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Research Article

Cop-edge critical generalized Petersen and Paley graphs

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Abstract: Cop Robber game is a two player game played on an undirected graph. In this game, the cops try to capture a robber moving on the vertices of the graph. The cop number of a graph is the least number of cops needed to guarantee that the robber will be caught. We study *cop-edge critical* graphs, i.e. graphs G such that for any edge e in $E(G)$ either $c(G-e) < c(G)$ or $c(G-e) > c(G)$. In this article, we study the edge criticality of generalized Petersen graphs and Paley graphs.

Keywords: Cops and Robbers, vertex-pursuit games, Petersen graphs, Paley graphs

AMS Subject classification: 05C57, 05C80, 05C10

1. Introduction

The game of Cops and Robbers is a vertex pursuit game played on a graph G. It was introduced by Nowakowski and Winkler [12]. It is an example of perfect information combinatorial game played by two players [6]. In this game we consider two players, one control a set of k cops (or searchers) C, where $k > 0$ is a fixed integer, and the other control robber R. Cops initiate the game placing themselves on a set of k vertices, then R occupies another vertex. In every round, the two players move alternatively, starting with player controlling cops. A move of a player's controlled element is considered as either moves to an adjacent vertex or stays on the same vertex. Therefore, in a cop move, each cop either a move to an adjacent vertex or

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stay on the same vertex. Analogously in a robber move, the robber either moves to an adjacent vertex or stays on the same vertex. Many cops are allowed to occupy a single vertex. The players know each others actual position (that is, the game is played with complete information). The cops win and the game ends if one of the cops move to the vertex occupied by the robber; if robber has a strategy to avoid cops indefinitely, then R wins. The minimum number of cops needed to catch the robber (regardless of robber's strategy) is called the cop number of G, and is denoted by $c(G)$. This parameter is well studied in literature (see [1, 6, 7]).
Throughout this paper we consider only simple graphs that is graphs without loops and parallel edges. For additional definitions on graphs we refer to the book [11] and for the game we refer to the book [7].
A graph G is said to be *cop-vertex/edge critical* if for any vertex/edge x in G either $c(G-x) < c(G)$ or $c(G-x) > c(G)$ (see [9]).
A graph G is said to be m-cop win if $c(G) = m$ and is said to be m-cop edge critical if G is edge critical and $c(G) = m$. In this paper we study the edge/vertex critical graphs with cop number 3. The Petersen graph, the dodecahedron and the Heawood graph are examples of edge critical graphs with cop number 3. In this paper, we present some more examples of edge critical cubic graphs with cop number 3 continuing the work started in [9]. In the second section we focus on generalized Petersen graph while in the third section we study the Paley graphs.

2. Generalized Petersen graphs

The length of the shortest cycle contained in a graph G is called the girth of G. Let us remind the following elementary but useful results.

Proposition 1 ([1]). *If G has girth at least 5, then $c(G) \geq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proposition 2 ([7]). *If G is planar, then $c(G) \leq 3$.*

The *generalized Petersen graph* $GP(n,k)$ also denoted $\mathcal{P}(n,k)$ for $n \geq 5$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ is a graph consisting of vertex set

$$V(\mathcal{P}(n,k)) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$$

and the edge set $\{(u_i, u_{i+1}),\ (u_i, v_i),\ (v_i, v_{i+k}) \mid i = 1, 2, \dots, n\}$ where subscript are taken modulo n. These graphs were introduced by Coxeter (see [10]). It is known that the Petersen graph $\mathcal{P}(5,2)$ is 3-cop edge critical [4], we also know that the graph $\mathcal{P}(6,2)$ is 2-cop win. Therefore we start with $\mathcal{P}(7,2)$. The following useful result can be found in [5].

Theorem 1 ([5]). *Let $\mathcal{P}$ be the generalized Petersen graph. Then $c(G) \leq 4$.*

Lemma 1. *Let G be the graph $\mathcal{P}(7,2)$. Then G is edge critical with cop number* 3.

Proof. Let G be the graph $\mathcal{P}(7,2)$. Since G is a graph with girth at least 5 therefore, from Proposition 1,

$$c(G) \geq 3. \tag{1}$$

If we place cop c_1 on the vertex 2, cop c_2 on the vertex 5 and c_3 on the vertex 14, then we can catch the robber in at most three rounds (see graph $\mathcal{P}(7,2)$ presented in Figure 1). This implies that

$$c(G) \leq 3. \tag{2}$$

Thus we have from (1) and (2) that $c(G) = 3$.

Claim. $c(G - e) = 2$ for any edge $e \in G$.

To prove this we delete three types of edges from graph G. i) An edge connecting two vertices in the inner cycle of the graph G. ii) an edge connecting two vertices in the outer cycle of graph G. iii) an edge connecting the vertices in the inner cycle and the outer cycle of graph G (see Figure 1).

