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Existence of Multiple Solutions for P-Laplacian Problems Involving Critical Exponents and Singular Cylindrical Potential

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Abstract

In this paper, we establish the existence of multiple solutions for p-Laplacian problems involving critical exponents and singular cylindrical potential, by using Ekeland's variational principle and mountain pass theorem without Palais-Smale conditions.

Keywords: P-Laplacian; Critical exponents; Cylindrical potential; Dimensional

Introduction

The aim of this paper is to establish the existence and multiplicity of solutions to the following quasilinear elliptic problem

$$(\mathcal{P}_{\lambda,\mu})\begin{cases}-\Delta_p u-\mu|y|^{-p}|u|^{p-2}u=h(y)|y|^{-s}|u|^{q-2}u+\lambda g(x) \text{ in } \mathbb{R}^N,\ y\neq 0\\ u\in\mathcal{D}_1^p(\mathbb{R}^N),\end{cases}$$

Where $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right), 1<p<k, k \text{ and } N$ are integers with N>p, $p\le k\le N$, $\mathbb{R}^N=\mathbb{R}^k\times\mathbb{R}^{N-k}$\$., the point $x\in\mathbb{R}^N$ can be written as $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}, -\infty<\mu<\bar{\mu}_{k,p}:=((k-p)/p)^p, 0\le s<p, q:=p^*(s)=p(N-s)/(N-p)$ is the critical Sobolev-Hardy exponent, λ and μ are positive parameters which we will specify later, g is a continuous function on $\mathbb{R}^N$ and h is a bounded positive function on $\mathbb{R}^k$.

Let $\mathcal{H}_\mu=\mathcal{D}_1^p(\mathbb{R}^N)$ be the space defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_p=\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{\frac{1}{p}}$.

When $\mu<\bar{\mu}_{k,p}$, Hardy type inequality implies that the norm

$$\|u\|=\|u\|_{\mu,p}=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^p-\mu|y|^{-p}|u|^p\right)dx\right)^{1/p},$$

is will defined in $\mathcal{H}_\mu$ and $\|.\|$ is equivalent to $\|\nabla.\|_p$; since the following inequalities hold: $\left(1-\left(\max(\mu,0)/\bar{\mu}_{k,p}\right)\right)^{1/p}\|\nabla u\|_p\le\|u\|\le\left(1-\left(\min(\mu,0)/\bar{\mu}_{k,p}\right)\right)^{1/p}\|\nabla u\|_p$, for all $u\in\mathcal{H}_\mu$

We define the weighted Sobolev space $\mathcal{D}:=\mathcal{H}_\mu\cap L^p(\mathbb{R}^N,|y|^{-s}dx)$ which is a Banach space with respect to the norm defined by $\mathcal{N}(u):=\|u\|_\mu+(\int_{\mathbb{R}^N}|y|^{-s}|u|^q dx)^{1/q}$.

Several existence results are available in the case p = 2 and k = N; we quote for example [1-3]; and the references therein. For more details, when $h\equiv 1$, $\mu=0$ and $q=2^*$, the regular problem $(\mathcal{P}_{1,0})$ has been considered, on the bounded domain Ω, by Tarantello [4]. She proved that for $g\in\left(H_0^1(\Omega)\right)'$ not identically zero and satisfying a suitable condition, the problem considered admits two solutions. Also, they are two nontrivial non-negative solutions when g is nonnegative. The problem $(\mathcal{P}_{\lambda,\mu})$ has been studied by Bouchekif and Matallah in [2], by using Ekeland.s variational principle and mountain pass theorem, they established the existence of two nontrivial solutions when $0<\mu\le\bar{\mu}_N, \lambda\in(0,\Lambda_*)$, where Λ_* is a positive constant and under sufficient conditions on functions g and h.

