Communications in Combinatorics and Optimization
Vol. 8, No. 1 (2023), pp. 115-125
DOI: 10.22049/CCO.2021.27443.1265

CCO
Commun. Comb. Optim.

Research Article

Covering total double Roman domination in graphs

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Received: 4 October 2021; Accepted: 6 November 2021
Published Online: 10 November 2021

Abstract: For a graph G with no isolated vertex, a covering total double Roman dominating function ($CTDRD$ function) f of G is a total double Roman dominating function ($TDRD$ function) of G for which the set $\{v \in V(G)|f(v) \neq 0\}$ is a vertex cover set. The covering total double Roman domination number $\gamma_{ctdR}(G)$ equals the minimum weight of an $CTDRD$ function on G. An $CTDRD$ function on G with weight $\gamma_{ctdR}(G)$ is called a $\gamma_{ctdR}(G)$-function. In this paper, the graphs G with small $\gamma_{ctdR}(G)$ are characterised. We show that the decision problem associated with $CTDRD$ is NP-complete even when restricted to planer graphs with maximum degree at most four. We then show that for every graph G without isolated vertices, $\gamma_{oitR}(G) < \gamma_{ctdR}(G) < 2\gamma_{oitR}(G)$ and for every tree T, $2\beta(T) + 1 \leq \gamma_{ctdR}(T) \leq 4\beta(T)$, where $\gamma_{oitR}(G)$ and $\beta(T)$ are the outer independent total Roman domination number of G, and the minimum vertex cover number of T respectively. Moreover we investigate the γ_{ctdR} of corona of two graphs.

Keywords: Covering, Roman domination, total double Roman domination

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, suppose that G be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For the terminologies and notations which are not defined here explicitly, we may use [10] as a reference. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u \mid uv \in E(G)\}$. The *closed neighborhood* of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$. The closed neighborhood of a set $S \subseteq V$ is the set $N[S] =$

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$N(S) \cup S = \cup_{v \in S} N[v]$. We denote the degree of v by $\deg(v) = \deg_G(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the *maximum degree* and *minimum degree* of a graph G, respectively. We write K_n, P_n and C_n for the complete graph, path and cycle of order n, respectively. A tree T is an *acyclic connected graph*. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \odot G_2$ formed from one copy of G_1 and $|G_1|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G. A set of vertices is *independent* if no two vertices in it are adjacent. The *independence number* $\alpha(G)$ is the maximum cardinality among all independent sets of G. A *vertex cover* of G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The *vertex cover number* $\beta(G)$ is the minimum cardinality among all vertex covers of G. An *independent dominating set* of G is a dominating set that is also independent in G. The *independent domination number* of G, denoted by $\gamma_i(G)$, is the minimum size of an independent dominating set in G. A *total dominating set* of a without isolate graph G is a set S of vertices such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The concept of total domination in graphs is now well studied (see [10, 11]).

Given a graph G and a positive integer m, assume that $g : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \ldots, V_m)$ is the ordered partition of V induced by g, where $V_i = \{v \in V | g(v) = i\}$ for $i \in \{0, 1, \ldots, m\}$. So we can write $g = (V_0, V_1, V_2, \ldots, V_m)$. A *Roman dominating function* (*RD function*) on a graph G is a function $f : V \rightarrow \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ with $f(w) = 2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of a Roman dominating function on G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [6] introduced the concept of Roman domination in graphs, and since then a lot of related variations and generalizations have been studied (see [3–5]).

The *total Roman dominating function* (*TRD function*) on a graph G with no isolated vertex is an *RD* function f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total Roman domination number $\gamma_{tR}(G)$ is the minimum weight of a *TRD* function on G. A *TRD* function on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function [1]. An *outer independent Roman dominating function* (*OIRD function*) of a graph G is a Roman dominating function $f : V \rightarrow \{0, 1, 2\}$ for which $V_0 = V_0^f$ is independent. The *outer independent Roman domination number* (*OIRD* number) $\gamma_{oiR}(G)$ is the minimum weight of an *OIRD* function of G. An *outer independent total Roman dominating function (OITRD function)* on a graph G is a *TRD* function for which V_0^f is independent. The outer independent total Roman domination number $\gamma_{oitR}(G)$