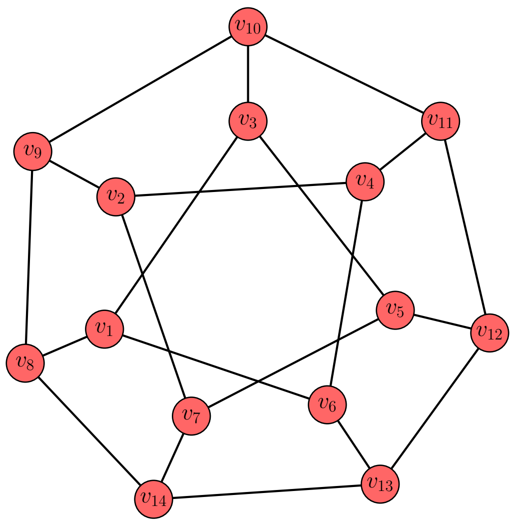

Figure 1. **The generalized Petersen graph $\mathcal{P}(7,2)$**

ad i) Label the vertices as shown in the Figure 2 and assume that edge e belongs to the outer cycle of graph $\mathcal{P}(7,2)$ (in the Figure 2 edges colored by the color blue represents inner cycle edges of the graph $\mathcal{P}(7,2)$), edges colored by the color green represents outer cycle edges of graph $\mathcal{P}(7,2)$ and edges colored by the color black represents the edges between inner and outer cycle of graph $\mathcal{P}(7,2)$. Assume that we always start the game by placing cop c_1 on the vertex v_3 and cop c_2 on vertex

v_2. Clearly the robber cannot start on $v_1, v_2, v_3, v_4, v_5, v_7, v_9$ and v_{10}. If the robber starts on vertex v_6, then we move cop c_1 to vertex v_1. The robber then must move to vertex v_{13}. Now the cop c_1 moves to vertex v_6 and cop c_2 moves to vertex v_7. Then the game finish in the next move. This shows that vertex v_6 is a bad starting point. (Now 5 vertices remain for the robber to start).

If the robber starts at vertex v_8, then we move cop c_1 to vertex v_1. The robber then must move to vertex v_{14}. Now cop c_2 moves to vertex v_7 and the robber must move to vertex v_{13}. Again cop c_2 moves to vertex v_{14} and then the game finish in the next move.

Suppose the robber starts at vertex v_{11}, then we move cop c_2 to vertex v_4. Now the robber must move to vertex 12. Again we move cop c_1 to vertex v_5 and the game finish in the next move.
If the robber starts at vertex v_{12}, then we move cop c_1 to vertex v_5 and cop c_2 to vertex v_4. Then the game finish in the next move.
If the robber starts at vertex v_{13}, then we move cop c_1 to vertex v_1 and cop c_2 to vertex v_7. Now the robber is trapped and we can finish the game in the next two moves. If the robber starts at vertex v_{14}, then we move cop c_1 to vertex v_1 and cop c_2 to vertex v_7. Robber then must move to vertex v_{13}. New position is equivalent to the previous one and we can finish the game in the next two moves. Therefore the graph depicted in Figure 2 is 2-cop win.

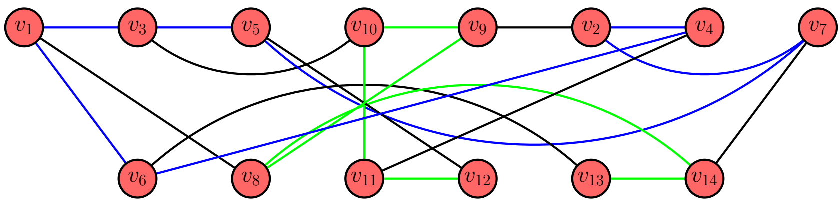

Figure 2. **The graph** $\mathcal{P}(7,2) - \{e\}$

ad ii) Assume that edge e lies between inner and outer cycles. Removal of any of these edges from G results the graph depicted in Figure 3.

Label the vertices as shown in Figure 3. As in the previous case assume that we always start the game by placing cop c_1 on vertex v_3 and cop c_2 on vertex v_2. Clearly, the robber cannot start on $v_1, v_2, v_3, v_4, v_5, v_7, v_9$ and v_{10}. If the robber starts on vertex v_6, then we move cop c_1 to vertex v_1. Then the game finish in the next move. This shows that vertex v_6 is a bad starting point. (Now 5 vertices remain for the robber to start).

If the robber starts at vertex v_8, then we move cop c_1 to vertex v_1. The robber then must move to vertex v_{14}. Now cop c_2 moves to vertex v_7 and cop c_1 moves to vertex v_8, then the robber must move to vertex v_{13}. Again the cop c_2 moves to vertex v_5 and cop c_1 moves to vertex v_{14}. Now the robber is trapped and we can finish the game in the next move. (Now 4 vertices remain for the robber to start).
Suppose the robber starts at vertex v_{11}, then we move cop c_2 to vertex v_4. The the robber then must move to vertex v_{12}. Again we move cop c_1 to vertex v_5. The robber must move to vertex v_{13}. Now we move cop c_1 to the vertex v_7 and cop c_2 to vertex v_{11}. Now the robber is trapped and we can finish the game in the next two moves.
If the robber starts at vertex v_{12}, then we move cop c_1 to vertex v_5 and cop c_2 to vertex v_4. Then the robber must move to vertex v_{13}. Again we move cop c_2 to vertex v_{11} and cop c_1 to vertex v_7. Now the robber is trapped and we can finish the game in the next two moves. (Now 2 vertices remain for the robber to start).
If the robber starts at vertex v_{13}, then we move cop c_1 to vertex v_5 and cop c_2 to vertex v_7. Now the robber is trapped and we can finish the game in the next two moves (at this point only one vertex remain for the robber to start).
If the robber starts at vertex v_{14}, then we move cop c_1 to vertex v_1 and cop c_2 to vertex v_7. The robber then must move to vertex v_{13}. Again the cop c_2 moves to vertex v_5 and cop c_1 moves to vertex v_8. Now the robber is trapped and we can finish the game in the next two moves. Therefore the graph depicted in the Figure 3 is 2-cop win.

