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Research Article

Outer-independent total 2-rainbow dominating functions in graphs

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Abstract: Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. An outer-independent total 2-rainbow dominating function of a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2\}$ such that the following conditions hold: (i) for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2\}$, (ii) the set of all vertices $v \in V(G)$ with $f(v) = \emptyset$ is independent and (iii) $\{v | f(v) \neq \emptyset\}$ has no isolated vertex. The outer-independent total 2-rainbow domination number of G, denoted by $\gamma_{oitr2}(G)$, is the minimum value of $\omega(f) = \sum_{v \in V(G)} |f(v)|$ over all such functions f. In this paper, we study the outer-independent total 2-rainbow domination number of G and classify all graphs with outer-independent total 2-rainbow domination number belonging to the set $\{2, 3, n\}$. Among other results, we present some sharp bounds concerning the invariant.

Keywords: Domination number; 2-rainbow domination number; total 2-rainbow domination number, outer-independent total 2-rainbow domination number

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1. Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v) = N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree*

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of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree 1 is called a *leaf* and its neighbor is a *support* vertex. Also, a support vertex is called a *strong support* vertex if it is adjacent to at least two leaves and *weak support* if it is adjacent to one leaf. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of a graph G, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of G. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, C_n for a *cycle* of order n and P_n for a *path* of order n. By a star we mean the graph $S_{1,m}$ where $m \geq 2$. Let $S_{r,t}$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves. The *girth* $g(G)$ of a graph G is the length of a shortest cycle. For terminology and notation on graph theory not defined here, the reader is referred to [11].

A set D of vertices in a graph G is called a *dominating set* if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A set D of vertices in a graph G is called a *total dominating set* if every vertex $v \in V(G)$ is adjacent to an element of D. The *domination number* of a graph G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in G. Respectively, the *total domination number* of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G.

A subset S of vertices is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of vertices $u, v \in S$. The 2-*packing number* $\rho(G)$ of a graph G is the maximum cardinality of a 2-packing in G.

A k-rainbow dominating function of a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2, \ldots, k\}$ such that for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, \ldots, k\}$. The 1-rainbow domination is the same as the ordinary domination. The k-rainbow domination problem is to determine the k-rainbow domination number $\gamma_{rk}(G)$ of a graph G, that is the minimum value of $\sum_{v \in V(G)} |f(v)|$ where f runs over all k-rainbow dominating functions of G. The concept of rainbow domination was introduced in [3] and has been studied extensively [1, 2, 4, 5, 7, 8, 13].

An *outer-independent* 2-*rainbow dominating function* of a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2\}$ such that the following conditions hold: (i) for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2\}$, (ii) the set of all vertices $v \in V(G)$ with $f(v) = \emptyset$ is independent. The *outer-independent* 2-*rainbow domination number* of G, denoted by $\gamma_{oir2}(G)$, is the minimum value of $\omega(f) = \sum_{v \in V(G)} |f(v)|$ over all such functions f. Outer independent 2-rainbow domination was introduced by Kang et al. in [9] in 2019. This concept has been studied by several authors, see for example [6, 10].

Lately, the interest in the domination theory in graphs has increased and a very high number of variants of domination parameters have been studied. Here we initiate *outer-independent total* 2-*rainbow dominating function* and continue the study in this context.

An *outer-independent total* 2-*rainbow dominating function* (OIt2RDF) on a graph G is a function f from $V(G)$ to the set of all subsets of $\{1, 2\}$ such that the following

conditions hold: (i) for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2\}$, (ii) the set of all vertices $v \in V(G)$ with $f(v) = \emptyset$ is independent and (iii) $\{v | f(v) \neq \emptyset\}$ has no isolated vertex. The *outer-independent total 2-rainbow domination number* of G, denoted by $\gamma_{oitr2}(G)$, is the minimum $\omega(f) = \sum_{v \in V(G)} |f(v)|$ over all such functions f. An outer-independent total 2-rainbow dominating function with weight $\gamma_{oitr2}(G)$ is called a $\gamma_{oitr2}(G)$-function of G. An outer-independent total 2-rainbow dominating function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ of $V(G)$ induced by f, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{12} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{12}|$.
Suppose that $G_1, G_2, \ldots, G_t$ are the components of G. Then

$$\gamma_{oitr2}(G) = \sum_{i=1}^{t} \gamma_{oitr2}(G_i).$$

Therefore, in the rest of the text, we assume, without loss of generality, that G is a connected graph.
In the next section, we investigate some basic properties of the outer-independent total 2-rainbow dominating functions and we determine exact values for some classes of graphs. Then in Section 3, we obtain the relationship between $\gamma_{oitr2}(G)$ and some other domination parameters. At the end, in Section 4, we present some sharp bounds for outer-independent total 2-rainbow domination number.