equals the minimum weight of an $OITRD$ function of G [14].
A *double Roman dominating function* on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:
(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.
The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function (DRD function) f on G is called *double Roman domination number* of G. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function of G. Double Roman domination was studied in [2, 12, 13, 15, 18] and elsewhere.
The *total double Roman dominating function* ($TDRD$ function) on a graph G with no isolated vertex is a DRD function f on G with the additional property that the subgraph of G induced by the set $\{v \in V | f(v) \neq 0\}$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a $TDRD$ function on G. A $TDRD$ function on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-function [8, 9]. Another invariant of double Roman dominating function is defined as follows.
A covering total double Roman dominating function (CTDRD function) on a graph G with no isolated vertex is a $TDRD$ function for which $V \setminus V_0 = \{v \in V | f(v) \neq 0\}$ is a vertex cover set or $V_0 = V_0^f = \{v \in V | f(v) = 0\}$ is an independent set. The covering total double Roman domination number $\gamma_{ctdR}(G)$ equals the minimum weight of a $CTDRD$ function of G (See [17]).
The paper is organized as follows. In Section 2, the graphs G with small $\gamma_{ctdR}(G)$ are characterised. We show that the decision problem associated with $CTDRD$ is NP-complete even when restricted to planer graphs with maximum degree at most four in Section 3. Then we show that for every graph G $\gamma_{oitR}(G) < \gamma_{ctdR}(G) < 2\gamma_{oitR}(G)$ and for every tree T, $2\beta(T) + 1 \leq \gamma_{ctdR}(T) \leq 4\beta(T)$, where $\gamma_{oitR}(G)$ is the outer independent total Roman domination number of G and $\beta(T)$ is the minimum vertex cover number of T. Moreover, we investigate the γ_{ctdR} of corona of two graphs in Section 4.

2. Connected graphs with small CTDRD numbers

In this section, we characterize the family of all connected graphs G for which $\gamma_{ctdR}(G) \in \{3, 4, 5\}$. To this end, let $\mathcal{G}$ be the family of all graphs of the form G_1, G_2 depicted in the Figure 1. In the figure, the number of vertices $w_1, \cdots, w_k$ in G_1, G_2 is at least 1 and 0, respectively.

We next define other four necessary families of graphs, that is, the families $\mathcal{F}_i$, $1 \leq i \leq 4$. To this end, we shall use the following conventions. For a given set of vertices $\{v_1, \ldots, v_k\}$ with $k \geq 1$, let $V_{v_1,\ldots,v_k} = \{v \in V(G) | N(v) = \{v_1, \ldots, v_k\}\}$.

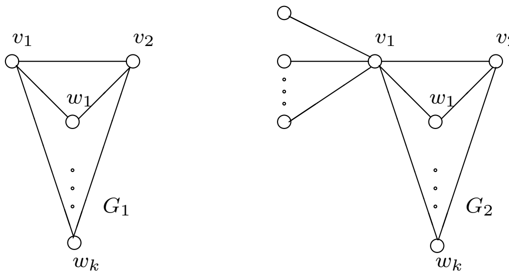

Figure 1. **The graphs G_1 and G_2**

Such convention shall be used also while proving some results.

- $\mathcal{F}_1$: We begin with a path $P = abc$. Then we add three sets $V_{a,b}$, $V_{b,c}$ and $V_{a,b,c}$ such that one of the following conditions holds.
(a_1): $V_{a,b} = \emptyset = V_{b,c}$ and $|V_{a,b,c}| \geq 2$.
(b_1): only one of the sets $V_{a,b}$ and $V_{b,c}$ is of order 0, and $|V_{a,b,c}| \geq 1$.
(c_1): $|V_{a,b}| \geq 1$, $|V_{b,c}| \geq 1$.

- $\mathcal{F}_2$: We begin with a cycle of order three $C = abca$ and proceed as above, by adding the sets $V_{a,b}$, $V_{b,c}$ and $V_{a,b,c}$. Then, one of the following situations holds.
(a_2): $|V_{a,b,c}| \geq 1$.
(b_2): $|V_{a,b}| \geq 1$, $|V_{b,c}| \geq 1$.

- $\mathcal{F}_3$: We begin with a path $P = abc$. Then we add sets $V_{a,b}$ and $V_{a,b,c}$ such that one of the following conditions holds.
(a_3): $|V_{a,b}| = 0$ and $|V_{a,b,c}| \geq 2$.
(b_3): $|V_{a,b}| \geq 1$, $|V_{a,b,c}| \geq 1$.