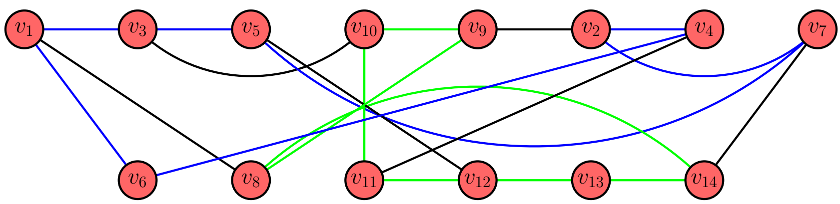

Figure 3. **The graph** $\mathcal{P}(7,2) - \{e'\}$

ad iii) Assume that edge e lies in the inner cycle. Removal of any of these edges from G results the graph depicted in Figure 4.
Label the vertices as shown in the Figure 4. Assume that we always start the game by placing cop c_1 on vertex v_{14} and cop c_2 on vertex v_{11}. Clearly the robber cannot start on $v_8, v_7, v_{14}, v_{13}, v_4, v_{11}, v_{12}$ and v_{10}. If the robber starts on vertex v_1, then we move cop c_1 to vertex v_8 and cop c_2 to vertex v_4. Now the robber is trapped and we can finish the game in the next move. This shows that vertex v_1 is a bad starting point. (Now 5 vertices remain for the robber to start).
If the robber starts at vertex v_2, then we move cop c_2 to vertex v_4. The robber then must move to vertex v_9. Now cop c_1 moves to vertex v_8. The robber then must move to vertex v_{10}. Again cop c_1 moves to the vertex v_9 and cop c_2 moves to vertex

v_{11}, then the robber must move to the vertex v_3. Now we move the cop c_1 to vertex v_{10} and cop c_2 to vertex v_{12}, then the game finish in the next move.(Now 4 vertices remain for the robber to start).
Suppose the robber starts at vertex v_3, then we move cop c_2 to vertex v_{10} and cop c_1 to vertex v_7. Now robber is trapped and we can finish the game in the next two moves. (Now 3 vertices remain for the robber to start).
If the robber starts at vertex v_5, then we move cop c_1 to vertex v_7. Now the robber must move to vertex v_3. Again we move cop c_2 to vertex v_{10}. Now the robber is trapped and we can finish the game in the next move. (Now two vertices remain for the robber to start).
If the robber starts at vertex v_6, then we move the cop c_2 to vertex v_4. The robber then must move to vertex v_1. Again the cop c_1 moves to vertex v_8. Now the robber is trapped and we can finish the game in the next two move. (Now one more vertex remain for the robber to start).
If the robber starts at vertex v_9, then we move cop c_1 to vertex v_8 and cop c_2 to vertex v_4. The robber then must move to vertex v_{10}. Again cop c_1 moves to vertex v_9 and cop c_2 moves to vertex v_{11}. The robber then must move to vertex v_3. Now we move cop c_1 to vertex v_{10} and cop c_2 to vertex v_{12}. Now the robber is trapped and we can finish the game in the next move. Therefore, the graph depicted in the Figure 4 is 2-cop win.

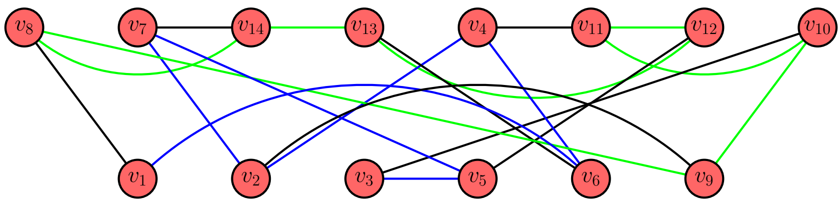

Figure 4. **The graph** $\mathcal{P}(7,2) - \{e''\}$

□

We know that graph $\mathcal{P}(8,2)$ is 2-cop win [5]. Therefore next graph to analyze is $\mathcal{P}(9,2)$.

Lemma 2. *Let G be the graph $\mathcal{P}(9,2)$. Then G is edge critical with cop number* 3.

Proof. Let G be the graph $\mathcal{P}(9,2)$. Since G is a graph with girth at least 5 therefore, from Proposition 1,

$$c(G) \geq 3. \tag{3}$$

If we place cop c_1 on vertex 10, cop c_2 on vertex 4 and c_3 on vertex 12, then we can catch the robber in at most five rounds. This implies that

$$c(G) \leq 3. \tag{4}$$

Now we have from (3) and (4), $c(G) = 3$.

Claim. $c(G - e) = 2$ for any edge $e \in G$. Assume that the edge e is in the outer cycle (see Figure 5). We always start the game by placing the cop c_1 on vertex 1 and cop c_2 on vertex 9. These two cops dominates the vertices 1, 3, 11, 8, 9, 10, 7 and 2, the remaining unguarded vertices in G are 12, 13, 14, 15, 4, 5, 6, 16, 17 and 18. If we place the robber R on these vertices, then we get the following strategies. Since strategy includes more steps we use the following notations for the movements of the cops and the robber. Cop c_1 moves to vertex v by $c_1 \to v$. Cop c_2 moves to vertex v by $c_2 \to v$. Robber R moves to vertex v by $R \to v$. The robber stay on vertex v by $R = v$. Cop c_1 stay on vertex v by $c_1 = v$ and cop c_2 stay on vertex v by $c_2 = v$, where $v = \{1, 2, \ldots, 18\}$ (see Figure 5).

Case-I : Assume $R = 12$.
Now the cops can catch the robber in at most 5 rounds.

R1 : $c_1 = 1$, $c_2 \to 2$, $R \to 13$.

R2 : $c_1 \to 3$, $c_2 = 2$, $R \to 14$.

R3 : $c_1 = 3$, $c_2 \to 4$, $R \to 15$.

