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Refined Estimates on Conjectures of Woods and Minkowski-I

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Abstract

Let ^ be a lattice in R^n reduced in the sense of Korkine and Zolotare having a basis of the form $(A_1, 0, 0, \ldots, 0)$, $(a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}, A_n)$ where $A_1, A_2, \ldots$, An are all positive. A well known onjecture of Woods in Geometry of Numbers asserts that if $A_1 A_2 \ldots A_{n=1}$ and $A_i \leq A_1$ for each i then any closed sphere in R^n of radius $\sqrt{n/2}$ contains a point of ^. Woods' Conjecture is known to be true for $n \leq 9$. In this paper we obtain estimates on the Conjecture of Woods for n=10; 11 and 12 improving the earlier best known results of Hans-Gill et al. These lead to an improvement, for these values of n, to the estimates on the long standing classical conjecture of Minkowski on the product of n non-homogeneous linear forms.

MSC: 11H46; 11 - 04; 11J20; 11J37; 52C15.

Keywords: Lattice; Covering; Non-homogeneous; Product of linear forms; Critical determinant; Korkine and Zolotare reduction; Hermite's constant; Centre density

Introduction

Let $L_i=a_{i1}x_1+\ldots+a_{in}x_n$; $1 \leq i \leq n$ be n real linear forms in n variables x1; : : : ; xn and having determinant $\Delta = \det(a_{ij}) \neq 0$ The following conjecture is attributed to H. Minkowski:

Conjecture I: For any given real numbers c1; : : : ; cn, there exists integers x1; : : : ; xn such that

$$|(L_1+c_1)\ldots(L_n+c_n)| \leq \frac{1}{2n}|\Delta| \tag{1.1}$$

Equality is necessary if and only if after a suitable unimodular transformation the linear forms L_i have the form $2c_i x_l$ for $1 \leq i \leq n$

This result is known to be true for $n \leq 9$ For a detailed history and the related results,

Minkowski's Conjecture is equivalent to saying that [1]

$$M_n \leq \frac{1}{2n}|\Delta|$$

where $M_n=M_n(\Delta)$ is given by

$$M_n = \sup_{L_1,\ldots,L_n} \ \sup_{(c_1,\ldots,c_n)\in R^n} \ \inf_{(u_1,\ldots,u_n)\in z^n} \prod_{i=1}^{n} |L_i(u_1,\ldots,u_n)+c_i|$$

Chebotarev proved the weaker inequality

$$M_n \leq \frac{1}{2^{n/2}}|\Delta| \tag{1.2}$$

Since then several authors have tried to improve upon this estimate. The bounds have been obtained in the form

$$M_n \leq \frac{1}{v_n 2^{n/2}}|\Delta| \tag{1.3}$$

where Vn>1. Clearly $v_n \leq 2^{n/2}$ by considering the linear forms Li=xi and $c_i = \frac{1}{2}$ for $1 \leq i \leq n$ During 1949-1986, many authors such as Davenport, Woods, Bombieri, Gruber, Skubenko, Andrijasjan, Il'in and Malyshev obtained V_n for large n. obtained $v_n = 4-2(2-3\sqrt{2/4})^n - 2^{-n/2}$ for all n [2-4] improved Mordell's estimates for $6 \leq n \leq 31$ Hans-Gill et al. [12,14] got improvements on the results of [5-8] for $9 \leq n \leq 31$ Since recently V_n $9=2^{9/2}$ has been established by the authors [9], we study Vn for $10 \leq n \leq 33$ in a series of three papers.

In this paper we obtain improved estimates on Minkowski's Conjecture for n=10; 11 and 12. In next papers [10-12], we shall derive improved estimates on Minkowski's Conjecture for n=13; 14; 15 and for $16 \leq n \leq 33$ respectively [13-16]. For sake of comparison, we give results by our improved Vn in Table 1.

We shall follow the Remak-Davenport approach. For the sake of convenience of the reader we give some basic results of this approach. Minkowski's Conjecture can be restated in the terminology of lattices as : Any lattice ^ of determinant d(^) in Rn is a covering lattice for the set

$$S : |x_1 x_2 \ldots x_n| \leq \frac{d(\wedge)}{2^n}$$

The weaker result (1.3) is equivalent to saying that any lattice ^ of determinant d(^) in Rn is a covering lattice for the set

$$S : |x_1 x_2 \ldots x_n| \leq \frac{d(\wedge)}{v_n 2^{n/2}}$$

Define the homogeneous minimum of ^ as

$$m_H(\wedge) = \inf\{|x_1 x_2 \ldots x_n| : X = (x_1, x_2, \ldots x_n) \in \wedge, X \neq o\}$$

Proposition 1. Suppose that Minkowski Conjecture has been proved for dimensions 1, 2,…., n - 1: Then it holds for all lattices ^ in Rn for which MH(^)=0.

Proposition 2. If ^ is a lattice in Rn for $n \geq 3$ with MH(^) ≠ 0 then there exists an ellipsoid having n linearly independent points of ^ on its boundary and no point of ^ other than O in its interior.

It is well known that using these results, Minkowski's Conjecture would follow from

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	Estimates by Mordell	Estimates by Il'in	Estimates by Hans-Gill et al	Our improved Estimates
n	V_n	V_n	V_n	V_n
10	2.899061	3.47989	24.3627506	27.60348
11	2.973102	3.52291	29.2801145	33.47272
12	3.040525	3.55024	32.2801213	39.59199
13	3.102356	3.57856	34.8475153	45.40041
14	3.159373	3.60209	37.8038391	51.26239
15	3.21218	3.61116	40.905198	57.00375
16	3.261252	3.61908	44.3414913	57.4702
17	3.306972	3.63924	47.2339309	57.67598
18	3.349652	3.66176	46.7645724	57.38876
19	3.389556	3.66734	47.2575897	60.09339
20	3.426907	3.67236	46.8640155	58.48592
21	3.461897	3.67692	46.0522028	56.42571
22	3.494699	3.68408	43.6612034	53.94142
23	3.525464	3.68633	37.8802374	50.98842
24	3.55433	3.68978	32.5852958	47.74632
25	3.581421	3.69295	27.8149432	42.39088
26	3.606852	3.69589	23.0801951	38.8657
27	3.630729	3.70012	17.3895105	31.93316
28	3.653149	3.70263	12.9938763	26.10663
29	3.674203	3.70497	9.5796191	19.96254
30	3.693976	3.70867	6.7664335	16.06884
31	3.712547	3.72558	4.745972	11.23872
32	3.729989			8.325879
33	3.746371			5.411488

Table 1: The weaker result.

Conjecture II. If ^ is a lattice in Rn of determinant 1 and there is a sphere |X| <R which contains no point of ^ other than O in its interior and has n linearly independent points of ^ on its boundary then ^ is a covering lattice for the closed sphere of radius $\sqrt{n/4}$ Equivalently, every closed sphere of radius $\sqrt{n/4}$ lying in Rn contains a point of ^.

They formulated a conjecture from which Conjecture-II follows immediately. To state Woods' conjecture, we need to introduce some terminology [17,18].

Let L be a lattice in Rn. By the reduction theory of quadratic forms introduced by a cartesian co-ordinate system may be chosen in Rn in such a way that L has a basis of the form [19-22],

$(A_1; 0; 0; : : : ; 0); (a_{2;1};A_2; 0; : : : ; 0); : : : ; (a_{n;1}; a_{n;2}; : : : ; a_{n;n-1},A_n);$

where A1;A2; : : : ;An are all positive and further for each i=1; 2; : : : ; n any two points of the lattice in R^{n-i+1} with basis

$(A_i; 0; 0; : : : ; 0); (a_{i+1;i};A_{i+1}; 0; : : : ; 0); : : : ; (a_{n;i}; a_{n;i+1}; : : : ; a_{n;n-1};A_n)$

are at a distance atleast Ai apart. Such a basis of L is called a reduced basis [23].

Conjecture III (Woods): If $A_1A_2 \ldots A_n$=1 and $A_i \le A_1$ for each i then any closed sphere in Rn of radius $\sqrt{n/2}$ contains a point of L.

Woods [10] proved this conjecture for $4 \le n \le 6$ Hans-Gill et al. [12] gave a unified proof of Woods' Conjecture for $n \le 6$ Hans-Gill et al. [12,14] proved Woods' Conjecture for n=7 and n=8 and thus completed the proof of Minkowski's Conjecture for n=7 and 8 Woods [10,24] proved Conjecture and hence Minkowski's Conjecture for n=9. With the assumptions as in Conjecture III, a weaker result would be that

If $w_n \ge n$ any closed sphere in R^n of radius $\sqrt{w_n/2}$ contains a point of L [25,26].

Hans-Gill et al. [12,14] obtained the estimates w_n on Woods' Conjecture for $n^3 \ge 9$ As w_9=9 has been established by the authors [17] recently, we study w_n for $n^3 \ge 10$ in a series of three papers. In this paper we obtain improved estimates w_n on Woods' Conjecture for n=10; 11 and 12. In next papers [18,19], we shall derive improved estimates w_n on Woods' Conjecture for n=13; 14; 15 and for $16 \le n \le 33$ respectively. Together with the following result of Hans-Gill et al. [12], we get improvements of w_n for $n^3 \ge 34$ also.

