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Differential Transform Method for Nonlinear Initial-Value Problems by Adomian Polynomials

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Abstract

In this paper, the differential transformation method is modified to be easily employed to solve wide kinds of nonlinear initial-value problems. In this approach, the nonlinear term is replaced by its Adomian polynomials for the index *k*, and hence the dependent variable components are replaced in the recurrence relation by their corresponding differential transform components of the same index. Thus the nonlinear initial-value problem can be easily solved with less computational effort. New theorems for product and integrals of nonlinear functions are introduced. In order to show the power and effectiveness of the present modified method and to illustrate the pertinent features of related theorems, several numerical examples with different types of nonlinearities are considered.

Keywords: Differential transform method; Adomian polynomials; Nonlinear equations

AMS Subject Classifications: 35Gxx, 45Jxx

Introduction

The Differential Transform Method (DTM) has been proved to be efficient for handling nonlinear problems, but the nonlinear functions used in these studies are restricted to polynomials and products with derivatives [1-5]. For other types of nonlinearities, the usual way to calculate their transformed functions as introduced by [6] is to expand the nonlinear function in an infinite power series then take the differential transform of this series. The problem with this approach is that the massive computational difficulties will arise in determining the differential transform of nonlinear function while working with this infinite series. Another approach for obtaining the differential transform of nonlinear terms is the algorithm in [7]. It is based on using the properties of differential transform and calculus to develop a canonical equation. Then this equation is solved for the required differential transform of nonlinear term. But, as seen in the simple examples in section 3 [7] the algorithm requires a sequence of differentiation, algebraic manipulations and computations of differential transform for other functions which is more difficult for the case of composite nonlinearities.

In this work, we introduce a comprehensive and more efficient approach for using the DTM to solve nonlinear initial-value problems; the idea is based on the methodology in [8]. The nonlinear function is replaced by its Adomian polynomials and then the dependent variable components are replaced by their corresponding differential transform components of the same index. This technique benefits the properties of the Adomian polynomials and the efficient algorithm to generate them quickly as in the work [9-11]. Numerical simulations of some nonlinear equations with different types of nonlinearity are treated and the proposed technique has provided good results.

Differential Transform Method

The transformation of the kth derivative of a function $y(x)$ is as follows

$$Y(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0}, \tag{1}$$

and the inverse transformation is defined by

$$y(x)=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k \tag{2}$$

In this work, we use lower case letters for the original functions and upper case letters stand for the transformed functions.

5.1 Theorem 1. If $y(x)=f(x)\pm h(x)$, then $Y(k)=F(k)\pm H(k)$.

5.2 Theorem 2. If $y(x)=cf(x)$, then $Y(k)=cF(k)$, where c is a constant.

5.3 Theorem 3. If $y(x)=f^{(n)}(x)$, then $Y(k)=\frac{(k+n)!}{k!}F(k+n)$.

5.4 Theorem 4. If $y(x)=f(x)h(x)$, then

$$Y(k)=\sum_{k_1=0}^{k}F(k_1)H(k-k_1).$$

5.5 Theorem 5. If $y(x)=x^m$, then $Y(k)=\delta(k-m)$,

where $\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$.

The above theorems can be deduced from equations (1) and (2).

The Modified Differential Transform Method

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of a nonlinear function $g(y(x))$. The Adomian polynomials of this nonlinear function are determined formally as follows [12,13].

$$\tilde{A}_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[g\left(\sum_{i=0}^{\infty}\lambda_i y_i\right)\right]\right]_{\lambda=0}, n\geq 0 .$$

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That is, the Adomian polynomials of $g(y(x))$ are

$\tilde{A}_0 = g(y_0)$,

$\tilde{A}_1 = y_1 g^{(1)}(y_0)$,

$\tilde{A}_2 = y_2 g^{(1)}(y_0) + \frac{1}{2!} y_1^2 g^{(2)}(y_0)$,

$\tilde{A}_3 = y_3 g^{(1)}(y_0) + y_1 y_2 g^{(2)}(y_0) + \frac{1}{3!} y_1^3 g^{(3)}(y_0)$,