R4 : $c_1 \to 5$, $c_2 \to 14$, $R = 15$. Then the game finish in the next step.

Case-II : Assume $R = 13$.
Now the cops can catch the robber in at most 3 rounds.

R1 : $c_1 \to 3$, $c_2 \to 2$, $R \to 14$.

R2 : $c_1 = 3$, $c_2 \to 4$, $R \to 15$.

R3 : $c_1 \to 5$, $c_2 \to 14$, $R = 15$. Then the game finish in the next step.

Case-III : Assume $R = 14$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 3$, $c_2 \to 2$. Now either $R = 14$ or $R \to 15$. Suppose $R = 14$.

R2 : $c_1 = 3$, $c_2 \to 4$, $R \to 15$.

R3 : $c_1 \to 5$, $c_2 \to 14$, $R = 15$. Then the game finish in the next step. Suppose after the first round $R \to 15$.

R2 : $c_1 \to 5$, $c_2 \to 4$, $R = 15$. Then the game finish in the next step.

Case-IV : Assume $R = 15$.
Now the cops can catch the robber in at most 3 rounds.

R1 : $c_1 \to 3$, $c_2 \to 2$. Now either $R = 15$ or $R \to 14$. Suppose $R = 15$.

R2 : $c_1 \to 5$, $c_2 \to 4$, $R = 15$. Then the game finish in the next step. Suppose after the first round $R \to 14$.

R2 : $c_1 = 3$, $c_2 \to 4$, $R \to 15$. Then the game finish in the next step.

Case-V : Assume $R = 4$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 8$, $c_2 \to 2$, $R \to 14$.

R2 : $c_1 \to 1$, $c_2 = 2$. Now the robber has three choices $R = 14$, $R \to 13$ and $R \to 15$. Suppose $R = 14$.

R3 : $c_1 \to 3$, $c_2 \to 4$, $R \to 15$.

R4 : $c_1 \to 5$, $c_2 \to 14$, $R = 15$. Then the game finish in the next step. Suppose after the second round $R \to 13$.

R3 : $c_1 \to 3$, $c_2 = 2$, $R \to 14$.

R4 : $c_1 = 3$, $c_2 \to 4$, $R \to 15$.

R5 : $c_1 \to 5$, $c_2 \to 14$. Then the game finish in the next step. Again assume that after the second round $R \to 15$.

R3 : $c_1 \to 3$, $c_2 \to 4$, $R = 15$.

R4 : $c_1 \to 5$, $c_2 \to 14$, $R = 15$. Then the game finish in the next step.

Case-VI : Assume $R = 18$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 8$, $c_2 = 9$, $R \to 17$.

R2 : $c_1 = 8$, $c_2 \to 7$, $R \to 16$.

R3 : $c_1 \to 6$, $c_2 = 7$, $R = 16$. Then the game finish in the next step.

Case-VII : Assume $R = 17$.
Now the cops can catch the robber in at most 3 rounds.

R1 : $c_1 \to 8$, $c_2 \to 7$, $R \to 16$.

R2 : $c_1 \to 6$, $c_2 \to 17$, $R = 16$. Then the game finish in the next step.

Case-VIII : Assume $R = 16$.
Now the cops can catch the robber in at most 3 rounds.

R1 : $c_1 \to 8$, $c_2 \to 7$, $R = 16$.

R2 : $c_1 \to 6$, $c_2 \to 17$, $R = 16$. Then the game finish in the next step.

Case-IX : Assume $R = 6$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 8$, $c_2 \to 2$, $R \to 16$.

R2 : $c_1 \to 18$, $c_2 \to 4$, $R = 16$.

R3 : $c_1 \to 17$, $c_2 \to 6$, $R = 16$. Then the game finish in the next step.

Case-X : Assume $R = 5$.
Now the cops can catch the robber in at most 3 rounds.

R1 : $c_1 \to 3$, $c_2 \to 7$, $R \to 15$.

R2 : $c_1 \to 13$, $c_2 \to 5$, $R = 15$. Then the game finish in the next step.

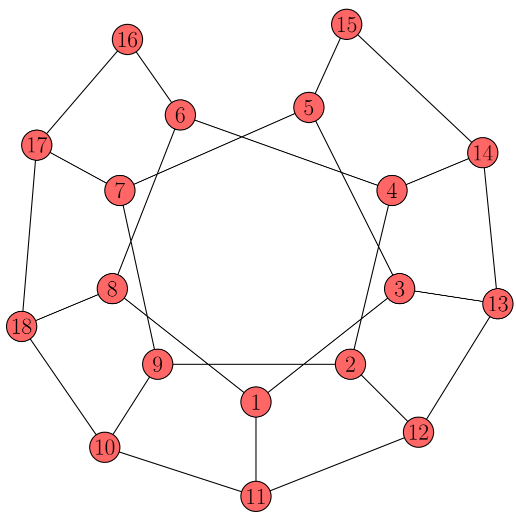

Figure 5. **The graph $\mathcal{P}(9,2) - \{e\}$**

Assume that the edge e is in between the inner cycle and the outer cycle (see Figure 6). We always start the game by placing cop c_1 on vertex 4 and cop c_2 on vertex 5. These two cops dominates the vertices 4, 5, 7, 6, 3, 2, 14, and 15, the remaining unguarded vertices in G are 12, 13, 11, 10, 18, 17, 16, 8, 9 and 1. If we place robber R on these vertices, then we get the following strategies. We follow notations from the previous cases.

Case-I : Assume $R = 12$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 2$, $c_2 \to 3$, $R \to 11$.

