By Proposition 2 (ii), there exists a $(PS)_{c^-}$ sequence for I,$(u_n)_n$ in $\mathcal{N}^-$ for all $0 < \lambda < \Lambda_3 = \min(\Lambda_1, \Lambda_2)$. From Lemmas 7; 8 and 4 (ii), for $\lambda \in (0, \Lambda_*)$,I satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Therefore, there exist a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and $u_2 \in \mathcal{N}^-$ such that u_n converges to u_2 strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for all $\lambda \in (0, \Lambda_4)$. Finally, by using the same arguments as in the proof of the Proposition 3, for all $\lambda \in (0, \Lambda_1)$, we have that u2 is a solution of (1.1).

Now, we complete the proof of Theorem 2. By Propositions 3 and 4, we obtain that $(\mathcal{P})$ has two solutions u_1 and u_2 such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}\%^- = \emptyset$, this implies that u_1 and u_2 are distinct.

Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem on the constraint defined by the Nehari manifold $\mathcal{N}$, which are solutions of our problem. Under some sufficient conditions on coefficients of equation of (1.1), we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$ respectively. In the sections 3 and 4 we have proved the existence of at least two nontrivial solutions on $\mathcal{H}_\mu$ for all $0 < \lambda < \Lambda_3 = \min(\Lambda_1, \Lambda_2)$.

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