For the case p=2 and k<N, there are much less studies in the literature at our knowledge. We cite for example [4-6], and the references therein. As noticed in [6] considered the minimization problem

$$S(p)=S(N,p,k,s)$$
$$=\inf\left\{\int_{\mathbb{R}^N}|\nabla u|^p, u\in c_c^\infty((\mathbb{R}^k\setminus\{0\}\times\mathbb{R}^{N-k}) \text{ and } \int_{\mathbb{R}^N}|y|^{-s}|u|^q dx=1\right\}$$

and in [6], solutions which are radially symmetric in the x-variable receive impor-tance with regard to certain elliptic equations on the n=N-k+1 dimensional hyperbolic space $\mathbb{H}^n$. In particular, Musina in [6] has considered the problem $(\mathcal{P}_{0,\mu})$ with $h\equiv 1$. She established the existence of ground state solution when $0<\mu<\bar{\mu}_k$ and $2<k\le N$ and the support of the ground state solution is a half-space when k = 1 and $N\ge 4$

In case p>2 and 1<k<N, equations with cylindrical potentials were also studied by many people [1,4,7-10]. For instance, in [11], Xuan studied the multiple weak solutions for p-Laplace equation with singularity and cylindrical symmetry in bounded domains. However, they only considered the equation with sole critical Hardy-Sobolev term.

Since our approach is variational, we de.ne the functional $I_{\lambda,\mu}$ on $\mathcal{D}$ by

$$I_{\lambda,\mu}(u):(1/p)\|u\|^p-(1/q)\int_{\mathbb{R}^N}h(y)|y|^{-s}|u|^q dx-\lambda\int_{\mathbb{R}^N}g(x)udx.$$

Throughout this work, we consider the following assumptions

(G) g $\in\mathcal{H}_\mu'$(dual of $\mathcal{H}_\mu$),

(H) $\lim_{|y|\to 0}h(y)=\lim_{|y|\to\infty}h(y)=h_0>0,\ h(y)\ge h_0, y\in\mathbb{R}^k$.

In our work, we prove the existence of at least two distinct critical points of $I_{\lambda,\mu}$.

One by the Ekeland variational principle in [10] with negative energy, and the other by mountain pass theorem in [7] without Palais-

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Smale conditions with positive energy.

Our main result is given as follows

Theorem 1: Suppose that $p<k<N, 0\le s<p, \mu<\bar{\mu}_{k,p}$, hypothesis (H) holds, $g\in\mathcal{H}_\mu^{'}\cap C(\mathbb{R}^N)$ and $g\ne 0$. Then there exists $\Lambda_*>0$ such that the problem $(\mathcal{P}_{\lambda,\mu})$ has at least two solutions for any $\lambda\in(0,\Lambda_*)$.

This paper is organized as follows. In Section 2, we give some preliminaries.

Section 3 is devoted to the proof of Theorem 1.

Preliminaries

We start by recalling the following definition and properties from the paper [6].

The first inequality that we need is the Hardy inequality

$$\int_{\mathcal{R}^N}|\nabla u|^p dx \ge \bar{\mu}\%_{k,p}\int_{\mathcal{R}^N}|y|^{-p}|u|^p\, dx, \text{ for all } u\in\mathcal{D}_1^p(\mathbb{R}^N), \tag{4.1}$$

the constant $\bar{\mu}_{k,p}:=((k-p)/p)^p$ is sharp but not achieved [2].

Definition 1: An entire solution v to $(\mathcal{P}_{\lambda,\mu})$ is a ground state solution if it achieves the best constant

$$S_{\mu,p}=S_{\mu,p}(k,N):=\lim_{v\in\mathcal{H}_\mu((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})}\frac{\int_{\mathbb{R}^N}\left(\int_{\mathcal{R}^N}(|\nabla u|^p-\mu|y|^{-p}|u|^p)\,dx\right)dx}{\left(\int_{\mathbb{R}^N}|y|^{-s}|v|^q\,dx\right)^{p/q}}, \tag{4.2}$$

Lemma 1: Assume [6] that $p<k<N, 0\le s<p$ and $\mu<\overline{\mu}_{k,p}$. Then, the in fimum $S_{\mu,p}$ is achieved on $\mathcal{H}_\mu((\mathbb{R}^k\setminus\{0\}\setminus\{0\}\times\mathbb{R}^{N-k})$.

Lemma 2: Let $(u_n)\subset\mathcal{D}$ be a Palais-Smale sequence $[(PS)_c$ in short$]$ of $I_{\lambda,\mu}$, i.e.,

$$I_{\lambda,\mu}(u_n)\to c \text{ and } I'_{\lambda,\mu},(u_n)\to 0 \text{ in } \mathcal{D}'(duqlof\mathcal{D}) \text{ as n}\to\infty \quad \text{for some } c\in\mathbb{R}. \tag{4.3}$$

Then, $u_n\rightharpoonup u$ in $\mathcal{D}$ and $I'_{\lambda,\mu}(u)=0$.