$\tilde{A}_4 = y_4 g^{(1)}(y_0) + (y_1 y_3 + \frac{1}{2!} y_2^2) g^{(2)}(y_0) + \frac{1}{2!} y_1^2 y_2 g^{(3)}(y_0) + \frac{1}{4!} y_1^4 g^{(4)}(y_0)$,

$\tilde{A}_5 = y_5 g^{(1)}(y_0) + (y_2 y_3 + y_1 y_4) g^{(2)}(y_0) + \frac{1}{2!}(y_1^2 y_3 + y_1 y_2^2) g^{(3)}(y_0) + \frac{1}{3!} y_1^3 y_2 g^{(4)}(y_0) + \frac{1}{5!} y_1^5 g^{(5)}(y_0)$, and so on.

6.1 Lemma: If $f(x) = g(y(x))$, then $F(k) = A_k$ where A_k are the Adomian polynomials $\tilde{A}_k$ but with replacing y_k by $Y(k), k = 0,1,2,\cdots$.

6.2 *Proof*: The differential transforms of $f(x)$ are computed by utilizing (1) as

$$F(0) = \frac{1}{0!}\{g(y(x))\}_{x=x_0} = g(y(x_0)) = g(Y(0)) = A_0 ,$$

$$F(1) = \frac{1}{1!}\left\{\frac{d}{dx} g(y(x))\right\}_{x=x_0}$$

$$= y^{(1)}(x_0) g^{(1)}(y(x_0)) = Y(1) g^{(1)}(Y(0)) = A_1 ,$$

$$F(2) = \frac{1}{2!}\left\{\frac{d^2}{dx^2} g(y(x))\right\}_{x=x_0}$$

$$= \frac{1}{2!}\left\{y^{(2)}(x_0) g^{(1)}(y(x_0)) + (y^{(1)}(x_0))^2 g^{(2)}(y(x_0))\right\}$$

$$= Y(2) g^{(1)}(Y(0)) + \frac{1}{2!}(Y(1))^2 g^{(2)}(Y(0)) = A_2 ,$$

$$F(3) = \frac{1}{3!}\left\{\frac{d^3}{dx^3} g(y(x))\right\}_{x=x_0}$$

$$= \frac{1}{3!}\left\{\begin{matrix} y^{(3)}(x_0) g^{(1)}(y(x_0)) + 3y^{(1)}(x_0) y^{(2)} \\ (x_0) g^{(2)}(y(x_0)) + (y^{(1)}(x_0))^3 g^{(3)}(y(x_0)) \end{matrix}\right\}$$

$$\begin{matrix} = Y(3) g^{(1)}(Y(0)) + Y(1) Y(2) g^{(2)}(Y(0)) + \\ \frac{1}{3!}(Y(1))^3 g^{(3)}(Y(0)) \end{matrix} = A_3 ,$$

In general we have, $F(k) = A_k$.

Consequently, the inverse transform of the nonlinear function can be written as

$$f(x) = g(y(x)) = \sum_{k=0}^{\infty} A_k (x - x_0)^k , \tag{3}$$

where, A_k are the differential transform of $f(x) = g(y(x))$.

The advantage of using this algorithm for computing differential transformation of nonlinear functions comparing with the algorithm suggested in [7], is this algorithm dealing directly with nonlinear function of the problem in hand in its form without any differentiation or algebraic manipulations or even there is no need to compute the differential transform of other functions to obtain the required one. This will be clear throughout the following theorems.

6.3 Theorem 6. If $f(x) = h(x) g(y(x))$, then $F(k) = \sum_{k_1=0}^{k} H(k_1) A_{k-k_1}$.

6.4 *Proof*: By utilizing definition (1), we can get

$$F(0) = \frac{1}{0!}\{h(x) g(y(x))\}_{x=x_0} = h(x_0) g(y(x_0))$$

$$= H(0) g(Y(0)) = H(0) A_0 ,$$

$$F(1) = \frac{1}{1!}\left\{\frac{d}{dx}\left[h(x) g(y(x))\right]\right\}_{x=x_0}$$

$$= h^{(1)}(x_0) g(y(x_0)) + h(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0))$$

$$= H(1) A_0 + H(0) A_1 ,$$

$$F(2) = \frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[h(x) g(y(x))\right]\right\}_{x=x_0}$$