2. Basic properties and examples

In this section we present some basic properties of the outer-independent total 2-rainbow domination. We have the following simple results.

Observation 1. *For any connected graph G with $n \geq 2$, $\gamma_{oitr2}(G)$ is well defined and $2 \leq \gamma_{oitr2}(G) \leq n$.*

We give now, the characterizations of all connected graphs G for which $\gamma_{oitr2}(G) \in \{2, 3, n\}$.

Proposition 1. *Let G be a graph of order $n \geq 2$. Then $\gamma_{oitr2}(G) = 2$ if and only if $G = \overline{K_{n-2}} \vee P_2$.*

Proof. If $G = \overline{K_{n-2}} \vee P_2$, then clearly $\gamma_{oitr2}(G) = 2$. Conversely, assume that $\gamma_{oitr2}(G) = 2$ and f is a $\gamma_{oitr2}(G)$-function. Since $\{v \mid f(v) \neq \emptyset\}$ has no isolated vertex, no vertex of G has label $\{1, 2\}$. Thus there are two adjacent vertices u, v such that $|f(u)| = |f(v)| = 1$ and the other vertices must be independent with label $\emptyset$ and adjacent with u, v. Therefore, for $n = 2$, $G = P_2$ and for $n > 2$, $G = \overline{K_{n-2}} \vee P_2$. $\square$

To continue the characterization, we need to define some family of graphs.

- Let $\mathcal{F}_1$ be the family of graphs obtained from a path uv by first adding $t \geq 1$ pendant edges at u and then adding $s \geq 0$ new vertices and connecting them to u and v.
- Let $\mathcal{F}_2$ be the family of graphs G obtained from a path uvw by first adding $t \geq 0$ new vertices and connecting them to u and v, and then adding $s \geq 0$ new vertices and connecting them to u and w.
- Let $\mathcal{F}_3$ be the family of graphs G obtained from a path uvw by first adding $t \geq 0$ new vertices and connecting them to u and v, and then adding $s \geq 0$ new vertices and connecting them to u and w, and adding $\ell \geq 1$ new vertices and connecting them to u, v and w.
- Let $\mathcal{F}_4$ be the family of graphs G obtained from a triangle $uvwu$ by first adding $t \geq 1$ new vertices and connecting them to u and v, and then adding $s \geq 1$ new vertices and connecting them to u and w.
- Let $\mathcal{F}_5$ be the family of graphs G obtained from a triangle $uvwu$ by first adding $t \geq 0$ new vertices and connecting them to u and v, and then adding $s \geq 0$ new vertices and connecting them to u and w, and adding $\ell \geq 1$ new vertices and connecting them to u, v and w.
- Let $\mathcal{F}_6$ be the family of graphs G obtained from a path uvw by first adding $t \geq 0$ new vertices and connecting them to v and u, and then adding $s \geq 0$ new vertices and connecting them to v and w.
- Let $\mathcal{F}_7$ be the family of graphs G obtained from a path uvw by first adding $t \geq 0$ new vertices and connecting them to v and u, and then adding $s \geq 0$ new vertices and connecting them to v and w, and adding $\ell \geq 1$ new vertices and connecting them to u, v and w.

Let $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_7\}$.

Proposition 2. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{oitr2}(G) = 3$ if and only if $G \in \mathcal{F}$.*

Proof. Obviously $\gamma_{oitr2}(G) = 3$ if $G \in \mathcal{F}$.
Conversely, assume that $\gamma_{oitr2}(G) = 3$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{oitr2}(G)$-function such that $|V_{1,2}|$ is maximized. We consider the following cases.

Case 1. There is a vertex $v \in V(G)$ such that $f(v) = \{1, 2\}$.
Then there is a vertex $u \in N(v)$ with $|f(u)| = 1$ and the other vertices must be independent with label $\emptyset$ and adjacent with v. So $G \in F_1$.

Case 2. $V_{1,2} = \emptyset$ and there are three vertices $v, u, w \in V(G)$ such that $|f(u)| = |f(v)| = |f(w)| = 1$.
Without loss of generality, let $f(u) = f(w) = \{1\}$ and $f(v) = \{2\}$. The other vertices must be independent with label $\emptyset$ and adjacent with v. Consider two following subcases:

Subcase 2.1. The subgraph of G induced by $\{u, v, w\}$ is the path uwv.
If all vertices with label $\emptyset$ are adjacent with u but not with w, then $G = K_{2,n-2} \in \mathcal{F}_2$. If each vertex with label $\emptyset$ is adjacent to either u or w, then $G \in \mathcal{F}_2$. If some vertices with label $\emptyset$ are adjacent with u and some of them are adjacent with w and some of them are adjacent with both, then $G \in \mathcal{F}_3$. If all vertices with label $\emptyset$ are adjacent with w but not with u, then $G \in \mathcal{F}_1$.