- $\mathcal{F}_4$: We begin with a path $P = abc$. Then we add the set $V_{a,c}$ and add the set $V_{a,b,c}$.
(a_4): $|V_{a,c}| = 0$ and $|V_{a,b,c}| \geq 2$.
(b_4): $|V_{a,c}| \geq 1$.

Proposition 1. *Let G be a connected graph of order $n \geq 2$. Then,*
(i) $\gamma_{ctdR}(G) = 3$ *if and only if* $G = K_2$.
(ii) $\gamma_{ctdR}(G) = 4$ *if and only if* $G \in \mathcal{G}$.
(iii) $\gamma_{ctdR}(G) = 5$ *if and only if* $G \in \cup_{i=1}^{4} \mathcal{F}_i$.

Proof. (i) It is clear.
(ii) Assume that $G \in \mathcal{G}$. If we assign the value 3 to the vertex v_1, the value 1 to the

vertex v_2 and 0 to the other vertices, then $\gamma_{ctdR}(G) \leq 4$.
Conversely, let G be a graph with $\gamma_{ctdR}(G) = 4$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{ctdR}(G)$-function. We note that the assumption $(|V_0|, |V_1|, |V_2|, |V_3|) = (0, 2, 1, 0)$ is possible if and only if $G = K_3$. So, G is of the form of G_1.
Now assume that there are two adjacent vertices $\{v_1, v_2\} \subseteq V_2$. So, G is of the form of G_1 with $k \geq 1$. Let v_1 and v_2 in G such that $v_1 \in V_3$ and $v_2 \in V_1$. If $k \geq 1$, then G is of the form of G_1 or G_2 and if $k = 0$, then G is of the form of G_2. Therefore, such a graph should be of the form G_1 or G_2 in $\mathcal{G}$.
(iii) If $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, then $(f(a), f(b), f(c)) = (1, 3, 1)$ and $f(v) = 0$ otherwise, is a $CTDRD$ function of G, thus $\gamma_{ctdR}(G) \leq 5$. If $G \in \mathcal{F}_4$, then $(f(a), f(b), f(c)) = (2, 1, 2)$ and $f(v) = 0$ otherwise, is a $CTDRD$ function of G, thus $\gamma_{ctdR}(G) \leq 5$. By items (i) and (ii), we get the desired equalities.
Conversely, we assume that $f : V(G) \to \{0, 1, 2, 3\}$ is a $\gamma_{ctdR}(G)$-function of weight 5. If $V_1 = \emptyset$, then there exist two adjacent vertices a and b such that $(f(a), f(b)) = (3, 2)$ and f assigns 0 to the other vertices. Each vertex v with $f(v) = 0$ must be adjacent only to the vertex a or to the vertices a and b, but not only to the vertex b. So, we can assign value 1 to b and therefore $\gamma_{ctdR}(G) \leq 4$, that is a contradiction.
Now assume that $V_1 \neq \emptyset$. Let $|V_1| = 1$ and b is the only member of V_1. Therefore, there exist two vertices a and c assigned 2 under f. We first consider b is adjacent to both a and c. Note that the remaining vertices must be adjacent to both a and c, as well. If $V_{a,c} - \{b\} = \emptyset$ and $|V_{a,b,c}| \leq 1$, then we have $\gamma_{ctdR}(G) \leq 4$. Thus $|V_{a,b,c}| \geq 2$. If $V_{a,c} - \{b\} \neq \emptyset$ then $\gamma_{ctdR}(G) = 5$. This shows that $G \in \mathcal{F}_4$.
Let b be adjacent to only one vertex in $\{a, c\}$, say c. Therefor a must be adjacent to c. Then the other vertices belong to $V_{a,c} \cup V_{a,b,c}$. If $V_{a,c} = \emptyset$ and $|V_{a,b,c}| \leq 1$, then $\gamma_{ctdR}(G) \leq 4$, so $|V_{a,b,c}| \geq 2$. If $V_{a,c} \neq \emptyset$ and $V_{a,b,c} = \emptyset$, then $\gamma_{ctdR}(G) \leq 4$. Therefore, $V_{a,b,c} \neq \emptyset$. In such case, $G \in \mathcal{F}_3$.
We now consider a situation in which $|V_1| = 2$. Let $V_1 = \{a, c\}$. Then both a and c must be adjacent to a vertex b assigned 3 under f. It follows that, the other vertices belong to $V_b \cup V_{a,b} \cup V_{b,c} \cup V_{a,b,c}$. We need to consider two possibilities depending on the adjacency between a and c. First, let $ac \notin E(G)$ and assume that $V_{a,b} = V_{b,c} = \emptyset$. If $|V_{a,b,c}| \leq 1$, then we have $\gamma_{ctdR}(G) \leq 4$, and so $|V_{a,b,c}| \geq 2$. If only one of the sets $V_{a,b}$ and $V_{b,c}$ is empty, $V_{a,b,c} = \emptyset$, then $\gamma_{ctdR}(G) \leq 4$. Thus, $V_{a,b,c} \neq \emptyset$. We note that if $V_{a,b}, V_{b,c} \neq \emptyset$, then we have no conditions on the set $V_{a,b,c} \neq \emptyset$. This argument guarantees that $G \in \mathcal{F}_1$.
On the other hand, let $ac \in E(G)$. Hence, we have a cycle $abca$. If at least one of the sets $V_{a,b}$ and $V_{b,c}$ is empty, then we must have $V_{a,b,c} \neq \emptyset$, for otherwise $\gamma_{ctdR}(G) \leq 4$. If both $V_{a,b}$ and $V_{b,c}$ are nonempty, then we have no conditions on the set $V_{a,b,c}$. Therefore, $G \in \mathcal{F}_2$.
Finally, in the case $|V_1| = 3$, we have $V_0 = V_3 = \emptyset$ and only one vertex is assigned 2 under f. In such situation, $G \cong K_4 \in \mathcal{F}_2$. This completes the proof. $\square$