R2 : $c_1 \to 9$, $c_2 \to 13$, $R = 11$.

R3 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

Case-II : Assume $R = 13$.
Now the cops can catch the robber in at most 5 rounds.

R1 : $c_1 = 4$, $c_2 \to 3$, $R \to 12$.

R2 : $c_1 \to 2$, $c_2 = 3$, $R \to 11$.

R3 : $c_1 \to 9$, $c_2 \to 13$, $R = 11$.

R4 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

Case-III : Assume $R = 11$.
Now the cops can catch the robber in at most 5 rounds.

R1 : $c_1 \to 2$, $c_2 \to 7$. Now either $R = 11$ or $R \to 10$. Suppose $R = 11$.

R2 : $c_1 = 2$, $c_2 \to 9$, $R = 11$.

R3 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step. Suppose after the first round $R \to 10$.

R2 : $c_1 = 2$, $c_2 \to 17$. Now either $R = 10$ or $R \to 11$. Suppose $R = 10$.

R3 : $c_1 = 2$, $c_2 \to 18$, $R \to 11$.

R4 : $c_1 \to 12$, $c_2 \to 10$, $R = 11$. Then the game finish in the next step. Suppose $R \to 11$ in the previous round.

R3 : $c_1 \to 12$, $c_2 \to 18$, $R = 11$. Then the game finish in the next step.

Case-IV : Assume $R = 10$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 6$, $c_2 \to 7$. Now the robber has three choices $R = 10$, $R \to 18$ and $R \to 11$. Suppose $R = 10$.

R2 : $c_1 \to 8$, $c_2 \to 9$, $R \to 11$.

R3 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R4 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step. Suppose after the first round $R \to 18$.

R2 : $c_1 \to 8$, $c_2 = 7$, $R \to 10$.

R3 : $c_1 = 8$, $c_2 \to 9$, $R \to 11$.

R4 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R5 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step. Suppose after the first round $R \to 11$.

R2 : $c_1 \to 4$, $c_2 \to 9$. Now either $R = 11$ or $R \to 12$. Suppose after the second round $R = 11$.

R3 : $c_1 \to 2$, $c_2 \to 10$. Then the game finish in the next step. Suppose after the second round $R \to 12$.

R3 : $c_1 \to 14$, $c_2 = 9$. Now either $R = 12$ or $R \to 11$. In both cases we keep c_2 on the vertex 9 and move c_1 along the vertices 13 and 12 and catch the robber in at most three rounds.

Case-V : Assume $R = 18$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 6$, $c_2 \to 7$. Now either $R = 18$ or $R \to 10$. Suppose $R = 18$.

R2 : $c_1 \to 8$, $c_2 = 7$, $R \to 10$.

R3 : $c_1 = 8$, $c_2 \to 9$, $R \to 11$.

R4 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R5 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step. Suppose after the first round $R \to 10$.

R2 : $c_1 \to 8$, $c_2 \to 9$, $R \to 11$.

R3 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R4 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

Case-VI : Assume $R = 17$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 6$, $c_2 \to 7$, $R \to 18$.

R2 : $c_1 \to 8$, $c_2 = 7$, $R \to 10$.

R3 : $c_1 = 8$, $c_2 \to 9$, $R \to 11$.

R4 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R5 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

Case-VII : Assume $R = 16$.
Now the cops can catch the robber in at most 7 rounds.

R1 : $c_1 \to 6$, $c_2 = 5$, $R \to 17$.

R2 : $c_1 = 6$, $c_2 \to 7$, $R \to 18$.

R3 : $c_1 \to 8$, $c_2 = 7$, $R \to 10$.

R4 : $c_1 = 8$, $c_2 \to 9$, $R \to 11$. After this round we follow CASE IV ROUND 4 onwards. Therefore we can catch the robber in at most three more rounds and finish the game.

Case-VIII : Assume $R = 1$.
Now the cops can catch the robber in at most 2 rounds.

R1 : $c_1 \to 6$, $c_2 \to 3$, $R = 1$. Then the game finish in the next step.

Case-IX : Assume $R = 9$.
Now the cops can catch the robber in at most 5 rounds.

R1 : $c_1 \to 6$, $c_2 \to 7$. Now either $R \to 2$ or $R \to 10$. Suppose $R \to 2$.

R2 : $c_1 \to 4$, $c_2 \to 9$, $R \to 12$.

R3 : $c_1 \to 14$, $c_2 = 9$. Now either $R \to 11$ or $R = 12$. In both cases we keep c_2 on the vertex 9 and move c_1 along the vertices 13 and 12 and catch the robber in at most three rounds. Now assume that after the first round $R \to 10$.

R2 : $c_1 \to 8$, $c_2 \to 9$, $R \to 11$.

R3 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R4 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

Case-X : Assume $R = 8$.
Now the cops can catch the robber in at most 7 rounds.

R1 : $c_1 \to 6$, $c_2 \to 7$. Now either $R \to 1$ or $R \to 18$. Suppose $R \to 1$.

R2 : $c_1 \to 8$, $c_2 \to 5$, $R = 1$. Then the game finish in the next step. Now assume that after the first round $R \to 18$.

R3 : $c_1 \to 8$, $c_2 = 5$, $R \to 10$.

R4 : $c_1 = 8$, $c_2 \to 9$, $R \to 11$.

R5 : $c_1 \to 18$, $c_2 \to 2$, $R = 11$.

R6 : $c_1 \to 10$, $c_2 \to 12$, $R = 11$. Then the game finish in the next step.