Proposition 3. Let L be a lattice in R^n with $A_1A_2 \ldots A_n$=1 and $A_i \le A_1$ for each i. Let $0 < l_n \le A_n^2 \le m_n$ where l_n and m_n are real numbers. Then L is a covering lattice for the sphere $|x| \le \sqrt{w_n/2}$ where Wn is defined inductively by

$$w_n = \max\{w_{n-1} l_n^{-1/l_{n-1}} + l_n, w_{n-1} m_n^{-1/m_{n-1}} + m_n\}$$

Here we prove

Theorem 1. Let n=10; 11; 12. If d(L)=A1 : : :An=1 and $A_i \le A_1$ for i=2;….; n, then any closed sphere in Rn of radius $\sqrt{w_n/2}$ contains a point of L, where $w_{10} = 10.3, w_{11} = 11.62$ and $w_{12} = 13$.

The earlier best known values were w_{10}=10:5605061, w_{11}=11:9061976 and w_{12}=13:4499927.

To deduce the results on the estimates of Minkowski's Conjecture we also need the following generalization of Proposition 1

Proposition 4. Suppose that we know

$$M_j \le \frac{1}{v_j 2^{j/2} |\Delta|} for 1 \le j \le n-1$$

Let v_n<min V_{k1} V_{k2}…V_{ks} , where the minimum is taken over all (k_1; k_2; ; k_s) such that n=k1+k2+ : : :+ks, ki positive integers for all i and $s^3 \geq 2$. Then for all lattices in Rn with homogeneous minimum MH(<)=0, the estimate V_n holds for Minkowski's Conjecture.

Since by arithmetic-geometric inequality the sphere $\{X \in R^n : |X| \leq \frac{\sqrt{w_n}}{2}\}$ is a subset of $\{X : |x_1 x_2 \ldots x_n| \leq \frac{1}{2^{n/2}}(\frac{w_n}{2_n})^{n/2}\}$ Propositions 2 and 4 immediately imply

Theorem 2: The values of Vn for the estimates of Minkowski's Conjecture can be taken as $(\frac{2n}{w_n})^{n/2}$

For $10 \leq n \leq 33$ these values are listed in Table 1. In Section 2 we state some preliminary results and in Sections 3-5 we prove Theorem 1 for n=10; 11 and 12.

Preliminary Results and Plan of the Proof

Let L be a lattice in Rn reduced in the sense of Korkine and Zolotare. Let (Sn) denotes the critical determinant of the unit sphere Δ Sn with centre O in R^n i.e.

$\Delta(S_n) = Inf\{d(\wedge) : \wedge$ has no point other than O in the interior of $S_n\}$

Let γ_n be the Hermite's constant i.e. γ_n is the smallest real number such that for any positive de nite quadratic form Q in n variables of determinant D, there exist integers u_1; u_2;…; u_n not all zero satisfying

$$Q(u_1, u_2, \ldots u_n) \leq \gamma_n D^{1/n}$$

It is well known that We write A^2_i=Bi.

We state below some preliminary lemmas. Lemmas 1 and 2 are due to Woods [25], Lemma 3 is due to Korkine and Zolotare [21] and Lemma 4 is due to Pendavingh and Van Zwam [24]. In Lemma 5, the cases n=2 and 3 are classical results of Lagrange and Gauss; n=4 and 5 are due to Korkine and Zolotare [21] while n=6; 7 and 8 are due to Blichfeldt [3].

Lemma 1. If $2\Delta(S_{n+1})A_1^n \geq d(l)$ then any closed sphere of radius

$$R = A_1(1 - \{A_1^n \Delta(S_{n+1}) / d(L)\}^2)^{1/2}$$

in R^n contains a point of L.

Lemma 2. For a Fixed integer i with $1 \leq i \leq n-1$ denote by L_1 the lattice in R^i with reduced basis

$$(A_1, 0, \ldots, 0), (a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{i,1}, a_{i,2}, \ldots, a_{i,i-1}, A_i)$$

and denote by L2 the lattice in R^{n-i} with reduced basis

(A_{i+1}; 0; ; 0); ($a_{i+2;i+1;}A_{i+2}$; 0; ; 0); ; ($a_{n;i+1}$; $a_{n;i+2}$; ; $a_{n;n-1}$;A_n).

If any closed sphere in R_i of radius r1 contains a point of L_1 and if any closed sphere in R_{n-i} of radius r_2 contains a point of L_2 then any closed sphere in Rn of radius $(r_1^2 + r_2^2)^{1/2}$ contains a point of L:

Lemma 3. For all relevant i,

$$B_{i+1} \geq \frac{3}{4} B_i \text{ and } B_{i+2} \geq \frac{2}{3} B_i \tag{2.1}$$

Lemma 4. For all relevant i,

$$B_{i+4} \geq (0.46873) B_i \tag{2.2}$$

Throughout the paper we shall denote 0.46873 by ε .

Lemma 5. $\Delta(S_n) = \sqrt{3}/2, 1/\sqrt{2}, 1/2\sqrt{2}, \sqrt{3}/8, 1/8$ and $1/16$ for n=2; 3; 4; 5; 6; 7 and 8 respectively:

Lemma 6. For any integer s; $1 \leq s \leq n-1$

$$B_1 B_2 \ldots B_{s-1} B_s^{n-s+1} \leq \gamma_{n-s+1}^{n-s+1} \quad \text{and}$$

$$B_1 B_2 \ldots B_s \leq (\gamma_n^{\frac{1}{n-1}} \gamma_{n-1}^{\frac{1}{n-2}} \ldots \gamma_{n-s+1}^{\frac{1}{n-s}})^{n-s} \tag{2.4}$$

This is Lemma 4 of Hans-Gill et al. [12].

Lemma 7.

$$\{(8.5337)^{\frac{1}{5}} \gamma_n^{\frac{1}{n-1}} \gamma_{n-1}^{\frac{1}{n-2}} \ldots \gamma_6^{\frac{1}{5}}\}^{-1} \leq B_n \leq \gamma_{n-1}^{\frac{n-1}{n}} \tag{2.5}$$

This is Lemma 6 of Hans-Gill et al. [14].

Remark 1. Let

δ_n =the best centre density of packings of unit spheres in R^n;

δ_n^* =the best centre density of lattice packings of unit spheres in R^n:

Then it is known that

$$\gamma_n = 4(\delta_n^*)^{\frac{2}{n}} \leq 4(\delta_n)^{\frac{2}{n}} \tag{2.6}$$

δ_n^* and hence δ_n is known for $n \leq 8$ Also γ_{24} =4 has been proved by Cohn and Kumar [6]. For $9 \leq n \leq 12$ using the bounds on δ_n given by Cohn and Elkies [5] and inequality (2.6) we find that $\gamma_9 \leq 2.1326324$, $\gamma_{10} \leq 2.2636302$, $\gamma_{11} \leq 2.3933470$, $\gamma_{12} \leq 2.5217871$

We assume that Theorem 1 is false and derive a contradiction. Let L be a lattice satisfying the hypothesis of the conjecture. Suppose that there exists a closed sphere of radius $\sqrt{w_n/2}$ in R^n that contains no point of L in R^n.

Since B_i=A^2_i and d(L)=1; we have $B_1 B_2$: : :B_n=1:

We give some examples of inequalities that arise. Let L1 be a lattice in R4 with basis (A_1; 0; 0; 0), ($a_{2;1}$;A_2; 0; 0); ($a_{3;1}$; $a_{3;2}$;A_3; 0); ($a_{4;1}$; $a_{4;2}$; $a_{4;3}$;A_4); and L_i for $2 \leq i \leq n$ be lattices in R1 with basis (Ai+3). Applying Lemma 2 repeatedly and using Lemma 1 we see that if $2\Delta(S_5)A_1^4 \geq A_1 A_2 A_3 A_4$ then any closed sphere of radius

$$(A_1^2 - \frac{A_1^{10}\Delta(S_5)^2}{A_1^2 A_2^2 A_3^2 A_4^2} + \frac{1}{4}A_5^2 + \ldots + \frac{1}{4}A_n^2)^{1/2}$$

contains a point of L: By the initial hypothesis this radius exceeds $\sqrt{w_n}/2$ Since $\Delta(S_5) = 1/2\sqrt{2}$ and $B_1 B_2 \ldots B_n = 1$ this results in the conditional inequality : if $B_1^4 B_5 B_6 \ldots B_n \geq 2$ then

$$4B_1 - \frac{1}{2} B_1^5 B_5 B_6 \ldots B_n + B_5 + B_6 + \ldots + B_n > w_n \tag{2.7}$$

We call this inequality (4; 1;…; 1); since it corresponds to the ordered partition (4; 1;…; 1) of n for the purpose of applying Lemma 2. Similarly the conditional inequality (1;…; 1; 2; 1;…; 1) corresponding to the ordered partition (1;…; 1; 2; 1;…; 1) is : if $2B_i \geq B_{i+1}$ then

$$B_1 + \ldots + B_{i-1} + 4B_i - \frac{2B_i^2}{B_{i+1}} + B_{i+2} + \ldots + B_n > w_n \tag{2.8}$$

Since $4B_i - \frac{2B_i^2}{B_{i+1}} \leq 2B_{i+1}$, (2.8) gives

$B_1 + \ldots + B_{i-1} + 2B_{i+1} + B_{i+2} + \ldots + B_n > W_n$:

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One may remark here that the condition $2B_i \geq B_{i+1}$ is necessary only if we want to use inequality (2.8), but it is not necessary if we want to use the weaker inequality (2.9). This is so because if $2B_i<B_{i+1}$, using the partition (1; 1) in place of (2) for the relevant part, we get the upper bound 2Bi + B_{i+1} which is clearly less than $2B_{i+1}$. We shall call inequalities of type (2.9) as weak inequalities and denote it by $(1;\ldots;1;2;1;\ldots;1)_w$.