$$= \frac{1}{2!}\left\{ h^{(2)}(x_0) g(y(x_0)) + 2h^{(1)}(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0)) + \begin{matrix} h(x_0)[y^{(2)}(x_0) g^{(1)}(y(x_0)) + \\ (y^{(1)}(x_0))^2 g^{(2)}(y(x_0))] \end{matrix}\right\}$$

$$= H(2) A_0 + H(1) A_1 + H(0) A_2 ,$$

$$F(3) = \frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[h(x) g(y(x))\right]\right\}_{x=x_0}$$

$$= \frac{1}{3!}\left\{\begin{matrix} h^{(3)}(x_0) g(y(x_0)) + \\ 3h^{(2)}(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0)) \\ +3h^{(1)}(x_0)[y^{(2)}(x_0) g^{(1)}(y(x_0)) \\ +(y^{(1)}(x_0))^2 g^{(2)}(y(x_0))] + h(x_0)[y^{(3)}(x_0) g^{(1)}(y(x_0)) \\ +3y^{(1)}(x_0) y^{(2)}(x_0) g^{(2)}(y(x_0)) + \\ (y^{(1)}(x_0))^3 g^{(3)}(y(x_0))] \end{matrix}\right\}$$

$$= H(3) A_0 + H(2) A_1 + H(1) A_2 + H(0) A_3 ,$$

In general we have, $F(k) = \sum_{k_1=0}^{k} H(k_1) A_{k-k_1}$

6.5 Theorem 7. If $f(x) = \int_{x_0}^{x} g(y(t)) dt$, then $F(k) = \frac{A_{k-1}}{k}, k \geq 1$

6.6 *Proof*: By using (3), the transform of the integral can be found as

$$f(x) = \int_{x_0}^{x} \sum_{k=0}^{\infty} A_k (t - x_0)^k dt$$

$$= \sum_{k=0}^{\infty} A_k \int_{x_0}^{x} (t - x_0)^k dt = \sum_{k=1}^{\infty} \frac{A_{k-1}}{k}(t - x_0)^k ,$$

Again utilizing (3), we get $F(k) = \frac{A_{k-1}}{k}$, where $k \geq 1$ and $F(0) = f(x_0) = 0$.

6.7 Theorem 8. If $f(x) = h(x) \int_{x_0}^{x} g(y(t)) dt$,

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then $F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}H(k-k_1)A_{k_1-1}$, $k\geq 1$

6.8 ***Proof*****:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{h(x)\int_{x_0}^{x} g(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[h(x)\int_{x_0}^{x} g(y(t))dt\right]\right\}_{x=x_0}$$

$$=\left\{h^{(1)}(x)\int_{x_0}^{x} g(y(t))dt+h(x)g(y(x))\right\}_{x=x_0}=h(x_0)g(y(x_0))$$

$$=H(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[h(x)\int_{x_0}^{x} g(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{2!}\left\{2h^{(1)}(x_0)g(y(x_0))+h(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right\}$$

$$=H(1)A_0+H(0)A_1/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[h(x)\int_{x_0}^{x} g(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{3!}\left\{3h^{(2)}(x_0)g(y(x_0))+3h^{(1)}(x_0)y^{(1)}(x_0)g^{(1)}(y(x_0))\right.$$
$$\left.+h(x_0)[y^{(2)}(x_0)g^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g^{(2)}(y(x_0))\right\}$$

$$=H(2)A_0+\frac{1}{2}H(1)A_1+\frac{1}{3}H(0)A_2,$$

In general we have, $F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}H(k-k_1)A_{k_1-1}$, where $k\geq 1$.

6.9 Theorem 9. If $f(x)=\int_{x_0}^{x} g_1(t)g_2(y(t))dt$,

then $F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)A_{k-k_1-1}$, $k\geq 1$.

6.10 ***Proof*****:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=g_1(x_0)g_2(y(x_0))=G_1(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{2!}\left\{g_1^{(1)}(x_0)g_2(y(x_0))+g_1(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))\right\}$$

$$=[G_1(1)A_0+G_1(0)A_1]/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{3!}\left\{g_1^{(2)}(x_0)g_2(y(x_0))+2g_1^{(1)}(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))\right.$$
$$\left.+g_1(x_0)[y^{(2)}(x_0)g_2^{(1)}(y(x_0))+(y^{(1)}(x_0))^2g_2^{(2)}(y(x_0))]\right\}$$

$$=[G_1(2)A_0+G_1(1)A_1+G_1(0)A_2]/3,$$

In general we have, $F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)A_{k-k_1-1}$, where $k\geq 1$.