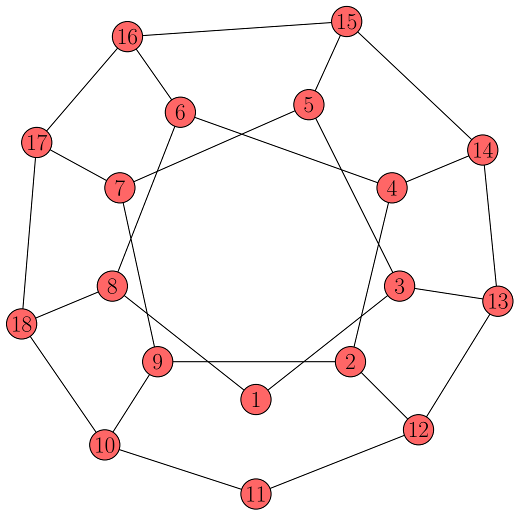

Figure 6. **The graph $\mathcal{P}(9,2) - \{e'\}$.**

Assume that the edge e is in the inner cycle (see Figure 7). We always start the game by placing cop c_1 on the vertex 14 and cop c_2 on vertex 18. These two cops dominates the vertices 4, 15, 13, 14, 10, 8, 17 and 18, the remaining unguarded vertices in G are 1, 2, 3, 5, 6, 7, 9, 11, 12 and 16. If we place robber R on these vertices, then we get the following strategies. We follow notations from the previous cases.

Case-I : Assume $R = 12$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_2 \to 10$, $c_1 \to 13$, $R \to 2$.

R2 : $c_2 \to 11$, $c_1 \to 14$, $R = 2$.

R3 : $c_1 \to 12$, $c_2 \to 4$, $R = 2$. Then the game finish in the next step.

Case-II : Assume $R = 11$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 13$, $c_2 \to 8$. Now either $R \to 10$ or $R = 11$. Suppose $R \to 10$.

R2 : $c_1 \to 12$, $c_2 \to 18$, $R \to 9$.

R3 : $c_1 \to 11$, $c_2 \to 17$, $R = 9$.

R4 : $c_1 \to 10$, $c_2 \to 7$, $R = 9$. Then the game finish in the next step. Assume after the first round $R = 11$.

R2 : $c_1 \to 12$, $c_2 = 8$, $R \to 10$.

R3 : $c_1 \to 11$, $c_2 \to 18$, $R \to 9$.

R4 : $c_1 \to 10$, $c_2 \to 17$, $R = 9$. Then the game finish in the next step.

Case-III : Assume $R = 16$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 15$, $c_2 = 18$, $R \to 6$.

R2 : $c_1 = 15$, $c_2 \to 8$, $R \to 4$.

R3 : $c_1 \to 14$, $c_2 = 8$, $R \to 2$.

R4 : $c_1 = 14$, $c_2 \to 1$. Now either $R \to 12$ or $R = 2$. Suppose $R \to 12$.

R5 : $c_1 = 14$, $c_2 \to 11$, $R \to 2$.

R6 : $c_1 = 14$, $c_2 \to 12$, $R = 2$. Then the game finish in the next step. Assume after the fourth round $R = 2$.

R5 : $c_1 \to 4$, $c_2 \to 11$, $R = 2$. Then the game finish in the next step.

Case-IV : Assume $R = 6$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 15$, $c_2 \to 8$, $R \to 4$.

R2 : $c_1 \to 14$, $c_2 = 8$, $R \to 2$.

R3 : $c_1 = 14$, $c_2 \to 1$. Now either $R \to 12$ or $R = 2$. Suppose $R \to 12$.

R4 : $c_1 = 14$, $c_2 \to 11$, $R \to 2$.

R5 : $c_1 = 14$, $c_2 \to 12$, $R = 2$. Then the game finish in the next step. Assume after the third round $R = 2$.

R4 : $c_1 \to 4$, $c_2 \to 11$, $R = 2$. Then the game finish in the next step.

Case-V : Assume $R = 5$.
Now the cops can catch the robber in at most 8 rounds.

R1 : $c_1 \to 13$, $c_2 \to 17$. Now either $R \to 15$ or $R = 5$. Suppose $R \to 15$.

R2 : $c_1 = 13$, $c_2 \to 16$, $R \to 5$.

R3 : $c_1 \to 3$, $c_2 = 16$, $R \to 7$.

R4 : $c_1 = 3$, $c_2 \to 17$, $R \to 9$.

R5 : $c_1 \to 1$, $c_2 = 17$. Now either $R \to 10$ or $R = 9$. In both cases we keep c_2 on the vertex 17 and move c_1 along the vertices 11 and 10 and catch the robber in at most three rounds. Now assume that after the first round $R = 5$.

R2 : $c_1 \to 3$, $c_2 \to 16$, $R \to 7$.

R3 : $c_1 = 3$, $c_2 \to 17$, $R \to 9$.

R4 : $c_1 \to 1$, $c_2 = 17$. Now either $R \to 10$ or $R = 9$. In both cases we keep c_2 on the vertex 17 and move c_1 along the vertices 11 and 10 and catch the robber in at most three rounds. Now assume that after the first round $R = 5$.

Case-VI : Assume $R = 7$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 15$, $c_2 = 18$. Now either $R \to 9$ or $R = 7$. Suppose $R \to 9$.

R2 : $c_1 \to 5$, $c_2 \to 10$, $R = 9$. Then the game finish in the next step. Assume after the first round $R = 7$.

R2 : $c_1 \to 5$, $c_2 = 18$, $R \to 9$.

R3 : $c_1 \to 7$, $c_2 \to 10$, $R = 9$. Then the game finish in the next step.