If $(\lambda_1,\lambda_2,\ldots,\lambda_s)$ is an ordered partition of n, then the conditional inequality arising from it, by using Lemmas 1 and 2, is also denoted by $(\lambda_1,\lambda_2,\ldots,\lambda_s)$ If the conditions in an inequality $(\lambda_1,\lambda_2,\ldots,\lambda_s)$ are satisfied then we say that $(\lambda_1,\lambda_2,\ldots,\lambda_s)$ holds. Sometimes, instead of Lemma 2, we are able to use induction. The use of this is indicated by putting (*) on the corresponding part of the partition. For example, if for n=10, B5 is larger than each of B6;B7;….;B10, and if $\frac{B_1^3}{B_1B_3B_4}>2$ the inequality (4; 6*) gives

$$4B_1-\frac{1}{2}\frac{B_1^3}{B_1B_3B_4}+6(\mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\mathrm{B}_4)^{-1/6}>w_{10} \tag{2.10}$$

In particular the inequality ((n-1)*; 1) always holds. This can be written as

$$w_{n-1}(\mathrm{B}_n)\frac{-1}{(\mathrm{n}-1)}+B_n>W_n \tag{2.11}$$

Also we have $B_1 \geq 1$ because if $B_1<1$, then $B_i \leq B_1 < 1$ for each I contradicting B1B2:::Bn=1.

Using the upper bounds on and the inequality (2.5), we obtain numerical lower and upper bounds on Bn, which we denote by ln and mn respectively. We use the approach of Hans-Gill et al. [14], but our method of dealing with

Is somewhat different. In Sections 3-5 we give proof of Theorem 1 for n=10; 11 and 12 respectively. The proof of these cases is based on the truncation of the interval [ln;mn] from both the sides.

In this paper we need to maximize or minimize frequently functions of several variables. When we say that a given function of several variables in x; y; is an increasing/decreasing function of x; y;…., it means that the concerned property holds when function is considered as a function of one variable at a time, all other variables being fixed.

Proof of Theorem 1 for n=10

Here we have W_{10}=10:3, $B_1< \gamma_{10}$<2:2636302. Using (2.5), we have l10=0:4007<B10<1:9770808=m_{10}.

The inequality (9*; 1) gives $9(\mathrm{B10})^{\frac{-1}{9}}$ + B10<10:3. But for 0:4398 B10 1:9378, this inequality is not true. Hence we must have either B10<0:4398 or B10>1:9378. We will deal with the two cases 0:4007< B10<0:4398 and 1:9378<B10<1:9770808 separately:

0:4007<B_{10}<0:4398

Using the Lemmas 3 & 4 we have:

$$\begin{cases} B_9 \leq \frac{4}{3}B_{10} < 0.5864 \quad B_8 \leq \frac{3}{2}B_{10} < 0.6597 \quad B_7 \leq 2B_{10} < 0.8796 \\ B_6 \leq \frac{B_{10}}{\varepsilon} < 0.9383 \quad B_5 \leq \frac{4}{3}\frac{B_{10}}{\varepsilon} < 1.2511 \quad B_4 \leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.4075 \\ B_3 \leq \frac{2B_{10}}{\varepsilon} < 1.8766 \quad B_2 \leq \frac{B_{10}}{(\varepsilon)^2} < 2.0018 \end{cases}$$

Claim(i) B_2>1:7046

The inequality (2; 2; 2; 2; 2)w gives $2B_2+2B_4+2B_6+2B_8+2B_{10}$>10:3. Using (3.1), we find that this inequality is not true for $B_2 \leq$ 1:7046. Hence we must have B_2>1:7046.

Claim(ii) B_2<1:8815

Suppose $B_2 \geq 1.8815$ then using (3.1) and that $B_6 \geq \varepsilon B_2$ we find that $\frac{B_2^3}{B_3B_4B_5}>2$ and $\frac{B_6^3}{B_7B_8B_9}>2$ So the inequality (1,4,4,1) holds, i.e. B_1+4B_2-

$\frac{1}{2}\frac{B_2^4}{B_3B_4B_5}+4B_6-\frac{1}{2}\frac{B_2^4}{B_7B_8B_9}+B_{10}>10.3$ Applying AM-GM inequality we get

$B_1+4B_2+4B_6+B_{10}-\sqrt{B_2^5B_6^5B_1B_{10}}>10.3$ Now since

$\varepsilon^2B_2 \leq B_{10} < 0.4398$ $B_6 \geq \varepsilon B_2, B_1 \geq B_2$ and $B_2 \geq 1.8815$ we find that the left side is a decreasing function of B_{10} and B_6. So replacing B_{10} by $\varepsilon^2 B_2$ and by εB_2 we get $\varnothing_1 = B_1+(4+4\varepsilon+\varepsilon^2)\mathrm{B}_2-\sqrt{(\varepsilon)^7B_2^{11}B_1}>10.3$ Now the left side is a decreasing function of B2, so replacing B2 by 1.8815 we find that $\varnothing_1<10.3$ for 1<B_1<2:2636302, a contradiction. Hence we must have B_2<1:8815.

Claim (iii) B_3<1:5652

Suppose $B_3 \geq 1.5652$ From (3.1) we have $B_4B_5B_6$<1:6524 and $B_8B_9B_{10}$ <0:1702, so we find that $\frac{B_3^3}{B_4B_5B_6}>2$ and $\frac{B_7^3}{B_8B_9B_{10}} \geq \frac{(\varepsilon B_3)^3}{B_8B_9B_{10}}>2$ for B_3>1:49.

Applying AM-GM to inequality (2,4,4) we get $4B_1-\frac{2B_1^2}{B_2}+4B_3+4B_7-\sqrt{B_3^5B_7^5B_1B_2}>10.3$ Since $B_1 \geq B_2 > 1.7046, B_7 \geq \varepsilon B_3$ and $B_3 \geq 1.5652$ we find that left side is a decreasing function of B_1 and B_7. So we replace B_1 by B_2, B_7 by εB_3 and get that $\varnothing_2 = 2B_2+4(1+\varepsilon)\mathrm{B}_3-\sqrt{(\varepsilon)^5B_3^{10}B_2^2}>10.3$.

But left side is a decreasing function of B3, so replacing B3 by 1.5652 we find that $\varnothing_2<10.3$ for 1:7046<B_2<1:8815, a contradiction. Hence we must have B_3<1:5652.

Claim (iv) B1>1:9378

Suppose $B_1 \leq 1.9378$ Using (3.1) and that B3<1:5652, B2>1:7046, we find that B_2 is larger than each of B_3; B4;…;B10. So the inequality (1; 9,*) holds. This gives $B_1+9(\mathrm{B}_1)^{-1/9}>10.3$ which is not true for $B_1 \leq 1.9378$ So we must have B_1>1:9378.

Claim (v) B3<1:5485

Suppose $B_3 \geq 1.5485$ We proceed as in Claim(iii) and replace B_1 by 1.9378 and B_7 by εB_3 to get that

$$\varnothing_3 = 4(1.9378)-\frac{2(1.9378)^2}{B_2}+4(1+\varepsilon)\mathrm{B}_3-\sqrt{(\varepsilon)^5(1.9378)\mathrm{B}_3^{10}B_2}>10.3$$

One easily checks that $\varnothing_3<10.3$ for $1.5485 \leq$ B_3<1:5652 and 1:7046< B_2<1:8815. Hence we have B_3<1:5485.

Claim (vi) B_1<2:0187

Suppose $B_1 \geq 2.0187$ Using (3.1) and Claims (ii), (v) we have $B_2B_3B_4$<4:11. Therefore $\frac{B_1^3}{B_2B_3B_4}>2$ As $B_5 \geq \varepsilon B_1 > 0.9462$ we see using (3.1) that B_5 is larger than each of $B_6;B_7,\dots;B_{10}$. Hence the inequality (4; 6,*) holds. This gives $\varnothing_4 = 4B_1 - \frac{1}{2}\frac{B_1^4}{B_2B_3B_4} + 6(B_1B_2B_3B_4)^{-1/6} > 10.3$ Left side is an increasing function of $B_2B_3B_4$ and decreasing function of B1. So we can replace $B_2B_3B_4$ by 4:11 and B1 by 2.0187 to find $\varnothing_4 < 10.3$ a contradiction. Hence we have B_1<2:0187.