Subcase 2.2. The subgraph of G induced by $\{u, v, w\}$ is the path uvw.
All vertices with label $\emptyset$ must be adjacent with u or with w. If all vertices with label $\emptyset$ are adjacent with u but not with w, then $G \in \mathcal{F}_1$. If some vertices with label $\emptyset$ are adjacent with u and some of them are adjacent with w, then $G \in \mathcal{F}_6$. If some vertices with label $\emptyset$ are adjacent with u, some of them are adjacent with w and some of them are adjacent with both, then $G \in \mathcal{F}_7$.
Subcase 2.3. The subgraph of G induced by $\{u, v, w\}$ is the triangle $uvwu$.
If all vertices with label $\emptyset$ are adjacent with u but not with w, then $G = \overline{K_{n-2}} \vee P_2$ which is a contradiction by Proposition 1. Thus some vertices with label $\emptyset$ are adjacent with u, some of them are adjacent with w, some of them are adjacent with u and w. Hence $G \in \mathcal{F}_4 \cup \mathcal{F}_5$. □

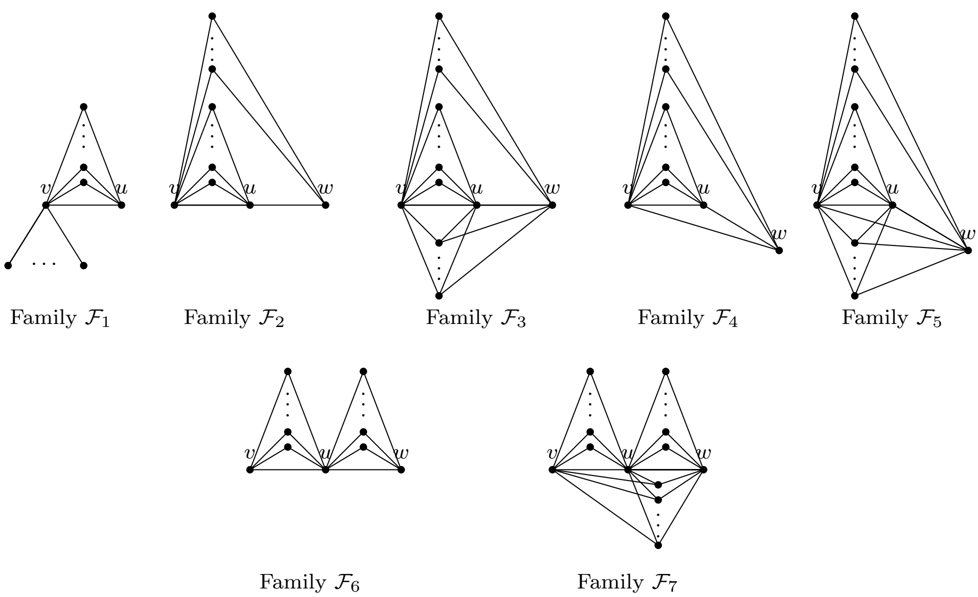

Figure 1. **The families of graphs G with $\gamma_{oitr2}(G) = 3$**

We now characterize the graphs attaining the upper bound from Observation 1.

Proposition 3. *Let G be a graph of order $n \geq 2$. Then $\gamma_{oitr2}(G) = n$ if and only if $G = P_2$ or $G = P_3$ or $G = P_4$ or every non-leaf vertex of G is a weak support vertex.*

Proof. Necessary is clear. For the sufficiency let $\gamma_{oitr2}(G) = n$. If $\delta \geq 2$, then clearly $\gamma_{oitr2}(G) \leq n-1$. So we can assume that $\delta = 1$. It is easy to see that if $2 \leq n \leq 4$ and $\gamma_{oitr2}(G) = n$, then $G = P_2$ or $G = P_3$ or $G = P_4$. So assume that $n \geq 5$. Obviously, if there is a strong support vertex, then $\gamma_{oitr2}(G) \leq n-1$. Thus we can suppose that every support vertex is weak. We claim that every non-leaf vertex of G is a weak support vertex. To see this by contradiction, suppose that there is a non-leaf vertex say v which is not a support vertex. Hence we have $deg(v) \geq 2$. Let $u, w \in N(v)$ such that $\deg(u) \geq 2, \deg(w) \geq 2$. Define $f : V(G) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(v) = \emptyset$, $f(u) = 1$, $f(w) = 2$ and $f(x) = 1$ otherwise. It is easy to see that f is an outer-independent total 2-rainbow dominating function on G. Thus $\gamma_{oitr2}(G) \leq n-1$, a contradiction. □

Next we determine the outer-independent total 2-rainbow domination number of some special graphs.