3. Decision problem associated with $CTDRD$

We first consider the problem of deciding whether a graph G has the CTDRD number at most a given integer. That is stated in the following decision problem. Note that Mojdeh et al. [16] proved that the problem of computing the OIDRD number of graphs is NP-hard, even when restricted to planar graphs of maximum degree at most four.

CTDRD problem
INSTANCE: A graph G and an integer $k \leq 2|V(G)|$.
QUESTION: Is $\gamma_{ctdR}(G) \leq k$?

Our aim is to show that the CTDRD problem is NP-complete for planer graphs with maximum degree at most four. To this end, we make use of the well-known INDEPENDENCE NUMBER PROBLEM (IN problem) which is known to be NP-complete from [7].

IN problem
INSTANCE: A graph G and an integer $k \leq |V(G)|$.
QUESTION: Is $\alpha(G) \geq k$?

Moreover, the problem above remains NP-complete even when restricted to some planer graphs. Indeed, we have the following result.

Theorem 1. ([7]) *The IN problem is NP-complete even when restricted to planer graphs with maximum degree at most three.*

Theorem 2. *The $CTDRD$ problem is NP-complete even when restricted to planer graphs with maximum degree at most four.*

Proof. Let G be a planer graph with $V(G) = \{v_1, \ldots, v_n\}$ and maximum degree $\Delta(G) \leq 3$. Let H be a graph with 4 vertices u, a, b, c such that u is adjacent to the vertices a, b, c and a is adjacent to b. For any $1 \leq i \leq n$, we add a copy of the H with vertices u_i, a_i, b_i, c_i. We now construct a graph G' by joining v_i to u_i, for each $1 \leq i \leq n$. Clearly, G' is a planer graph, $|V(G')| = 5n$ and $\Delta(G') \leq 4$.
Let f be $\gamma_{ctdR}(G')$-function. Since u_i is adjacent to three vertices, f must assign a weight of at least four to u_i together with the three vertices adjacent to it. So, without loss of generality, we may consider that $f(u_i) = 3$, and that f assigns 0 to a_i, c_i and 1 to b_i for each $1 \leq i \leq n$. Since V_0^f is independent, it follows that the number of vertices $v_i \in V(G)$ which can be assigned 0 under f is at most $\alpha(G)$. Furthermore, the other vertices of $V(G)$ are assigned at least 1 under f. Consequently, we obtain that $\gamma_{ctdR}(G') \geq 4n + (n - \alpha(G)) = 5n - \alpha(G)$.