Claim (vii) B_4<1:337

Suppose $B_4 \geq 1.337$ then using (3.1) we get $\frac{B_4^3}{B_5B_6B_7} > 2$ Applying AMGM to inequality (1,2,4,2,1) we have

$$B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 + 4B_8 + B_{10} - 2\sqrt{B_4^5B_8^5B_1B_2B_3B_{10}} > 10.3$$

Since B_2>1:7046 $B_3 \geq \frac{3}{4}B_2, B_4 \geq 1.337 B_8 \geq \varepsilon B_4$ and $B_{10} \geq \frac{2\varepsilon}{3}B_4$ we find that left side is a decreasing function of B_2, B_8 and B_{10}. So we can replace B_2 by 1.7046; B_8 by εB_4 and B_{10} by $\frac{2\varepsilon}{3}B_4$ to get

$$\varnothing_5 = B_1 + 4(1.7046) - \frac{2(1.7046)^2}{B_3} + (4 + 4\varepsilon + \frac{2\varepsilon}{3})B_4 - 2\sqrt{\frac{2}{3}(\varepsilon)^4(1.7046)B_4^9B_1B_3} > 10.3$$

Now left side is a decreasing function of B4, replacing B4 by 1:337, we find that $\varnothing_5 < 10.3$ for 1<B_1<2:0187 and 1<B_3<1:5485, a contradiction. Hence we have B_4<1:337.

Claim (viii) B_5<1:1492

Suppose $B_5 \geq 1.1492$ Using (3.1), we get $B_6B_7B_8$<0:5445: Therefore $\frac{B_2^3}{B_6B_7B_8} > 2$ Also using Lemma 3 & 4, 2 $B_9 \geq 2(\varepsilon B_5)$ >1:077>B_{10}. So the inequality (4*; 4; 2) holds, i.e. 4 $(\frac{1}{B_5B_6B_7B_8B_9B_{10}})^{1/4} + 4B_5 - \frac{1}{2}\frac{B_5^4}{B_6B_7B_8} + 4B_9 - \frac{2B_9^2}{B_{10}} > 10.3$ Now left side is a decreasing function of B_5 and B_9. So we replace B_5 by 1.1492 and B_9 by 1.1492ε and get that $\varnothing_6(x, B_{10}) = 4(\frac{1}{(\varepsilon)(1.1492)^2 X b_{10}})^{1/4} + 4(1+\varepsilon)(1.1492) - \frac{1}{2}\frac{(1.1492)^4}{x} - \frac{2(1.1492\varepsilon)^2}{B_{10}} > 10.3$ where x=$B_6B_7B_8$. Using Lemma 3 & 4 we have x=$B_6B_7B_8$ $\geq \frac{B_5^3}{4} \geq \frac{(1.1492)^3}{4}$ and $B_{10} \geq \frac{3\varepsilon}{4}B_5 \geq \frac{3\varepsilon}{4}(1.1492)$ It can be verified that $\varnothing_6(x, B_{10}) < 10.3$ for $\frac{(1.1492)^3}{4} \leq x < 0.5445$ and $\frac{3\varepsilon}{4}(1.1492) \leq B_{10} < 0.4398$ giving thereby a contradiction. Hence we must have B_5<1:1492.

Claim (ix) B_2<1:766.

Suppose $B_2 \geq 1.766$ We have $B_3B_4B_5$<2:3793. So $\frac{B_2^3}{B_3B_4B_5} > 2$ Also $B_6 \geq \varepsilon B_2 > 0.8277$ Therefore B6 is larger than each of $B_7, B_8, B_9 B_{10}$ Hence the inequality (1; 4; 5,*) holds. This gives $B_1 + 4B_2 - \frac{1}{2}\frac{B_2^4}{B_3B_4B_5} + 5(\frac{1}{B_1B_2B_3B_4B_5})^{\frac{1}{5}} > 10.3$ Left side is an increasing function of $B_3B_4B_5$, a decreasing function of B_2 and an increasing function of B_1. One easily checks that this inequality is not true for B_1<2:0187;

$B_2 \geq 1.766$ and $B_3B_4B_5$<2:3793: Hence we have B_2<1:766.

Final contradiction

As 2(B_2+B_4+B_6+B_8+B_{10})<2(1:766+1:337+0:9383+0:6597+0:4398)<10:3, the weak inequality (2; 2; 2; 2; 2)w gives a contradiction.

9378<B_{10}<1:9770808

Here $B_1 \geq B_{10} > 1.9378$ and B_2=$(B_1B_3\dots B_{10})^{-1}$ $\leq (B_1B_2B_4\dots B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^3 B_3^6B_1^2B_{10})^{-1} = (\frac{1}{16}\varepsilon^4 B_2^7B_1B_{10})^{-1}$

Which implies $(B_2)^8 \leq (\frac{1}{16}\varepsilon^4(1.9378)^2)^{-1}$ i.e. B2<1:75076.

Similarly

$$B_3 = (B_1B_2B_4\dots B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^3 B_3^6B_1^2B_{10})^{-1}$$

$$B_4 = (B_1B_2B_3B_5\dots B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^2 B_4^5B_1^3B_{10})^{-1}$$

$$B_6 = (B_1\dots B_5B_7B_8B_9B_{10})^{-1} \leq (\frac{1}{16}\varepsilon B_6^3B_1^3B_{10})^{-1}$$

$$B_8 = (B_1\dots B_7B_9B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^3 B_8B_1^7B_{10})^{-1}$$

These respectively give B_3<1:46138, B_4<1:22883, B_6<0:896058 and B_8<0:721763. So we have $B_1^4B_5B_6B_7B_8B_9B_{10} = \frac{B_1^3}{B_2B_3B_4} > 2$ Also $2B_5 \geq 2(\varepsilon B_1) > 1.8166 > B_6$ and $2B_7 \geq 2(\frac{2\varepsilon}{3}B_1) > B_8$ Applying AM-GM to inequality (4,2,2,1,1) we have 4B_1 + 4B_5 + 4B_7 + B_9 +B_{10} -3 $(2B_1^5B_5^3B_7^3B_9B_{10})^{\frac{1}{3}} > 10.3$ We find that left side is a decreasing function of B_7 and B_5, so can replace B_7 by $\frac{2}{3}\varepsilon B_1$ and B_5 by εB_1 then it is a decreasing function of B_1, so replacing B_1 by B_{10} we have 4 $(1+\varepsilon+\frac{2}{3}\varepsilon)B_{10} + B_9 + B_{10} - 2^{\frac{4}{3}}(\varepsilon)^2(B_{10})4(B_9)^{\frac{1}{3}} > 10.3$ which is not true for $(1.9378)\varepsilon^2 < B_9 \leq B_1 < 2.2636302$ and 1:9378<B10<1:9770808. Hence we get a contradiction.

Proof of Theorem 1 for n=11

Here we have w_{11}=11.62, $B_1 \leq \gamma_{11} < 2.393347$ Using (2.5), we have l_{11}=0:3673<B_{11}<2:1016019=m_{11}.

The inequality (10^*; 1) gives 10:3 $(B_{11})^{\frac{-1}{10}} + B_{11} > 11.62$ But for $0.4409 \leq B_{11} \leq 2.018$ this inequality is not true. So we must have either $B_{11} < 0.4409$ or B_{11}>2:018.

0:3673<B11<0:4409

Claim (i) B10<0:4692

Suppose $B_{10} \geq 0.4692$ then 2B_{10}>B_{11}, so (9^*; 2) holds, i.e. 9 $(\frac{1}{B_{10}B_{11}})^{\frac{1}{9}} + 4B_{10} - \frac{2B_{10}^2}{B_{11}} > 11.62$ As left side is a decreasing function of B_{10}, we can replace B_{10} by 0.4692 and find that it is not true for 0:3673<B_{11}<0:4409.

Hence we must have $B_{10<}$0:4692.

Using Lemmas 3 and 4 we have:

$$B_9 \leq \frac{4}{3}B_{10} < 0.6256, B_8 \leq \frac{3}{2}B_{10} < 0.7038, B_7 \leq \frac{B_{11}}{\varepsilon} < 0.94063$$

$$B_6 \leq \frac{B_{10}}{\varepsilon} < 1.00.., B_5 \leq \frac{4}{3}\frac{B_{10}}{\varepsilon} < 1.3347, B_4 \leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.50151$$

$$B_3 \le \frac{B_{11}}{\varepsilon^2} < 2.0068, B_2 \le \frac{B_{10}}{\varepsilon^2} < 2.13557 \qquad (4.1)$$

Claim (ii) B2>1:913

The inequality (2; 2; 2; 2; 2; 1) $_w$ gives $2B_2+2B_4+2B_6+2B_8+2B_{10}+B_{11}>$ 11:62. Using (4.1) we find that this inequality is not true for $B_2 \le 1.913$ so we must have B_2>1:913.

Claim(iii) B3<1:761

Suppose $B_3 \ge 1.761$ then we have $\frac{B_3^3}{B_4B_5B_6} > 2$ and $\frac{B_7^3}{B_8B_9B_{10}} > \frac{(\varepsilon B_3)^3}{B_8B_9B_{10}} >$ 2. Applying AM-GM to the inequality (2,4,4,1) we get $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 + 4B_7 + B_{11} - \sqrt{B_3^5B_7^5B_1B_2B_{11}} > 11.62$ One easily finds that it is not true for $B_1 \ge B_2 > 1.913, B_3 \ge 1.761, B_7 \ge \varepsilon B_3, B_{11} \ge \varepsilon^2 B_3, 1.913 < B_2 < 2.13557$ and $1.761 \le B_3 < 2.0068$ Hence we must have B_3<1:761:

Claim (iv) B1<2:2436

Suppose $B_1 \ge 2.2436$ As $B_2B_3B_4$< 2:13557×1:761×1:50151<5:6468, we have $\frac{B_1^3}{B_2B_3B_4} > 2$ Also $B_5 \ge \varepsilon B_1 > 1.051$ so B5 is larger than each of B6;B7…;B11. Hence the inequality (4; 7,*) holds. This gives $4B_1 - \frac{1}{2}\frac{B_1^4}{B_2B_3B_4} + 7(\frac{1}{B_1B_2B_3})^{\frac{1}{7}} > 11.62$ Left side is an increasing function of $B_2B_3B_4$ and decreasing function of B_1. One easily checks that the inequality is not true for $B_2B_3B_4$<5:6468 and B1 $\ge$ 2:2436. Hence we have B_1<2:2436.