Observation 2. *For $n \geq 3$, $\gamma_{oitr2}(K_n) = n-1$.*

By Observation 2, one has the following fact.

Corollary 1. *For any integer $t \geq 2$, there is a graph G such that $\gamma_{oitr2}(G) = t$.*

Observation 3. *For $r, t \geq 2$, $\gamma_{oitr2}(S_{r,t}) = 4$.*

Proposition 4. *For $2 \leq n \leq m$, $\gamma_{oitr2}(K_{n,m}) = n+1$.*

Proof. Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be two partite sets of $K_{n,m}$. Define $f : V(K_{n,m}) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(u_1) = f(u_i) = f(v_1) = \{1\}$ for $3 \leq i \leq n$, $f(u_2) = 2$ and $f(v_i) = \emptyset$ for $2 \leq i \leq m$. Clearly f is an outer-independent total 2-rainbow dominating function and $\omega(f) \leq n+1$. So $\gamma_{oitr2}(K_{n,m}) \leq n+1$. The inverse inequality is obvious. □

Proposition 5. For $n \geq 2$, $\gamma_{oitr2}(P_n) = \begin{cases} \lceil \frac{2n}{3} \rceil, & n = 3k+2 \\ \lceil \frac{2n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$

Proof. Clearly $\gamma_{oitr2}(P_2) = 2$. Assume that $n \geq 3$ and let $P_n = v_1 v_2 \ldots v_n$ be a path on n vertices. If $n = 3k+2$ for some non-negative integer k, then define the function $f : V(P_n) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(v_{3i+1}) = \{1\}$, $f(v_{3i+2}) = \{2\}$ and $f(v_{3i}) = \emptyset$

for $0 \le i \le k$. Clearly, f is an outer-independent total 2-rainbow dominating function of P_n with $\omega(f) \le \lceil \frac{2n}{3} \rceil$.
If $n = 3k+1$ for some integer k, then define $f : V(P_n) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(v_{3i+1}) = \{1\}$, $f(v_{3i+2}) = \{2\}$, $f(v_{3i}) = \emptyset$ for $1 \le i \le k-1$ and $f(v_{n-1}) = f(v_n) = 1$. Clearly, f is an outer-independent total 2-rainbow dominating function and $\omega(f) \le \lceil \frac{2n}{3} \rceil + 1$. So $\gamma_{oitr2}(P_n) \le \lceil \frac{2n}{3} \rceil + 1$.
If $n = 3k$ for some integer k, then define $f : V(P_n) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(v_{3i+1}) = \{1\}$, $f(v_{3i+2}) = \{2\}$, $f(v_{3i}) = \emptyset$ for $1 \le i \le k-1$ and $f(v_n) = 1$. Clearly, f is an outer-independent total 2-rainbow dominating function and $\omega(f) \le \lceil \frac{2n}{3} \rceil + 1$. So $\gamma_{oitr2}(P_n) \le \lceil \frac{2n}{3} \rceil + 1$.
Conversely, assume that g is a $\gamma_{oitr2}(P_n)$-function. It is easy to verify that $|g(v_1)| + |g(v_2)| \ge 2$, $|g(v_{n-1})| + |g(v_n)| \ge 2$ and $|g(v_i)| + |g(v_{i+1})| + |g(v_{i+2})| \ge 2$ for $1 \le i \le n-2$.
If $n = 3k+2$, then we deduce that

$$\omega(g) = \sum_{i=1}^{3k} |g(v_i)| + |g(v_{n-1})| + |g(v_n)| \ge 2k+2 = \left\lceil \frac{2n}{3} \right\rceil .$$

If $n = 3k+1$, then we have

$$\begin{aligned} \omega(g) &= |g(v_1)| + |g(v_2)| + \textstyle\sum_{i=3}^{3k-1} |g(v_i)| + |g(v_{n-1})| + |g(v_n)| \\ &\ge 2 + 2(k-1) + 2 = 2k+2 \\ &= \lceil \tfrac{2n}{3} \rceil + 1. \end{aligned}$$

Assume that $n = 3k$. If $|g(v_1)| \ge 1$, then we conclude that

$$\omega(g) = |g(v_1)| + \sum_{i=2}^{3k-2} |g(v_i)| + |g(v_{n-1})| + |g(v_n)| \ge 1 + 2(k-1) + 2 = 2k+1 = \left\lceil \frac{2n}{3} \right\rceil + 1.$$