Case-VII : Assume $R = 9$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 \to 15$, $c_2 = 18$. Now either $R \to 7$ or $R = 9$. Suppose $R \to 7$.

R2 : $c_1 \to 5$, $c_2 = 18$, $R \to 9$.

R3 : $c_1 \to 7$, $c_2 \to 10$, $R = 9$. Then the game finish in the next step. Suppose after the first round $R = 9$.

R2 : $c_1 \to 5$, $c_2 \to 10$. Then the game finish in the next step.

Case-VIII : Assume $R = 1$.
Now the cops can catch the robber in at most 5 rounds.

R1 : $c_1 \to 13$, $c_2 \to 8$, $R \to 11$.

R2 : $c_1 \to 12$, $c_2 = 8$, $R \to 10$.

R3 : $c_1 \to 11$, $c_2 \to 18$, $R \to 9$.

R4 : $c_1 \to 10$, $c_2 \to 17$, $R = 9$. Then the game finish in the next step.

Case-IX : Assume $R = 2$.
Now the cops can catch the robber in at most 4 rounds.

R1 : $c_1 = 14$, $c_2 \to 10$. Now either $R \to 12$ or $R = 2$. Suppose $R = 2$.

R2 : $c_1 = 14$, $c_2 \to 11$, $R = 2$.

R3 : $c_1 \to 4$, $c_2 \to 12$, $R = 2$. Then the game finish in the next step. Suppose after the first round $R \to 12$.

R2 : $c_1 = 14$, $c_2 \to 11$, $R \to 2$.

R3 : $c_1 \to 12$, $c_2 \to 4$, $R = 2$. Then the game finish in the next step.

Case-X : Assume $R = 3$.
Now the cops can catch the robber in at most 6 rounds.

R1 : $c_1 \to 15$, $c_2 \to 8$. Now either $R = 3$ or $R \to 13$. Suppose $R \to 13$.

R2 : $c_1 \to 14$, $c_2 \to 1$, $R \to 12$.

R3 : $c_1 = 14$, $c_2 \to 11$, $R \to 2$.

R4 : $c_1 \to 4$, $c_2 \to 12$, $R = 2$. Then the game finish in the next step. Suppose after the first round $R = 3$.

R2 : $c_1 = 15$, $c_2 \to 1$, $R \to 13$.

R3 : $c_1 \to 14$, $c_2 = 1$, $R \to 12$.

R4 : $c_1 = 14$, $c_2 \to 11$, $R \to 2$.

R5 : $c_1 \to 4$, $c_2 \to 12$, $R = 2$. Then the game finish in the next step.

□

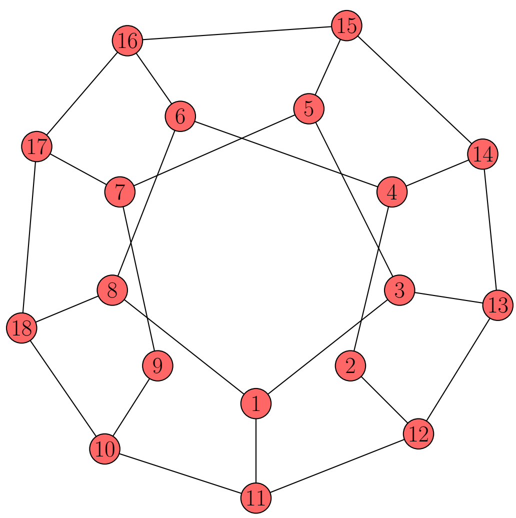

Figure 7. **The graph** $\mathcal{P}(9,2) - \{e''\}$.

3. Paley graphs

Paley graphs are graphs constructed from the members of a suitable finite field by connecting pairs of elements that differ by a quadratic residue.

Definition 1. Let q and r be two positive integers with $gcd(q, r) = 1$. Then r is a quadratic residue of q if and only if $x^2 \equiv r \pmod{q}$ has a solution, and r is a quadratic nonresidue of q if and only if $x^2 \equiv r \pmod{q}$ has no solution. We denote the set of quadratic residues by $(\mathbb{F}_p^*)^2$.

Definition 2. Let p be a prime number and n be a positive integer such that $p \equiv 1 \pmod{4}$. The graph $P_{p^n} = (V, E)$ with

$$V(P) = \mathbb{F}_{p^n} \text{ and } E(P) = \{\{x, y\} \mid x, y \in \mathbb{F}_{p^n},\ x - y \in (\mathbb{F}_{p^n}^*)^2\}$$

is called the Paley graph of order p^n.

The list of integers which can be considered as an order of the Paley graph starts with 5, 9, 13, 17, 25, 29, 37, 41, etc. Paley graphs have very nice properties, namely they are : connected, symmetric (they are vertex and edge-transitive), self-complementary (complement of P_x is P_x), and strongly regular (they are regular and every two adjacent vertices have the same number of common neighbours, as well as every two non-adjacent vertices) [2, 3, 8].
In [9] authors define the G^{Ξ} as a graph obtained by connecting the corresponding vertices in G and its complement $\overline{G}$. Notice that P_5 is just a 5-cycle and therefore Petersen graph is a P_5^{Ξ}.

Remark 1. P_5 is 2 cop-edge critical. P_5^{Ξ} is 3 cop-edge critical.

We prove that P_{17} is 3-cop edge critical graph.

Lemma 3. *Let G be the graph P_{17}. Then G is cop-edge critical with cop number* 3.