Claim (v) B4<1.4465 and B2>1:9686

Suppose $B_4 \ge$ 1.4465 We have B5B6B7<1:2569 and $B_9B_{10}B_{11}$<0:1295. Therefore for B_4>1:36, we have $\frac{B_4^3}{B_5B_6B_7} > 2$ and $\frac{B_8^3}{B_9B_{10}B_{11}} > \frac{(\varepsilon B_4)^3}{B_9B_{10}B_{11}} > 2$ So the inequality (1,2,4,4) holds. Applying AM-GM to inequality(1,2,4,4), we get $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_8 - \sqrt{B_4^5B_8^5B_1B_2B_3} > 11.62$ A simple calculation shows that this is not true for $B_1 \ge B_2 > 1.913$, $B_4 \ge 1.4465$, $B_8 \ge \varepsilon B_4 \ge 1.4465, B_1 < 2.2436$ and $B_3 < 1.761$ Hence we have B4<1:4465.

Further if $B_2 \le 1.9686$ then $2B_2+2B_4+2B_6+2B_8+2B_{10}+B_{11}$<11:62. So the inequality (2; 2; 2; 2; 2; 1)$_w$ gives a contradiction.

Claim (vi) B4<1:4265 and B2>1:9888

Suppose $B_4 \ge 1.4265$ We proceed as in Claim (v) and get a contradiction with improved bounds on B_2 and B_4.

Claim (vii) B1<2:2056

Suppose $B_1 \ge 2.2056$ As B3B4B5<1:761 × 1:4265 × 1:3347<3:3529, we have $\frac{B_2^3}{B_3B_4B_5} > 2$ Also $B_6 \ge \varepsilon B_2 > 0.9491$ so B6 is larger than each of B7;B8,…,B11. Hence the inequality (1; 4; 6*) holds, i.e. B1 + 4B2 - $\frac{1}{2}\frac{B_2^4}{B_3B_4B_5} + 6(\frac{1}{B_1B_2B_3B_4B_5})^{\frac{1}{6}} > 11.62$

Claim (ix) B1<2:1669

Suppose $B_1 \ge 2.1669$ We proceed as in Claim(iv) and get a contradiction with improved bounds on B_1, B_2 and B_4.

Claim (x) B4<1:403 and B2>2:012

Suppose $B_4 \ge 1.403$ We proceed as in Claim(v) and get a contradiction with improved bounds on B2 and B4.

Final Contradiction:

As now B3B4B5<1:761×1:403 1:3347<3:2977, we have $\frac{B_2^3}{B_3B_4B_5} > 2$ for B2>2:012. Also $B_6 \ge \varepsilon B_2 > 0.943 >$ each of B7; B8; B11. Hence the inequality (1; 4; 6) holds. Proceeding as in Claim (viii) we find that this inequality is not true for B_1<2:1669; B_2>2:012 and $B_3B_4B_5$<3:2977; giving thereby a contradiction.

2:018<B11<2:1016019

Here $B_1 \ge B_{11} > 2.018$ Therefore using Lemmas 3 & 4 we have $B10=(B1\ \ B9B11)^{-1}$

$$\le (B_1.\frac{3}{4}B_1.\frac{2}{3}B_1\frac{1}{2}B_1\,\varepsilon B_1\frac{3}{4}\varepsilon B_1\frac{2}{3}\varepsilon B_1\frac{1}{2}\varepsilon B_1\,\varepsilon^2 B_1.B_1)^{-1}$$

$$= (\frac{1}{16}\varepsilon^6 B_1^9 B_{11})^{-1} < (\frac{1}{16}\varepsilon^6(2.018)^{10})^{-1} < 1.34702$$

Similarly

$B_4 = (B_1B_2B_3B_4 \ldots B_{11})^{-1} \le (\frac{1}{16}\varepsilon^3 B_1^3 B_{11})^{-1}$ which gives B4<1:37661.

Claim (i) B10<0:4402

The inequality (9*; 1; 1) gives $9(\frac{1}{B_{10}B_{11}})^{\frac{1}{9}} + B_{10} + B_{11} > 11.62$ But this inequality is not true for $0.4402 \le B_{10}$<1:34702 and 2:018<B11<2:1016019. Hence we must have B10<0:4402.

Now we have $B_9 \le \frac{4}{3}B_{10}$ < 0:58694, $B_8 \le \frac{3}{2}B_{10} < 0.6603, B_7 \le 2B_{10} < 0.8804$ and $B_6 \le \frac{B_{10}}{\varepsilon} < 0.93914$

Claim (ii) B7<0:768

Suppose $B_7 \ge 0.768$ Then $\frac{B_7^3}{B_8B_9B_{10}} > 2$ so (6*; 4; 1) holds. This gives $\varnothing_7(x) = 6(x)^{1/6} + 4B_7 - \frac{1}{2}B_7^5B_{11}x + B_{11} > 11.62$ where $x=B_1B_2$: : :B_6. The function $\varnothing_7(x)$ has its maximum value at $x = (\frac{2}{B_7^5B_{11}})^{6/5}$ Therefore $\varnothing_7(x) \le \varnothing_7((\frac{2}{B_7^5B_{11}})^{6/5})$ which is less than 11:62 for $0.768 \le B_7 < 0.8804$ 2:018<B11<2:1016019. This gives a contradiction.

Now $B_5 \le \frac{3}{2}B_7 < 1.1521$ and $B_3 \le \frac{B_7}{\varepsilon} < 1.6385$

Claim (iii) B2<1:795

Suppose $B_2 \ge 1.795$ then $\frac{B_2^3}{B_3B_4B_5} > 2$ and $\frac{B_6^3}{B_7B_8B_9} > 2$ Applying AMGM to the inequality (1,4,4,1,1) p we get B1 + 4B2 + 4B6 + B10 + B11 - $\sqrt{B_2^5B_6^5B_1B_{10}B_{11}} > 11.62$ We find that left side is a decreasing function of B_6, so we first replace B_6 by εB_2 then it is a decreasing function of B_2, so we replace B_2 by 1.795 and get that

$$\varnothing_8(B_{11}) = B_1 + 4(1+\varepsilon)(1.795) + B_{10} + B_{11} - \sqrt{(\varepsilon)^5(1.795)^{10}B_1B_{10}B_{11}} > 11.62$$

Now $\varnothing_8(B_{11})>0$ so $\varnothing_8(B_{11})<\max\{\varnothing_8(2.018),\varnothing_8(2.1016019)\}$ which can be verified to be at most 11.62 for $(\varepsilon)^2(1.795)\le B_{10}<0.4402$ and 2:018<B1<2:393347, giving thereby a contradiction.

Claim (iv) B5<0:98392

Suppose $B_5\ge 0.98392$ We have $\frac{B_1^3}{B_2B_3B_4}>2$ and $\frac{B_5^3}{B_6B_7B_8}>2$ Also $2B_9\ge 2(\varepsilon B_5)>B_{10}$ Applying AM-GM to the inequality (4; 4; 2; 1) we get $4B_1+4B_5+4B_9-\frac{2B_9^2}{B_{10}}+B_{11}-\sqrt{B_1^5B_5^5B_9B_{10}B_{11}}>11.62$ One can easily check that left side is a decreasing function of B_9 and B_1 so we can replace B_9 by εB_5 and B1 by B11 to get $\varnothing_9=5B_{11}+4(1+\varepsilon)B_5-\frac{2(\varepsilon B_5)^2}{B_{10}}-\sqrt{\varepsilon B_{11}^6B_5^6B_{10}}>11.62$ Now the left side is a decreasing function of B5, so replacing B5 by 0.98392 we see that $\varnothing_9<11.62$ for $\frac{3\varepsilon}{4}(0.98392)<B_{10}<0.4409$ and 2:018<B11 < 2:1016019, a contradiction.

Final Contradiction:

As in Claim(iv), we have $\frac{B_1^3}{B_2B_3B_4}>2$ Also $B_5\ge\varepsilon B_1>0.9458$ each of $B_6;B_7,\ldots,B_{10}$. Therefore the inequality (4; 6*; 1) holds, i.e. $\varnothing_{10}=4B_1\frac{1}{2}\frac{B_1^4}{B_2B_3B_4}+6(\frac{1}{B_1B_2B_3B_4B_{11}})^{\frac{1}{6}}+B_{11}>11.62$ Left side is an increasing function of $B_2B_3B_4$ and B_{11} and decreasing function of B_1. Using B_5<0:98392, we have $B_3\le\frac{3}{2}B_5<1.47588$ and $B_4\le\frac{4}{3}B_5<1.311894$ One easily checks that $\varnothing_{10}<11.62$ for $B_2B_3B_4$<1:795 × 1:47588 × 1:311894, B11<2:1016019 and $B_1\ge 2.018$ Hence we have a contradiction.