If $|g(v_1)| = 0$, then we observe that $|g(v_1)| + |g(v_2)| + |g(v_3)| \ge 3$ and we obtain

$$\omega(g) = |g(v_1)| + |g(v_2)| + |g(v_3)| + \sum_{i=4}^{3k} |g(v_i)| \ge 3 + 2(k-1) = 2k+1 = \left\lceil \frac{2n}{3} \right\rceil + 1.$$

□

Proposition 6. *For $n \ge 3$, $\gamma_{oitr2}(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proof. Define $f : V(C_n) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(v_i) = \{1\}$ if $i \equiv 1 \pmod 3$, $f(v_i) = \{2\}$ if $i \equiv 2 \pmod 3$ and $f(v_i) = \emptyset$ if $i \equiv 0 \pmod 3$, for $1 \le i \le n$. Clearly f is an outer-independent total 2-rainbow dominating function with $\omega(f) \le \lceil \frac{2n}{3} \rceil$. So $\gamma_{oitr2}(C_n) \le \lceil \frac{2n}{3} \rceil$. Similar to the proof of Proposition 5, the inverse inequality arises. □

3. Outer-independent total 2-rainbow domination number and other graph parameters

Here we are interested in the relationship between $\gamma_{oitr2}(G)$ and several other domination parameters. For instance any outer-independent total 2-rainbow dominating function is an outer-independent 2-rainbow dominating function, so one has

$$\gamma_{oir2}(G) \leq \gamma_{oitr2}(G).$$

Also we have the following straightforward observation.

Observation 4. *Let $f = (V_0, V_1, V_2, V_{12})$ be a $\gamma_{oitr2}(G)$-function. Then $V_1 \cup V_2 \cup V_{12}$ is a total dominating set in G and $\gamma_t(G) \leq \gamma_{oitr2}(G)$.*

On the other hand, clearly $\gamma(G) \leq \gamma_t(G)$ and so we have the following relation by Observation 4.

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_{oitr2}(G).$$

A subset S of $V(G)$ is called *independent* if no two vertices in S are adjacent. An independent set of maximum cardinality is a maximum independent set of G. The *independence number* of G is the cardinality of a maximum independent set of G and is denoted by $\alpha(G)$. An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$-set. A set of vertices S is a vertex cover set if every edge of G is incident with a vertex of S. A vertex cover set of minimum cardinality is a minimum vertex cover set of G. The *vertex cover number* of G is the cardinality of a minimum vertex cover set of G and is denoted by $\beta(G)$. A vertex cover of cardinality $\beta(G)$ is called a $\beta(G)$-set.

Proposition 7. *If G is a graph of order $n \geq 2$, then $\beta(G) \leq \gamma_{oitr2}(G)$. Moreover, this bound is sharp.*

Proof. Let $f = (V_0, V_1, V_2, V_{12})$ be a $\gamma_{oitr2}(G)$-function. Since V_0 is an independent set, we deduce that $|V_0| \leq \alpha(G)$. Using this fact and Gallai's theorem ($\alpha(G)+\beta(G) = n$), we obtain

$$\begin{aligned} n - \gamma_{oitr2}(G) &= n - (|V_1| + |V_2| + 2|V_{12}|) \\ &\leq n - (|V_1| + |V_2| + |V_{12}|) \\ &= |V_0| \\ &\leq \alpha(G) \\ &= n - \beta(G) \end{aligned}$$

and the proof is complete. For the sharpness, consider $G = \overline{K_{n-2}} \vee P_2$, $n \geq 2$, by Proposition 1 and the graphs F_2, F_3, F_4 and F_5. $\square$

Corollary 2. *If G is a graph of order $n \geq 2$, then $\delta(G) \leq \beta(G) \leq \gamma_{oitr2}(G)$.*

Proof. If I is an $\alpha(G)$-set, then it is well known that $V(G) \setminus I$ is a $\beta(G)$-set. If $v \in I$, then this implies that $\delta(G) \leq \deg(v) \leq |V(G) \setminus I| = \beta(G)$. The other inequality follows from Proposition 7, and the proof is complete. □

The minimum degree bound $\delta(G) \leq \beta(G)$ can be found in the Thesis of W. Willis [12]. For completeness we have given the short proof.

Corollary 3. *If $G = K_{n_1,n_2,\ldots,n_p}$ is the complete p-partite graph with $n_1 \leq n_2 \leq \ldots \leq n_p$ and $p \geq 3$, then*

$$\delta(G) = \beta(G) = \gamma_{oitr2}(G) = n_1 + n_1 + \ldots + n_{p-1}.$$

Proof. Corollary 2 implies $\gamma_{oitr2}(G) \geq \beta(G) \geq \delta(G) = n_1 + n_2 + \ldots + n_{p-1}$. Let S_i be the partite set with $|S_i| = n_i$ for $1 \leq i \leq p$. Define the function $f = (V_0, V_1, V_2, V_{12})$ by $V_0 = S_p$, $V_1 = S_1$, $V_2 = S_2 \cup S_3 \cup \ldots \cup S_{p-1}$ and $V_{12} = \emptyset$. Then f is an outer-independent total 2-rainbow dominating function on G and thus $\gamma_{oitr2}(G) \leq n_1 + n_2 + \ldots + n_{p-1} = \beta(G) = \delta(G)$. □

Corollary 3 demonstrates the sharpness of Corollary 2 and Proposition 7.