6.11 Theorem 10. If $f(x)=h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt$,

then $F(k)=\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}G_1(k_1-1)A_{k_2-k_1}H(k-k_2)$.

6.12 ***Proof*****:** Utilizing the definition of the transform, we can get

$$F(0)=\frac{1}{0!}\left\{h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right\}_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\left\{\frac{d}{dx}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=h(x_0)g_1(x_0)g_2(y(x_0))=H(0)G_1(0)A_0,$$

$$F(2)=\frac{1}{2!}\left\{\frac{d^2}{dx^2}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

$$=\frac{1}{2!}\left\{\begin{array}{l}2h^{(1)}(x_0)g_1(x_0)g_2(y(x_0))+\\ h(x_0)[g_1^{(1)}(x_0)g_2(y(x_0))+g_1(x_0)y^{(1)}\\ (x_0)g_2^{(1)}(y(x_0))]\end{array}\right\}$$

$$=H(1)G_1(0)A_0+[H(0)G_1(1)A_0+H(0)G_1(0)A_1]/2,$$

$$F(3)=\frac{1}{3!}\left\{\frac{d^3}{dx^3}\left[h(x)\int_{x_0}^{x} g_1(t)g_2(y(t))dt\right]\right\}_{x=x_0}$$

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$$=\frac{1}{3!}\begin{cases}3h^{(2)}(x_0)g_1(x_0)g_2(y(x_0))+\\3h^{(1)}(x_0)[g_1^{(1)}(x_0)g_2(y(x_0))+\\g_1(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))]\end{cases}$$

$$+h(x_0)[g_1^{(2)}(x_0)g_2(y(x_0))+$$
$$2g_1^{(1)}(x_0)y^{(1)}(x_0)g_2^{(1)}(y(x_0))]$$

$$\left.\begin{matrix}+g_1(x_0)[y^{(2)}(x_0)g_2^{(1)}(y(x_0))\\+(y^{(1)}(x_0))^2g_2^{(2)}(y(x_0))]\end{matrix}\right\}$$

$$=H(2)G_1(0)A_0+$$
$$\begin{matrix}[H(1)G_1(1)A_0+\\H(1)G_1(0)A_1]\end{matrix}\Big/2+\begin{matrix}[H(0)G_1(2)A_0+\\H(0)G_1(1)A_1+H(0)G_1(0)A_2]\end{matrix}\Big/3,$$

In general we have, $F(k)=\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}G_1(k_1-1)A_{k_2-k_1}H(k-k_2)$.

Applications and Numerical Results

In this section, we implement the proposed method on some different examples with different types of nonlinearity.

7.1 Example 1. Consider the nonlinear Volterra integro-differential equation

$$y''(x)+y'(x)y(x)+y(x)=\cos 2x+x^3-x^2\int_0^x\frac{1+\sin 2t}{y^2(t)}dt \quad ,0\le x\le 1 \tag{4}$$

with the initial conditions

$$y(0)=1 \text{ and } y'(0)=1. \tag{5}$$

The differential transformation of equation (4) and the initial conditions (5) are

$$Y(k+2)=\frac{k!}{(k+2)!}\left[\frac{2^k}{k!}\cos(\pi k/2)+\delta(k-3)-\sum_{m=0}^{k}(m+1)Y(m+1)Y(k-m)-Y(k)\right.$$
$$\left.-\frac{A_{k-3}}{k-2}-\frac{1}{k-2}\sum_{m=1}^{k-2}\frac{2^{m-1}}{(m-1)!}\sin\left(\frac{\pi(m-1)}{2}\right)A_{k-m-2}\right],$$