Proof: From (4.3);

We have

$$(1/p)\|u_n\|^p-(1/q)\int_{\mathbb{R}^N}h(y)|y|^{-s}|u_n|^q\,dx-\lambda\int_{\mathbb{R}^N}g(x)u_n dx=c+o_n(1)$$

and

$$\|u_n\|^p-\int_{\mathbb{R}^N}h(y)|y|^{-s}|u_n|^q\,dx-\lambda\int_{\mathbb{R}^N}g(x)u_n dx=o_n(1), \text{ for } n \text{ large},$$

where $o_n(1)$ denotes $o_n(1)\to 0$ as $n\to\infty$. Then,

$$c+o_n(1)=I_{\lambda,\mu}(u_n)-(1/q)\langle I'_{\lambda,\mu}(u_n),u_n\rangle$$

$$\ge ((q-p)/pq)\|u_n\|^p-\lambda((q-1)/q)\|g\|\mathcal{H}'_\mu\|u_n\|,$$

(u_n) is bounded in $\mathcal{D}$. Up to a subsequence if necessary, we obtain that

$u_n \rightharpoonup u$ in $\mathcal{D}$

$u_n\rightharpoonup u$ in $L_q(\mathbb{R}^N;|y|^{-s})$

$u_n\to u$ a.e in $\mathbb{R}^N$.

Consequently, we get

$I'_{\lambda,\mu}(u)=0.$

Lemma 3: Let $(u_n)\subset\mathcal{D}$ be a Palais-Smale sequence $(PS)_c\, I_{\lambda,\mu}$ for some $c\in\mathbb{R}$.

Then, $u_n\rightharpoonup u$ in $\mathcal{D}$ and either $u_n\to u$ or $c\ge I_{\lambda,\mu}(u)+((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)}$ for all $q\in(p,p^*(0)]$

Proof: We know that (u_n) is bounded in $\mathcal{D}$ Up to a subsequence if necessary, we have that

$u_n\rightharpoonup u$ in $\mathcal{D}$

$u_n\to u$ a.e in $\mathbb{R}^N$.

Denote $v_n=u_n-u$, then $v_n\rightharpoonup 0$. As in Brézis and Lieb [2]; we have $|v_n|_q^p=|u_n|_q^p-|u|_q^p$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(y)\left(|y|^{-s}|u_n|^q-|y|^{-s}|u_n-u|^q\right)dx=\int_{\mathbb{R}^N}h(y)|y|^{-s}|u|^q\,dx.$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(y)\,|y|^{-s}|u_n|^q\,dx=h_0\lim_{n\to\infty}\int_{\mathbb{R}^N}|y|^{-s}|u_n|^q\,dx.$$

Then, we get

$$I_{\lambda,\mu}(u_n)=I_{\lambda,\mu}(u)+(1/p)\|v_n\|^p-(h_0/q)\int_{\mathbb{R}^N}|y|^{-s}|v_n|^q+o_n(1)$$

and

$$\langle I'_{\lambda,\mu}(u_n),u_n\rangle=\|v_n\|^p-h_0\int_{\mathbb{R}^N}|y|^{-b}|u_n|^q+o_n(1).$$

Then we can assume that

$$\lim_{n\to\infty}\|v_n\|^p=h_0\lim_{n\to\infty}\int_{\mathbb{R}^N}|y|^{-s}|u_n|^q=l\ge 0.$$

Assume $l>0$, we have by definition of $S_{\mu,q}$

$$l\ge S_{\mu,q}\left(lh_0^{-1}\right)^{p/q},$$

and so that

$$l\ge\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)}.$$

Thus we get

$$c=I_{\lambda,\mu}(u)+((q-p)/pq)l$$

$$\ge I_{\lambda,\mu}(u)+((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)}.$$

Proof of Theorem 1

The proof of Theorem 1 is given in two parts.

Existence of a local minimizer

We prove that there exists $\lambda_*>0$ such that for any $\lambda\in(0,\lambda_*), I_{\lambda,\mu}$ can achieve a local minimizer. First, we establish the following result.