4. Bounds

Our aim in this section is to determine some bounds on the OIt2RD number of graphs. First, we obtain an upper bound for graphs G of girth $g(G) \geq 5$.

Theorem 5. If G is a graph of order n with $g(G) \geq 5$ and $\delta(G) \geq 2$, then

$$\gamma_{oitr2}(G) \leq n - \Delta(G) + 1,$$

and this bound is sharp.

Proof. Let v be a vertex of maximum degree $\Delta = \Delta(G)$, and let $u_1, u_2, \ldots, u_\Delta$ be the neighbors of v. Define the function f by $f(v) = f(u_1) = \{1\}$, $f(u_2) = f(u_3) = \ldots = f(u_\Delta) = \emptyset$ and $f(x) = \{2\}$ otherwise.
As G is triangle-free, $\{u_2, u_3, \ldots, u_\Delta\}$ is an independent set. The condition $\delta(G) \geq 2$ implies that each vertex u_i has a neighbor w with $f(w) = \{2\}$ for $2 \leq i \leq \Delta$. As $f(v) = \{1\}$, we obtain $\bigcup_{x \in N(u_i)} f(x) = \{1, 2\}$ for each $2 \leq i \leq \Delta$. In addition, we deduce from $g(G) \geq 5$ and $\delta(G) \geq 2$ that each vertex x with $f(x) = \{2\}$ has a neighbor with $f(w) = \{2\}$. Consequently, f is an outer-independent total 2-rainbow dominating function on G of weight $n - \Delta + 1$ and thus $\gamma_{oitr2}(G) \leq n - \Delta + 1$.
Let H be consist of a subdivide star with the leaves $x_1, x_2, \ldots, x_{2p}$ such that x_{2i-1} and x_{2i} are adjacent for $1 \leq i \leq p$. Then it is easy to verify that $\gamma_{oitr2}(H) = 2p + 2 = n(H) - \Delta(H) + 1$. This family of graphs show that this upper bound is sharp, □

Next we present a lower bound for outer-independent total 2-rainbow domination number with regard to the maximum and minimum degree.

Theorem 6. Let G be a connected graph of order $n \geq 2$ with minimum degree δ and maximum degree Δ. Then

$$\gamma_{oitr2}(G) \geq \left\lceil \frac{\delta n}{\Delta + \delta - 1} \right\rceil.$$

Moreover, this bound is sharp.

Proof. The results is trivial if $n = 2$ or $\gamma_{oitr2}(G) = n$. So assume that $n \geq 3$ and $\gamma_{oitr2}(G) < n$. Let $f = (V_0, V_1, V_2, V_{12})$ be a $\gamma_{oitr2}(G)$-function and $V_0 = \{x_1, x_2, \ldots, x_t\}$. Since V_0 is an independent set, every vertex x_i, for $1 \leq i \leq t$, has at least δ neighbors in $V_1 \cup V_2 \cup V_{12}$. On the other hand, every vertex in $V_1 \cup V_2 \cup V_{12}$ has at most $\Delta - 1$ neighbors in V_0, since $\{v \mid f(v) \neq \emptyset\}$ has no isolated vertex. So we obtain

$$\delta|V_0| \leq (\Delta - 1)(|V_1| + |V_2| + |V_{12}|).$$

Using this inequality and the fact that $n = |V_0| + |V_1| + |V_2| + |V_{12}|$, we have

$$\begin{aligned}
\delta n &\leq (\Delta - 1)(|V_1| + |V_2| + |V_{12}|) + \delta(|V_1| + |V_2| + |V_{12}|) \\
&= (\Delta - 1 + \delta)(|V_1| + |V_2| + |V_{12}|) \\
&\leq (\Delta - 1 + \delta)(|V_1| + |V_2| + 2|V_{12}|) \\
&= (\Delta - 1 + \delta)\gamma_{oitr2}(G).
\end{aligned}$$

Therefore $\gamma_{oitr2}(G) \geq \lceil \frac{\delta n}{\Delta+\delta-1} \rceil$, because $\gamma_{oitr2}(G)$ is an integer. For the sharpness, consider cycles by Proposition 6. □

As an immediate consequence of Theorem 6, we have the following corollaries.