On the other hand, let I be an $\alpha(G)$-set. It is easy to observe that the function

$$g(v) = \begin{cases} 3, & \text{if } v \in \{u_1, \ldots, u_n\}, \\ 0, & \text{if } v \in \{a_1, c_1, \ldots, a_n, c_n\} \, or \, v \in I, \\ 1, & \text{otherwise.} \end{cases}$$

is a $CTDRD$ function of G' with weight $5n - \alpha(G)$, which leads to the equality $\gamma_{ctdR}(G') = 5n - \alpha(G)$. Now, by taking $j = 5n - k$, it follows that $\gamma_{ctdR}(G') \leq j$ if and only if $\alpha(G) \geq k$, which completes the reduction. Since the IN problem is NP-complete for planer graphs of maximum degree at most three, we deduce that the $CTDRD$ problem is NP-complete for planer graphs of maximum degree at most four. □

As a consequence of Theorem 2, we conclude that the problem of computing the $CTDRD$ number even when restricted to planer graphs with maximum degree at most four is NP-hard.

4. CTDRD versus other parameters in graphs

In consequence, it would be desirable to bound the $CTDRD$ number in terms of several different invariants of the graph.

Theorem 3. *For any nontrivial connected graph G, $\gamma_{ctdR}(G) < 2\gamma_{oitR}(G)$.*

Proof. If $f = (V_0, V_1, V_2)$ is a $\gamma_{oitR}(G)$, it is easy to observe that $g = (V_0^g = V_0, V_1^g = \emptyset, V_2^g = V_1, V_3^g = V_2)$ is a $CTDRD$ function of G. Therefore,

$$\gamma_{ctdR}(G) \leq 2|V_1| + 3|V_2| \leq 2|V_1| + 4|V_2| = 2\gamma_{oitR}(G). \tag{1}$$

We now let $\gamma_{ctdR}(G) = 2\gamma_{oitR}(G)$. This equality along with the inequality chain (1) imply that $V_2 = \emptyset$, and since f is an $OITRD$ function of G, $V_0 = V_0^g = \emptyset$ as well. Therefore, all vertices of G are assigned 2 under g. Now assigning value 1 to one vertex of G and value 2 to the remaining vertices introduces a CTDRD function of weight $2\gamma_{oitR}(G) - 1$ which is a contradiction. Thus $\gamma_{ctdR}(G) < 2\gamma_{oitR}(G) - 1$. □

As an immediate consequence of the equation (1), we have the following result.

Corollary 1. *If G is a connected graph and $f = (V_0, V_1, V_2)$ is a $\gamma_{oitR}(G)$-function, then $\gamma_{ctdR}(G) \leq 2\gamma_{oitR}(G) - |V_2|$.*

For the equality in the upper bound given in Corollary 1, consider the double stars.

Lemma 1. *For every graph G, $\gamma_{oitR}(G) < \gamma_{ctdR}(G)$.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be any $\gamma_{ctdR}(G)$-function. If $V_3 \neq \emptyset$, then $g = (V_0^g = V_0, V_1^g = V_1, V_2^g = V_2 \cup V_3)$ is an $OITRD$ function on G, therefore $\gamma_{oitR}(G) \leq \omega(g) < \omega(f) = \gamma_{ctdR}(G)$ that is, $\gamma_{oitR}(G) < \gamma_{ctdR}(G)$. Hence, assume that $V_3 = \emptyset$. Since $V_2 \cup V_3$ dominates G, it follows that $V_2 \neq \emptyset$. Thus, all vertices are assigned the values $0, 1$ or 2 under f and all vertices in V_0 must have at least two neighbors in V_2 and all vertices in V_1 must have at least one neighbor in V_2. In such case, at least one vertex in V_2 can be reassigned the value 1 under g and the resulting function g will be an $OITRD$ function of G, as well. Therefore, $\gamma_{oitR}(G) < \gamma_{ctdR}(G)$. If $V_2 = V$, then $g = (\emptyset, V_1, \emptyset)$ is an $OITRD$. It is clear $\gamma_{oitR}(G) < \gamma_{ctdR}(G)$. All in all $\gamma_{oitR}(G) < \gamma_{ctdR}(G)$. □

Corollary 2. *For any nontrivial connected graph G, $\gamma_{oitR}(G) < \gamma_{ctdR}(G) < 2\gamma_{oitR}(G)$.*

The following observation has routine proof and so its proof is left.

Observation 4. For every graph G, $\gamma_{oidR}(G) \leq \gamma_{ctdR}(G)$.

According to Observation 4, any lower bound for outer-independent double Roman domination number is a lower bound for covering total double Roman domination number.
The following theorem is from [16].