Proof of Theorem 1 for n=12

Here we have w_{12}=13, $B_1\le\gamma_{12}<2.5217871$ Using (2.5), we have l12 =0:3376<B_{12}<2:2254706=m_{12} and using (2.3) we have $B_1B_2^{11}\le\gamma_{11}^{11}$ i.e $B_2\le\gamma_{11}^{\frac{11}{12}}<2.2254706$

The inequality (11*; 1) gives 11:62$(B_{12})^{-1/11}$ +B12>13. But this is not true for $0.4165\le B_{12}\le 2.17$ So we must have either B12<0:4165 or B12>2:17.

0:3376<B12<0:4165

Claim (i) B11<0:459

Suppose $B_{11}\ge 0.459$ then $B_{12}\ge\frac{3}{4}B_{11}>0.34425$ and 2B11>B_{12}, so (10*; 2) holds, i.e. $\phi_{11}=10.3(\frac{1}{B_{11}B_{12}})^{\frac{1}{10}}+4B_{11}-\frac{2B_{11}^2}{B_{12}}>13$ Left side is a decreasing function of B11, so we can replace B11 by .459 to find that ϕ_{11} <13 for 0:34425<B12<0:4165, a contradiction. Hence we have B_{11}<0:459.

Claim (ii) B10<0:5432

Suppose $B_{10}\ge 0.5432$ From Lemma 3, $B_{11}B_{12}\ge\frac{1}{2}B_{10}^2$ and $B_{10}\le\frac{3}{2}B_{12}$. Therefore $\frac{1}{2}(0.5432)^2\le B_{11}B_{12}<0.1912$ and $B_{10}^2>B_{11}B_{12}$ so the inequality (9*; 3) holds, i.e. $9\ (\frac{1}{B_{10}B_{11}B_{12}})^{\frac{1}{9}}+4B_{10}-\frac{B_{10}^3}{B_{11}B_{12}}>13$ One easily checks that it is not true noting that left side is a decreasing function of B_{10}. Hence we must have B_{10}<0:5432.

Claim (iii) B9<0:6655

Suppose $B_9\ge 0.6655$ then $\frac{B_9^3}{B_{10}B_{11}B_{12}}>2$ So the inequality (8*; 4) holds. This gives $\phi_{12}(x)^{1/8}+4B_9-\frac{1}{2}B_9^5x>13$ where x=B_1B_2 … B_8. The function $\phi_{12}(x)$ has its maximum value at $x=(\frac{2}{B_9^5})^{\frac{8}{7}}$ so $\phi_{12}(x)<\phi_{12}((\frac{2}{B_9^5})^{\frac{8}{7}})<13$ for $0.6655\le B_9-\frac{1}{2}B_9^5x>13$ where x=B_1B_2... B8. The function $\phi_{12}(x)$ has its maximum value at $x=(\frac{2}{B_9^5})^{\frac{8}{7}}$ so $\phi_{12}(x)<x=(\frac{2}{B_9^5})^{\frac{8}{7}}$<13 for 0:6655 $\le B_9\le\frac{3}{2}B_{11}<0.6885$ This gives a contradiction.

Using Lemmas 3 & 4 we have:

$$B_8\le\frac{3}{2}B_{10}<0.8148,B_7\le\frac{B_{11}}{\varepsilon}<0.9793,B_6\le\frac{B_{10}}{\varepsilon}<1.1589$$

$$B_5\le\frac{B_9}{\varepsilon}<1.4198,B_4\le\frac{3}{2}\frac{B_{10}}{\varepsilon}<1.7384,B_3\le\frac{B_{11}^{\varepsilon}}{\varepsilon^2}<2.0892$$

Claim (iv) B2>1:828, B4>1:426, B6>1:019 and B8>0:715

Suppose $B_2\le 1.828$ Then 2(B2+B4+B6+B8+B10+B12)<2(1:828+ 1:7384+1:1589+0:8148+0:5432+0:4165)<13, giving thereby a contradiction to the weak inequality (2; 2; 2; 2; 2; 2) w.

Similarly we obtain lower bounds on B_4;B_6 and B_8 using (2; 2; 2; 2; 2; 2)w.

Claim(v) B2>2:0299

Suppose $B_2\le 2.0299$ Consider following two cases:

Case (i) B3>B4

We have B3>B4>1:426>each of B5,….,B12. So the inequality (2; 10*) holds, i.e. $4B_1-\frac{2B_1^2}{B_2}+10.3(\frac{1}{B_1B_2})^{\frac{1}{10}}>13$ The left side is a decreasing function of B_1, so replacing B_1 by B_2 we get 2B2 + 10:3 $(\frac{1}{B_2^2})^{\frac{1}{10}}>13$ which is not true for $B_2\le 2.0299$

Case (ii) $B_3\le B_4$

As B4>1:426>each of B5,….,B12, the inequality (3; 9*) holds, i.e. $\phi_{13}(X)=4B_1-\frac{B_1^3}{x}+9(\frac{1}{B_1x})^{\frac{1}{9}}>13$ where X=B2B3<min $\{B_1^2,(2.0299)(1.7384)\}=\alpha$ say. Now $\phi_{13}(X)$ is an increasing function of X for $B_1\ge B_2>1.828$ and So $\phi_{13}(x)<\phi_{13}(X)$ which can be seen to be less than 13. Hence we have B2>2:0299.

Claim (vi) B1>2:17 and B3<1:9517

Using (2.3) we have $B_3\le(\frac{\gamma_{10}^{10}}{B_1B_2})^{\frac{1}{10}}<1.9648$ Therefore B_2>2:0299>each of $B_3,\ldots,B_{12}$. So the inequality (1; 11*) holds, i.e. B_1 + 11:62 $(\frac{1}{B_1})^{\frac{1}{11}}>13$ But this is not true for $B_1\le 2.17$ So we must have B1>2:17: Again using (2.3) we have $B_3<(\frac{2.2636302}{2.17\times 2.0299})^{\frac{1}{10}}<1.9517$

Claim (vii) B4<1:646

Suppose $B_4\ge 1.646$ From (5.1) and Claims (i)-(iii), we have $\frac{B_4^3}{B_5B_6B_7}>2$ and $\frac{B_8^3}{B_9B_{10}B_{11}}>\frac{(\varepsilon B_4)^3}{B_9B_{10}B_{11}}>2$ Applying AM-GM to the inequality (1,2,4,4,1) we get $\phi_{14}=B_1+4B_2-\frac{2B_2^2}{B_3}+4B_4+4B_8+B_{12}-\sqrt{B_4^5B_8^5B_1B_2B_3B_{12}}>13$ We find that left side is a decreasing function of B2, B8 and B12. So we can replace B2 by 2:0299, B8 by "B4 and B12 by $\varepsilon^2 B_4$. Then it turns a decreasing function of $\varepsilon^2 B_4$, so can replace B4 by 1.646 to find that ϕ_{14}<13 for B_1<2:52178703 and B_3<1:9517, a contradiction. Hence we have B_4<1:646.

Claim (viii) B1<2:4273

Suppose B1 $\geq$ 2:4273. Consider following two cases:

Case (i) $B_5>B_6$

Here B_5>each of $B_{6,\dots,}B_{12}$ as $B_5 \geq \varepsilon B_1$>1.137 >each of B7,…, B12. Also B2B3B4<2:2254706×1:9517×1:646<7:15. So $\frac{B_1^3}{B_2B_3B_4}>2$ Hence the inequality (4; 8*) holds. This gives $4B_1-\frac{1}{2}\frac{B_1^4}{B_2B_3B_4}+8(B_1B_2B_3B_4)^{-1/8}>13$

Left side is an increasing function of $B_2B_3B_4$ and decreasing function of B_1. So we can replace $B_2B_3B_4$ by 7.15 and B_1 by 2.4273 to get a contradiction.

Case (ii) $B_5 \leq B_6$

Using (5.1) we have $B_5 \leq B_6$<1:1589 and so $B_4 \leq \frac{4}{3}B_5 < 1.5452$ Therefore $\frac{B_2^3}{B_3B_4B_5}>2$ as B2>2:0299 and B_3<1:9517. Also from Claim (iv), B_6>1:019>each of $B_7,\dots,B_{12}$. Hence the inequality (1; 4; 7*) holds. This gives $B_1+4B_2-\frac{1}{2}\frac{B_2^4}{B_4B_5B_6}+7$ $7(B_1B_2B_3B_4B_5)^{-1/7}$>13: Left side is an increasing function of $B_3B_4B_5$ and B_1 and a decreasing function of B_2. One can check that inequality is not true for $B_3B_4B_5$<1:9517×1:5452 ×1:1589, B_1<2:5217871 and for B_2>2:0299: Hence we must have B_1<2:4273:

Claim (ix) B5<1:396

Suppose $B_5 \geq 1.396$ From (5.1), $B_6B_7B_8$<0:925 and $B_{10}B_{11}B_{12}$<0:104, so we have $\frac{B_5^3}{B_6B_7B_8}>2$ and $\frac{B_9^3}{B_{10}B_{11}B_{12}}>\frac{(\varepsilon B_5)^3}{B_{10}B_{11}B_{12}}>2$ Applying AMGM to the inequality (1,2,1,4,4) we get B_1 + $4B_2$ - $\frac{2B_2^2}{B_3}$+ B_4 + $4B_5$ + $4B_9$- $\sqrt{B_5^5B_9^5B_1B_2B_3B_4}>13$ We find that left side is a decreasing function of B2 and B9. So we replace B2 by 2:0299 and B_9 by εB_5. Now it becomes a decreasing function of B_5 and an increasing function of B_1 so replacing

B_5 by 1.396 and B_1 by 2.4273, we find that above inequality is not true for 1:522<B_3<1:9517 and 1:426<B_4<1:646, giving thereby a contradiction. Hence we must have B_5<1:396.