Corollary 4. *Let G be an r-regular graph of order $n \geq 2$. Then $\gamma_{oitr2}(G) \geq \lceil \frac{rn}{2r-1} \rceil$.*

Corollary 5. *Let G be a graph of order $n \geq 2$ with $\delta = 1$. Then $\gamma_{oitr2}(G) \geq \lceil \frac{n}{\Delta} \rceil$, specially for every tree T, $\gamma_{oitr2}(T) \geq \lceil \frac{n}{\Delta} \rceil$.*

We propose a so called Nordhaus-Gaddum type inequality for the outer-independent total 2-rainbow domination number of regular graphs.

Theorem 7. *Let G be an r-regular graph of order $n \geq 4$. Then*

$$\gamma_{oitr2}(G) + \gamma_{oitr2}(\overline{G}) \geq \frac{n(n-1)}{n-2}.$$

Proof. Since G is r-regular, the complement $\overline{G}$ is $(n-r-1)$-regular. By Corollary 4, one has

$$\gamma_{oitr2}(G)+\gamma_{oitr2}(\overline{G})\geq \frac{rn}{2r-1}+\frac{(n-r-1)n}{2(n-r-1)-1}.$$

The function $f(x)=\frac{xn}{2x-1}+\frac{(n-x-1)n}{2(n-x-1)-1}$ gets its minimum at $x=\frac{n-1}{2}$. So we have

$$\gamma_{oitr2}(G)+\gamma_{oitr2}(\overline{G})\geq \frac{\frac{n-1}{2}n}{2\frac{n-1}{2}-1}+\frac{(n-\frac{n-1}{2}-1)n}{2(n-\frac{n-1}{2}-1)-1}=\frac{n(n-1)}{n-2}.$$

□

In the following proposition, an upper bound is given for outer-independent total 2-rainbow domination number with regard to the 2-packing.

Proposition 8. Let G be a graph of order n with $\delta\geq 2$. Then $\gamma_{oitr2}(G)\leq n-\rho$. Moreover, this bound is sharp.

Proof. Suppose that $A=\{v_1,v_2,\ldots,v_\rho\}$ is a 2-packing set of G and define $f: V(G)\longrightarrow \mathcal{P}(\{1,2\})$ by $f(v_i)=\emptyset$, $f(u_{i1})=\{1\}$, $f(u_{i2})=\{2\}$ for $1\leq i\leq \rho$ where u_{it} are neighbors of v_i for $t=1,2$ and $f(x)=\{1\}$ otherwise. Since $\delta\geq 2$, f is an outer-independent total 2-rainbow dominating function and so $\gamma_{oitr2}(G)\leq n-\rho$. For the sharpness, consider the complete graph K_n by Observation 2. □

Next, we present an upper bound in terms of the diameter of a graph using Proposition 8.

Proposition 9. Let G be a graph of order n with $\delta\geq 2$. Then

$$\gamma_{oitr2}(G)\leq n-1-\left\lfloor\frac{\operatorname{diam}(G)}{3}\right\rfloor.$$

Moreover, this bound is sharp.

Proof. Suppose that $P=v_0v_1\ldots v_{\operatorname{diam}(G)}$ is a diametral path, $\operatorname{diam}(G)=3t+r$ with integers $t\geq 0$ and $0\leq r\leq 2$. It is easy to see that $A=\{v_0,v_3,\ldots,v_{3t}\}$ is a 2-packing set of G such that $|A|=1+\lfloor\frac{\operatorname{diam}(G)}{3}\rfloor$. Then we have $\rho\geq |A|$. So by Proposition 8, one has

$$\gamma_{oitr2}(G)\leq n-\rho\leq n-|A|\leq n-1-\left\lfloor\frac{\operatorname{diam}(G)}{3}\right\rfloor.$$

For the sharpness, let $P_{3t+1} = v_1 v_2 \ldots v_{3t+1}$ be a path of diameter $3t$ for an integer $t \geq 2$, and let the graph H consists of P_{3t+1}, the vertices u and w and the edges uv_1, uv_2, wv_{3t} and wv_{3t+1}. We have

$$\gamma_{oitr2}(H) \leq 3t + 3 - 1 - \left\lfloor \frac{\mathrm{diam}(H)}{3} \right\rfloor = 2t + 2.$$

If g is a $\gamma_{oitr2}(H)$-function, then we observe that $g(v_1) + g(v_2) + g(u) \geq 2$, $g(v_{3t}) + g(v_{3t+1}) + g(w) \geq 2$ and $g(v_{3i}) + g(v_{3i+1}) + g(v_{3i+2}) \geq 2$ for $1 \leq i \leq t-1$. Therefore $\gamma_{oitr2}(H) \geq \frac{2n(H)}{3} = 2t + 2$. Also, consider the complete graph K_n for $n \geq 3$ by Observation 2. $\square$

Let $S(T)$ and $L(T)$ be the set of support vertices and the set of leaves of a tree T, respectively. We use the notations $s(T) = |S(T)|$ and $\ell(T) = |L(T)|$. In the following proposition we give an upper bound for $\gamma_{oitr2}(T)$ using $s(T)$ and $\ell(T)$.