where $[2^k\cos(\pi k/2)]/k!$ and $[2^k\sin(\pi k/2)]/k!$ are the differential transforms of $\cos(2x)$ and $\sin(2x)$, respectively and A_k are the differential transform of the nonlinear function $g(y)=y^{-2}$, and $Y(0)=Y(1)=1$. Using the Lemma, A_k are: $A_0=g(Y(0))=1$, $A_1=-2Y(1)$, $A_2=-2Y(2)+3Y^2(1)$, $A_3=-2Y(3)+6Y(1)Y(2)-4Y^3(1)$,

$$A_4=-2Y(4)-2Y(1)Y(3)-Y^2(2)+3Y^2(1)Y(2)+5Y^4(1),\ldots$$

Utilizing the recurrence relation, the transformed initial conditions and A_k, $Y(k)$ are evaluated. Hence using the inverse transformation formula, the following series solution up to $O(x^{10})$ can be obtained

$$y(x)=1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}-\frac{x^7}{7!}+\frac{x^8}{8!}+\frac{x^9}{9!}+O(x^{10}).$$

For sufficiently large number of terms, the closed form of the solution is $y(x)=\sin x+\cos x$, which is the exact solution. Table 1 shows the absolute relative error obtained for three various numbers of terms and at some test points.

x	Abs. rel. err., (5 Terms)	Abs. rel. err., (10 Terms)	Abs. rel. err., (15 Terms)
0.2	7.75099E-08	0	0
0.4	4.5762E-06	7.74161E-13	0
0.6	5.0297E-05	6.19746E-11	0
0.8	0.000283721	1.4145E-09	1.25621E-15

Table 1: Numerical comparison of results in example 1.

7.2 Example 2. Consider the nonlinear Volterra integro-differential equation

$$6(x^2+1)y'(x)=(x^3+3x^2+6x+6)e^{-x}+\int_0^x t^3e^{-\tan y(t)}dt \quad ,0\le x\le 1, \tag{6}$$

with the initial condition

$$y(0)=0. \tag{7}$$

The differential transformation of equation (6) and the initial condition (7) are

$$Y(k+1)=-\frac{k-1}{k+1}Y(k-1)+\frac{1}{6(k+1)}\left[\frac{(-1)^k(6-11k+6k^2-k^3)}{k!}+\frac{A_{k-4}}{k}\right], \tag{8}$$

where $\lambda^k/k!$ are the differential transforms of $e^{\lambda x}$, A_k are the differential transforms the nonlinear function $g(y)=e^{-\tan y}$ and $Y(0)=0$.

If we put $x=0$ into equation (6), we can get $y'(0)=1$ and hence $Y(1)=1$.

The following system for $k=1,2,3,\cdots,8$ is obtained from (8)

$$Y(2)=0,$$
$$Y(3)=-\frac{1}{3}Y(1),$$
$$Y(4)=-\frac{2}{4}Y(2),$$
$$Y(5)=-\frac{3}{5}Y(3)+\frac{1}{6(5)}\left[-\frac{6}{4!}+\frac{A_0}{4}\right],$$
$$Y(6)=-\frac{4}{6}Y(4)+\frac{1}{6(6)}\left[\frac{24}{5!}+\frac{A_1}{5}\right],$$
$$Y(7)=-\frac{5}{7}Y(5)+\frac{1}{6(7)}\left[-\frac{60}{6!}+\frac{A_2}{6}\right],$$

$$Y(8) = -\frac{6}{8}Y(6) + \frac{1}{6(8)}\left[\frac{120}{7!} + \frac{A_3}{7}\right],$$

$$Y(9) = -\frac{7}{9}Y(7) + \frac{1}{6(9)}\left[-\frac{210}{8!} + \frac{A_4}{8!}\right],$$

where differential transform components A_k are: $A_0 = e^{-\tan(Y(0))} = 1$, $A_1 = Y(1)$, $A_2 = Y^2(1) - Y(2)$, $A_3 = -(1/2)Y^3(1) + Y(1)\ Y(2)\ - Y(3)$,

$$A_4 = Y(1)Y(3) + (1/2)Y^2(2) - (3/2)Y^2(1)Y(2) + (3/8)Y^4(1) - Y(4).$$

By solving the above systems for $Y(k)$, the series solution of problem (6) and (7) up to $O(x^{10})$ is given by

$$y(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + O(x^{10}).$$

For sufficiently large of terms, the closed form of the solution is $y(x) = \tan^{-1} x$, which is the exact solution. Table 2 shows the absolute relative error obtained for three various numbers of terms and at some test points.