Proof. P_{17} has 68 edges and is a strongly regular with parameters $(17, 8, 3, 4)$, i.e. it has 17 vertices, the degree of every vertex is 8, for any two adjacent vertices, the number of vertices adjacent to both is 3, for any two non-adjacent vertices, the number of vertices adjacent to both is 4. Quadratic residues of 17 are $\{1, 2, 4, 8, 9, 13, 15, 16\}$ (this implies that vertex with label 0 is connected to vertices $1, 2, 4, \ldots, 16$ and vertex with label 1 to vertices $2, 3, 5, \ldots, 0$ etc.).

Notice that no two vertices of P_{17} dominates a neighbourhood of any other vertex. It is easy to prove as there is a gap of three consecutive vertices in the neighbourhood of every vertex (for 0 it is $10, 11, 12$) but on the other hand no vertex is connected to three consecutive vertices. That means (since P_{17} is vertex-transitive) that only 0 and 10, 11 or 12 could create the required pair. But since 10 is not connected to 3, 11 is not connected to 14 and 12 is not connected to 5 the condition is not true. Therefore two cops are not sufficient to catch robber in that graph.

On the other hand we can find 3 vertices (for example $\{0, 10, 14\}$) that dominate whole graph. Hence three cops are enough to guard the P_{17} graph. Thus we showed that the cop number of P_{17} is equal to 3.

Now we need to show that removing any edge from P_{17} decrease its cop number. We show that in P_{17} there are always two vertices that dominate any vertex and its neighbourhood without a single vertex. Let us choose a vertex 0 and edge $\{0, 4\}$. The two vertices with that property are 10 an 15 (they are not connected only with $\{3, 4, 5\}$). Notice that 0 is not connected to 3 and 5 thus after removing the edge $\{0, 4\}$, vertices 10 and 15 dominates whole neighbourhood of 0, and 0 itself. In general there is a cycle $0-4-8-12-16-3-7-11-15-2-6-10-14-1-5-9-13-0$ on which we can restrict the robber movement with proper positioning of the cops (when robber moves from vertex 0 to vertex 4 we move cops from 10, 15, to 14 and 2 etc.). Whenever some edge from this cycle is missing we will then catch the robber when he/she will reach it. Since Paley graphs are edge and vertex transitive the argument is true for every edge of P_{17} (after applying proper automorphism).

□

Lemma 4. *Let G be the graph P_{17}^{Ξ}. Then G is edge critical with cop number* 4.

Proof. Graph G^{Ξ} is obtained by taking a graph G and its complement (on the separate set of vertices) and by connecting the corresponding vertices in both copies with an edge. We will address the copies of the graph as P_{17} part and $\overline{P_{17}}$ part. We start with showing that cop number of P_{17}^{Ξ} is 4. We start the game by putting three cops in $\overline{P_{17}}$ part of the graph so that they dominate the whole graph and fourth robber in the P_{17} part. Then robber has to put him/herself on the part with only one cop. We then keep one cop in the complement part of the graph in order to prevent robber from changing the parts and place other cops so that they dominates whole P_{17}. Thus in at most three rounds robber would get caught. This implies that 4 cops are enough to catch the robber.

On the other hand, if we have only 3 cops then we have two options for the robber to move to safe position in every step: the move is either to move in the same part he is at the moment or to switch parts. As we know from previous proof we can not stop the robber on one part with only two cops. If we move three cops on one part, then robber can always change parts and continue the game. If we use only two cops then (again by previous proof) we can force the robber to choose between a single vertex from part he is in at the moment and changing the parts. Assume robber is at vertex v and his choices are vertex u and $\overline{v}$. The third cop can not guard both u and $\overline{v}$ as $\overline{u}$ is not connected with $\overline{v}$, therefore the robber would always have a "safe" vertex to move to and continue the game. Thus the cop number of P_{17}^{Ξ} is 4.

If we remove a single edge from either of parts then by previous proof we can stop robber on one part of the graph with just two cops and use the third one in order to prevent the robber from changing the sides. The game would look as follows: We start by putting all cops on one part, then after first round choose one cop to prevent the robber from changing parts while other two catch him.
If the missing edge is the edge between the parts (denote it as edge $\{v, \overline{v}\}$) then we force robber on the vertex v using the same approach as just described (where one cop only focus on preventing the robber from switching parts). Once the robber is forced by other two cops onto vertex v the third cop moves into vertex $\overline{u}$ which is the vertex copy of the only choice robber has on his part that is not guarded by other two cops. Then robber is caught in the next step. We need to show that such move by the third cop is always possible. Again we would use the example from the previous proof. Assume missing edge is $\{4, \overline{4}\}$ and robber is on vertex 0. Then cop which prevents him from moving onto $\overline{0}$ can move to any vertex which is connected to both $\overline{0}$ and $\overline{8}$, which are $\{\overline{3}, \overline{5}, \overline{11}, \overline{14}\}$. Since every vertex of $\overline{P_{17}}$ is connected to one of them, the cop can do it from every position his current was. Then in next step robber is forced on vertex 4, while third cop moves to $\overline{8}$ and robber is caught in the next round. Again due to P_{17} being vertex and edge transitive the same proof can be repeated for every edge concludes the proof that P_{17}^{Ξ} is edge critical with cop number 4. □

We propose two conjectures regarding Paley graphs:

Conjecture 2. There are infinitely many cop edge critical graphs among family $\{P_x^{\Xi} : x \in \mathbb{N}\}$.

Conjecture 3. If P_x is k-cop edge critical, then P_x^{Ξ} is k+1-cop edge critical.

4. Open Problems

In this paper we have found three more graphs which are 3-cop edge critical and have given a very first example of a graph which is 4-cop edge critical. Moreover we formulate a very interesting conjectures regarding edge criticality among Paley graphs. Characterization of graphs which are 3-cop edge critical still remains open.

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