Claim (x) B3>1:7855

Suppose $B_3 \leq 1.7855$ We have B4>1:426>each of $B_5;B_6,\dots,B_{12}$, hence the inequality (1; 2; 9*) holds. It gives $\phi_{15}=B_1+4B_2-\frac{2B_2^2}{B_3}+9(\frac{1}{B_1B_2B_3})^{\frac{1}{9}}>13$ It is easy to check that left side of above inequality is a decreasing function of B2 and an increasing function of B1 and B3. So replacing B_1 by 2.4273, B_3 by 1.7855 and B_2 by 2.0299 we get -15<13; a contradiction. Hence we have B_3>1:7855.

Claim (xi) $B_2>2.0733$

Suppose $B_2 \leq 2.0733$ We have B3>1:7855>each of $B_4;B_5,\dots,B_{12}$, hence the inequality (2; 10*) holds. It gives $\phi_{16}=4B_1-\frac{2B_1^2}{B_2}+10.3(\frac{1}{B_1B_2})^{\frac{1}{10}}>13$ The left side is a decreasing function of B_1 and an increasing function of B_2, so replacing B_1 by 2:17 and B_2 by 2.0733 we get $\phi_{16}<13$ a contradiction.

Claim (xii) B7<0:92 and B5<1:38

Suppose $B_7 \geq 0.92$ Here we have $B_4B_5B_6$<2:67 and $B_8B_9B_{10}$<0:295, so $\frac{B_3^3}{B_4B_5B_6}>2$ and $\frac{B_7^3}{B_8B_9B_{10}}>2$ Also $2B_{11} \geq 2\varepsilon B_7 > B_{12}$ Applying AM-GM to the inequality (2,4,4,2) we get $\phi_{17}=4B_1-\frac{2B_1^2}{B_2}+4B_3+4B_7-\sqrt{B_3^5B_7^5B_1B_2B_{11}B_{12}}+4B_{11}-\frac{2B_{11}^2}{B_{12}}>13$ We find that left side is a decreasing function of B1 and B11. So we can replace B1 by 2:17 and B11 by εB_7. Then left side becomes a decreasing function of B_7 and an increasing function of B_2, so can replace B_7 by 0.92 and B_2 by 2.2254706 to see that ϕ_{17}<13 for 1:7855<B_3<1:9517 and 0:3376<B_{12}<0:4156, a contradiction. Hence B_7<0:92. Further $B_5 \leq \frac{3}{2}B_7$ gives B_5<1:38.

Claim (xiii) B6<1:097

Suppose $B_6 \geq 1.097$ Here we have B3B4B5<4:44 and B7B8B9<0:5, so $\frac{B_2^3}{B_3B_4B_5}>\frac{(2.0733)^3}{4.44}>2$ and $\frac{B_6^3}{B_7B_8B_9}>2$ Also $2B_{10} \geq 2\varepsilon B_6 > B_{11}$ Applying AM-GM to the inequality (1,4,4,2,1) we get $\phi_{18}=B_1+4B_2+4B_6-\sqrt{B_2^5B_6^5B_1B_{10}B_{11}B_{12}}+4B_{10}-\frac{2B_{10}^2}{B_{11}}+B_{12}>13$ We find that left side is a decreasing function of B_{10}, B_{12} and B_{11}. So we can replace B_{10} by εB_6 and B_{12} by 0.3376 and B_{11} by $\frac{3\varepsilon}{4}B_6$. Then left side becomes a decreasing function of B_6, so we can replace B_6 by 1.097 to find that ϕ_{18}<13, for 2:17<B_1<2:4273 and 2:0733<B_2<2:2254706, a contradiction. Hence we must have B_6<1:097.

Claim (xiv) $B_5>B_6$ and $\frac{B_1^3}{B_2B_3B_4}<2$

First suppose B5 $\leq$ B6, then B4B5B6<1:646 × 1:0972<1:981 and $\frac{B_3^3}{B_4B_5B_6}>2$ Also $B_7 \geq \varepsilon B_3 > 0.83 >$ eachof$B_8,\dots,B_{12}$.Hencetheinequality (2; 4; 6*) holds, i.e. $4B_1-\frac{2B_1^2}{B_2}+4B_3-\frac{1}{2}\frac{B_3^4}{B_4B_5B_6}+6(\frac{1}{BlB_2B_3B_4B_5B_6})^{\frac{1}{6}}>13$ Now the left side is a decreasing function of B1 and B3 as well; also it is an increasing function of B_2 and $B_4B_5B_6$. But one can check that this inequality is not true for B_1>2:17, B_3>1:7855, B_2<2:2254706 and $B_4B_5B_6$<1:981, giving thereby a contradiction. Further suppose $\frac{B_1^3}{B_2B_3B_4} \geq 2$ then as B5>B6>1:019>each of B7,…, B12, the inequality (4; 8*) holds. Now working as in Case (i) of Claim (viii) we get contradiction for B1>2:17 and B2B3B4<2:2254706 × 1:9517 × 1:646<7:14934.

Claim (xv) B3<1:9 and B1<2:4056

Suppose $B_3 \geq 1.9$, then for $B_4B_5B_6$<1:646×1:38×1:097<2:492, $\frac{B_3^3}{B_4B_5B_6}>2$ Also B7$\geq \varepsilon$ B3>0:89>each of $B_8,\dots,B_{12}$. Hence the inequality (2; 4; 6*) holds. Now working as in Claim (xiv) we get contradiction for B_1>2:17, B_2<2:2254706, B_3>1:9 and $B_4B_5B_6$<2:492. So B_3<1:9. Further if $B_1 \geq$ 2:4056, then $\frac{B_1^3}{B_2B_3B_4}>\frac{(2.4056)^3}{2.2254706 \times 1.9 \times 1.646}>2$ contradicting Claim (xiv).

Claim (xvi) B_4<1:58 and B_1<2:373

Suppose $B_4 \geq 1.58$ thenforB_5B_6B<1:38×1:097×0:92<1:393, $\frac{B_4^3}{B_5B_6B_7}>2$ Also $B_8 \geq \varepsilon B_4$>0:74>each of $B_9,\dots,B_{12}$. Hence the inequality (1; 2; 4; 5*) holds, i.e. -19=B_1 + 4B_2 - $\frac{2B_2^2}{B_3}+4B_4-\frac{1}{2}\frac{B_4^4}{B_5B_6B_7}+5(\frac{1}{B_1B_2B_3B_4B_5B_6B_7})^{\frac{1}{5}}>13$

Left side is a decreasing function of B_2 and B_4.

So we replace B_2 by 2.0733 and B_4 by 1.58. Then it becomes an increasing function of B_1, B_3 and $B_5B_6B_7$. So we replace B_1 by 2.4056, B_3 by 1.9 and $B_5B_6B_7$ by 1.393 to find that -19<13, a contradiction. Further if B_1 $\geq$ 2:373, then $\frac{B_1^3}{B_2B_3B_4}>2$ contradicting Claim (xiv).

Final Contradiction:

We have B_3B_4B<1:9×1:58×1:38<4:15. Therefore $\frac{B_2^3}{B_3B_4B_5}$ >2 Also B6>1:019>each of B_7,…,B_{12}. Hence the inequality (1; 4; 7*) holds. Now we get contradiction working as in Case (ii) of Claim (viii).

5.2 2:17<B12<2:2254706

Here $B_1\geq B_{12}>2.17$ Using Lemma 3 and 4, we have

B^{11}=$(B_1B_2\ldots B_{10}B_{12})^{-1}$ $<(\frac{3}{64}\varepsilon^8\,B_1^{10}B_{12})^{-1}<1.8223$

Claim (i) Either B11<0:4307 or B11>1:818

Suppose 0:4307 $\leq B_{11}\leq$ 1.818 The inequality (10*; 1; 1) gives 10:3 $(\frac{1}{B_{11}B_{12}})^{\frac{1}{10}}+B_{11}+B_{12}>13$ which is not true for 0:4307 $\leq$ $B_{11}\leq$ 1.818 and 2:17<B12<2:2254706. So we must have either B_{11}<0:4307 or B_{11}>1:818.

Claim (ii) B11<0:4307

Suppose B_{11} $\geq$ 0.4307 then using Claim(i) we have B11>1:818. Now we have using Lemmas 3 & 4,

$B_2=(B_1B_2\ldots B_{12})^{-1}<(\frac{1}{16}\varepsilon^6\,B_2^8B_1B_{11}B_{12})^{-1}$ This gives B2<1:777.

$B_3=(B_1B_2B_4\ldots B_{12})^{-1}<(\frac{3}{64}\varepsilon^4\,B_3^7B_1^2B_{11}B_{12})^{-1}$ This gives B3<1:487

$B_4=(B_1B_2B_4\ldots B_{12})^{-1}<(\frac{1}{16}\varepsilon^3\,B_4^6B_1^3B_{11}B_{12})^{-1}$ This gives B4<1:213.