7.3 Example 3. Let us consider the nonlinear Volterra integro-differential equation

$$y''(x) - 2y(x)y'(x) = -x + \int_0^x \frac{y'(t)}{1+y^2(t)}dt,\ 0 \le x \le 1, \tag{9}$$

with the initial conditions

$$y(0) = 0\text{, and } y'(0) = 1. \tag{10}$$

The differential transformation of this equation and its initial conditions are

$$Y(k+2) = \frac{k!}{(k+2)!}\left[\begin{array}{l} 2\sum_{m=0}^{k}(m+1)Y(m+1)Y(k-m) - \delta(k-1) + \\ \frac{1}{k}\sum_{m=0}^{k-1}(m+1)Y(m+1)A_{k-m-1} \end{array}\right],$$

$Y(0) = 0$ and $Y(1) = 1$.

A_k can be obtained by using Lemma as: $A_0 = (1+Y^2(0))^{-1} = 1, A_1 = 0$, $A_2 = -Y^2(1)$, $A_3 = -2Y(1)Y(2)$, $A_4 = -2(Y(1)Y(3) + Y^2(2)) + Y^4(1)$, … .

By solving for $Y(k)$, the series solution of problem (9) and (10) up to $O(x^{10})$ is given by

$$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + O(x^{10}).$$

For sufficiently large number of terms, the closed form of the solution is $y(x) = \tan x$, which is the exact solution. Table 3 shows the absolute relative error obtained for three various numbers of terms and at some test points.

7.4 Example 4. Consider the initial-value problem of Bratu-type [7]

$$y''(x) - 2e^{y(x)} = 0,\ \ 0 \le x \le 1, \tag{11}$$

$$y(0) = 0\text{, and } y'(0) = 0. \tag{12}$$

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*, (10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	9.12337E-09	0	0
0.4	8.82908E-06	4.80303E-10	0
0.6	0.000468447	1.45549E-06	5.92E-10
0.8	0.007532013	0.000411299	2.95854E-07

Table 2: Numerical comparison of results in example 2.

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*, (10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	9.10218E-10	0	0
0.4	9.40244E-07	7.01122E-14	6.49916E-15
0.6	5.50308E-05	5.3086E-10	4.48908E-12
0.8	0.000998396	3.04049E-07	2.23E-10

Table 3: Numerical comparison of results in example 3.

The differential transformation of this equation and its initial conditions are

$$Y(k+2) = \frac{2A_k}{(k+1)(k+2)},\ Y(0) = 0 \text{ and } Y(1) = 0,$$

where A_k are: $A_0 = e^{Y(0)}$, $A_1 = Y(1)\,e^{Y(0)}$,

$$A_2 = [Y(2) + (Y(1))^2/2]\,e^{Y(0)},$$

$$A_3 = [Y(3) + Y(1)Y(2) + (Y(1))^3/6]\,e^{Y(0)},\ \cdots$$

The following differential transform components are obtained:

$Y(2) = 1$, $Y(3) = 0$, $Y(4) = 1/6$, $Y(5) = 0$, $Y(6) = 2/45$, $Y(7) = 0$, $Y(8) = 17/1260$, $Y(9) = 0$, ⋯ .

The series solution of problem (11) and (12) up to $O(x^{10})$ is given by

$$y(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + O(x^{10}).$$

This is the same result with that obtained by [7,14]. The closed form solution of this problem is $y(x) = -2\ln(\cos x)$.

7.5 **Example 5.** Consider the nonlinear initial-value problem [7]

$$y''(x) = 2y(x) + 4y(x)\ln(y(x)),\ \ x > 0, \tag{13}$$

$$y(0) = 1\text{, and } y'(0) = 0. \tag{14}$$

The differential transformation of this equation and its initial conditions are

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[2Y(k) + 4\sum_{k_1=0}^{k}Y(k_1)A_{k-k_1}\right],\ Y(0) = 1 \text{ and } Y(1) = 0.$$

where A_k are: $A_0 = \ln Y(0)$, $A_1 = Y(1)/Y(0)$,

$$A_2 = Y(2)/Y(0) - (Y(1)/Y(0))^2/2,$$

$$A_3 = Y(3)/Y(0) + Y(1)Y(2)/(Y(0))^2 - (Y(1)/Y(0))^3/3,$$

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$$A_4 = Y(4)/Y(0) - [2Y(1)Y(3) + (Y(2))^2]/(Y(0))^2 + (Y(1))^2/(Y(0))^3 - [Y(1)/Y(0)]^4/4 \quad \cdots$$