Proposition 1: Suppose that $p<k<N, 0\le s<p, \mu<\bar{\mu}_{k,p}$, hypothesis (H) holds, $g\in\mathcal{H}'_\mu\cap C(\mathbb{R}^N)$ and $g\ne 0$. Then there exists λ_*,ϱ and δ such that for all $\lambda\in(0,\lambda_*)$ we have

$$I_{\lambda,\mu}(u)\ge\delta>0 \text{ for } \|u\|=\varrho \tag{5.1}$$

Proof: By the Holder inequality and the definition of we get for all $u\in\mathcal{D}\setminus\{0\}$ and $\varepsilon>0$

$$I_{\lambda,\mu}(u) \ := (1/p)\ \|u\|^p - (1/q)\int_{\mathbb{R}^N} h(y)|y|^{-s}|u|^q\,dx - \lambda\int_{\mathbb{R}^N} g(x)u\,dx,$$

$$\geq\ (1/p)\|u\|^p - (h_\infty/q)S_{\mu,q}\|u\|^q - \lambda\|g\|\mathcal{H}'_\mu\|u\|,$$

$$\geq (1/p-\varepsilon)\|u\|^p - (|h|_\infty/p)S_{\mu,q}\|u\|^q - -C_\varepsilon\|\lambda g\|H'_\mu.$$

Taking $\varepsilon < 1/p$ and $\varrho = \|u\|_\mu$, then there exist $\varrho > 0$ small enough and a positive constant λ_* such that

$$I_{\lambda,\mu}(u) \geq \delta > 0 \text{ for } \|u\|_\mu = \varrho \text{ and } \lambda \in (0,\lambda_*). \tag{5.2}$$

Since g is a continuous function on $\mathbb{R}^N$, not identically zero, we can choose $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\int_{\mathbb{R}^N} g(x)\phi dx > 0$. It follows that for $t>0$ small,

$$I_{\lambda,\mu}(t\phi) := (t^p/p)\|\phi\|^p - (t^q/q)\int_{\mathbb{R}^N} h(y)|y|^{-s}|\phi|^q\,dx - \lambda t\int_{\mathbb{R}^N} g(x)\phi dx < 0. \tag{5.3}$$

We also assume that t is so small enough such that $\|t\phi\|_\mu < \varrho$. Thus, we have $c_1 = \inf\{I_{\lambda,\mu}(u) : u \in B\varrho\} < 0$, where $B_\varrho = \{u \in \mathcal{D}, \mathcal{N}(u) \leq \varrho\}$. (5.4)

Using the Ekeland's variational principle, for the complete metric space $\overline{B}_\varrho$ with respect to the norm of $\mathcal{D}$, we can prove that there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \overline{B}_\varrho$ such that $u_n \rightharpoonup u_1$ for some u_1 with $\mathcal{N}(u_1) \leq \varrho$.

Now, we claim that $u_n \to u_1$. If not, by Lemma??; we have

$$c_1 \geq I_{\lambda,\mu}(u_1) + ((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)}$$

$$\geq c_1 + ((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)}$$

$$> c_1,$$

which is a contradiction.

Then we obtain a critical point u_1 of $I_{\lambda,\mu}$ for all $\lambda \in (0,\lambda_*)$ satisfying $c_1 = I_{\lambda,\mu}(u_1) < 0$.

On the other hand we have

$$c_1 = ((q-p)/pq)\|u_1\|^p - ((q-1)/q)\int_{\mathbb{R}^N} \lambda g(x)u_1 dx$$

$$\geq -(1/pq)(q-1)^p(q-p)^{-1}\lambda^p\|g\|^p_{\mathcal{H}'_\mu}. \tag{5.5}$$

Thus u1 is a nontrivial solution of our problem with negative energy.

Existence of mountain pass type solution

We use the mountain pass theorem without Palais-Smale conditions to prove the existence of a nontrivial solution with positive energy. For this, we need the following Lemma.

Lemma 4: Let $\lambda^* > 0$ such that

$c^*_{\lambda,p} > 0$ for all $\lambda \in (0,\lambda^*)$.

Then, there exist $\Lambda \in (0,\lambda^*)$ and $\varphi_\varepsilon \in \mathcal{D}$ for $\varepsilon > 0$ such that $\sup_{t\geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < c^*_{\lambda,p}$, for all $\lambda \in (0,\Lambda)$.