$B_6=(B_1..B_5B_7\ldots B_{12})^{-1}<(\frac{1}{16}\varepsilon^2\,B_6^4B_1^5B_{11}B_{12})^{-1}$ This gives B6<0:826.

$B_7=(B_1..B_6B_8\ldots B_{12})^{-1}<(\frac{3}{64}\varepsilon^2\,B_7^3B_1^6B_{11}B_{12})^{-1}$ This gives B7<0:697.

$B_8=(B_1..B_7B_9\ldots B_{12})^{-1}<(\frac{1}{16}\varepsilon^3\,B_8^2B_1^7B_{11}B_{12})^{-1}$ This gives B8<0:559.

$B_9=(B_1..B_8B_{10}B_{11}B_{12})^{-1}<(\frac{3}{64}\varepsilon^3\,B_9B_1^7B_{11}B_{12})^{-1}$ This gives B9<0:478.

$B_{10}=(B_1..B_9B_{11}B_{12})^{-1}<(\frac{1}{16}\varepsilon^6\,B_1^9B_{11}B_{12})^{-1}<0.359$

Therefore we have $\frac{B_1^3}{B_2B_3B_4}>2$ and $B_5\geq\varepsilon B_1>1.01>$ each of B_6,…,B_{10}. So the inequality (4; 6*; 1; 1) holds, i.e. 4B_1 $-\frac{1}{2}\frac{B_1^4}{B_2B_3B_4}+6$ $(B_1B_2B_3B_4B_{11}B_{12})^{-1/6}$ +B_{11}+B_{12}>13 Now the left side is an increasing function of B2B3B4, B11 and of B12 as well. Also it is a decreasing function of B1. So we replace $B_2B_3B_4$ by 1:777 × 1:487 × 1:213, B_{11} by 1.8223, B_{12} by 2.2254706 and B_1 by 2.17 to arrive at a contradiction. Hence we must have B_{11}<0:4307.

Claim (iii) B10<0:445

Suppose $B_{10}\geq 0.445$ then 2B_{10}>B_{11}. So the inequality (9*; 2; 1) holds, i.e. $\phi_{20}=9(\frac{1}{B_{10}B_{11}B_{12}})^{\frac{1}{9}}+4B_{10}-\frac{2B_{10}^2}{B_{11}}+B_{12}>13$ $B_{11}\geq\frac{3}{4}$ B10 and B12>2:2254706, the left side is an increasing function of B12 and a decreasing function of B_{10}, so replacing B_{12} by 2.2254706 and B_{10} by 0.445 we find that ϕ_{20}<13, for 3 4(0:445)<B_{11}<0:4307, a contradiction. Hence we must have B_{10}<0:445.

Using Lemmas 3 and 4 we have:

$$B_9\leq\frac{4}{3}B_{10}<0.594, B_8\leq\frac{3}{2}B_{10}<0.67, B_7\leq 0.89$$

$$B_6\leq\frac{B_{10}}{\varepsilon}<0.9494, B_5\leq\frac{4}{3}\frac{B_{10}}{\varepsilon}<1.266, B_4\leq\frac{3}{2}\frac{B_{10}}{\varepsilon}<1.4242$$

$$B_3\leq\frac{2B_{10}}{\varepsilon}<1.899, B_2\leq\frac{B_{10}}{(\varepsilon)^2}<2.0255$$

Claim (iv) B3<1:62

Suppose $B_3\geq 1.62$ From (5.2), we have B4B5B6<1:712 and B8B9B10<0:178, so $\frac{B_3^3}{B_4B_5B_6}>2$ and $\frac{B_7^3}{B_8B_9B_{10}}>2$ Applying AM-GM to the inequality (2,4,4,1,1) we get $\phi_{21}=4B_1-\frac{2B_1^2}{B_2}+4B_3+4B_7-\sqrt{B_3^5B_7^5B_1B_2B_{11}B_{12}}+B_{11}+B_{12}>13$ We find that left side is a decreasing function of B_1, B_7 and B_{11}. So we can replace B_1 by B_{12}, B_7 by ε B_3 and B_{11} by ε^2B_3. Then it becomes a decreasing function of B_3, so replacing B_3 by 1.62 we find that ϕ 21<13; for 1:6275<B_2<2:0255 and 2:17<B_{12}<2:2254706, a contradiction. Hence we must have B_3<1:62.

Claim (v) B12>2:196

Suppose $B_{12}\leq 2.196$ From (5.2), we have $B_2B_3B_4$<4:674 and $\frac{B_1^3}{B_2B_3B_4}>2$ Also $B_5\geq\varepsilon B_1>1.01>$ each of B6,…, B11. Therefore the inequality (4; 7*; 1) holds, i.e. ϕ_{22} $\phi_{22}=4B_1-\frac{1}{2}\frac{B_1^4}{B_2B_3B_4}+7(B_1B_2B_3B_4B_{12})^{-1/7}+B_{12}>13$ Left side is an increasing function of B2B3B4 and of B_{12} as well. Also it is a decreasing function of B_1. So we can replace $B_2B_3B_4$ by 4.674, B_{12} by 2.196 and B_1 by 2.17 to get ϕ 22<13, a contradiction. Hence we must have B_{12}>2:196.

Final Contradiction

Now we have B_1 $\geq$ B_{12}>2:196. We proceed as in Claim(v) and use (4; 7*; 1). Here we replace $B_2B_3B_4$ by 4.674, B_{12} by 2.2254706 and B_1 by 2.196 to get ϕ $_{22}$<13, a contradiction.

References

1. Bambah RP, Dumir VC, Hans-Gill RJ (2000) Non-homogeneous prob-lems: Conjectures of Minkowski and Watson, Number Theory, Trends in Mathematics, Birkhauser Verlag, Basel 15-41.

2. Birch BJ, Swinnerton-Dyer HPF (1956) On the inhomogeneous min-imum of the product of n linear forms, Mathematika 3: 25-39.

3. Blichfeldt HF (1934) The minimum values of positive quadratic forms in six, seven and eight variables, Math Z 39: 1-15.

4. Cebotarev N (1940) Beweis des Minkowski'schen Satzes uber lineare inhomo-gene Formen, Vierteljschr. Naturforsch. Ges. Zurich, 85 Beiblatt 27-30.

5. Cohn H, Elkies N (2003) New upper bounds on sphere packings, I. Ann of Math 157: 689-714.

6. Cohn H, Kumar A (2004) The densest lattice in twenty-four dimensions, Electron. Res Announc Amer Math Soc 10: 58-67.

7. Conway JH, Sloane NJA (1993) Sphere packings, Lattices and groups, Springer-Verlag, Second edition, New York.

8. Gruber P (2007) Convex and discrete geometry, Springer Grundlehren Series 336.

9. Gruber P, Lekkerkerker CG (1987) Geometry of Numbers, Second Edi-tion, North Holland, 37.

10. Hans-Gill RJ, Raka M, Sehmi R, Sucheta (2009) A uni ed simple proof of Woods' conjecture for n 6, J Number Theory 129: 1000-1010.

11. Hans-Gill RJ, Raka M, Sehmi R (2009) On conjectures of Minkowski and Woods for n=7, J Number Theory 129: 1011-1033.

12. Hans-Gill RJ, Raka M, Sehmi R (2010) Estimates On Con-jectures of Minkowski and Woods, Indian Jl Pure Appl Math 41: 595-606.

13. Hans-Gill RJ, Raka M, Sehmi R (2011) On Conjectures of Minkowski and Woods for n=8, Acta Arithmetica 147: 337-385.

14. Hans-Gill RJ, Raka M, Sehmi R (2011) Estimates On Con-jectures of Minkowski and Woods II, Indian Jl. Pure Appl. Math. 42: 307-333.

15. Il'in IV (1986) A remark on an estimate in the inhomogeneous Minkowski conjecture for small dimensions,) 90, Petrozavodsk. Gos. Univ., Petrozavodsk 24-30.

16. IV (1991) Chebotarev estimates in the inhomogeneous Minkowski con-jecture for small dimensions, Algebraic systems, Ivanov. Gos. Univ, Ivanovo 115-125.

17. Kathuria L, Raka M (2014) On Conjectures of Minkowski and Woods for n=9.

18. Kathuria L, Raka M (2011) Refined Estimates on Conjectures of Woods and Minkowski-II, To be Submitted.

19. Kathuria L, Raka M (2014) Generalization of a result of Birch and Swinnerton-Dyer, To be Submitted.

20. Korkine A, Zolotare G (1877) Sur les formes quadratiques, Math. Ann. 366-389; Sur les formes quadratiques positives. Math Ann 11: 242-292.

21. McMullen CT (2005) Minkowski's conjecture, well rounded lattices and topo-logical dimension. J Amer Math Soc 18: 711-734.

22. Mordell LJ (1960) Tschebotare's Theorem on the product of Non-homogeneous Linear Forms (II). J London Math Soc 35: 91-97.

23. Pendavingh RA, Van Zwam SHM (2007) New Korkine-Zolotarev in-equalities. SIAM J Optim 18: 364-378.

24. Woods AC (1965) The densest double lattice packing of four spheres. Math-ematika 12: 138-142.

25. Woods AC (1965) Lattice coverings of ve space by spheres. Mathematika 12: 143-150.

26. Woods AC (1972) Covering six space with spheres. J Number Theory 4: 157-180.