The following differential transform components are obtained: $Y(2)=1$, $Y(4)=1/2$, $Y(6)=1/6$, $Y(8)=1/24$, $\cdots$. The series solution of problem (13) and (14) up to $O(x^{10})$ is given by

$$y(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + O(x^{10}).$$

This is the same result with that obtained by [7]. The closed form solution of this problem is $y(x) = e^{x^2}$

7.6 Example 6. Consider the nonlinear initial-value problem

$$y''(x) = \frac{9}{4}\sqrt{y(x)} + y(x), \tag{15}$$

$$y(0)=1, \text{ and } y'(0)=2. \tag{16}$$

The differential transformation of this problem are

$$Y(k+2) = \frac{9A_k + 4Y(k)}{4(k+1)(k+2)}, \quad Y(0)=1 \text{ and } Y(1)=2.$$

where A_k are: $A_0 = \sqrt{Y(0)}$, $A_1 = Y(1)/(2\sqrt{Y(0)})$,

$$A_2 = Y(2)/2\sqrt{Y(0)} - \frac{1}{8}(Y(1))^2/(Y(0))^{3/2},$$

$$A_3 = Y(3)/2\sqrt{Y(0)} - Y(1)Y(2)/(4(Y(0))^{3/2}) + (Y(1))^3/(16Y(0))^{5/2}),$$

$$A_4 = Y(4)/2\sqrt{Y(0)} - [2Y(1)Y(3) + (Y(2))^2]/(8(Y(0))^{3/2}) + 3(Y(1))^2Y(2)/(16(Y(0))^{5/2}) - 5(Y(1))^4/(128(Y(0))^{7/2}), \cdots$$

The following differential transform components are obtained:

$Y(2)=13/8, Y(3)=17/24,$

$Y(4)=149/768,$

$Y(5)=77/1920,$

$Y(6)=641/92160,$

$Y(7)=317/322560,$

$Y(8)=2609/20643840,$

$Y(9)=1277/92897280, \cdots.$

The series solution of problem (15) and (16) up to $O(x^{10})$ is given by

$$y(x) = 1 + 2x + \frac{13}{8}x^2 + \frac{17}{24}x^3 + \frac{149}{768}x^4 + \frac{77}{1920}x^5 + \frac{641}{92160}x^6 + \frac{317}{322560}x^7 + \frac{2609}{20643840}x^8 + \frac{1277}{92897280}x^9 + O(x^{10}).$$

The exact solution of this example is

$$y(x) = \frac{9}{4}\left(\frac{3}{2}e^{x/2} + \frac{1}{6}e^{-x/2} - 1\right)^2.$$

x	*Abs. rel. err.*, (5 Terms)	*Abs. rel. err.*, (10 Terms)	*Abs. rel. err.*, (15 Terms)
0.2	7.79747E-08	0	0
0.4	4.8256E-06	9.47535E-12	0
0.6	2.01E-06	2.69667E-09	6.74169E-10
0.8	1.28303E-05	3.5877E-08	2.44062E-09

Table 4: Numerical comparison of results in example 6.

Table 4 shows the absolute relative error obtained for three various numbers of terms and at some test points.

Conclusion

In this work, we present a new approach for applying the differential transform method for solving nonlinear initial-value problems. The differential transform of the nonlinear term is replaced in the recurrence relation by its Adomian polynomial of index *k*. Hence, the dependent variable components are replaced by their corresponding differential transforms of the same index. This technique benefits the properties of the Adomian polynomials and the efficient algorithm to generate them quickly. Also, this technique is dealing directly with nonlinear function of the problem in its form without any differentiation or algebraic manipulations or even there is no need to compute the differential transform of other functions to obtain the required one. The considered prototype examples include initial-value problems with different types of nonlinearity. These numerical examples have proved good results.

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