Proof: Let

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \\ \omega_\varepsilon(x-x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \mathbb{R}^N \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \in \mathbb{R}^N \end{cases}, \tag{5.6}$$

Where ω_ε veri.es (2:2)

Then, we claim that there is an ε_0 such that

$$\int_{\mathbb{R}^N} g(x)\varphi_\varepsilon(x) > 0, \text{for any } \varepsilon \in (0,\varepsilon_0). \tag{5.7}$$

In fact, $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, and (5.7) holds obviously. If there exists an $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, by the continuity of $g(x)$ there is an $\eta > 0$ such that $g(x) > 0$ for all $x \in B_\eta(x_0)$. Then, by the definition of $\omega_\varepsilon(x-x_0)$ it is easy to see that there exists an ε_0 small enough such that

$$\int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x-x_0) > 0,\ \textit{for any } \varepsilon \in (0,\varepsilon_0). \tag{5.8}$$

Now, we consider the following functions

$$f(t) = I_{\lambda,\mu}(t\varphi_\varepsilon)$$

and

$\tilde{f}(t) = (t^p/p)\ \|\varphi_\varepsilon(x)\|^p - (t^q/q)h_0\int_{\mathbb{R}^N} |y|^{-s}|\varphi_\varepsilon(x)|^q\,dx.$ Then, we get for all $\lambda \in (0,\lambda^*)$

$$0 = f(0) < c^*_{\lambda,p}.$$

By the continuity of f(t), there exists t_1 a sufficiently small positive quantity such that $f(t) < c^*_{\lambda,p}$, for all $t \in (0,t_1)$. On the other hand, we have

$$\max_{t\geq 0}\tilde{f}(t) = ((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)},$$

then, we obtain

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < ((q-p)/pq)\left(h_0^{-p/q}S_{\mu,q}\right)^{q/(q-p)} - \lambda t_1\int_{\mathbb{R}^N} |y|^{-s} g(x)\varphi_\varepsilon dx.$$

Taking $\lambda > 0$ such that

$$\lambda t_1\int_{\mathbb{R}^N} g(y)\varphi_\varepsilon dx > (1/pq)(q-1)(q-p)^{-1/p}\lambda^p\|g\|^p_{\mathcal{H}'_\mu}.$$

By (5.7) we get

$$0 < \lambda < Q.$$

Where

$$Q := (pq(q-p)^{1/p}(q-1)^{-1})t_1\left(\int_{\mathbb{R}^N} g(x)\varphi_\varepsilon dx\right)\|g\|^{-p}_{\mathcal{H}'_\mu}.$$

Set

$$\Lambda = \min\{\lambda^*, Q\}.$$

We deduce that

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < c^*_{\lambda,p},\ \text{for all } \lambda \in (0,\Lambda).$$

Since $\lim_{t\to\infty} I_{\lambda,\mu}(t\varphi_\varepsilon) = -\infty$, we can choose T>0 large enough such that $I_{\lambda,\mu}(T\varphi_\varepsilon) < 0$. From Proposition 1, we have $I_{\lambda,\mu|\partial B_\varrho} \geq \delta > 0$ for all $\lambda \in (0,\lambda_*)$. By mountain pass theorem without the Palais-Smale condition, there exists a $(PC)_{c_2}$ sequence (u_n) in $\mathcal{D}$ which is characterized by

$$c_2 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1],\mathcal{D}), \gamma(0) = 0, \gamma(1) = T\varphi_\varepsilon\}.$$

Then, (u_n) has a subsequence, still denoted by (u_n) such that $u_n \rightharpoonup u_2$ in $\mathcal{D}$.

By Lemma 3, if u_n doesn.t converge to u_2; we get

$$c_2 \geq I_{\lambda,\mu}(u_2) + ((q-p)/pq)\left(h_0^{-p/q} S_{\mu,q}\right)^{q/(q-p)} \geq c^*_{\lambda,p},$$

what contradicts the fact that, by Lemma 4, we have $\sup_{t\geq 0} I_{\lambda,\mu}(t\varphi_\varepsilon) < c^*_{\lambda,p}$, for all $\lambda \in (0,\Lambda)$. Then $u_n \to u_2$ in$\mathcal{D}$.

Thus, we obtain a critical point u_2 of $I_{\lambda,\mu}$ for all $\lambda \in (0,\lambda_*)$ with $\Lambda_* := \min\{\lambda_*, \Lambda\}$ satisfying $I_{\lambda,\mu}(u_2) > 0$.

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