Proposition 10. Let T be a tree of order $n \geq 3$ with $\mathrm{diam}(T) \geq 3$. Then $\gamma_{oitr2}(T) \leq n + s(T) - \ell(T)$. Moreover, this bound is sharp.

Proof. Define $f : V(T) \longrightarrow \mathcal{P}(\{1,2\})$ by $f(s) = \{1,2\}$ for every support vertex s, $f(u) = \emptyset$ for every leaf and $f(x) = 1$ otherwise. Clearly, f is a $\gamma_{oitr2}(T)$-function of T with

$$\omega(f) = 2s(T) + (n - s(T) - \ell(T)) = n + s(T) - \ell(T)$$

and the proof has been completed. For the sharpness, consider double stars $S_{r,t}$. $\square$

In the following theorem we prepare a lower bound in terms of the order and $\ell(T)$ for a tree T.

Theorem 8. Let T be a tree of order $n \geq 2$. Then

$$\gamma_{oitr2}(T) \geq \left\lceil \frac{2(n + 2 - \ell(T))}{3} \right\rceil.$$

Moreover, this bound is sharp.

Proof. We proceed by induction on n. The statement holds for all trees of order $n \leq 4$. Suppose that $n \geq 5$ and let the result hold for all non-trivial tree T of order less than n. Let T be a tree of order $n \geq 5$. If $\mathrm{diam}(T) = 2$, then T is a star, which yields $\gamma_{oitr2}(T) = 3 > \lceil \frac{2(n+2-\ell(T))}{3} \rceil = 2$ by Proposition 2. If $\mathrm{diam}(T) = 3$, then T is a double star and by Observation 3 we have $\gamma_{oitr2}(T) = 4 > 3 = \lceil \frac{2(n+2-\ell(T))}{3} \rceil$. Thus we may assume that $\mathrm{diam}(T) \geq 4$. Let $P = v_1 v_2 \ldots v_k$ be a diametral path in T and root T in v_k. Let f be a $\gamma_{oitr2}(T)$-function. We consider the following cases:

Case 1. $\deg_T(v_2) = t \geq 3$.
Let $T' = T - T_{v_2}$. It is not hard to see that $\gamma_{oitr2}(T) \geq \gamma_{oitr2}(T') + 2$ and $\ell(T) - (t-1) \leq \ell(T') \leq \ell(T) - (t-2)$ so we conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{oitr2}(T) &\geq \gamma_{oitr2}(T') + 2\\ &\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + 2\\ &\geq \lceil \frac{2((n-t) + 2 - \ell(T) + (t-2))}{3} \rceil + 2\\ &= \lceil \frac{2(n - \ell(T))}{3} \rceil + 2\\ &\geq \lceil \frac{2(n + 2 - \ell(T))}{3} \rceil,\end{aligned}$$

as desired.

Case 2. $\deg_T(v_2) = 2$.
If $\deg_T(v_3) \geq 3$, then let $T' = T - \{v_1, v_2\}$. Clearly $\gamma_{oitr2}(T) \geq \gamma_{oitr2}(T') + 2$ and $\ell(T') = \ell(T) - 1$ so we conclude from the induction hypothesis on T' that

$$\begin{aligned}\gamma_{oitr2}(T) &\geq \gamma_{oitr2}(T') + 2\\ &\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + 2\\ &= \lceil \frac{2((n-2) + 2 - \ell(T) + 1))}{3} \rceil + 2\\ &\geq \lceil \frac{2(n + 2 - \ell(T))}{3} \rceil,\end{aligned}$$

as desired.
If $\deg_T(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Clearly $\gamma_{oitr2}(T) \geq \gamma_{oitr2}(T') + 2$ and $\ell(T) - 1 \leq \ell(T') \leq \ell(T)$. We obtain from the induction hypothesis on T' that

$$\begin{aligned}\gamma_{oitr2}(T) &\geq \gamma_{oitr2}(T') + 2\\ &\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + 2\\ &\geq \lceil \frac{2((n-3) + 2 - \ell(T))}{3} \rceil + 2\\ &\geq \lceil \frac{2(n + 2 - \ell(T))}{3} \rceil.\end{aligned}$$

This completes the proof. By Proposition 5, paths of order $3k + 2$ attain this bound. $\square$

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