Theorem 5. ([16] Theorem 4) *For any connected graph G of order $n \geq 2$ with maximum degree Δ,*

$$\max\{\gamma(G), \frac{2}{\Delta}\alpha(G)\} + \beta(G) \leq \gamma_{oidR}(G) \leq 3\beta(G).$$

These bounds are sharp.

Similar to the Theorem 5 we have the result.

Corollary 3. *For any connected graph G of order $n \geq 2$ with maximum degree Δ,*

$$\max\{\gamma(G), \frac{2}{\Delta}\alpha(G)\} + \beta(G) \leq \gamma_{ctdR}(G) \leq 4\beta(G).$$

Proof. Let I be an $\alpha(G)$-set and C be a $\beta(G)$-set. Then, assigning 3 to the vertices of C, 1 to the only one vertex in $N(x)$ if $x \in C$ and $N(x) \cap C = \emptyset$ and 0 to the remaining vertices, introduces a CTDRD function f on G with $\omega(f) \leq 4(n - \alpha(G))$. Since $\alpha(G) + \beta(G) = n$, it follows that $\gamma_{ctdR}(G) \leq \omega(f) \leq 4\beta(G)$. This upper bound is sharp for stars.

On the other hand, by Observation 4 and Theorem 5, we obtain

$$\max\{\gamma(G), \frac{2}{\Delta}\alpha(G)\} + \beta(G) \leq \gamma_{ctdR}(G).$$

Thus the proof is complete. □

The following theorem is from [16].

Theorem 6. ([16] Theorem 5) *For any tree T, $2\beta(T) + 1 \leq \gamma_{oidR}(T) \leq 3\beta(T)$.*

As an immediate result from Corollary 3 and Theorem 6 we have.

Corollary 4. *For any tree T, $2\beta(T) + 1 \leq \gamma_{ctdR}(T) \leq 4\beta(T)$.*

Let G and H be graphs where $V(G) = \{v_1, \dots, v_n\}$. The corona $G \odot H$ of graphs G and H is obtained from the disjoint union of G and n disjoint copies of H, say $\{H_1, \dots, H_n\}$, such that for all $i \in \{1, \dots, n\}$, the vertex $v_i \in V(G)$ is adjacent to every vertex of H_i. We next present an exact formula for $\gamma_{ctdR}(G \odot H)$ when $\Delta(H) \leq |n(H)| - 2$.
In [16] it has been shown that:

Theorem 7. ([16] Theorem 7) *Let G be a graph of order n, and H be a graph such that $\Delta(H) \leq |n(H)| - 2$ and $\delta(H) \geq 1$. Then $\gamma_{oidR}(G \odot H)$ equals $min\{|V_0|(n(H) + \gamma(H)) + |V_1|(\gamma_{oidR}(H) + 1) + |V_2|(\gamma_{oiR}(H) + 2) + |V_3|(\beta(H) + 3)\}$, taken over all possible function $f_G = (V_0, V_1, V_2, V_3)$ over $V(G)$ for which the vertices labeled with 0 form an independent set.*

As a remark on the proof of Theorem 7 in [16], it is easy to see that the set of vertices with positive weight has no isolated vertex. Therefore we have.

Theorem 8. *Let G be a graph of order n, and H be a graph such that $\Delta(H) \leq |n(H)| - 2$ and $\delta(H) \geq 1$. Then $\gamma_{ctdR}(G \odot H)$ equals $min\{|V_0|(n(H) + \gamma(H)) + |V_1|(\gamma_{oidR}(H) + 1) + |V_2|(\gamma_{oiR}(H) + 2) + |V_3|(\beta(H) + 3)\}$, taken over all possible function $f_G = (V_0, V_1, V_2, V_3)$ over $V(G)$ for which the vertices labeled with 0 form an independent set.*

5. Conclusion and problems

In this paper, we studied the covering total double Roman domination of graphs. We characterize the graphs G with small $\gamma_{ctdR}(G)$. The complexity of $CTDRD$ of planar graph with maximum degree four was investigated. We also compared the covering total double Roman domination number to other parameter such as $OITR$, covering

and independence number of graphs. So for further, it is natural to pose the following open problems.
Problem 1. Characterise the graphs G with large $\gamma_{ctdR}(G)$.
Problem 2. For any graph G, obtain the lower and upper bound for $\gamma_{ctdR}(G)$.

Acknowledgements

The authors are grateful to the anonymous referees for helpful comments and advices.

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